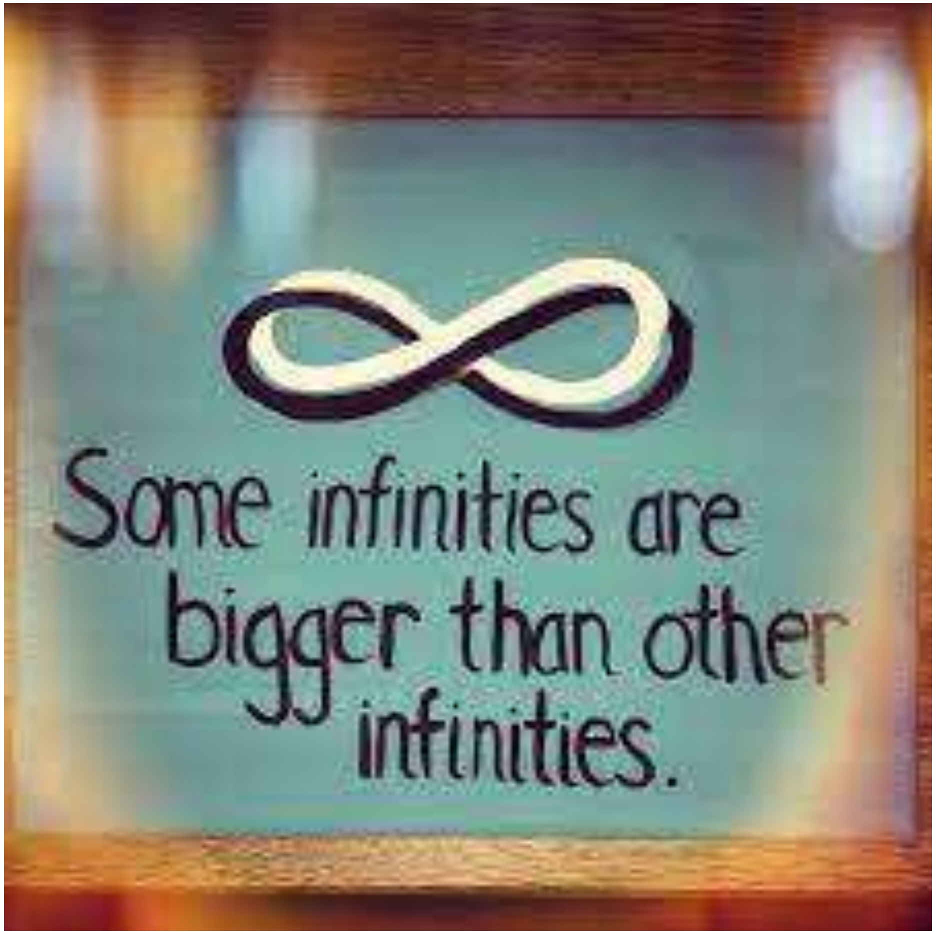
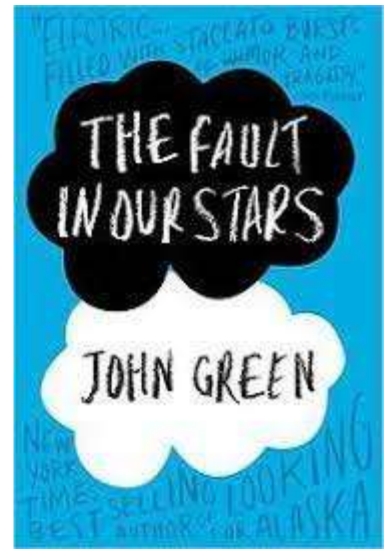
ACCEPTANCE PROBLEM FOR TMS

- The TM U recognizes A_{TM}
- $U =$ "On input $\langle M, w \rangle$ where M is a TM and w is a string:
  1. Simulate M on w
  2. If M ever enters its accepts state, *accept*; if M ever enters its reject state, *reject*.
- Note that if M loops on w, then U loops on $\langle M, w \rangle$, which is why it is NOT a decider!
- U can not detect that M halts on w.
- A_{TM} is also known as the Halting Problem
- U is known as the Universal Turing Machine because it can simulate every TM (including itself!)

- The proof of the undecidability of ATM uses a technique called *diagonalization* discovered by mathematician Georg Cantor in 1873.
- Cantor was concerned with the problem of measuring the sizes of infinite sets.
- If we have two infinite sets, how can we tell whether one is larger than the other or whether they are of the same size?


