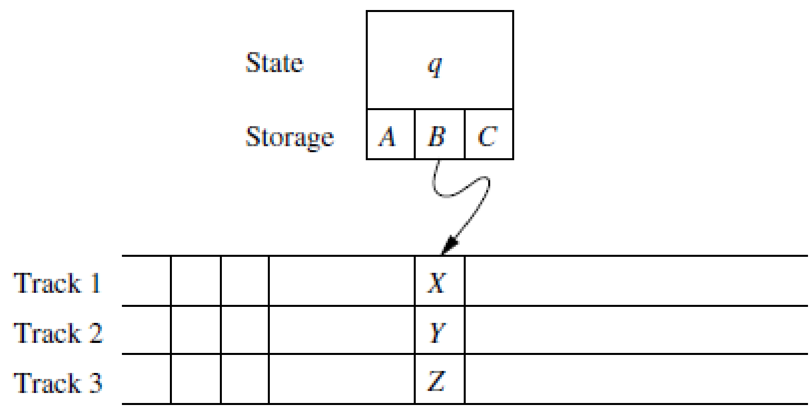
Variants of TM

Multi-track, multi-tape, NTM

https://www.andrew.cmu.edu/user/ko/pdfs/lecture-14.pdf

State storage & Multi-track



- In this example, the tape symbol is the triplet (X, Y, Z) and we can see the tape as a single-track one.

Example We shall design a TM

$$M = (Q, \{0, 1\}, \{0, 1, B\}, \delta, [q_0, B], B, \{[q_1, B]\})$$

that remembers in its finite control the first symbol (0 or 1) that it sees, and checks that it does not appear elsewhere on its input. Thus, M accepts the language $\mathbf{01}^* + \mathbf{10}^*$.

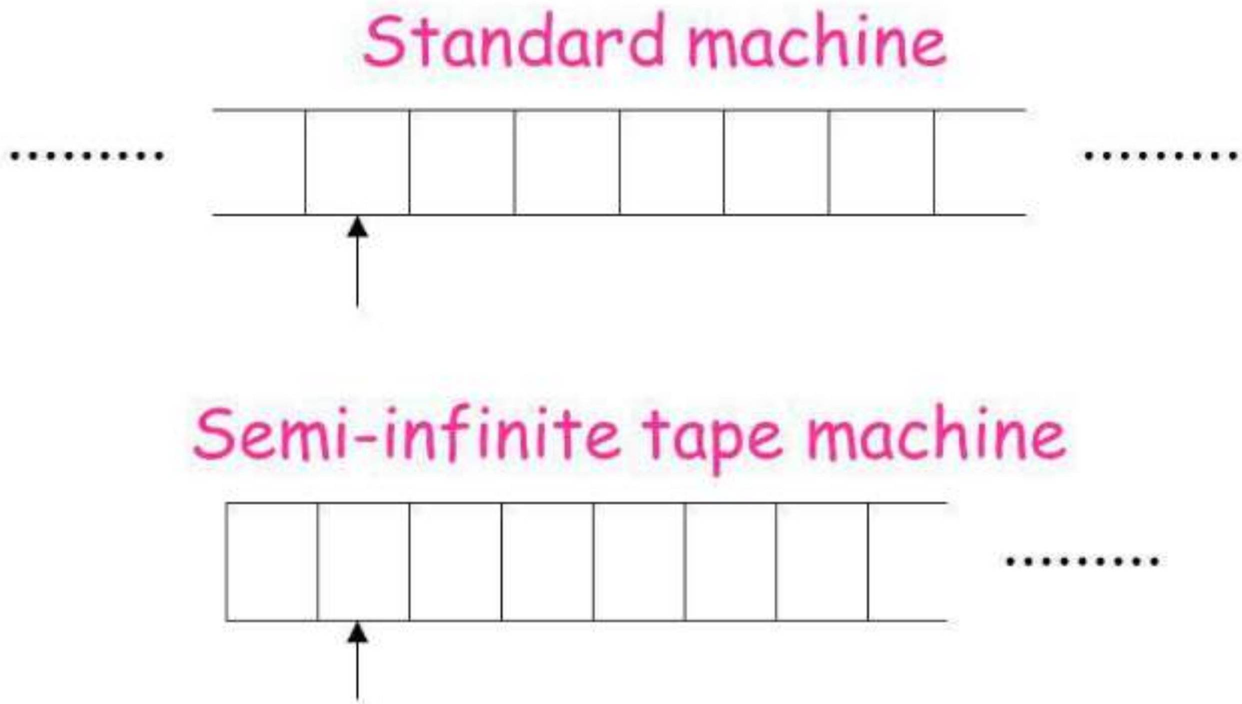
The transition function δ of M is as follows:

1. $\delta([q_0, B], a) = ([q_1, a], a, R)$ for $a = 0$ or $a = 1$. Initially, q_0 is the control state, and the data portion of the state is B. The symbol scanned is copied into the second component of the state, and M moves right, entering control state q_1 as it does so.

2. $\delta([q_1, a], \overline{a}) = ([q_1, a], \overline{a}, R)$ where $\overline{a}$ is the "complement" of a, that is, 0 if $a = 1$ and 1 if $a = 0$. In state q_1, M skips over each symbol 0 or 1 that is different from the one it has stored in its state, and continues moving right.

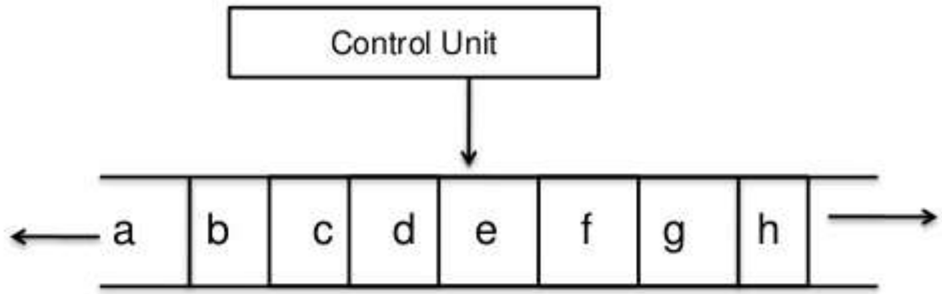
3. $\delta([q_1, a], B) = ([q_1, B], B, R)$ for $a = 0$ or $a = 1$. If M reaches the first blank, it enters the accepting state $[q_1, B]$.

