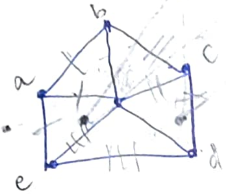
Covering: $G(V, E)$

A Covering of a graph G is a subset K of V such that every edge of G has atleast one end in K.

Note: A covering K is minimum covering if G has no covering K' with $|K'| < |K|$



matching is alternating path in G.
↓
no repeated vertices.

af bc ed → matching

Covering: $K = \{a, c, d, f\}$

$K' = \{a, c, d, e, f\}$ → not a minimum covering.

Remark:

1) If K is a covering of G and M is matching of G.
Then K contains atleast one end of each edge of M.
$\therefore |M| \leq |K|$

2) M^* - maximum matching
$\hat{K}$ - minimum covering, then $|M^*| \leq |\hat{K}|$

3) If G is bipartite, we do have $|M^*| = |\hat{K}|$

Lemma: Let M be a matching and K be a covering such that $|M| = |K|$. Then M is a maximum matching and K is a minimum covering.

$|M| = |K|$

proof: $|M| \leq |M^*|$ $|\hat{K}| \leq |K|$

$|M| \leq |K|$

$\boxed{|M^*| \leq |\hat{K}|}$

$\rightarrow |M| \leq |M^*| \leq |\hat{K}| \leq |K|$

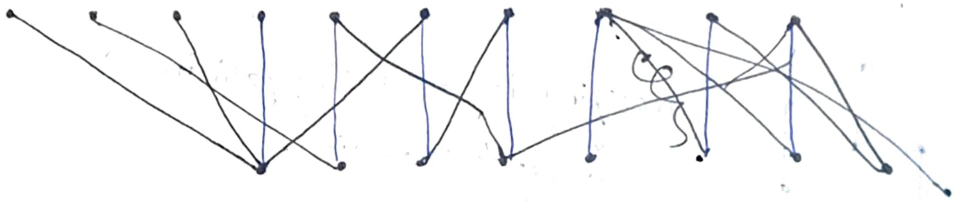
$|M^*| = |\hat{K}|$

Theorem:

In a bipartite graph G, the no. of edges in a maximum matching is equal to the number of vertices in a minimum covering.

$$|M| = |K|$$

Proof:



Covering :- $K \subseteq V$ ∋ every edge of G has atleast one end in K.

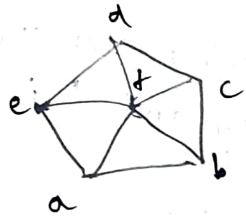
Covering number: $\beta(G)$ - Cardinality of minimum Covering

Edge Covering: A $L \subseteq E$ ∋ each vertex of G is an end of some edge in L.

Independant Set: A Subset S of V is called as independant set of G if no two vertices of S are adjacent in G.

An independant set is maximum if G has no independant set S' with $|S'| > |S|$.

P)

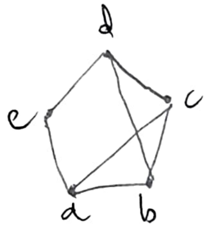


$\{a, c\} = S$ (independant)

$\{a\} = S$

Maximum independant set.

Clique:- A clique of a simple graph is a subset S of V such that $G[S]$ is complete.



$S = \{a\ b\ c\}$ $G[S] =$ Complete

$S = \{b\ c,\ d\}$ $G[S] =$ Complete

$\alpha(G)$: Independance number of G.
→ Cardinality of maximum independant set.

Theorem: A set $S \subseteq V$ is an independant set iff $V \setminus S$ is a covering of a G.

proof: S is independant set.
⟺ no edge of G has both edges in S.
⟺ each edge has atleast one end in $V \setminus S$
There fore $V \setminus S$ is a cover of G if S is independant set of G

Corollary: $\alpha + \beta = V$

proof:
S - Maximum independant set.
K - minimum covering.
Then $V \setminus K$ is an independant set.
$V \setminus S$ is a covering.

