Chapter 6

Pushdown Automata

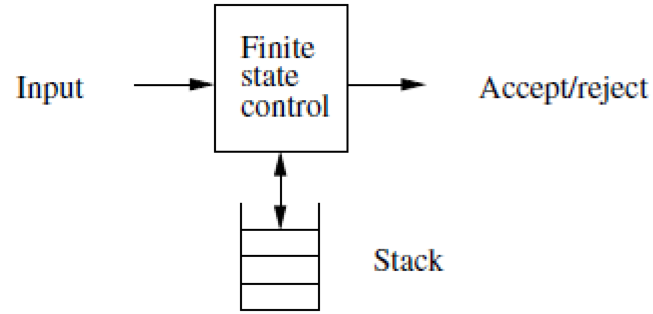
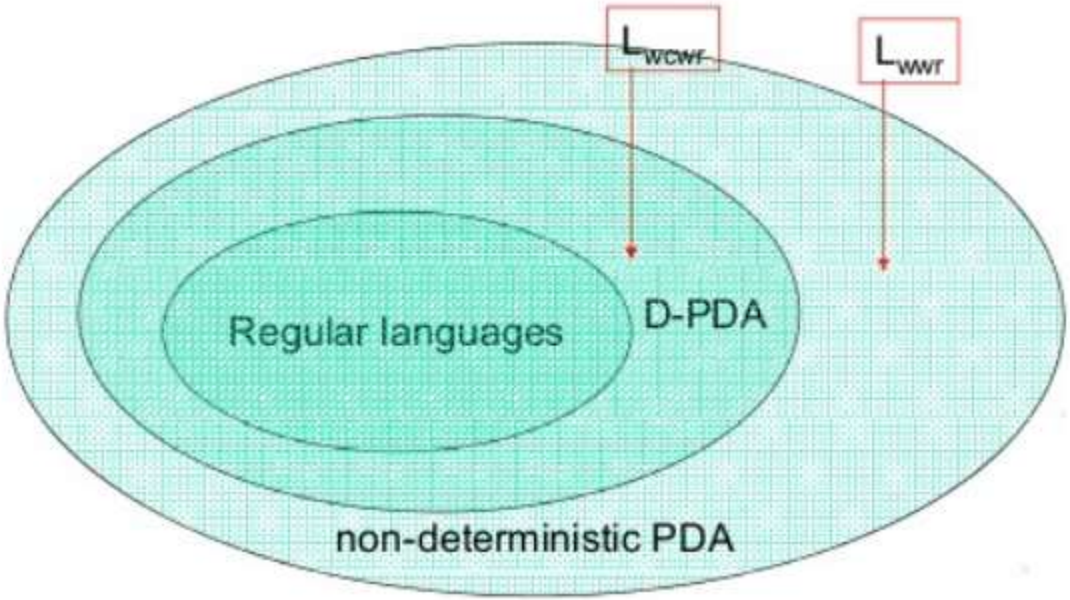


Figure 6.1: A pushdown automaton is essentially a finite automaton with a stack data structure

- PDA is an extension of nondeterministic finite automaton with a stack (of infinite size).
- DPDA -- deterministic version of PDA

  (but is not enough to recognize all CFLs).

  (the languages recognized by DPDAs are called "deterministic CFLs").



It holds that:

$$\left\{\begin{array}{c}\text{Deterministic}\\ \text{Context-Free}\\ \text{Languages}\\ \text{(DPDA)}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{PDAs}\end{array}\right\}$$

Since every DPDA is also a PDA

6.1.2 The Formal Definition of Pushdown Automata

Our formal notation for a *pushdown automaton* (PDA) involves seven components. We write the specification of a PDA P as follows:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$

The components have the following meanings:

Q: A finite set of *states*, like the states of a finite automaton.

Σ: A finite set of *input symbols*, also analogous to the corresponding component of a finite automaton.

Γ: A finite *stack alphabet*. This component, which has no finite-automaton analog, is the set of symbols that we are allowed to push onto the stack.

δ: The *transition function*.

q_0: The *start state*. The PDA is in this state before making any transitions.

Z_0: The *start symbol*. Initially, the PDA's stack consists of one instance of this symbol, and nothing else.

F: The set of *accepting states*, or *final states*.

δ: The *transition function.* As for a finite automaton, δ governs the behavior of the automaton. Formally, δ takes as argument a triple $\delta(q, a, X)$, where:

1. q is a state in Q.
2. a is either an input symbol in Σ or $a = \epsilon$, the empty string, which is assumed not to be an input symbol.
3. X is a stack symbol, that is, a member of Γ.

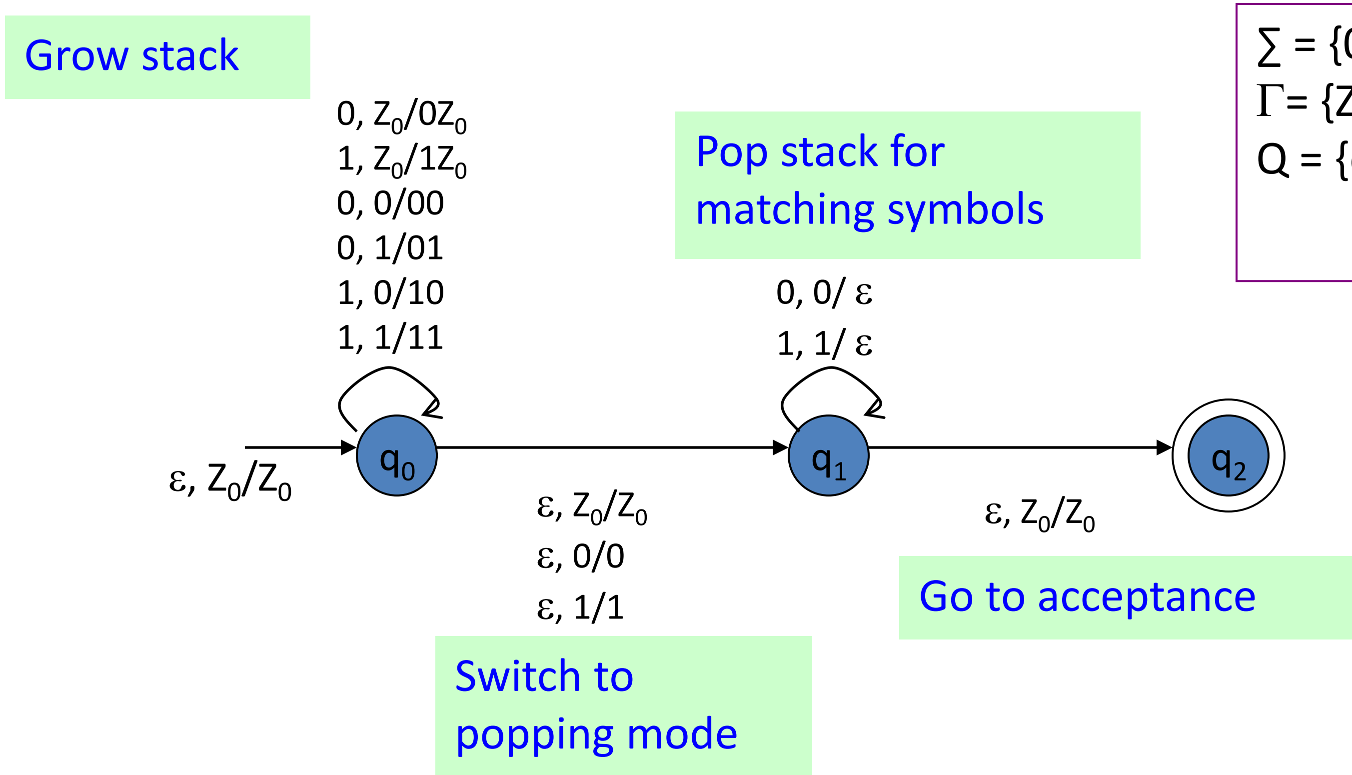
The output of δ is a finite set of pairs (p, γ), where p is the new state, and γ is the string of stack symbols that replaces X at the top of the stack. For instance, if $\gamma = \epsilon$, then the stack is popped, if $\gamma = X$, then the stack is unchanged, and if $\gamma = YZ$, then X is replaced by Z, and Y is pushed onto the stack.

- $\delta(q, a, X) = \{(p_1, \gamma_1), (p_2, \gamma_2), \ldots\}$ where $\gamma_i \in \Gamma^*$.
  - Note, it is possible that $\gamma_i = \epsilon$.
  - $p_i \in Q$.
- $a \in \Sigma \cup \{\epsilon\}$.
- $X \in \Gamma$
  - X can never be ϵ. PDA must read a symbol from stack.

$(p, \boxed{\gamma})$

If $\gamma = YZ$, then Y will be on the top of the stack.
If $\gamma = \epsilon$, then do not push anything on to the stack.