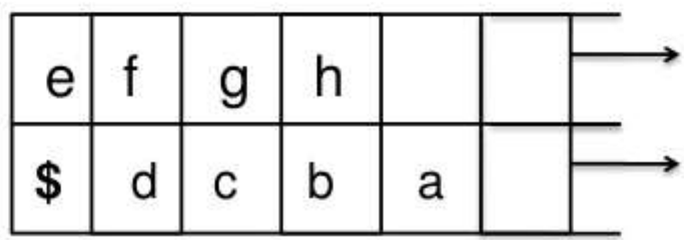
Semi-infinite tape



Simulation of two way infinite by semi-infinite tape

- Two way infinite tape simulated by semi -infinite tape





Multi-tape

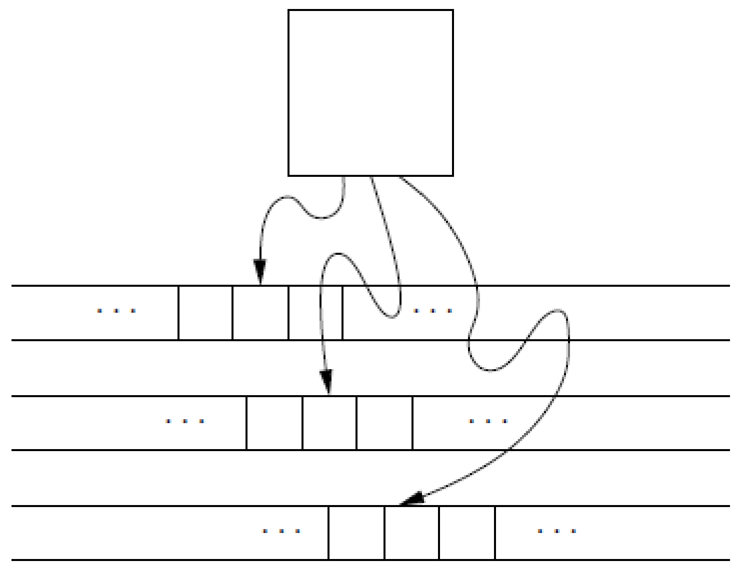


Figure 8.16: A multitape Turing machine

Simulation of multi-tape by one-tape

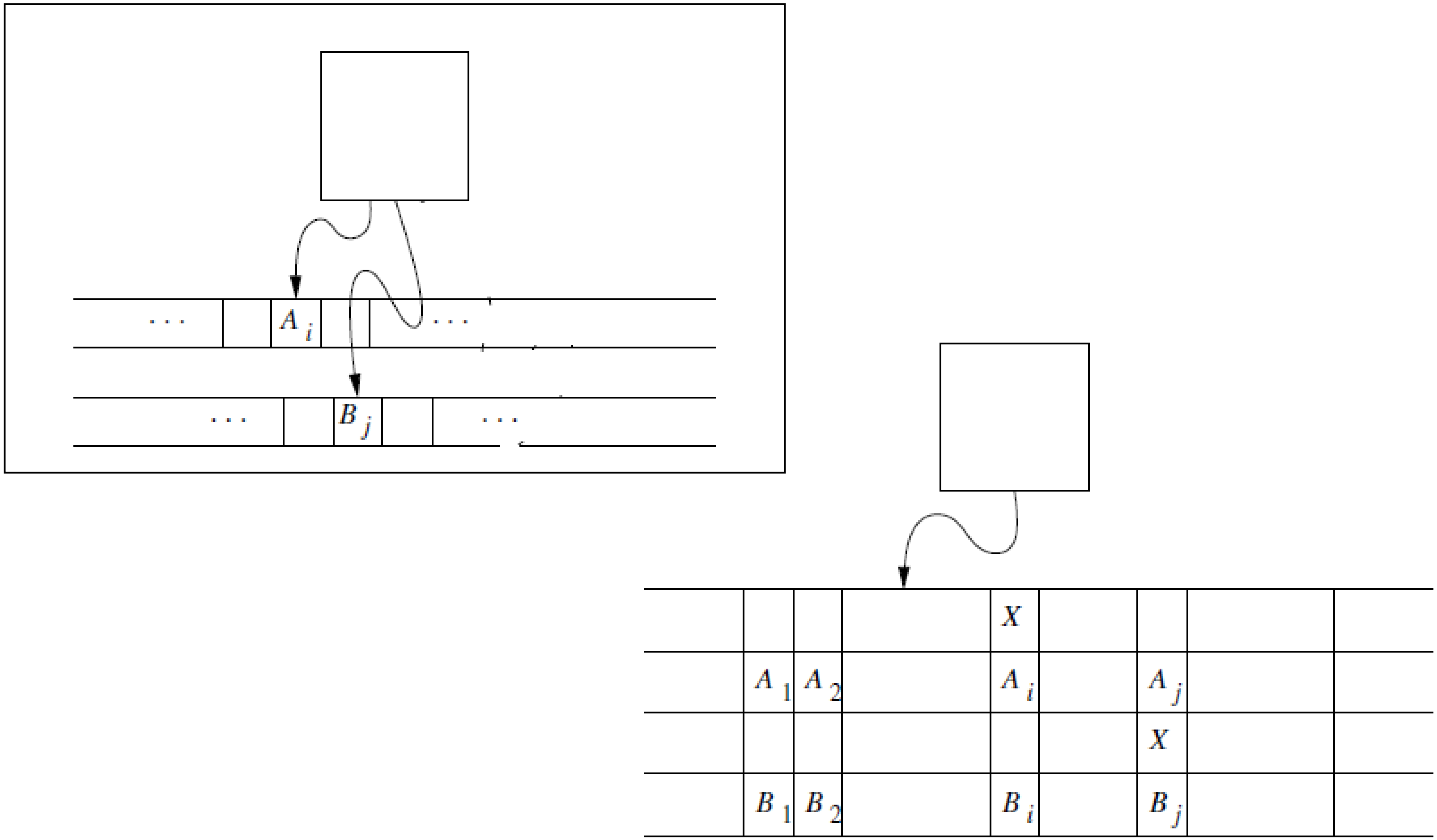


Figure 8.17: Simulation of a two-tape Turing machine by a one-tape Turing machine

NTM

NONDETERMINISTIC TM

- There is a choice in the next move.