Some infinities are
bigger than other
infinities.


THE FAULT
IN OUR STARS
JOHN GREEN

The Diagonalization Method

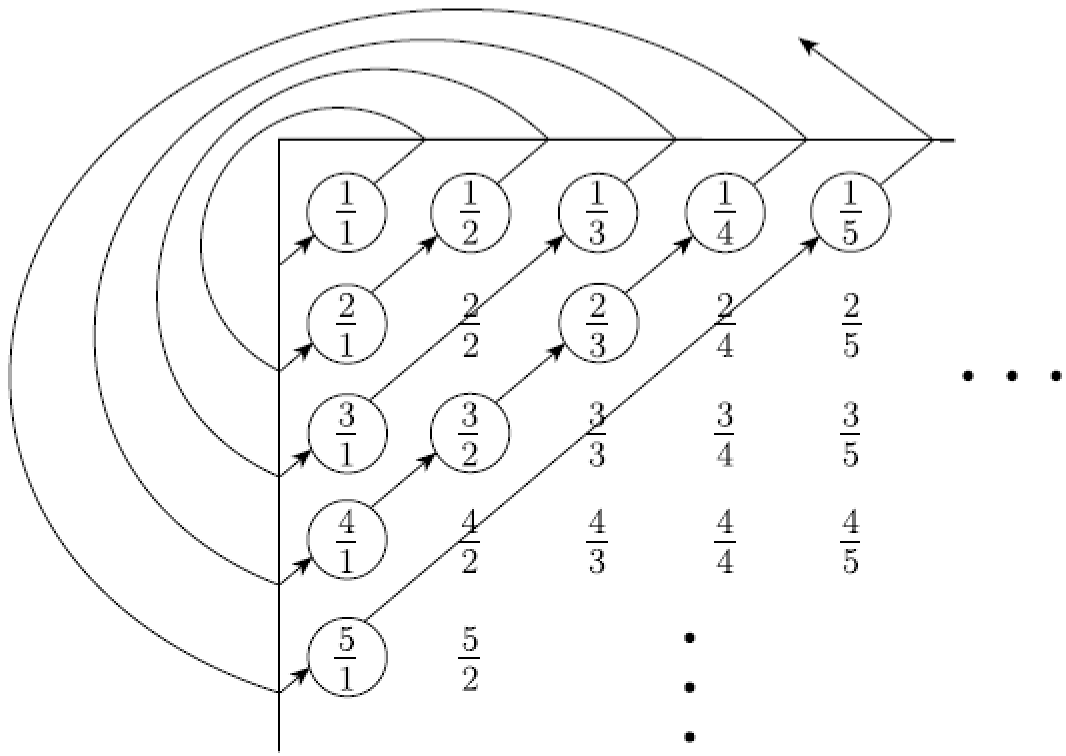
Some basic definitions

- Let A and B be any two sets (not necessarily finite) and f be a function from A to B.
- f is one-to-one if $f(a) \neq f(b)$ whenever $a \neq b$.
- f is onto if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.
- We say A and B are the same size if there is a one-to-one and onto function $f : A \longrightarrow B$.
- Such a function is called a correspondence for pairing A and B.
  - Every element of A maps to a unique element of B
  - Each element of B has a unique element of A mapping to it.

The Diagonalization Method

- Let $\mathcal{N}$ be the set of natural numbers $\{1, 2, \dots\}$ and let $\mathcal{E}$ be the set of even numbers $\{2, 4, \dots\}$.
- $f(n) = 2n$ is a correspondence between $\mathcal{N}$ and $\mathcal{E}$.
- Hence, $\mathcal{N}$ and $\mathcal{E}$ have the same size (though $\mathcal{E} \subset \mathcal{N}$).
- A set A is countable if it is either finite or has the same size as $\mathcal{N}$.
- $\mathcal{Q} = \{\frac{m}{n} \mid m, n \in \mathcal{N}\}$ is countable!
- Z the set of integers is countable:

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n+1}{2} & n \text{ odd} \end{cases}$$



1/1 1/2 1/3 1/4 1/5
2/1 2/2 2/3 2/4 2/5
3/1 3/2 3/3 3/4 3/5
4/1 4/2 4/3 4/4 4/5
5/1 5/2

THE DIAGONALIZATION METHOD

THEOREM

$\mathcal{R}$ is uncountable

PROOF.