PDA for L_{wwr}: Transition Diagram



$\Sigma = \{0, 1\}$
$\Gamma = \{Z_0, 0, 1\}$
$Q = \{q_0, q_1, q_2\}$

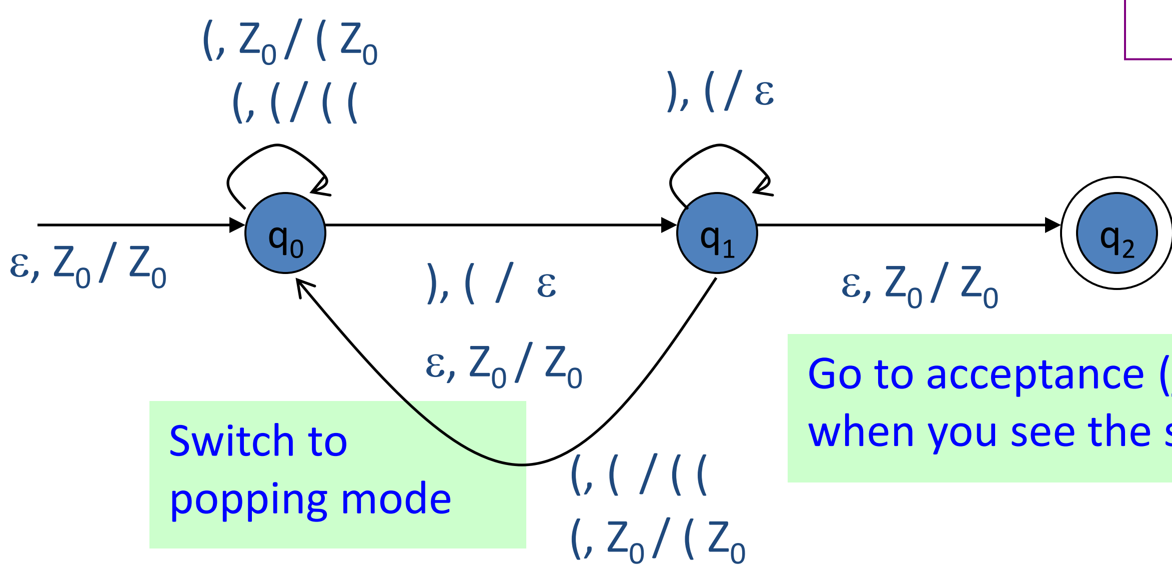
This would be a non-deterministic PDA

Example 2: language of balanced paranthesis

$\sum = \{ (,) \}$
$\Gamma = \{Z_0, (\}$
$Q = \{q_0,q_1,q_2\}$

Grow stack

Pop stack for matching symbols



Switch to popping mode

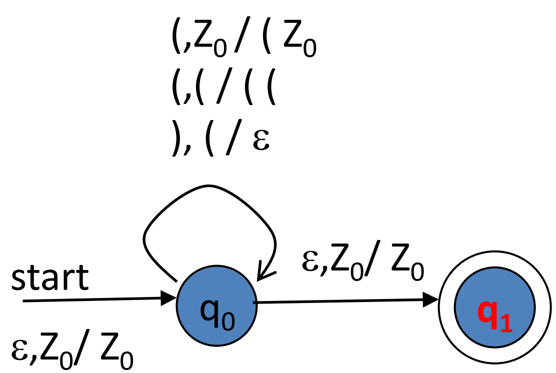
Go to acceptance (by final state) when you see the stack bottom symbol

To allow adjacent blocks of nested paranthesis

Finite Stack

- The power of a PDA with finite stack will be equivalent to that of NFA.
- It can recognize regular languages and not CFLs.

Example 2: language of balanced paranthesis (another design)



$\sum$ = { (,) }
Γ= {Z_0, (}
Q = {q_0,q_1}

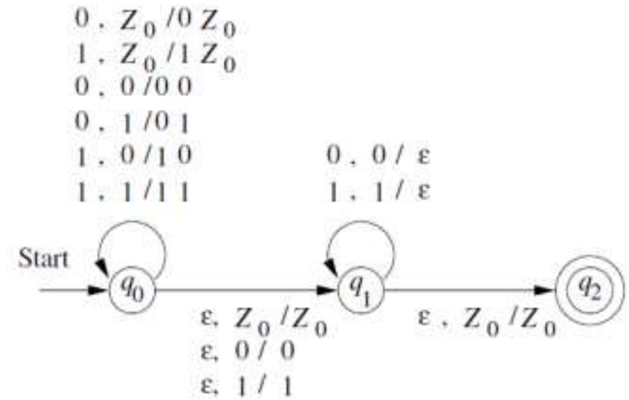


Figure 6.2: Representing a PDA as a generalized transition diagram

1. $\delta(q_0, 0, Z_0) = \{(q_0, 0Z_0)\}$ and $\delta(q_0, 1, Z_0) = \{(q_0, 1Z_0)\}$. One of these rules applies initially, when we are in state q_0 and we see the start symbol Z_0 at the top of the stack. We read the first input, and push it onto the stack, leaving Z_0 below to mark the bottom.

2. $\delta(q_0, 0, 0) = \{(q_0, 00)\}$, $\delta(q_0, 0, 1) = \{(q_0, 01)\}$, $\delta(q_0, 1, 0) = \{(q_0, 10)\}$, and $\delta(q_0, 1, 1) = \{(q_0, 11)\}$. These four, similar rules allow us to stay in state q_0 and read inputs, pushing each onto the top of the stack and leaving the previous top stack symbol alone.

3. $\delta(q_0, \epsilon, Z_0) = \{(q_1, Z_0)\}$, $\delta(q_0, \epsilon, 0) = \{(q_1, 0)\}$, and $\delta(q_0, \epsilon, 1) = \{(q_1, 1)\}$. These three rules allow P to go from state q_0 to state q_1 spontaneously (on ϵ input), leaving intact whatever symbol is at the top of the stack.

4. $\delta(q_1, 0, 0) = \{(q_1, \epsilon)\}$, and $\delta(q_1, 1, 1) = \{(q_1, \epsilon)\}$. Now, in state q_1 we can match input symbols against the top symbols on the stack, and pop when the symbols match.

5. $\delta(q_1, \epsilon, Z_0) = \{(q_2, Z_0)\}$. Finally, if we expose the bottom-of-stack marker Z_0 and we are in state q_1, then we have found an input of the form ww^R. We go to state q_2 and accept.

6.1.4 Instantaneous Descriptions of a PDA

we shall represent the configuration of a PDA by a triple (q, w, γ), where

1. q is the state,
2. w is the remaining input, and
3. γ is the stack contents.

One step, and several steps

Suppose $\delta(q, a, X)$ contains (p, α).

Then for all strings w in Σ^* and β in Γ^*: $\quad (q, aw, X\beta) \vdash (p, w, \alpha\beta)$

|--- sign is called a "***turnstile notation***" and represents one move
|---* sign represents a sequence of moves

$\overset{*}{\vdash}$ is reflexive and transitive closure of $\vdash$

How does the P… input…

All mov…

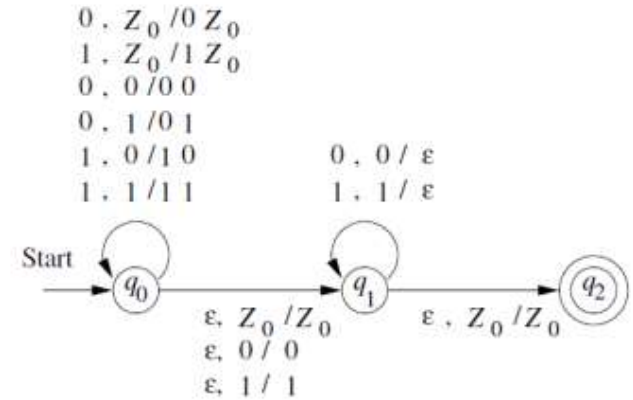


Figure 6.2: Representing a PDA as a generalized transition diagram

$(q_0,1111,Z_0)$ → $(q_1,1111,Z_0)$ → Path dies…

$(q_0,111,1Z_0)$ → $(q_1,111,1Z_0)$ → Path dies…

$(q_0,11,11Z_0)$ → $(q_1,11,11Z_0)$ → $(q_1,1,1Z_0)$ → (q_1, ε,Z_0) → (q_2, ε,Z_0)

$(q_0,1,111Z_0)$ → $(q_1,1,111Z_0)$ → $(q_1, \varepsilon,11Z_0)$ → Path dies…

$(q_0,\varepsilon,1111Z_0)$ → $(q_1, \varepsilon,1111Z_0)$ → Path dies…

Acceptance by final state:

= empty input AND *final state*

6.2 The Languages of a PDA

6.2.1 Acceptance by Final State

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $L(P)$, the *language accepted by P by final state*, is

$$\{w \mid (q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (q, \epsilon, \alpha)\}$$

for some state q in F and any stack string α.

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That is, starting in the initial ID with w waiting on the input, P consumes w from the input and enters an accepting state. The contents of the stack at that time is irrelevant.

6.2 The Languages of a PDA

6.2.1 Acceptance by Final State

Do we have some other acceptance criterion?!

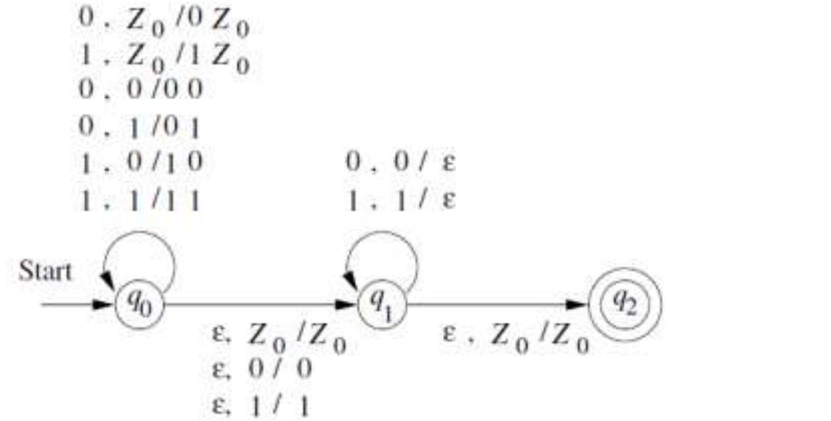
Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $L(P)$, the *language accepted by P by final state*, is

$$\{w \mid (q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (q, \epsilon, \alpha)\}$$

for some state q in F and any stack string α.

<u>Checklist:</u>
- input exhausted?
- in a final state?