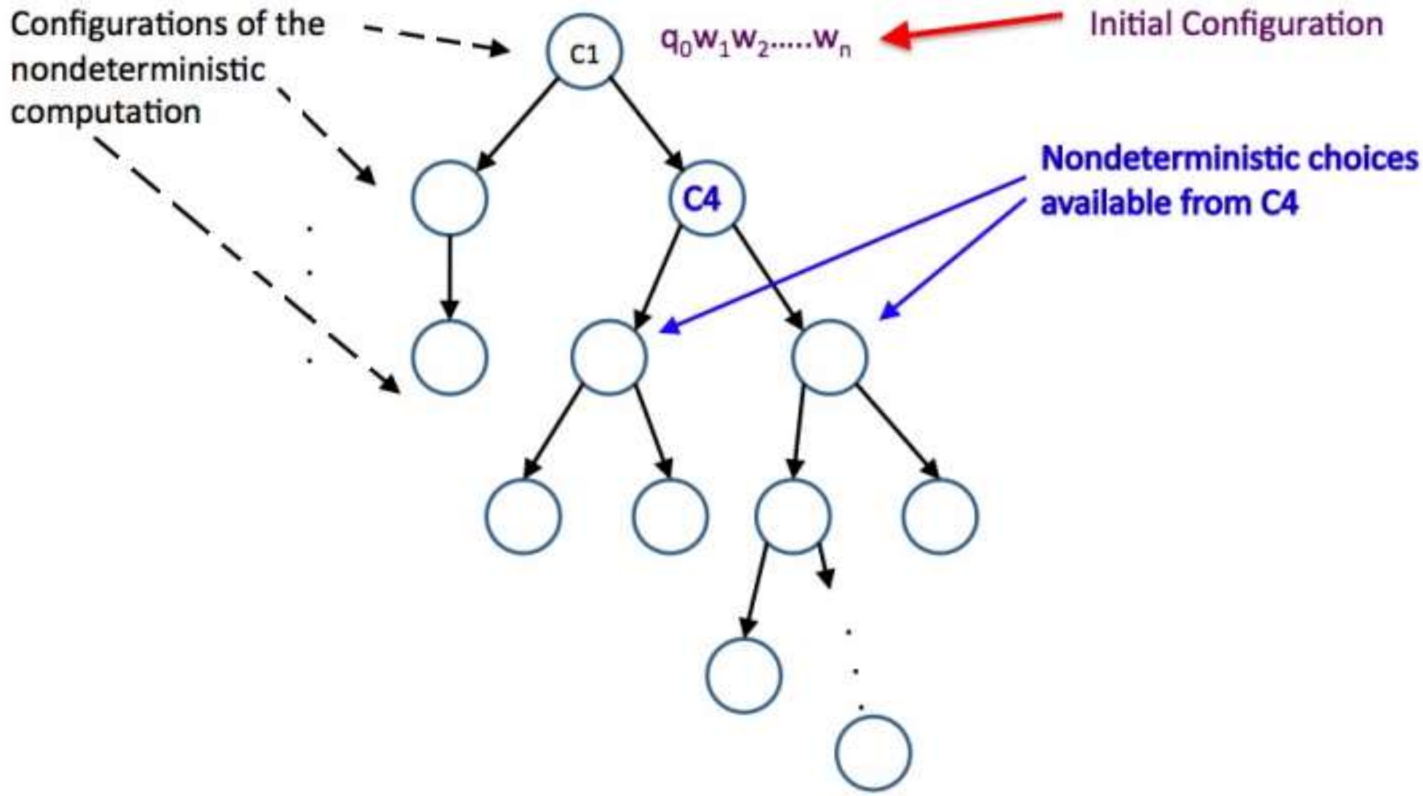
$$\delta(q, X) = \{(q_1, Y_1, D_1),\ (q_2, Y_2, D_2), \ldots, (q_k, Y_k, D_k)\}$$

- Here, Y_i is a tape symbol, and D_i is one from $\{L, R\}$, the direction of movement of the head.

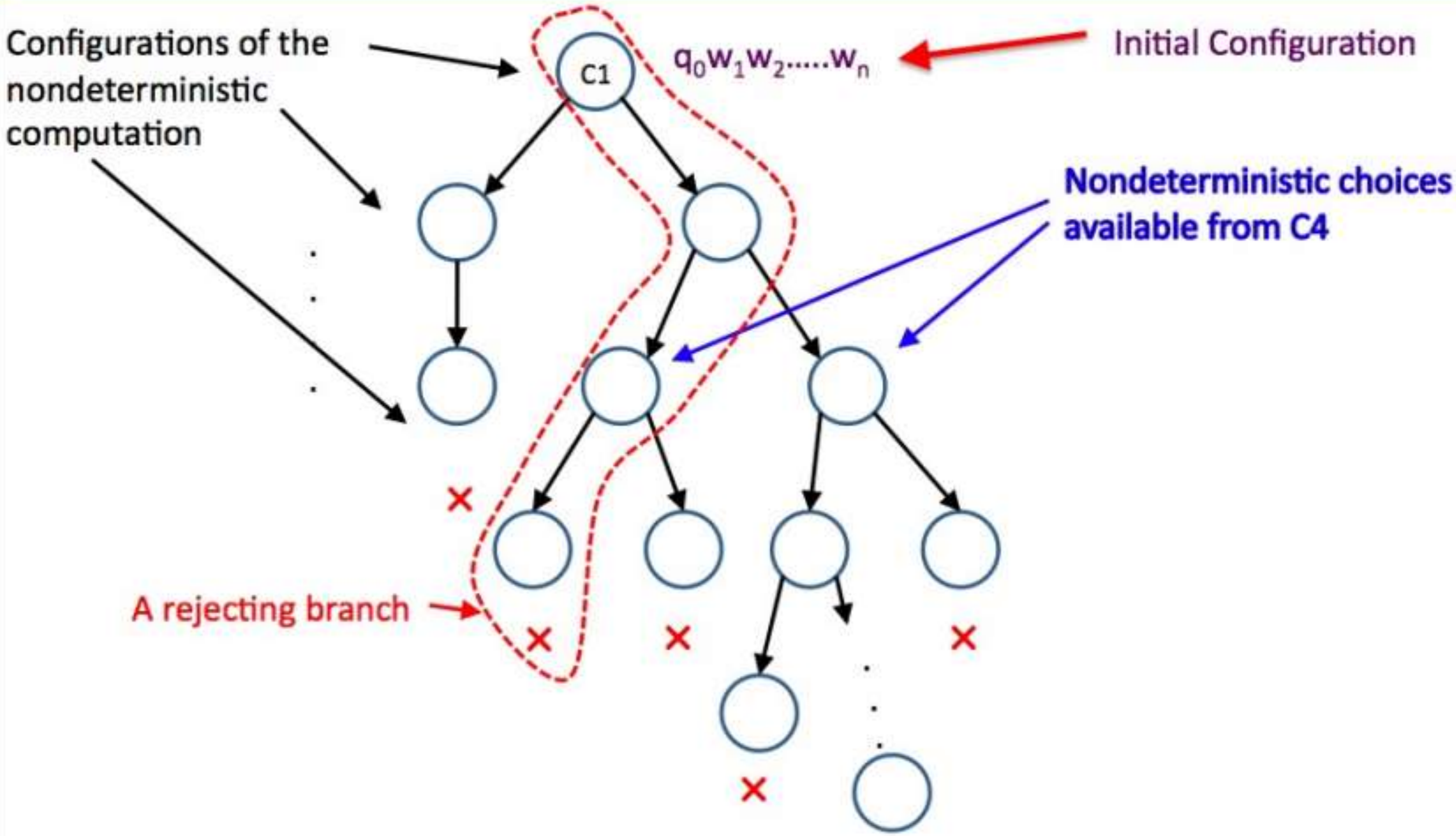
The transition function for a nondeterministic Turing machine has the form