- Assume f exists and every number in $\mathcal{R}$ is listed.
- Assume $x \in \mathcal{R}$ is a real number such that x differs from the j^{th} number in the j^{th} decimal digit.
- If x is listed at some position k, then it differs from itself at k^{th} position; otherwise the premise does not hold
- f does not exist

n	$f(n)$
1	3.14159...
2	55.77777...
3	0.12345...
4	0.50000...
$\vdots$	$\vdots$

$x = .4527\ldots$ defined as such, can not be on this list.

$\square$

- Pi: “ it keeps on going, forever, without ever repeating. Which means that contained within this string of decimals, is every single other number. Your birthdate, combination to your locker, your social security number, it's all in there, somewhere. And if you convert these decimals into letters, you would have every word that ever existed in every possible combination; the first syllable you spoke as a baby, the name of your latest crush, your entire life story from beginning to end, everything we ever say or do; all of the world's infinite possibilities rest within this one simple circle.”

Diagonalization over Languages

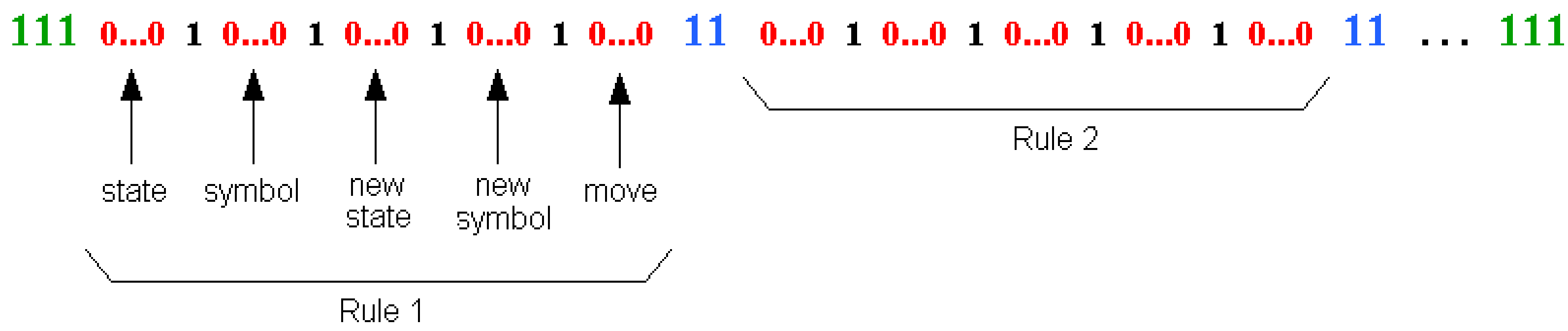
Corollary

Some languages are not Turing-recognizable.

Proof

- For any alphabet Σ, Σ^* is countable. Order strings in Σ^* by length and then alphanumerically, so $\Sigma^* = \{s_1, s_2, \ldots, s_i, \ldots\}$
- The set of all TMs is a countable language.
  - Each TM M corresponds to a string $\langle M \rangle$.
  - Generate a list of strings and remove any strings that do not represent a TM to get a list of TMs.

Encoding of a TM



Diagonalization over Languages

Proof (continued)

- The set of infinite binary sequences, $\mathcal{B}$, is uncountable. (Exactly the same proof we gave for uncountability of $\mathcal{R}$)
- Let $\mathcal{L}$ be the set of all languages over Σ.
- For each language $A \in \mathcal{L}$ there is unique infinite binary sequence $\mathcal{X}_A$
  - The i^{th} bit in $\mathcal{X}_A$ is 1 if $s_i \in A$, 0 otherwise.

$\Sigma^*=\{$	$\epsilon,$	0,	1,	00,	01,	10,	11,	000,	001,	$\cdots$	$\}$
$A=\{$		0,		00,	01,			000,	001,	$\cdots$	$\}$
$\mathcal{X}_A=\{$	0	1	0	1	1	0	0	1	1	$\cdots$	$\}$

Diagonalization over Languages

Proof (continued)

- The function $f : \mathcal{L} \longrightarrow \mathcal{B}$ is a correspondence. Thus $\mathcal{L}$ is uncountable.
- So, there are languages that can not be recognized by some TM. There are not enough TMs to go around.

A_{TM} is undecidable

THEOREM

$A_{TM} = \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$, is undecidable.