That is, starting in the initial ID with w waiting on the input, P consumes w from the input and enters an accepting state. The contents of the stack at that time is irrelevant.



Representing a PDA as a generalized transition diagram

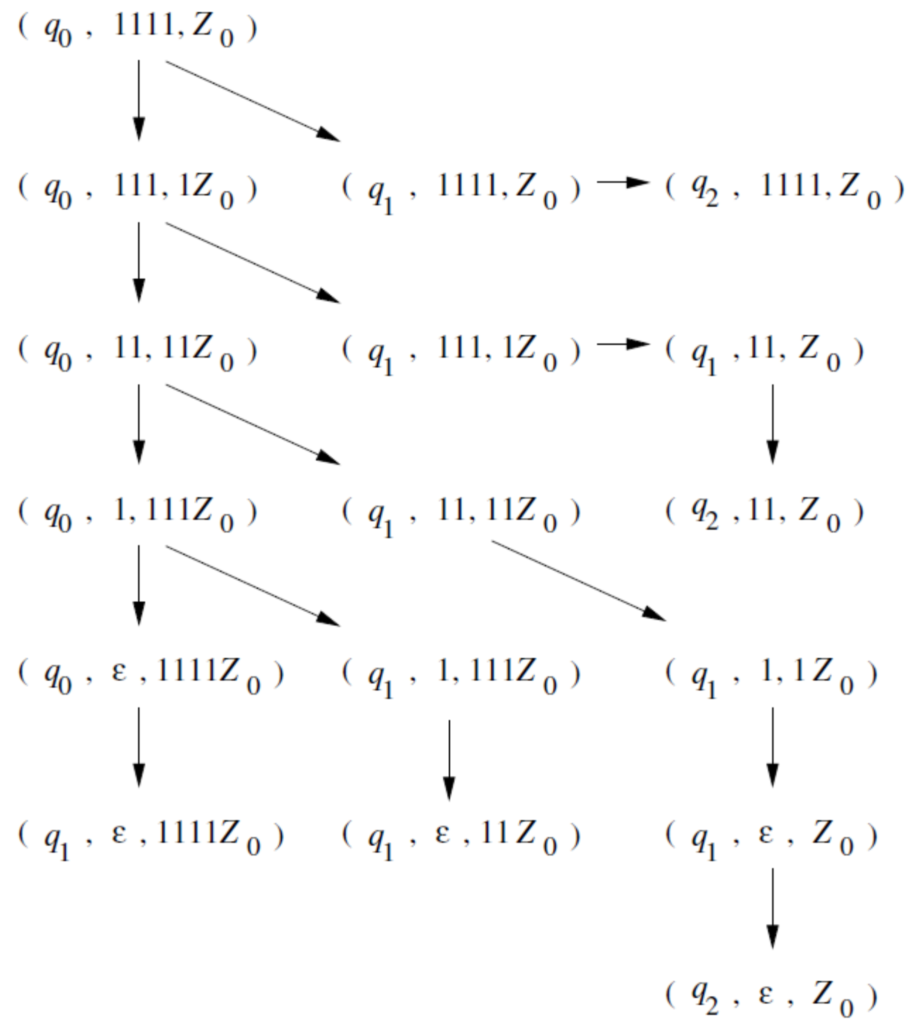


Figure 6.3: ID's of the PDA of Example 6.2 on input 1111

$$(q_0, 1111, Z_0) \vdash (q_0, 111, 1Z_0) \vdash (q_0, 11, 11Z_0) \vdash (q_1, 11, 11Z_0) \vdash (q_1, 1, 1Z_0) \vdash (q_1, \epsilon, Z_0) \vdash (q_2, \epsilon, Z_0).$$

$$(q_0, 1111, Z_0) \vdash (q_0, 111, 1Z_0) \vdash (q_0, 11, 11Z_0) \vdash (q_1, 11, 11Z_0) \vdash (q_1, 1, 1Z_0) \vdash (q_1, \epsilon, Z_0) \vdash (q_2, \epsilon, Z_0).$$

- This is same as

$$(q_0, 1111, Z_0) \overset{*}{\vdash} (q_2, \epsilon, Z_0).$$

- So, 1111 is in the language.

6.2 The Languages of a PDA

6.2.1 Acceptance by Final State

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $L(P)$, the *language accepted by* P *by final state*, is

$$\{w \mid (q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (q, \epsilon, \alpha)\}$$

for some state q in F and any stack string α.

6.2.2 Acceptance by Empty Stack

For each PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, we also define

$$N(P) = \{w \mid (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \epsilon)\}$$

for any state q. That is, $N(P)$ is the set of inputs w that P can consume and at the same time empty its stack.[2]

<u>Checklist:</u>
- input exhausted?
- is the stack empty?

$L(P)$ Vs. $N(P)$

- For the same PDA P there are now two languages !!

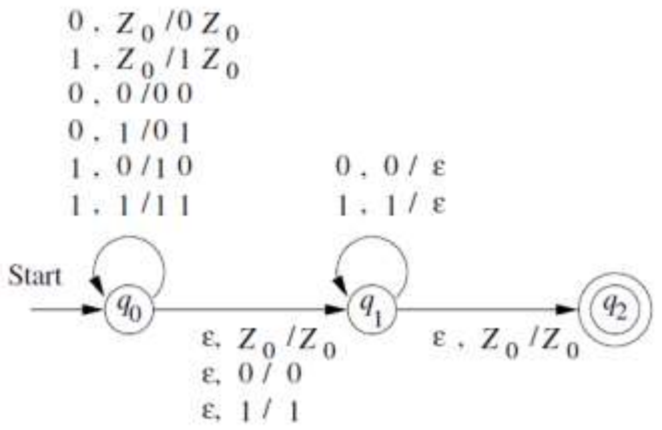


Figure 6.2: Representing a PDA as a generalized transition diagram

- $L(P) = \{ww^R | w \in (0+1)^*\} = L_{wwr}$
- $N(P) = ?$

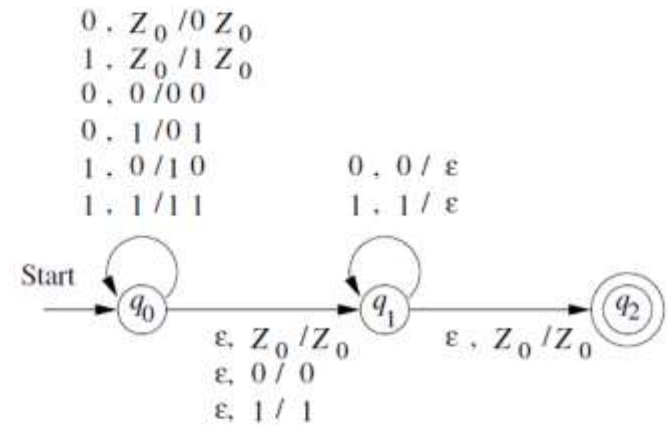
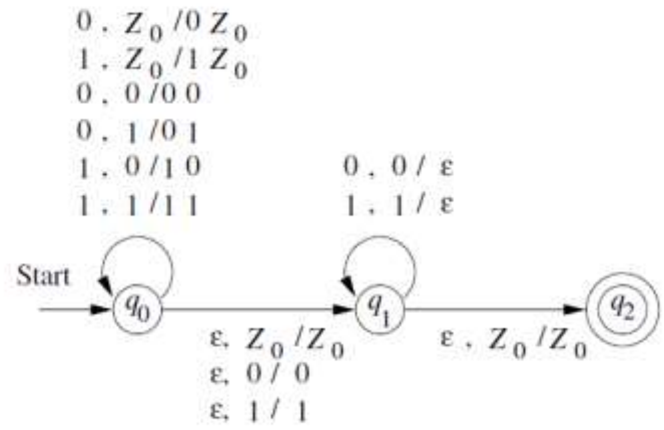


Figure 6.2: Representing a PDA as a generalized transition diagram

- $L(P) = \{ww^R | w \in (0+1)^*\} = L_{wwr}$
- $N(P) = \phi$
  - Stack never becomes empty
  - But, a small change can make $N(P) = L(P)$.
  - What is that?



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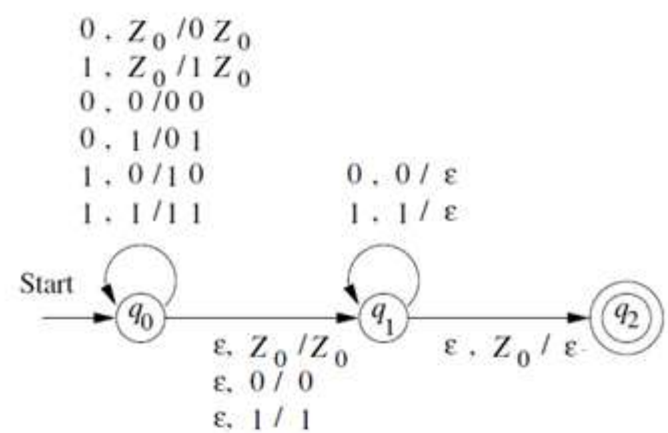

Figure 6.2(a) A modified PDA.

Acceptance by empty stack

Since the set of accepting states is irrelevant, we shall sometimes leave off the last (seventh) component from the specification of a PDA P.

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Thus, P can be written as a six-tuple $(Q, \Sigma, \Gamma, \delta, q_0, Z_0)$.

Do we have two types of PDAs?

- One with final states, other with empty stack …

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- NO.

Do we have two types of PDAs?

- One with final states, other with empty stack …
- NO.
- But, for any PDA there are two languages (both are CFLs) associated with that PDA.
  - These two (languages) may or may not be same.