$$\delta \colon Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}).$$

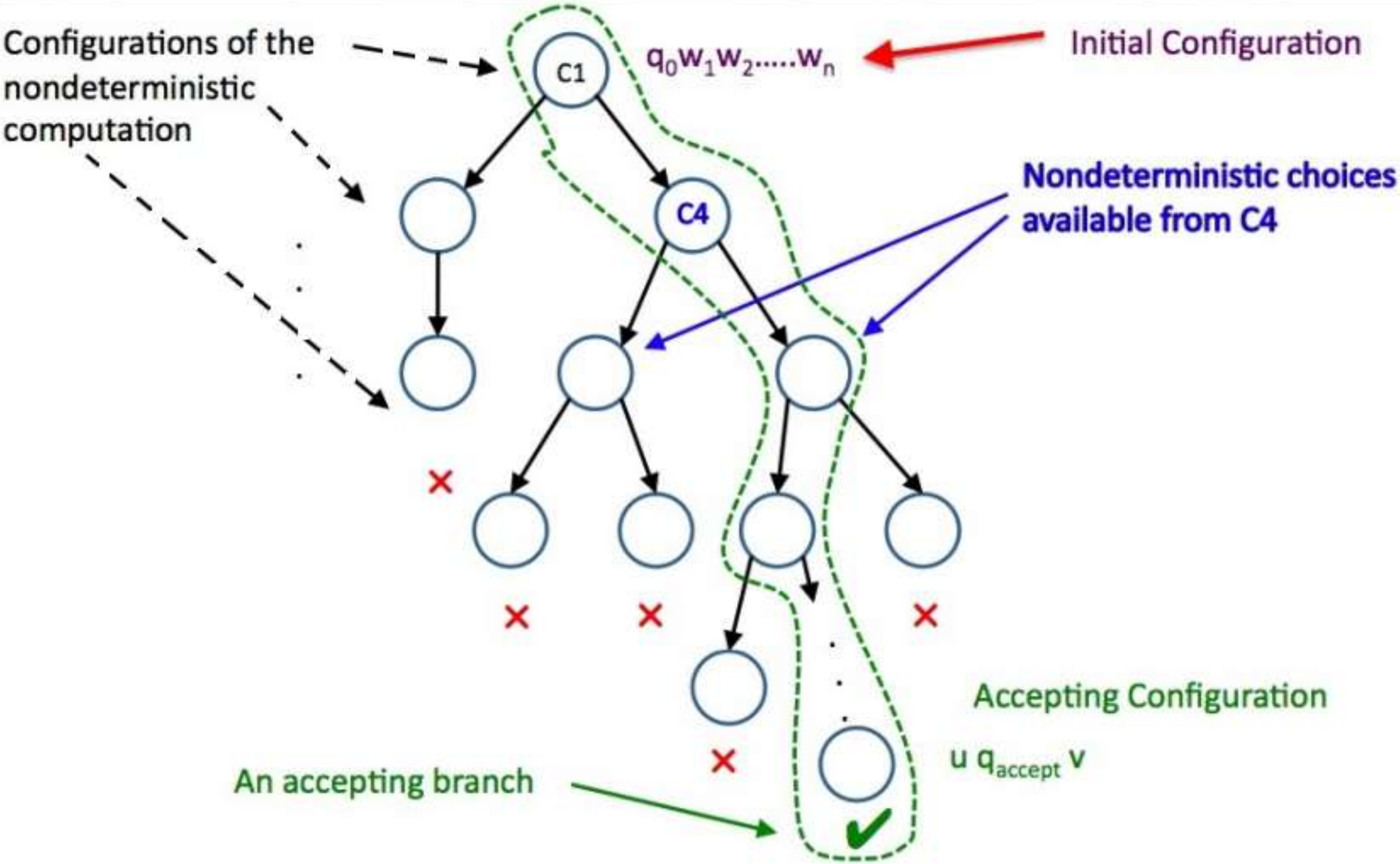
NONDETERMINISTIC COMPUTATION



NONDETERMINISTIC COMPUTATION

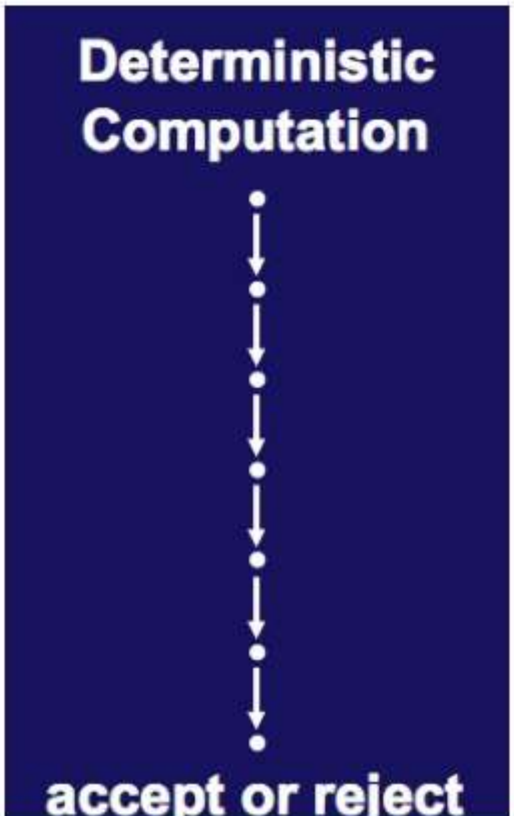


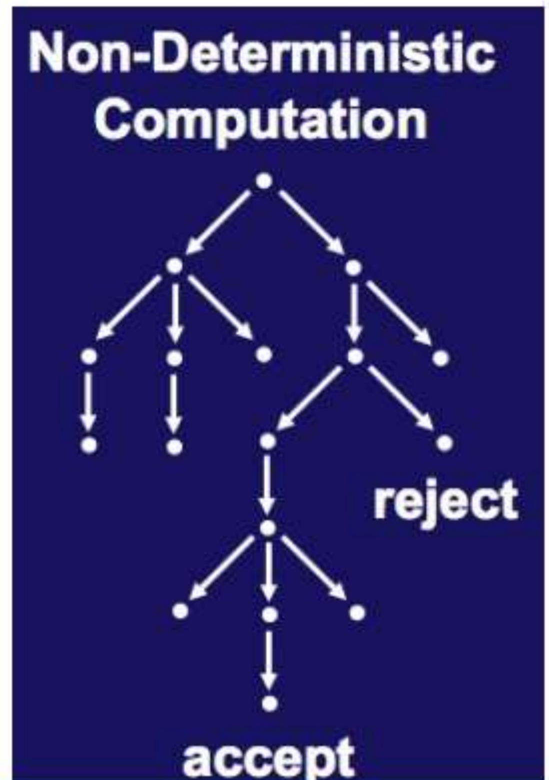
NONDETERMINISTIC COMPUTATION



NONDETERMINISTIC TURING MACHINES

- A computation of a Nondeterministic TM is a tree, where each branch of the tree is looks like a computation of an ordinary TM.





NONDETERMINISTIC TURING MACHINES

- If a single branch reaches the accepting state, the Nondeterministic TM accepts, even if other branches reach the rejecting state.
- What is the power of Nondeterministic TMs?