PROOF

- We assume A_{TM} is decidable and obtain a contradiction.
- Suppose H decides A_{TM}

$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

A_{TM} is undecidable

PROOF (CONTINUED)

- We now construct a new TM D
  $D =$ "On input $\langle M \rangle$, where M is a TM
  1. Run H on input $\langle M, \langle M \rangle \rangle$.
  2. If H accepts, *reject*, if H rejects, *accept*"
- So

$$D(\langle M \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M \rangle \end{cases}$$

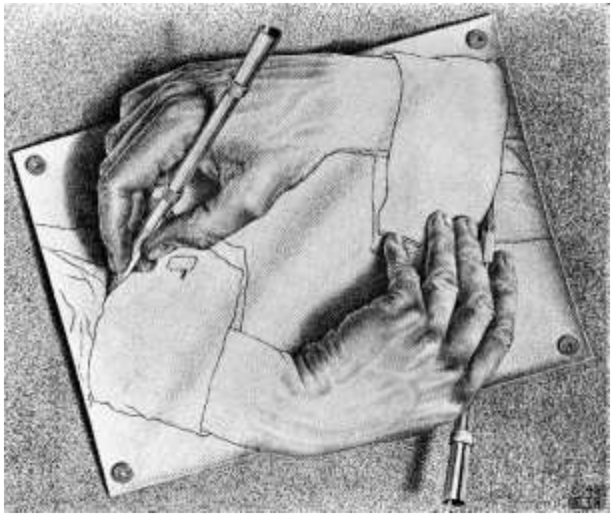
- When D runs on itself we get

$$D(\langle D \rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}$$

- Neither D nor H can exist.

Of Paradoxes & Strange Loops

E.g., Barber's paradox, Achilles & the Tortoise (Zeno's paradox)
MC Escher's paintings



A fun book for further reading:
"Godel, Escher, Bach: An Eternal Golden Braid"
by Douglas Hofstadter (Pulitzer winner, 1980)

What Happened to Diagonalization?

Consider the behaviour of all possible deciders:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	$\cdots$
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	$\cdots$
M_4	*accept*	*accept*	*reject*	*reject*	$\cdots$
$\vdots$		$\vdots$			$\ddots$

What Happened to Diagonalization?

Consider the behaviour of all possible deciders:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	$\cdots$	$\cdots$
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	$\cdots$	$\cdots$
M_4	*accept*	*accept*	*reject*	*reject*	$\cdots$	$\cdots$
$\vdots$		$\vdots$			$\ddots$	
						$\cdots$
$\vdots$		$\vdots$				$\ddots$

- D computes the opposite of the diagonal entries!

What Happened to Diagonalization?

Consider the behaviour of all possible deciders:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$ $\langle M_j \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	$\cdots$	*accept*	$\cdots$
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	$\cdots$	*reject*	$\cdots$
M_4	*accept*	*accept*	*reject*	*reject*	$\cdots$	*accept*	$\cdots$
$\vdots$		$\vdots$			$\ddots$		
$D = M_j$	*reject*	*reject*	*accept*	*accept*	$\cdots$	?	$\cdots$
$\vdots$		$\vdots$					$\ddots$

- D computes the opposite of the diagonal entries!

What Happened to Diagonalization?

Consider the behaviour of all possible deciders:

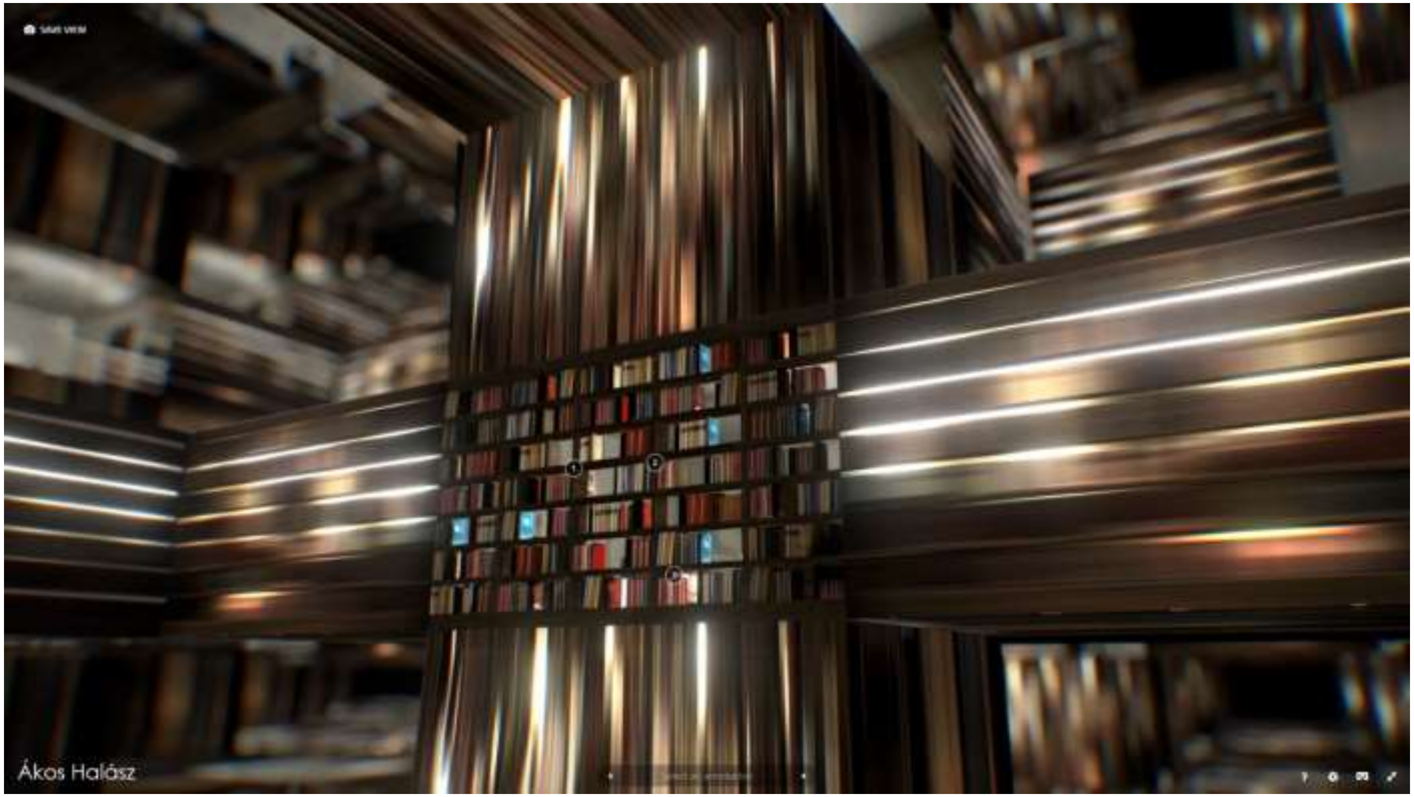
	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$ $\langle M_j \rangle$	$\cdots$
M_1	<u>*accept*</u>	*reject*	*accept*	*reject*	$\cdots$	*accept*	$\cdots$
M_2	*accept*	<u>*accept*</u>	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	<u>*reject*</u>	*reject*	$\cdots$	*reject*	$\cdots$
M_4	*accept*	*accept*	*reject*	<u>*reject*</u>	$\cdots$	*accept*	$\cdots$
$\vdots$		$\vdots$			$\ddots$		
$D = M_j$	*reject*	*reject*	*accept*	*accept*	$\cdots$	<u>?</u>	$\cdots$
$\vdots$		$\vdots$					$\ddots$

- D computes the opposite of the diagonal entries!

D cannot exist; After all D used only H; So if H exists then D exists; Hence H (decider for A_{TM}) cannot exist.

- Is Computer Technology stagnant?


AMY ADAMS
JEREMY RENNER
FOREST WHITAKER
WHY ARE THEY HERE?
FROM THE DIRECTOR OF SICARIO AND PRISONERS
ARRIVAL
IN THEATRES 11.11


Ákos Halász


MATTHEW McCONAUGHEY
ANNE HATHAWAY
JESSICA CHASTAIN
AND MICHAEL CAINE
GO FURTHER.
FROM THE DIRECTOR OF THE DARK KNIGHT TRILOGY AND INCEPTION
INTERSTELLAR
IN THEATRES AND IMAX
EVERYWHERE
NOVEMBER 7

Ludwig Wittgenstein

the limits of my language are the limits of my world

- *There are things in existence that are beyond our human ability to imagine or conceive*

https://oregonstate.edu/instruct/phl201/modules/Philosophers/Wittgenstein/wittgenstein.html