Crucial thing is…

- Set of languages accepted by final state is equal to the set of languages accepted by empty stack.
  - Proof of this one is by construction.

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- Set of languages accepted by final state is equal to the set of languages accepted by empty stack.
  - Proof of this one is by construction.
- So, power of PDA is same whether recognition happens either by final state or by empty stack.

6.2.3 From Empty Stack to Final State

Theorem 6.9: If $L = N(P_N)$ for some PDA $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_0)$, then there is a PDA P_F such that $L = L(P_F)$.

6.2.3 From Empty Stack to Final State

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PROOF: The idea behind the proof is in Fig. 6.4. We use a new symbol X_0, which must not be a symbol of Γ; X_0 is both the start symbol of P_F and a marker on the bottom of the stack that lets us know when P_N has reached an empty stack.

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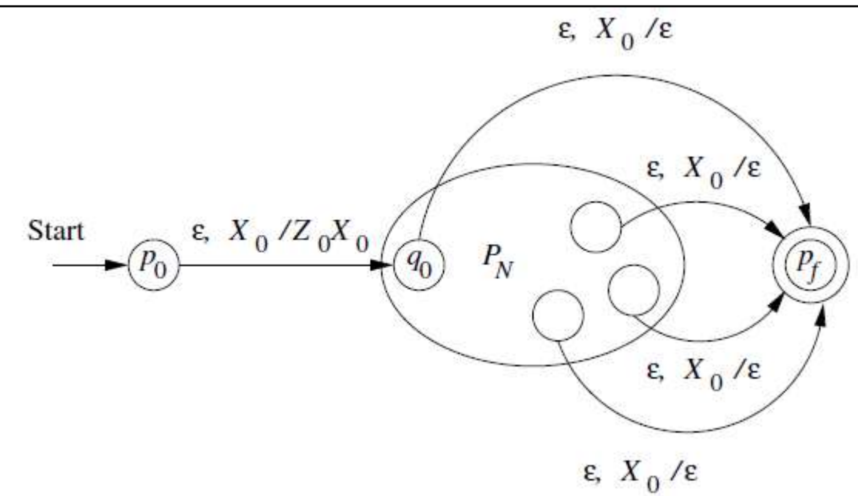


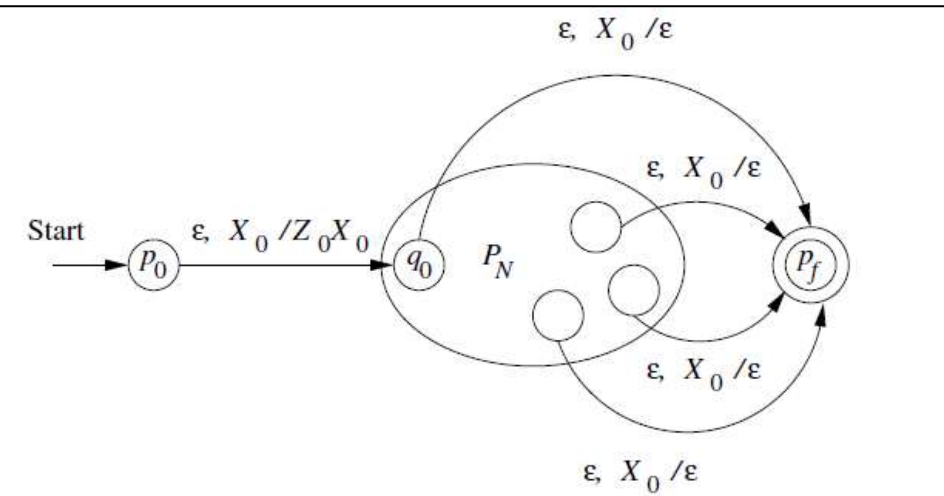
Figure 6.4: P_F simulates P_N and accepts if P_N empties its stack

$$P_F = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$$

1. $\delta_F(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$. In its start state, P_F makes a spontaneous transition to the start state of P_N, pushing its start symbol Z_0 onto the stack.

2. For all states q in Q, inputs a in Σ or $a = \epsilon$, and stack symbols Y in Γ, $\delta_F(q, a, Y)$ contains all the pairs in $\delta_N(q, a, Y)$.

3. In addition to rule (2), $\delta_F(q, \epsilon, X_0)$ contains (p_f, ϵ) for every state q in Q.



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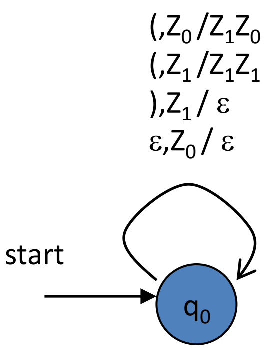
Example: Matching parenthesis "(" ")"

P_N: ($\{q_0\}$, $\{(,)\}$, $\{Z_0,Z_1\}$, δ_N, q_0, Z_0)

δ_N:

$\delta_N(q_0,(,Z_0) = \{ (q_0,Z_1Z_0) \}$
$\delta_N(q_0,(,Z_1) = \{ (q_0, Z_1Z_1) \}$
$\delta_N(q_0,),Z_1) = \{ (q_0, \varepsilon) \}$
$\delta_N(q_0, \varepsilon,Z_0) = \{ (q_0, \varepsilon) \}$



Accept by empty stack

P_f: ($\{p_0,q_0,p_f\}$, $\{(,)\}$, $\{X_0,Z_0,Z_1\}$, δ_f, p_0, X_0, p_f)

δ_f:

$\delta_f(p_0, \varepsilon,X_0) = \{ (q_0,Z_0) \}$
$\delta_f(q_0,(,Z_0) = \{ (q_0,Z_1 Z_0) \}$
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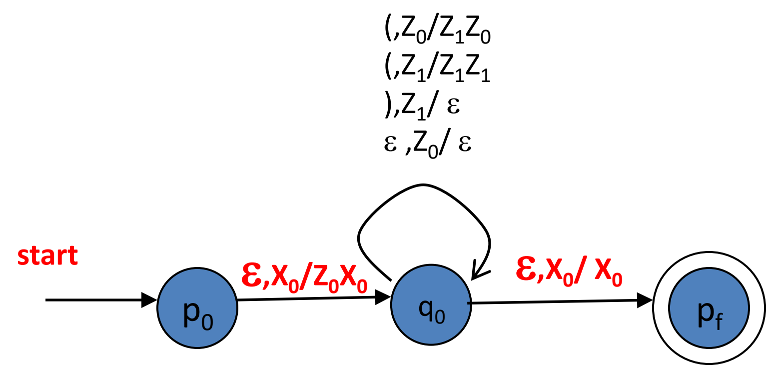

Accept by final state

6.2.4 From Final State to Empty Stack

Theorem 6.11: Let L be $L(P_F)$ for some PDA $P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_0, F)$. Then there is a PDA P_N such that $L = N(P_N)$.

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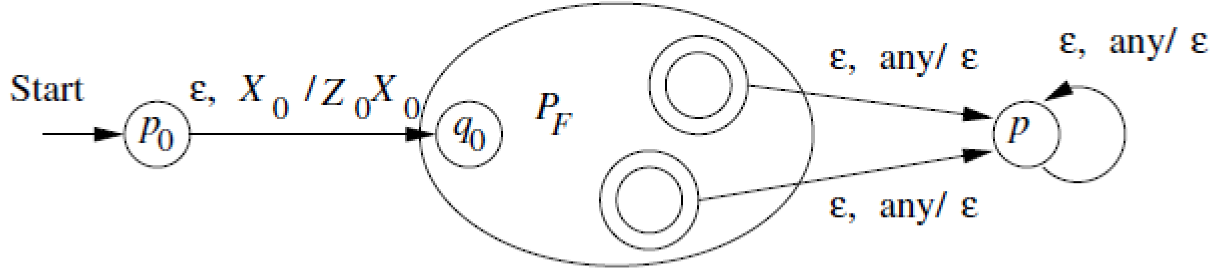

Figure 6.7: P_N simulates P_F and empties its stack when and only when P_N enters an accepting state

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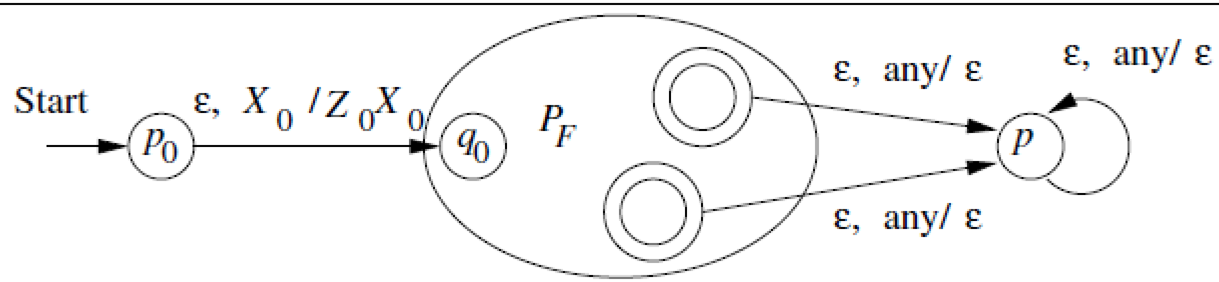

Figure 6.7: P_N simulates P_F and empties its stack when and only when P_N enters an accepting state

$$P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$$

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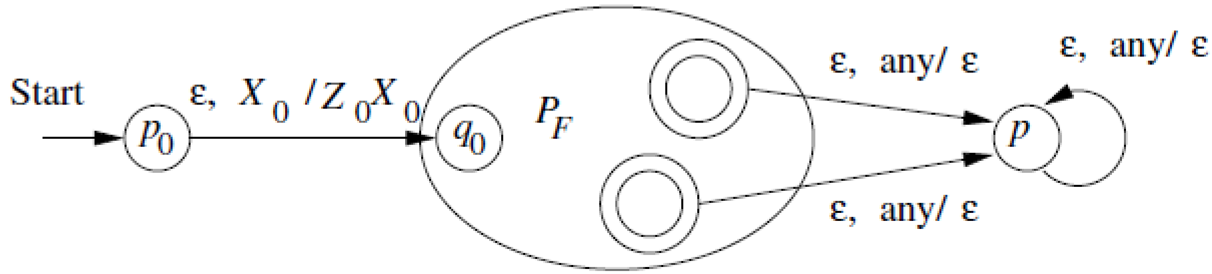

Figure 6.7: P_N simulates P_F and empties its stack when and only when P_N enters an accepting state

$$P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$$

To avoid simulating a situation where P_F accidentally empties its stack without accepting, P_N must also use a marker X_0 on the bottom of its stack.

