Context Free Languages

$$G_{pal} = (\{P\}, \{0,1\}, A, P)$$

$$\begin{array}{llll}
1. & P & \rightarrow & \epsilon \\
2. & P & \rightarrow & 0 \\
3. & P & \rightarrow & 1 \\
4. & P & \rightarrow & 0P0 \\
5. & P & \rightarrow & 1P1
\end{array}$$

A context-free grammar for palindromes

Prove that $L(G_{pal})$ is the set of palindromes over the given alphabet.

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1.	P	$\rightarrow$	ϵ
2.	P	$\rightarrow$	0
3.	P	$\rightarrow$	1
4.	P	$\rightarrow$	$0P0$
5.	P	$\rightarrow$	$1P1$

A context-free grammar for palindromes

- Proof [by induction on $|w|$]:

BASIS: We use lengths 0 and 1 as the basis.

If $|w| = 0$ or $|w| = 1$, then w is ϵ, 0, or 1.

Since there are productions $P \rightarrow \epsilon$, $P \rightarrow 0$, and $P \rightarrow 1$, we conclude that $P \overset{*}{\Rightarrow} w$ in any of these basis cases.

- Note, $w \in L(G_{pal})$ is same $P \overset{*}{\Rightarrow} w$
- Note, $(w = w^R)$ means w begins and ends with the same character.

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Inductive Hypothesis: Let for $|w| \leq k\ where\ (w = w^R)$, $P \overset{*}{\Rightarrow}$ w is true.

Inductive Step: We need to show for $|w| = k + 1, P \overset{*}{\Rightarrow}$ w is true.

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Note, $w = 0x0$ or w $= 1x1$, where $|x| = k - 1$.

Then, $P \Rightarrow 0P0 \overset{*}{\Rightarrow} 0x0$ (Since $|x| \leq k$, so $P \overset{*}{\Rightarrow} x$ is true).

So, $P \overset{*}{\Rightarrow}$ w is true. With a similar argument, $P \Rightarrow 1P1 \overset{*}{\Rightarrow} 1x1$

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This completes our proof for : $(w = w^R) \Rightarrow w \in L(G_{pal})$

$$w \in L(G_{pal}) \Rightarrow (w = w^R)$$

- Proof [by induction on number of steps in the derivation]:

BASIS: If the derivation is one step, then it must use one of the three productions that do not have P in the body. That is, the derivation is $P \Rightarrow \epsilon$, $P \Rightarrow 0$, or $P \Rightarrow 1$. Since ϵ, 0, and 1 are all palindromes, the basis is proven.

INDUCTION:

- Assume for n steps it is true.
- Then, show for (n+1) steps it must be true.

Left as an exercise.

Sentential Forms

$G = (V, T, P, S)$ is a CFG, then any string α in $(V \cup T)^*$ such that $S \stackrel{*}{\Rightarrow} \alpha$ is a *sentential form*.

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\begin{array}{llll}
1. & E & \rightarrow & I \\
2. & E & \rightarrow & E + E \\
3. & E & \rightarrow & E * E \\
4. & E & \rightarrow & (E) \\
\\
5. & I & \rightarrow & a \\
6. & I & \rightarrow & b \\
7. & I & \rightarrow & Ia \\
8. & I & \rightarrow & Ib \\
9. & I & \rightarrow & I0 \\
10. & I & \rightarrow & I1
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A context-free grammar for simple expressions

$$E \Rightarrow E * E \Rightarrow E * (E) \Rightarrow E * (E + E) \Rightarrow E * (I + E)$$

Sentential Forms

$G = (V, T, P, S)$ is a CFG, then any string α in $(V \cup T)^*$ such that $S \overset{*}{\Rightarrow} \alpha$ is a *sentential form.*

If $S \underset{lm}{\overset{*}{\Rightarrow}} \alpha$, then α is a *left-sentential form,*

and if $S \underset{rm}{\overset{*}{\Rightarrow}} \alpha$, then α is a *right-sentential form.*

Note that the language $L(G)$ is those sentential forms that are in T^*; i.e., they consist solely of terminals.

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- Is this sentential form left-sentential? Or right-sentential?

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A context-free grammar for simple expressions

$$
\boxed{E \Rightarrow E * E \Rightarrow E * (E) \Rightarrow E * (E + E) \Rightarrow E * (I + E)}
$$

- Is this sentential form left-sentential? Or right-sentential?
- It is a sentential form. But neither left nor right.

Exercise 5.1.2: The following grammar generates the language of regular expression $\mathbf{0}^*\mathbf{1}(\mathbf{0}+\mathbf{1})^*$:

$$\begin{array}{lcl} S & \rightarrow & A1B \\ A & \rightarrow & 0A \mid \epsilon \\ B & \rightarrow & 0B \mid 1B \mid \epsilon \end{array}$$

Give leftmost and rightmost derivations of the following strings:

* a) 00101.

b) 1001.

c) 00011.

Note, the given grammar is not a regular grammar (even-though it generates a regular language).

Can you find $L(G)$?

- $S \rightarrow aS|bS|a|b|\epsilon$

Can you find $L(G)$?

- $S \to aS|bS|a|b|\epsilon$

- Answer: All strings. Σ^*

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The answer:

$$a^{n_1} \mathrm{b}^{\mathrm{n}_1} a^{n_2} \mathrm{b}^{\mathrm{n}_2} \ldots a^{n_k} \mathrm{b}^{\mathrm{n}_\mathrm{k}} \in L(G)$$

$L(G) = (\{a^n b^n | n \geq 1\})^*$

Can you find $L(G)$?

$$S \rightarrow SS \mid [S] \mid (S) \mid [\,] \mid (\,)$$

Can you find $L(G)$?

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Set of all balanced parentheses with alphabet
$\{\ (,\ ),\ [,\ ]\ \}$

Can you find $L(G)$?

1. $S \to aB|bA$
2. $B \to b|bS|aBB$
3. $A \to a|aS|bAA$

Can you find $L(G)$?

1. $S \rightarrow SaSbS|SbSaS|\epsilon$