  - Is there a language that a Nondeterministic TM can accept but no deterministic TM can accept?

- Note, NTM rejects means
  – Each of the branch either explicitly rejects (by getting stuck in a non-final state), or goes in to an infinite loop.

Nondeterministic Turing Machines

Theorem

Every nondeterministic Turing machine has an equivalent deterministic Turing Machine.

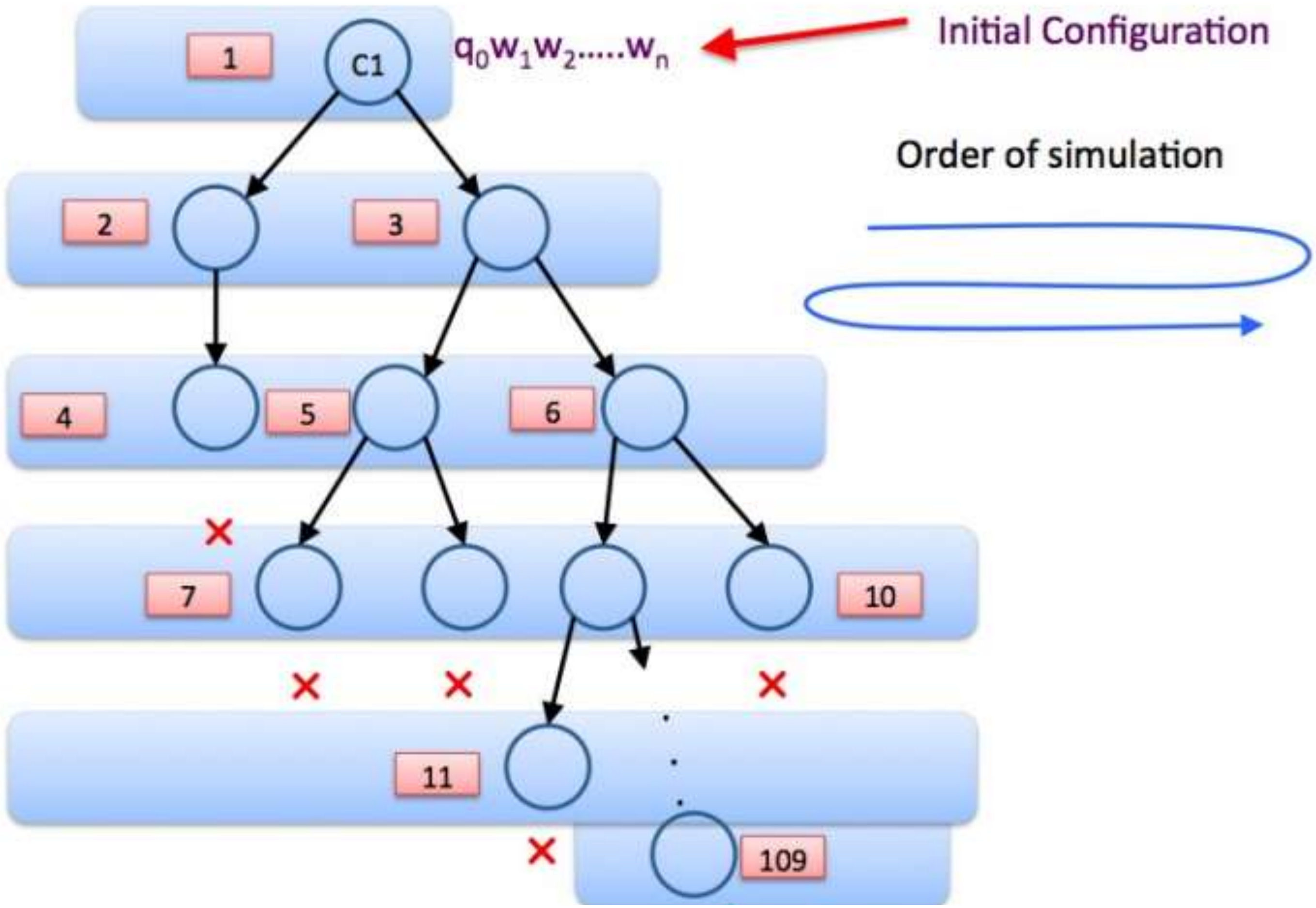
Proof Idea

- Timeshare a deterministic TM to different branches of the nondeterministic computation!
- Try out all branches of the nondeterministic computation until an accepting configuration is reached on one branch.
- Otherwise the TM goes on forever.