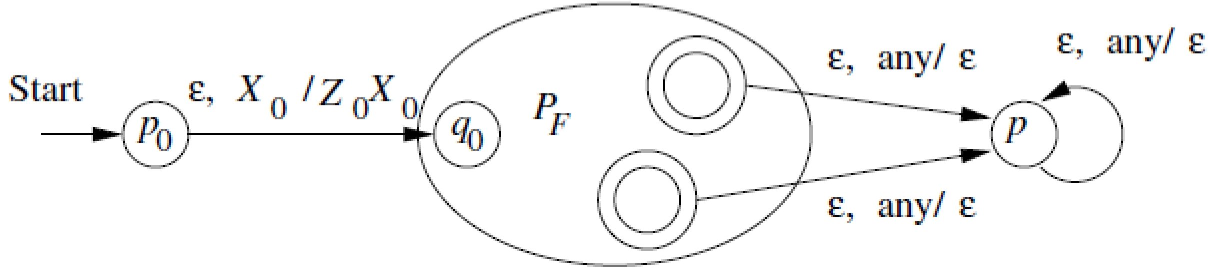

Figure 6.7: P_N simulates P_F and empties its stack when and only when P_N enters an accepting state

$$P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$$

1. $\delta_N(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$. We start by pushing the start symbol of P_F onto the stack and going to the start state of P_F.

2. For all states q in Q, input symbols a in Σ or $a = \epsilon$, and Y in Γ, $\delta_N(q, a, Y)$ contains every pair that is in $\delta_F(q, a, Y)$. That is, P_N simulates P_F.

3. For all accepting states q in F and stack symbols Y in Γ or $Y = X_0$, $\delta_N(q, \epsilon, Y)$ contains (p, ϵ). By this rule, whenever P_F accepts, P_N can start emptying its stack without consuming any more input.

4. For all stack symbols Y in Γ or $Y = X_0$, $\delta_N(p, \epsilon, Y) = \{(p, \epsilon)\}$. Once in state p, which only occurs when P_F has accepted, P_N pops every symbol on its stack, until the stack is empty. No further input is consumed.

One point ..

- In P_F if stack becomes empty (in between) then P_F gets stuck. (That PDA gets killed).

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- Otherwise, if the state is a non-final one, do we need to do something?

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- Now, in this situation P_N has X_0 in the stack, so will this be a problem?
- If the state is a final state no problem.
- Otherwise, if the state is a non-final one, do we need to do something? NO.
- Since there is no transition (on X_0 being stack top) that PDA gets killed.

- Which of the following are CFL?
  - $a^n b^n$
  - $a^n b^n c^n$
  - $a^{n+m} b^n c^m$
  - $a^n b^m c^n$
  - $a^n b^n c^n d^n$
  - $a^n b^m c^m d^n$
  - all strings with number of a’s is twice the number of b’s
  - $a^i b^j c^k$

Next …

- Equivalence of PDA’s and CFG’s

Next …

- Equivalence of PDA’s and CFG’s

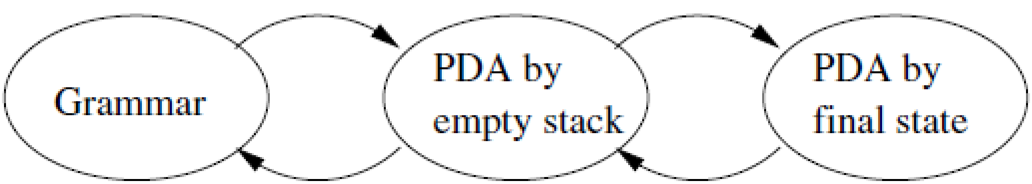

Figure 6.8: Organization of constructions showing equivalence of three ways of defining the CFL’s

CFG to PDA

Let $G = (V, T, Q, S)$ be a CFG.

CFG to PDA

Let $G = (V, T, Q, S)$ be a CFG.

Construct the PDA P that accepts $L(G)$ by empty stack as follows:

$$P = (\{q\}, T, V \cup T, \delta, q, S)$$

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Single state !

This is same as: "implementing a CFG using a PDA"

Converting a CFG into a PDA

<u>Main idea:</u> The PDA simulates the leftmost derivation on a given w, and upon consuming it fully it either arrives at acceptance (by <u>empty stack</u>) or non-acceptance.

<u>Steps:</u>

1. Push the right hand side of the production onto the stack, with leftmost symbol at the stack top
2. If stack top is the leftmost variable, then replace it by all its productions (each possible substitution will represent a *<u>distinct</u>* path taken by the non-deterministic PDA)
3. If stack top has a terminal symbol, and if it matches with the next symbol in the input string, then pop it

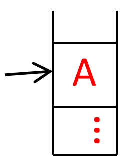
State is inconsequential (only one state is needed)

Formal construction of PDA from CFG

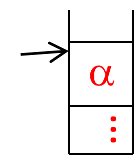
Note: Initial stack symbol (S) same as the start variable in the grammar

- Given: $G= (V,T,P,S)$
- Output: $P_N = (\{q\}, T, V \cup T, \delta, q, S)$
- δ:
  - For all $A \in V$, add the following transition(s) in the PDA:
    - $\delta(q, \varepsilon, A) = \{ (q, \alpha) \mid \text{“}A ==>\alpha\text{”} \in P\}$

Before:

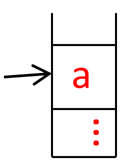


After:

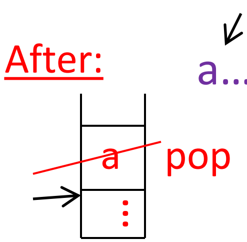


  - For all $a \in T$, add the following transition(s) in the PDA:
    - $\delta(q,a,a)= \{ (q, \varepsilon) \}$

Before:

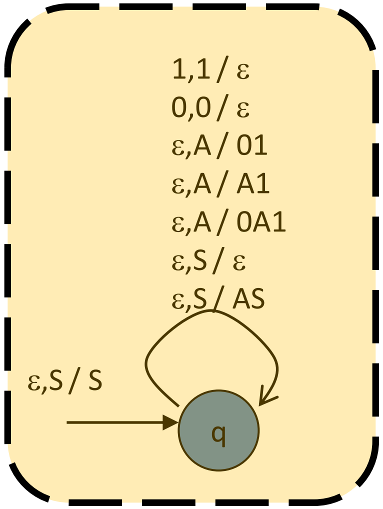


After:



Example: CFG to PDA

- $G = (\{S,A\}, \{0,1\}, P, S)$
- P:
  - $S \Longrightarrow AS \mid \varepsilon$
  - $A \Longrightarrow 0A1 \mid A1 \mid 01$
- $PDA = (\{q\}, \{0,1\}, \{0,1,A,S\}, \delta, q, S)$
- δ:
  - $\delta(q, \varepsilon, S) = \{ (q, AS), (q, \varepsilon)\}$
  - $\delta(q, \varepsilon, A) = \{ (q,0A1), (q,A1), (q,01) \}$
  - $\delta(q, 0, 0) = \{ (q, \varepsilon) \}$
  - $\delta(q, 1, 1) = \{ (q, \varepsilon) \}$



How will this new PDA work?

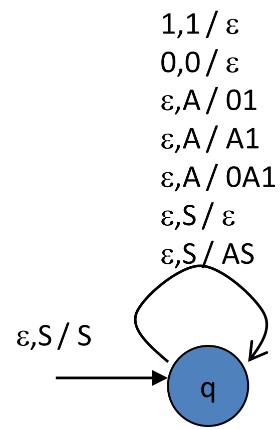
Lets simulate string <u>0011</u>

Simulating string 0011 on the new PDA

…

PDA (δ):

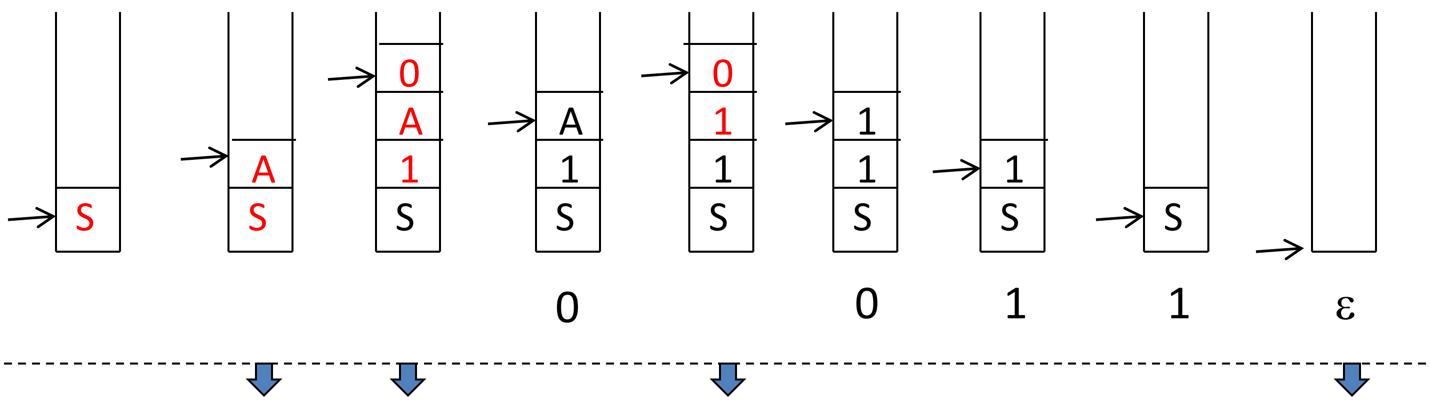
δ(q, ε , S) = { (q, AS), (q, ε)}
δ(q, ε , A) = { (q,0A1), (q,A1), (q,01) }
δ(q, 0, 0) = { (q, ε) }
δ(q, 1, 1) = { (q, ε) }



Leftmost deriv.:

S => AS
=> 0A1S
=> 0011S
=> 0011

Stack moves (shows only the successful path):



Accept by empty stack

S =>AS =>0A1S =>0011S => 0011

CFG to PDA

Let $G = (V, T, Q, S)$ be a CFG.