β = minimum Covering
α = maximum ind. set.

$V - \beta = |V \setminus K| \leq \alpha \rightarrow (1)$
$V - \alpha = |V \setminus S| \geq \beta \rightarrow (2)$

(1) and (2) implies theorem.

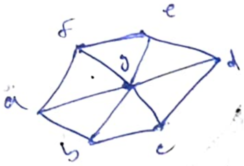
$K = \{f, a, c, d\}$
$V - K = \{b, e\}$
↓
independant set
↓
maximum.

$V - K = S \Rightarrow V - S = K$

α' : edge independance number.
Cardinality of maximum ~~independant~~ matching.

β': edge covering number.
Cardinality of minimum edge covering.

Theorem: If $\delta > 0$, then $\alpha' + \beta' = V$.



$\alpha' = \{fe, dc, ag\}$
3
$\beta' = \{ag, fe, dc, bg\}$
= 4

$3 + 4 = 7 = V$.

Independant Set $S \subseteq V$ Clique $K \subseteq G$ → G^c : $V(K)$ is independant set.

→ S is clique of G, iff S is independant set of G^c.

$r(s,t)$, $s, t \geq 1$

$G \cup G^c = K_n$

G G^c

n = 3 :

red - graph
black compliment (or) blue.

$r(2,1)$

(empty graph)

Ramsey's Theorem:

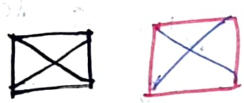
Def: The Smallest n such that every 2-coloring of K_n contain a monochromatic clique of order s or t.
Denoted by $r(s,t)$ known as Ramsey Number.

Remark:

1) For all possible integers, there exist $r(s,t)$ such tha if $n \geq r(s,t)$ and edges of K_n are coloured with red or blue then there is a "red k-clique" or "blue 2-clique".

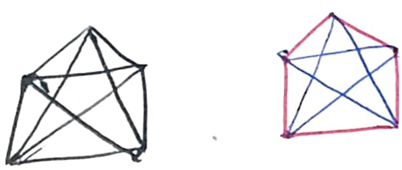
→ $r(\underset{R}{3}, \underset{B}{3})$

n = 4 :

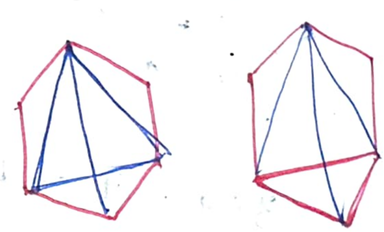


n = 3 :

n = 5 :



n = 6 :



3 blue clique. 3 red clique

$\rightarrow$ $r(s,1) = 1$
$r(1,t) = 1$ } trivial Case.

$r(2,t) = t$.
$r(s,2) = s$

$r(2,t)$
K_2 blue $\quad$ K_t red.

HW

$r(3,3) = 6$.

$\rightarrow r(3,4) : \leq r(\underline{2,4}) + r(3,3)$
$\qquad\qquad\quad \downarrow 4 \qquad\quad \downarrow 6$

≤ 10.

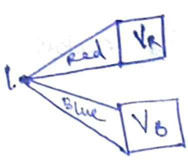
Statement: $\quad r(s,t) \leq r(s,t-1) + r(s-1,t)$.

Theorem: $r(s,t) \leq r(\underset{R\ B}{s-1,t}) + r(\underset{R\ B}{s,t-1})$

Recursion $\quad$ since it is a Ramsey number, it should either have a $s-1$ clique or t clique.

$\rightarrow$ 2 coloring K_n
Smallest number $r(K,L)$
K clique red. $\quad$ L independent set → Blue.

$N = r(K, l-1) + r(K-1, l)$



$\{(1,x) : x \in V$ red colored $\}$

$\{(1,x) : x \in V$ blue colored $\}$

$N-1 = |V_R| + |V_B|$

Note: If both $r(K,l-1)$ and $r(K-1,l)$ are even then strict inequality holds.