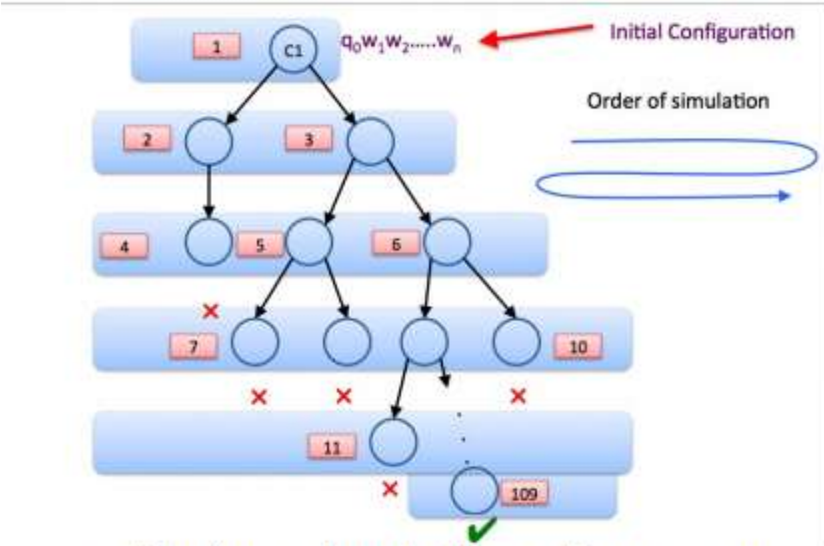
NONDETERMINISTIC TURING MACHINES

- Deterministic TM D simulates the Nondeterministic TM N.
- Some of branches of the N's computations may be infinite, hence its computation tree has some infinite branches.
- If D starts its simulation by following an infinite branch, D may loop forever even though N's computation may have a different branch on which it accepts.
- This is a very similar problem to processor scheduling in operating systems.
  - If you give the CPU to a (buggy) process in an infinite loop, other processes "starve".
- In order to avoid this unwanted situation, we want D to execute all of N's computations concurrently.

SIMULATING NONDETERMINISTIC COMPUTATION



SIMULATING NONDETERMINISTIC COMPUTATION

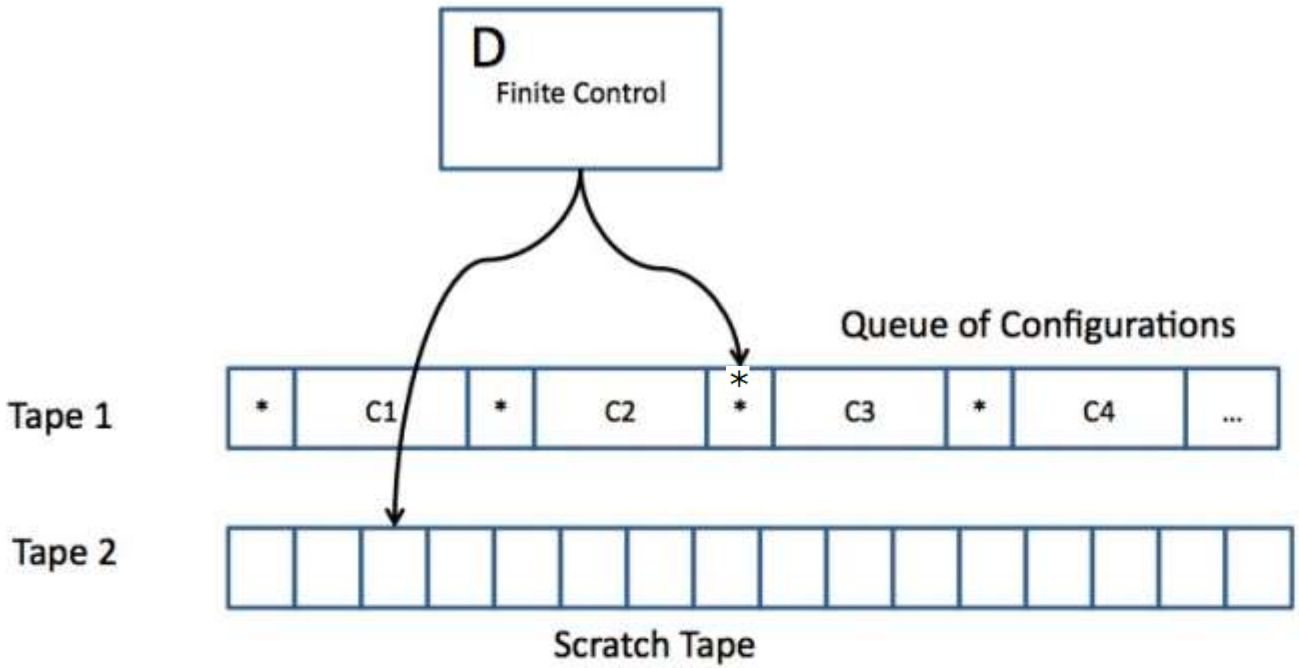


- During simulation, *D* processes the configurations of *N* in a breadth-first fashion.
- Thus *D* needs to maintain a queue of *N*'s configurations (Remember queues?)

- *D* gets the next configuration from the head of the queue.
- *D* creates copies of this configuration (as many as needed)
- On each copy, *D* simulates one of the nondeterministic moves of *N*.
- *D* places the resulting configurations to the back of the queue.