Construct the PDA P that accepts $L(G)$ by empty stack as follows:

$$P = (\{q\}, T, V \cup T, \delta, q, S)$$

where transition function δ is defined by:

1. For each variable A,

$$\delta(q, \epsilon, A) = \{(q, \beta) \mid A \rightarrow \beta \text{ is a production of } G\}$$

2. For each terminal a, $\delta(q, a, a) = \{(q, \epsilon)\}$.

Example 6.12: Let us convert the expression grammar of Fig. 5.2 to a PDA. Recall this grammar is:

$$E \rightarrow I \mid E * E \mid E + E \mid (E)$$
$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

- Can you identify (V,T,Q,S) of this CFG?

Example 6.12: Let us convert the expression grammar of Fig. 5.2 to a PDA. Recall this grammar is:

$$
\begin{array}{lcl}
E & \rightarrow & I \mid E * E \mid E + E \mid (E) \\
I & \rightarrow & a \mid b \mid Ia \mid Ib \mid I0 \mid I1
\end{array}
$$

- Can you identify (V,T,Q,S) of this CFG?
- $V = \{E, I\}$
- $T = \{\, a, b, 0, 1, +, *, (,)\, \}$
- $S = E$

$$E \rightarrow I \mid E * E \mid E + E \mid (E)$$

$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

- For the PDA stack alphabet is $\Gamma = \{E, I, a, b, 0, 1, +, *, (,)\}$
- Start symbol of the stack is E.
- We will have only one state, call it q.

$$E \rightarrow I \mid E * E \mid E + E \mid (E)$$
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- For the PDA stack alphabet is
  $$\Gamma = \{E, I, a, b, 0,1, +,*, (,)\}$$
- Start symbol of the stack is E.
- We will have only one state, call it q.

$$\delta(q, \epsilon, E) = \{(q, I),\ (q, E + E),\ (q, E * E),\ (q, (E))\}.$$

$$\delta(q, \epsilon, I) = \{(q, a),\ (q, b),\ (q, Ia),\ (q, Ib),\ (q, I0),\ (q, I1)\}.$$

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$$\delta(q, \epsilon, I) = \{(q, a),\ (q, b),\ (q, Ia),\ (q, Ib),\ (q, I0),\ (q, I1)\}.$$

$\delta(q, a, a) = \{(q, \epsilon)\}$; $\delta(q, b, b) = \{(q, \epsilon)\}$; $\delta(q, 0, 0) = \{(q, \epsilon)\}$; $\delta(q, 1, 1) = \{(q, \epsilon)\}$; $\delta(q, (, () = \{(q, \epsilon)\}$; $\delta(q,),)) = \{(q, \epsilon)\}$; $\delta(q, +, +) = \{(q, \epsilon)\}$; $\delta(q, *, *) = \{(q, \epsilon)\}$.

Have you noted?

- The PDA we constructed simulates, which derivation?

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- It is "left-most derivation"

Have you noted?

- The PDA we constructed simulates, which derivation?
- It is “left-most derivation”
- In compilers, these are top-down parsers
  - LL parsers;
    - but non-determinism is a problem.
    - Backtracking (to try the other choice)
    - Parse table (to find feasible choices; at that time)

Compilers

- We have parsers which are bottom-up
- Which will simulate right-most derivation
- These are called LR parsers.

- One notable drawback in all these parsers:

Compilers

- We have parsers which are bottom-up
- Which will simulate right-most derivation
- These are called LR parsers.

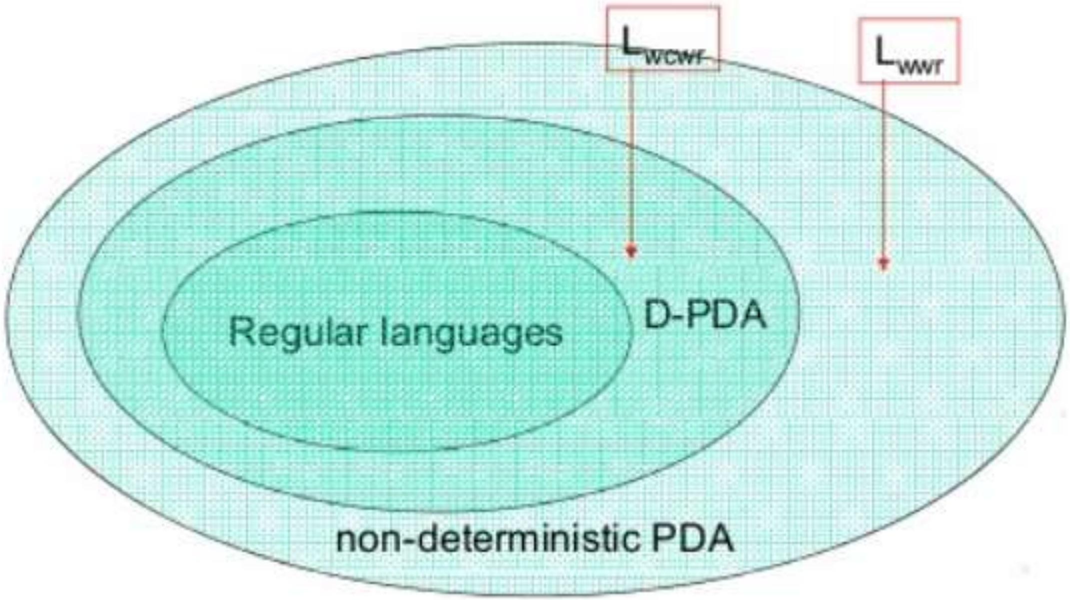
- One notable drawback in all these parsers: each have their own limitations, and works only for subclasses of CFLs;
  - Some superior, some inferior …

PDA to CFG

- We skip this in this basic course.

Deterministic PDA

- DPDA
- Can recognize a proper subset of CFLs
- Parsers (used in compilers), mostly are DPDAs.
  - Most of our programming languages are in the subclass which can be recognized by DPDAs.



It holds that:

$$\left\{\begin{array}{c}\text{Deterministic}\\ \text{Context-Free}\\ \text{Languages}\\ \text{(DPDA)}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{PDAs}\end{array}\right\}$$

Since every DPDA is also a PDA

DPDA

- Only one choice atmost.

DPDA

- Atmost one choice. But ϵ moves (should we remove them?).