$r(K,l) \leq r(K,l-1) + r(K-1,l) - 1$

$\rightarrow r(3,3) \leq r(2,3) + r(3,2) = 6$.

$r(4,5) \leq r(3,5) + r(4,4) \leq \underline{r(2,5) + r(3,4)} + r(3,4) + r(4,3)$

$r(3,5) \leq r(3,4) + r(2,5)$

$\leq r(2,4) + r(3,3) + \underline{5}$

$\leq 4 + 6 - 1 + 5$

$\leq 14.$

$r(4,4) \leq r(3,4) + r(4,3)$

$\leq r(3,3) + r(2,4) + r(3,3) + r(4,2)$

$\leq \cancel{6-1+6-1+4+2}$ $(6+4)-1 + (6+4)-1$

≤ 18

$r(4,5) \leq 32.$

(k,l) **Ramsey graph**: A graph on $r(k,l)-1$ vertices that Contains neither a k clique graph nor l- independant set.

<u>Note</u>: By definition of $r(k,l)$, the (k,l)- ramsey graph exist for $k \geq 2$, $l \geq 2$.

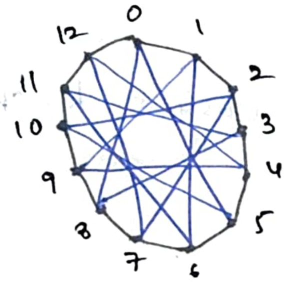
→ $(3,5)$ Ramsey graph : no. of vertices = 13.

Finite field $(Z_{13}, +_{13}, \times_{13})$

$Z_{13} = \{0, 1, 2 \dots 12\}$.

Relation between Ramsey number and ~~field~~ finite field:

Two vertices are adjacent if their difference is a cubic residue of modulo 13.



(4,4) - Ramsey graph.
17 vertices :
(a,b) is dif is quadratic residue.

→ Statement : $r(k,l) \leq \binom{k+l-2}{k-1}$

Proof: By induction
from $r(1,1) = r(k,l) = 1$.
$r(2,l) = l$ $\quad r(k,2) = k$.
theorem holds when $k+l \leq 5$. → base case.

$^{3+3-2}C_2 = {}^4C_2 = \frac{4\times 3}{2} = 6$

Ex: $\{1, 2, \cdots 13\}$
portions:
$\{1, 4, 10, 13\}$
$\{2, 3, 11, 12\}$
$\{5, 6, 7, 8, 9\}$
Mutually exclusive and union of all partitions give the original set

Schur's theorem:
Let $\{S_1, S_2, \ldots S_n\}$ be any partition of the set of integers $\{1, 2, \cdots r_n\}$ where r_n is Ramsey number, then for some i, S_i contains three integers x, y and z satisfy $x + y = z$.

Proof:
K_{r_n} with vertices $\{1, 2, \cdots r_n\}$
→ Color the edges of K_{r_n} in colours $1, 2 \cdots n$.
by the rule, the edge uv is assigned color j, iff
$|u-v| \in S_j$
By Ramsey's theorem, there exists a monochromatic triangle, that is there are 3 vertices, we can say a, b, c such that ab, bc, ca → colored with same color, say i.
$x = a - b \quad y = b - c \quad z = c - a$.
$x + y = z$.

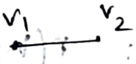
Statement: Every k Chromatic graph has atleast k-vertices of degree k-1.

Statement: For any positive integer k, there exists a k-chromatic graph containing no triangle.