Sapir–Whorf hypothesis

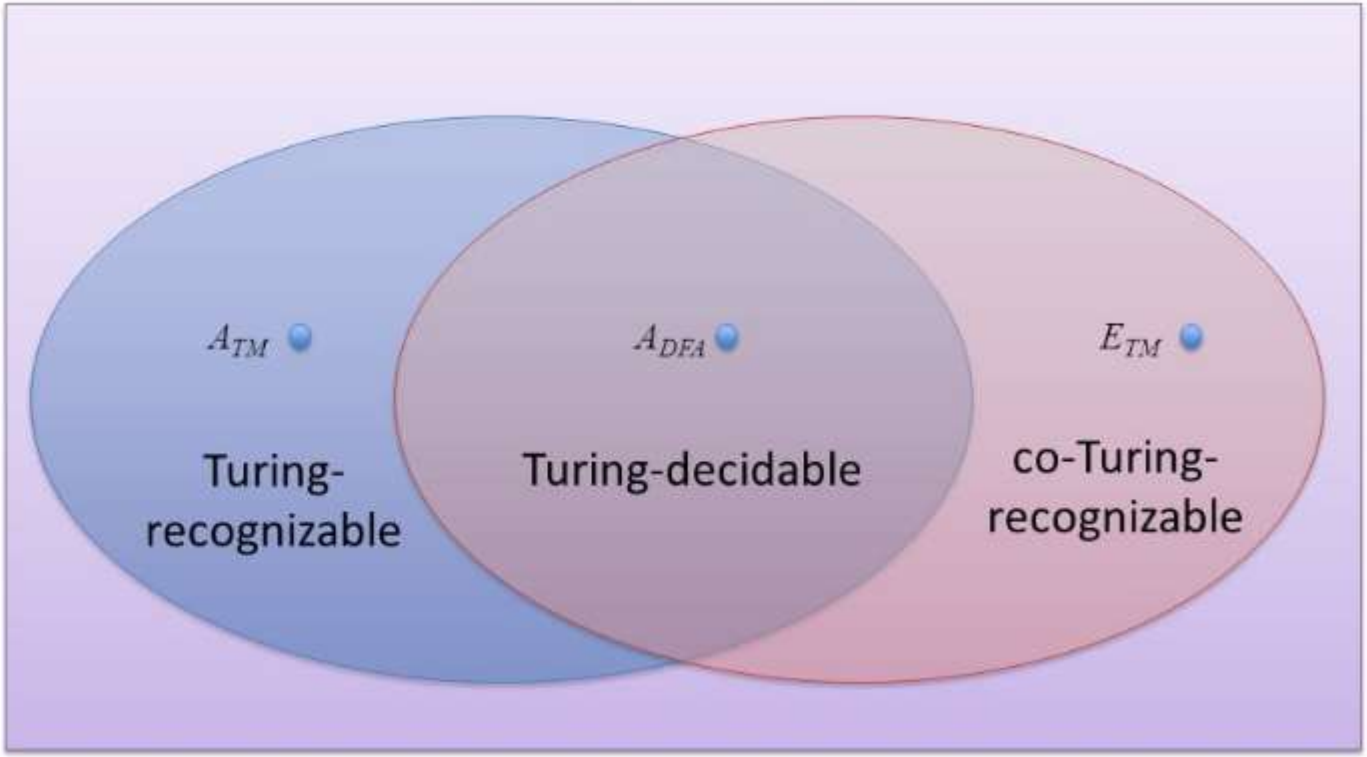
Linguistic relativity

The language you speak determines how you think

https://en.wikipedia.org/wiki/Linguistic_relativity

A Turing Unrecognizable Language

- A language is co-Turing-recognizable if it is the complement of a Turing-recognizable language.
- A language is decidable if it is Turing-recognizable and co-Turing-recognizable.
- $\overline{A_{TM}}$ is not Turing recognizable.
  - We know A_{TM} is Turing-recognizable.
  - If $\overline{A_{TM}}$ were also Turing-recognizable, A_{TM} would have to be decidable.
  - We know A_{TM} is not decidable.
  - $\overline{A_{TM}}$ must not be Turing-recognizable.



- Can you locate where $\overline{A_{TM}}$ will be?
- Can you locate $\overline{E_{TM}}$ will be?

Preview …

- Are there languages which are neither RE nor co-RE ?? (i.e. both the language and its complement are not in RE).

Preview …

- Are there languages which are neither RE nor co-RE ??
- Yes. $EQ_{TM} = \{\langle M_1, M_2\rangle \mid L(M_1) = L(M_2)\}$.
- But, to understand this we need the concept called "mapping reducibility" which is a binary relation between languages and is represented $\leq_m$

Preview …

- Are there languages which are neither RE nor co-RE ??

EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable

- Proof:

Show $A_{TM} \leq_m \overline{EQ_{TM}}$