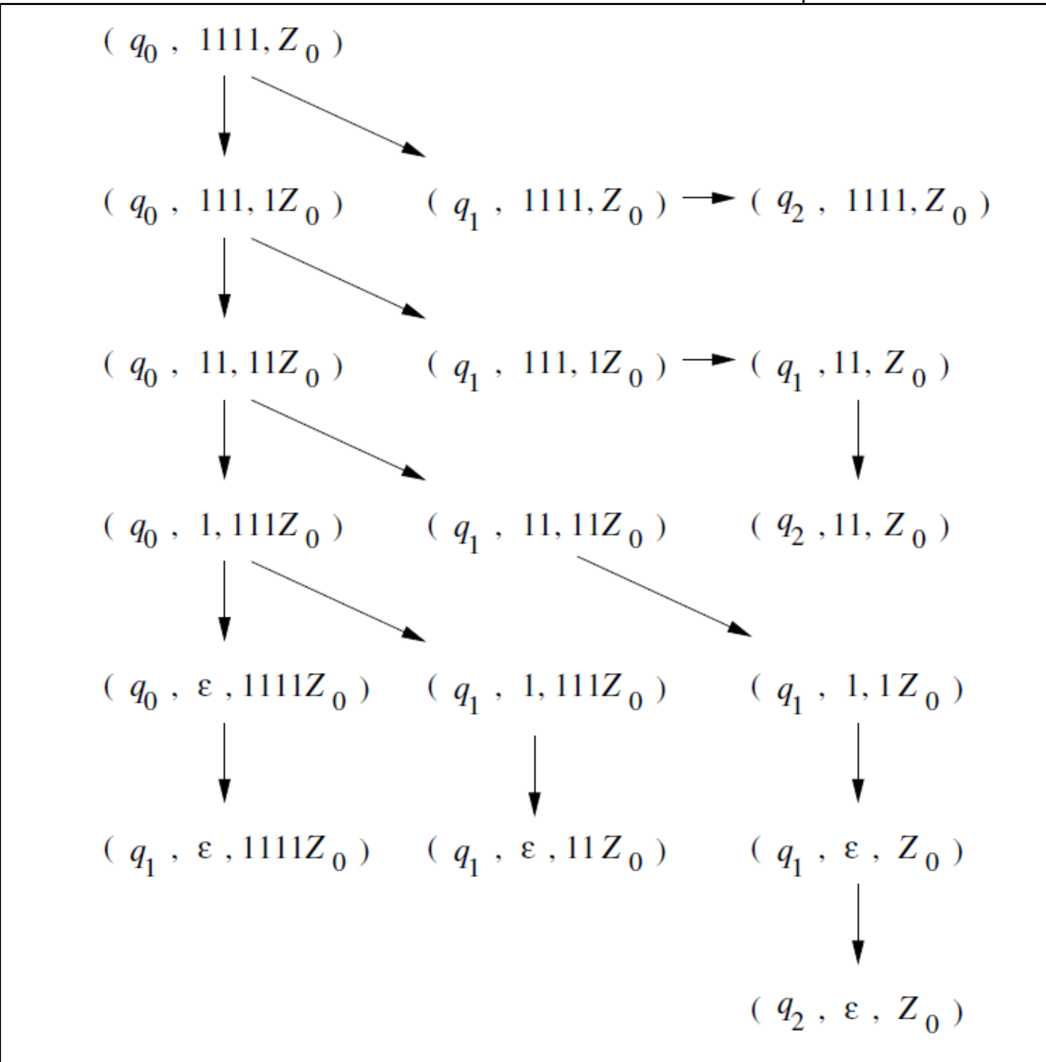
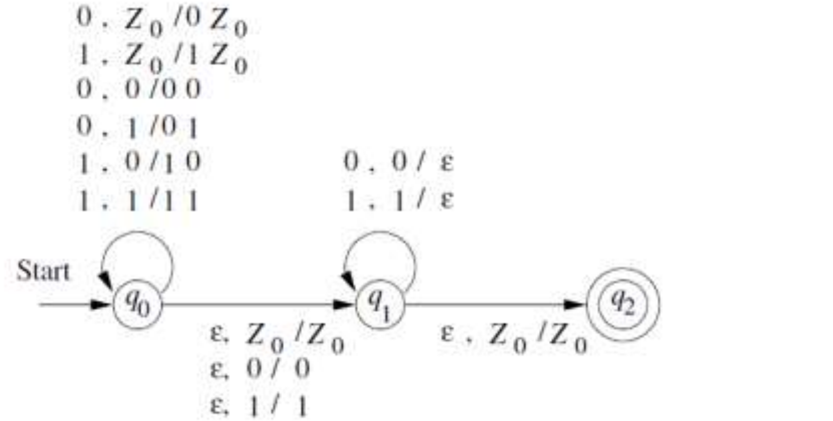


Figure 6.3: ID's of the PDA of Example 6.2 on input 1111



Representing a PDA as a generalized transition diagram

Corresponding NPDA for L_{wwr}

$$(q_0, 1111, Z_0) \vdash (q_0, 111, 1Z_0) \vdash (q_0, 11, 11Z_0) \vdash (q_1, 11, 11Z_0) \vdash (q_1, 1, 1Z_0) \vdash (q_1, \epsilon, Z_0) \vdash (q_2, \epsilon, Z_0).$$

DPDA

- Atmost one choice. But ϵ moves (should we remove them? No).
- For any $q \in Q, a \in \Sigma, or\ a = \epsilon, and\ X \in \Gamma,$ we have
  1) $|\delta(q, a, X)| \leq 1,$ and
  2) For any a except ϵ,
     $$(|\delta(q, a, X)| = 1) \quad \Rightarrow \quad (|\delta(q, \epsilon, X)| = 0).$$

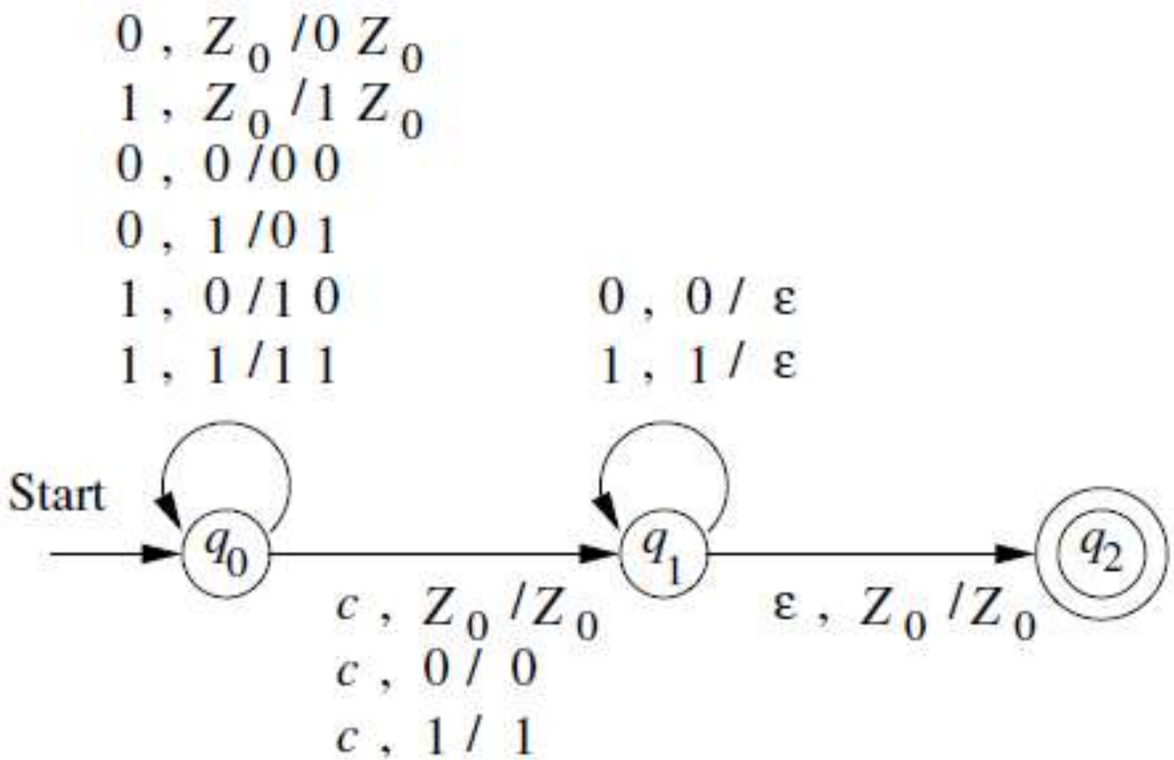


Figure 6.11: A deterministic PDA accepting L_{wcwr}

Regular Languages and DPDAs

Theorem 6.17: If L is a regular language, then $L = L(P)$ for some DPDA P.

Regular Languages and DPDAs

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- Proof is simple.
  - DFA is there for the regular language.
  - What about stack??

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Regular Languages and DPDAs

Theorem 6.17: If L is a regular language, then $L = L(P)$ for some DPDA P.

- Proof is simple.
  - DFA is there for the regular language.
  - What about stack??
  - Ignore the stack.

- Just see, the theorem is saying about $L(P)$ only.

Formally,

let $A = (Q, \Sigma, \delta_A, q_0, F)$ be a DFA. Construct DPDA

$$P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F)$$

Formally,

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$$P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F)$$

If $\delta_A(q, a) = p$ then $\delta_P(q, a, Z_0) = \{(p, Z_0)\}$ for all states p and q in Q.

Formally,

let $A = (Q, \Sigma, \delta_A, q_0, F)$ be a DFA. Construct DPDA

$$P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F)$$

If $\delta_A(q, a) = p$ then $\delta_P(q, a, Z_0) = \{(p, Z_0)\}$ for all states p and q in Q.

We claim that $(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, Z_0)$ if and only if $\hat{\delta}_A(q_0, w) = p$.

Exercise

- Create DFA for 0^*10^*
- Convert the DFA into PDA by final state.
  - Is this a DPDA?
- Convert the PDA into PDA by empty stack.

DPDA and *N(P)* ??

- For some regular languages, there can no DPDA by *empty stack* that recognizes the language.

DPDA and *N(P)* ??

- For some regular languages, there can no DPDA by *empty stack* that recognizes the language.
- But, for a proper subset of regular languages, it is possible to build a DPDA by empty stack.
  - These are characterized by "prefix property".
- DPDA by empty stack can recognize some non-regular languages also, provided they obey the property.

Prefix property

- A language L has the prefix property, if there are no two distinct strings x and y in L such that x is a prefix of y.

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- L_{wcwr} has this property.

Prefix property

- A language L has the prefix property, if there are no two distinct strings x and y in L such that x is a prefix of y.
- L_{wcwr} has this property.
- 0^* violates this property. See this is regular.

Exercise– which of the following has the prefix property

- 0*1
- (0+1)*
- Set of palindromes
- $\{0^n 1^n | n \geq 1\}$
- $\{0^n 1^n | n \geq 0\}$
- $\{\epsilon\}$
- $\{\epsilon, 0\}$

Exercise– which of the following has the prefix property

- 0*1 True
- (0+1)*
- Set of palindromes
- $\{0^n 1^n | n \geq 1\}$ True
- $\{0^n 1^n | n \geq 0\}$
- $\{\epsilon\}$ True
- $\{\epsilon, 0\}$

- Give DPDA by empty stack for these True cases.

Theorem 6.19: A language L is $N(P)$ for some DPDA P if and only if L has the prefix property and L is $L(P')$ for some DPDA P'. $\square$

Theorem 6.19: A language L is $N(P)$ for some DPDA P if and only if L has the prefix property and L is $L(P')$ for some DPDA P'. $\square$

See even for 0^* (a regular language) we cannot build a DPDA by empty-stack !!