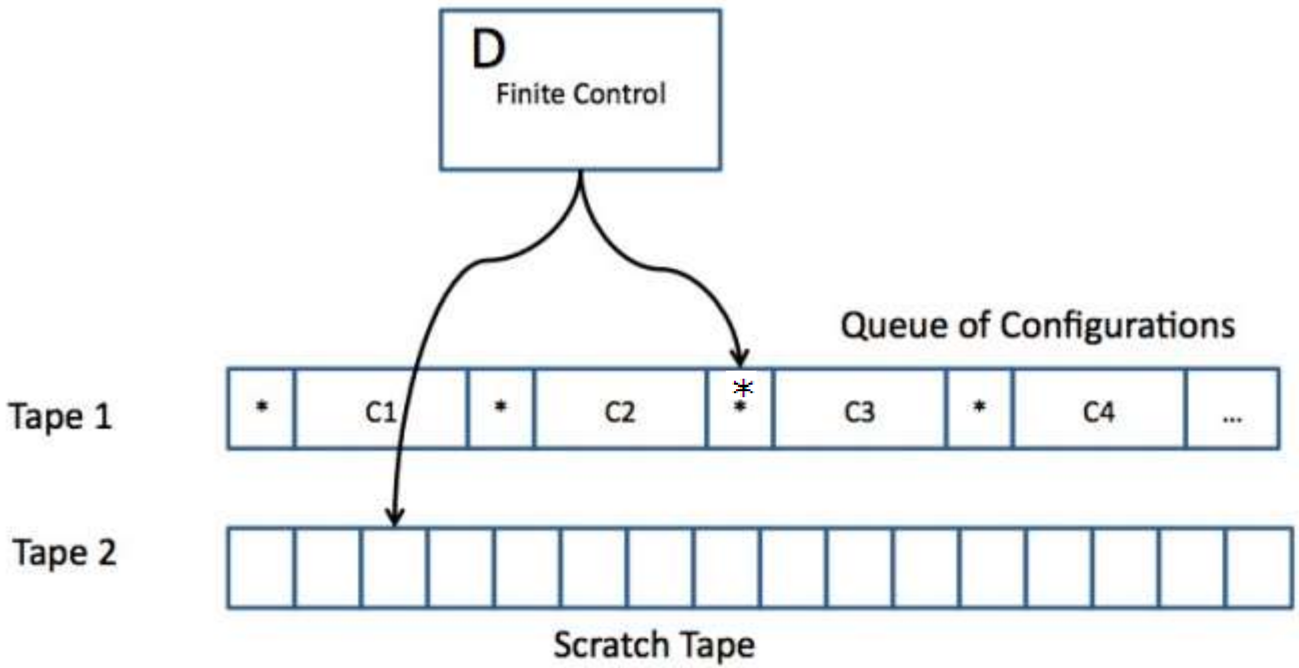
STRUCTURE OF THE SIMULATING DTM

- N is simulated with 2-tape DTM, D



STRUCTURE OF THE SIMULATING DTM

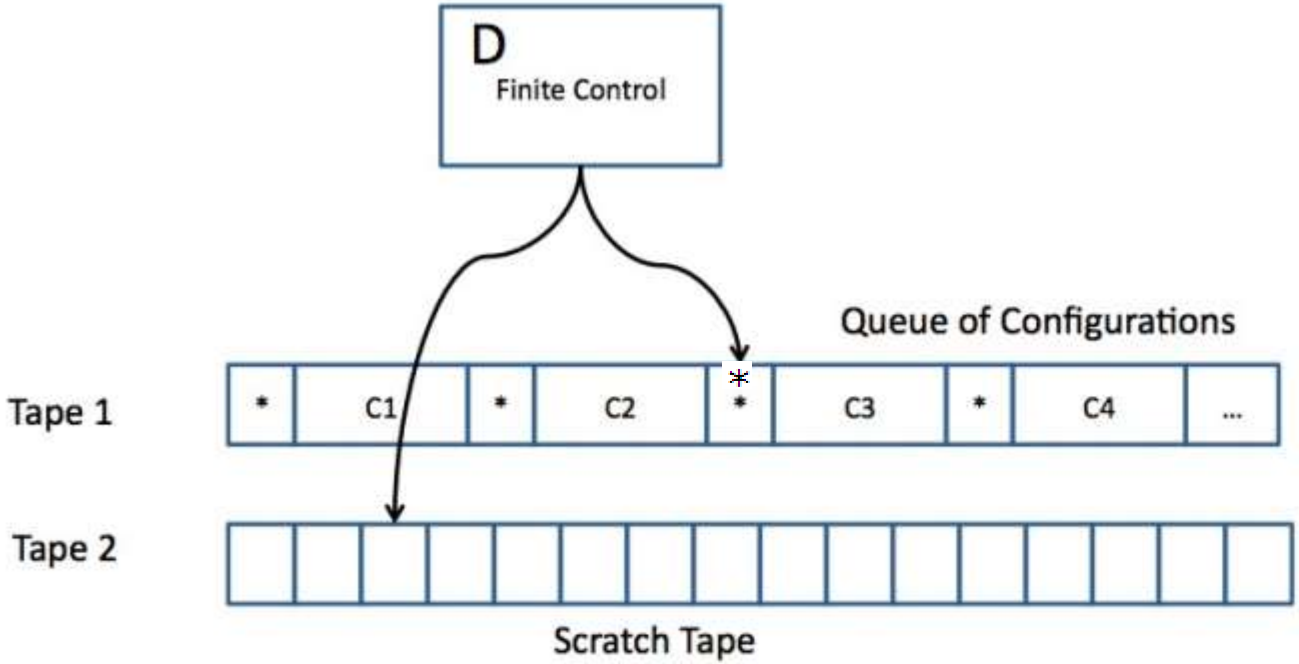
- N is simulated with 2-tape DTM, D



- Built into the finite control of D is the knowledge of what choices of moves N has for each state and input.

STRUCTURE OF THE SIMULATING DTM

- *N* is simulated with 2-tape DTM, *D*



1. *D* examines the state and the input symbol of the current configuration (right after the dotted separator)
2. If the state of the current configuration is the accept state of *N*, then *D* accepts the input and stops simulating *N*.

How D Simulates N

- Let m be the maximum number of choices N has for any of its states.
- Then, after n steps, N can reach at most $1 + m + m^2 + \cdots + m^n$ configurations (which is at most nm^n)
- Thus D has to process at most this many configurations to simulate n steps of N.
- Thus the simulation can take exponentially more time than the nondeterministic TM.
- It is not known whether or not this exponential slowdown is necessary.

IMPLICATIONS

COROLLARY

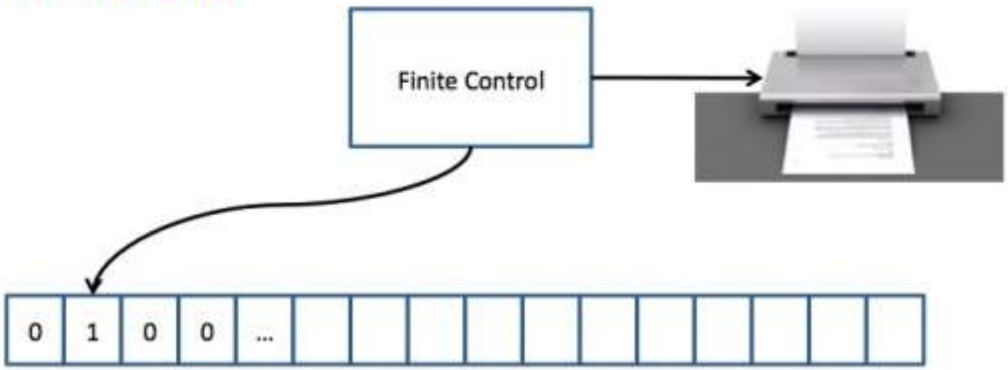
A language is Turing-recognizable if and only if some nondeterministic TM recognizes it.

COROLLARY

A language is decidable if and only of some nondeterministic TM decides it.

ENUMERATORS

- Remember we noted that some books used the term recursively enumerable for Turing-recognizable.
- This term arises from a variant of a TM called an enumerator.



- TM generates strings one by one.
- Everytime the TM wants to add a string to the list, it sends it to the printer.

Enumerators

- The enumerator E starts with a blank input tape.
- If it does not halt, it may print an infinite list of strings.
- The strings can be enumerated in any order; repetitions are possible.
- The language of the enumerator is the collection of strings it eventually prints out.

- We can assume that the enumerator E writes one string at a time over a tape (it can use a tape symbol # to separate strings).

ENUMERATORS

THEOREM

A language is Turing recognizable if and only if some enumerator enumerates it.

PROOF.

The If-part: If an enumerator E enumerates the language A then a TM M recognizes A.
$M =$ "On input w

1. Run E. Everytime E outputs a string, compare it with w.
2. If w ever appears in the output of E, *accept.*"

Clearly M accepts only those strings that appear on E's list.

$\square$

The TM M accepts w only when E produces w as one of its output strings.

ENUMERATORS

THEOREM

A language is Turing recognizable if and only if some enumerator enumerates it.

PROOF.

The Only-If-part: If a TM M recognizes a language A, we can construct the following enumerator for A.

- For each possible string $s \in \Sigma^*$ we can verify whether M accepts s, if so output s on to the tape .

Following attempt, does not work.

- Assume that there is an enumerator which enumerates Σ^* in a standard order.
- For each possible string $s \in \Sigma^*$ we can verify whether M accepts s, if so output s on to the tape .
- The problem with this is, for some s the TM M can enter in to an infinite loop and never returns.
  - There may be other strings which are accepted by M but will never gets a chance to be verified (and thus never is outputted).

- A feasible way of doing this (without falling in to an infinite loop) is given in the next slide.
- Basic idea:
- For example if there are two strings w1 and w2 and one of them makes the TM to loop infinitely. Do the following.

  1. **k = 1**
  2. **Run TM for k steps on w1 and if accept occurs then output "accept" and stop.**
  3. **Run TM for k steps on w2 and if accept occurs then output "accept" and stop.**
  4. **k++; goto step 2.**

ENUMERATORS

THEOREM

A language is Turing recognizable if and only if some enumerator enumerates it.

PROOF.

The Only-If-part: If a TM M recognizes a language A, we can construct the following enumerator for A. Assume $s_1, s_2, s_3, \ldots$ is a list of possible strings in Σ^*.

$E =$ "Ignore the input

1. Repeat the following for $i = 1, 2, 3, \ldots$
2. Run M for i steps on each input $s_1, s_2, s_3, \ldots s_i$.
3. If any computations accept, print out corresponding s_j."

If M accepts a particular string, it will appear on the list generated by E (in fact infinitely many times)

THE DEFINITION OF ALGORITHM - HISTORY

- in 1900, Hilbert posed the following problem:

  "Given a polynomial of several variables with integer coefficients, does it have an integer root – an assignment of integers to variables, that make the polynomial evaluate to 0"

- For example, $6x^3yz^2 + 3xy^2 - x^3 - 10$ has a root at $x = 5, y = 3, z = 0$.
- Hilbert explicitly asked that an algorithm/procedure to be "devised". He assumed it existed; somebody needed to find it!
- 70 years later it was shown that no algorithm exists.
- The intuitive notion of an algorithm may be adequate for giving algorithms for certain tasks, but was useless for showing no algorithm exists for a particular task.

This is known as Hilbert's Tenth Problem.

The Definition of Algorithm - History

- In early 20^{th} century, there was no formal definition of an algorithm.
- In 1936, Alonzo Church and Alan Turing came up with formalisms to define algorithms. These were shown to be equivalent, leading to the

Church-Turing Thesis

Intutitive notion of algorithms $\equiv$ Turing Machine Algorithms

THE DEFINITION OF AN ALGORITHM

- Let $D = \{p \mid p \text{ is a polynomial with integral roots}\}$
- Hilbert's 10^{th} problem in TM terminology is "Is D decidable?" (No!)
- However D is Turing-recognizable!
- Consider a simpler version
  $D_1 = \{p \mid p \text{ is a polynomial over } x \text{ with integral roots}\}$
- $M_1 =$ "The input is polynomial p over x.
  1. Evaluate p with x successively set to 0, 1, -1, 2, -2, 3, -3, ...
  2. If at any point, p evaluates to 0, *accept*."
- D_1 is actually decidable since only a finite number of x values need to be tested (math!)
- D is also recognizable: just try systematically all integer combinations for all variables.

- For D_1 (polynomial of single variable) there is a bound and we can abandon the search beyond that and declare "No".
- For D such a bound cannot exist (proof is given by Matijasevich (1971)).
  - So, if answer is "No" we enter in to an infinite loop.

In 1971, Yuri Matijasevich gave a resounding negative answer to Hilbert's tenth problem.

- This is called undecidability.
- Hilbert’s tenth problem is undecidable.

- We will see the theory behind this in the next …