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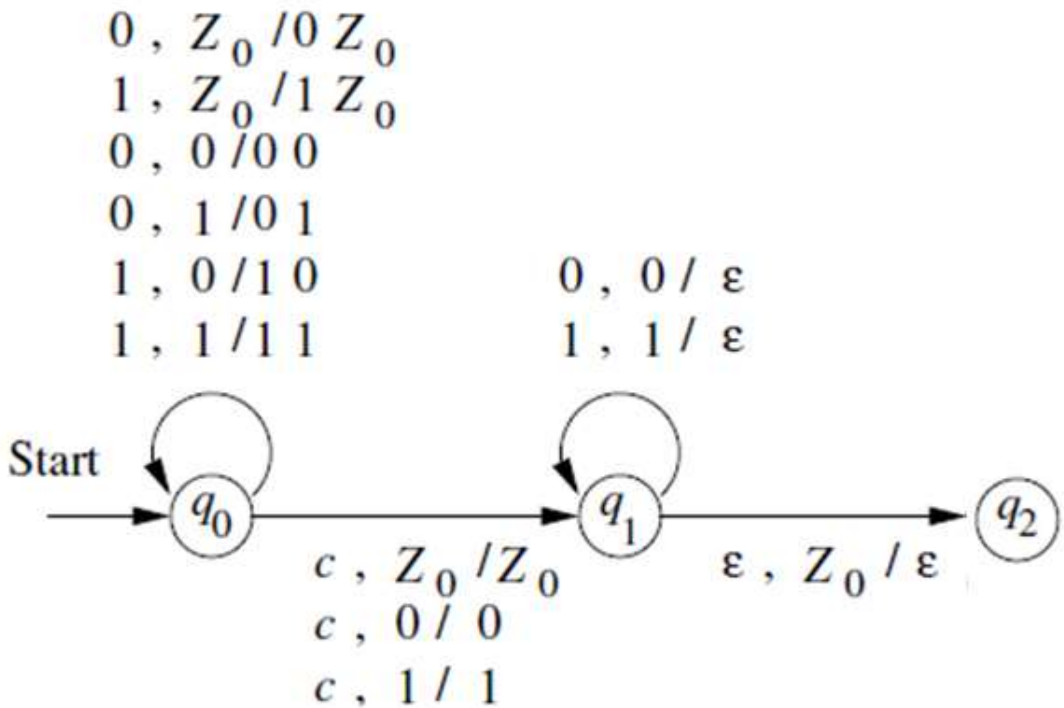
- But, we can build a DPDA by final state !
- Then, why not convert this to DPDA by empty stack by our construction principle?

See even for 0^* (a regular language) we cannot build a DPDA by empty-stack !!

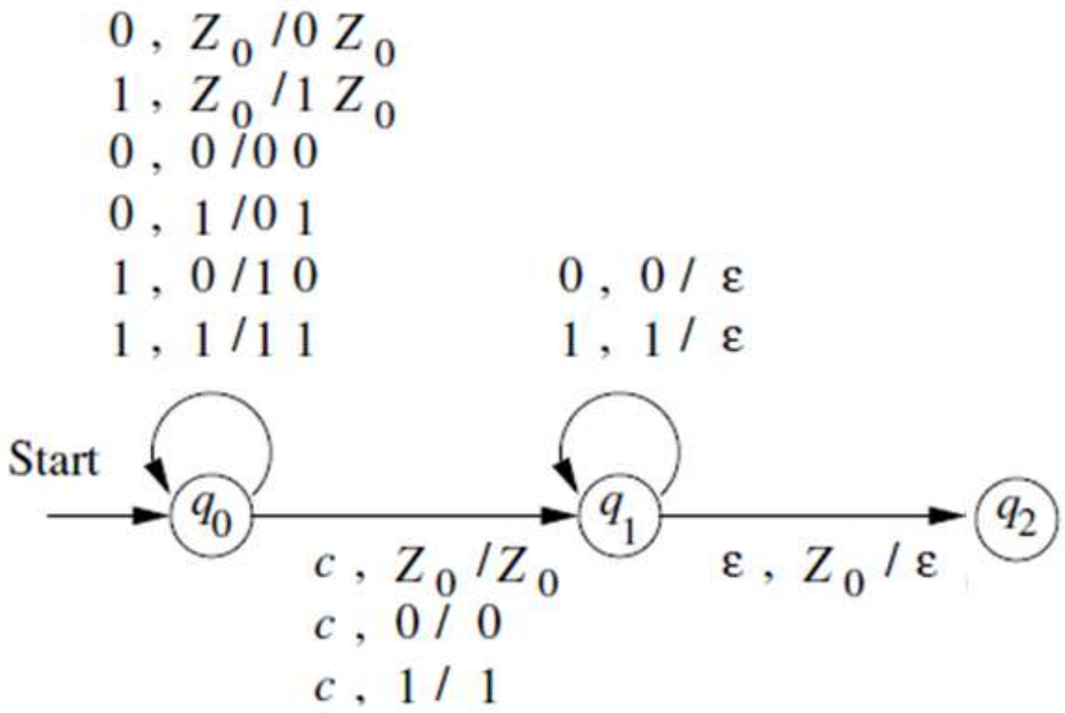
- But, we can build a DPDA by final state !
- Then, why not convert this to DPDA by empty stack by our construction principle?

This conversion gives us PDA, not DPDA !!

DPDA s.t. $N(P) = L_{wcwr}$

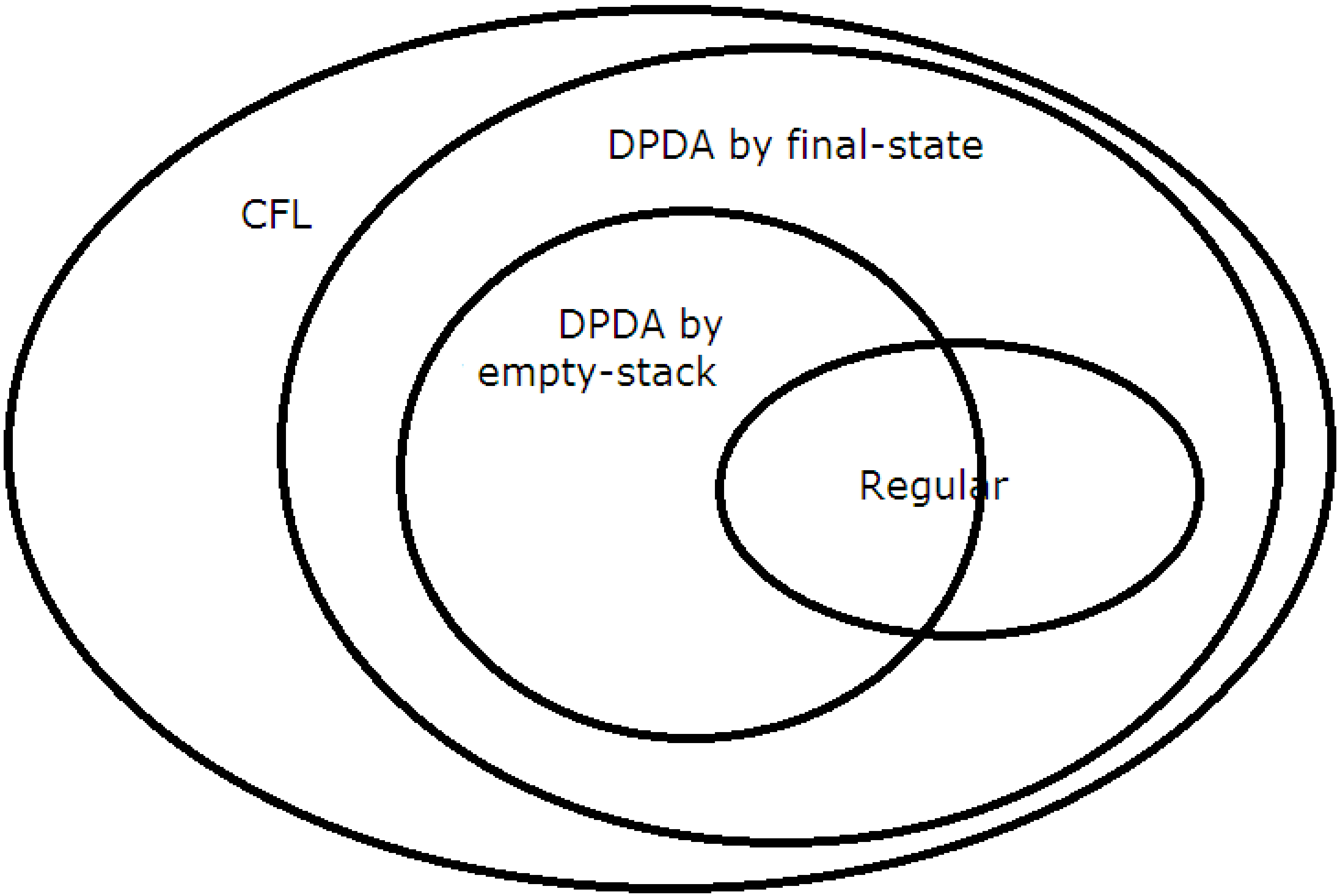


DPDA s.t. $N(P) = L_{wcwr}$



This is not a regular language.

- There is no DPDA to recognize the CFL L_{wwr}
- Proof is complex. But we can see the idea behind..


CFL
DPDA by final-state
DPDA by
empty-stack
Regular

- Some points to note,

Theorem 6.20: If $L = N(P)$ for some DPDA P, then L has an unambiguous context-free grammar.

Theorem 6.21: If $L = L(P)$ for some DPDA P, then L has an unambiguous CFG.

- We cannot make converse of these statements.