Examples: $G_2 = K_2$

~~$G_3 = K_3$~~

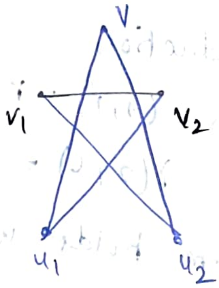
G_2 :

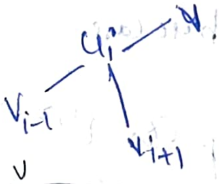


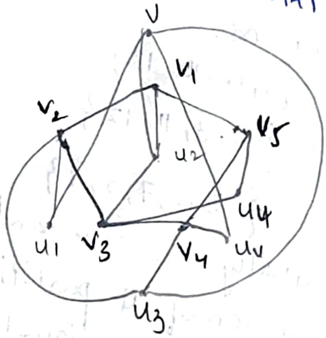
$v_1, v_2, \ldots v_n = G_k$

$u_1, u_2, \ldots u_n, v - G_{k+1}$

G_3 :





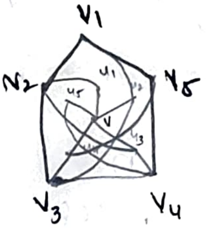


$u_1, u_2, u_3, u_4, u_5, v$

$u_1 - u_2, u_5, v.$ $\quad u_2 - u_1\ u_3\ v$

$u_3 - u_2\ u_4\ v$ $\quad u_4 : u_3\ u_5\ v$

$u_5 : u_4\ v$
u_1



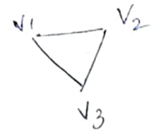
→ Planar Graph:

Def: A graph is said to be planar if it can be drawn in a plane so that its edges intersect only at their ends.

* drawn graph called planar graph $\tilde{G}$ of G and $\tilde{G} \cong G$.

prove: K_5 is non planar:

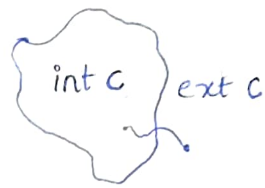
$v_1\ v_2\ v_3\ v_4\ v_5$



C: $v_1\ v_2\ v_3\ v_1$

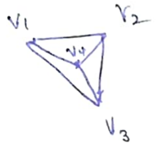
2 regions (int, ext)

Jordan curve: is a continuous non self intersecting curve whose origin and terminus coincide.



C - curve.

if v_4 inside C:

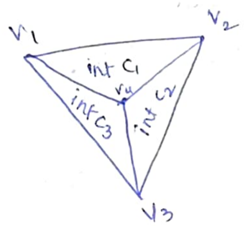


4 regions: 3 int, 1 ext.

Curve joining Interior and Exterior must intersect at a point on curve.

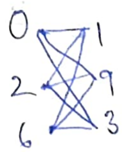
v_5 → ext : intersects curve → nonplanar

v_5 → int : when v_5 is placed in c_1 or c_2 or c_3, v_3, v_1, v_2 are external respectively.



So K_5 is Non planar.

Statement: A graph is nonpolar iff it contains subgraph turned to $K_{3,3}$ or K_5. (Kuratowski)

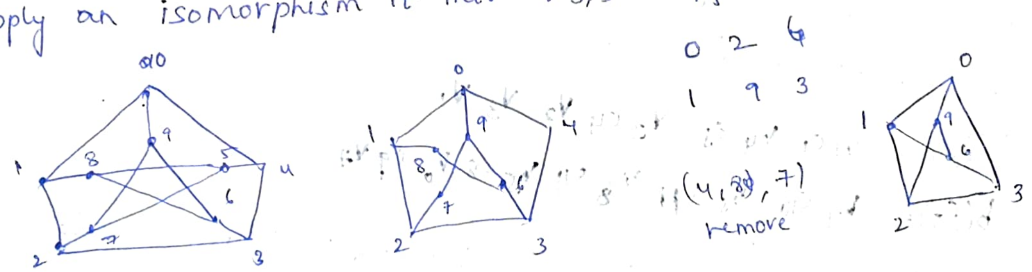


$\cong K_{3,3}$

1) Remove edges and vertices.

2) Collapse degree two vertices into a single edge

3) Apply an isomorphism it into $K_{3,3}$ or K_5.



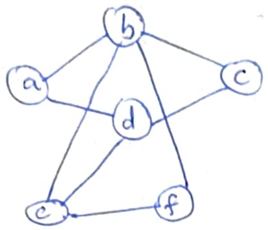
$K_{3,3}$

§ Eulers Formula:

Let $G(V, E)$ be a connected planar graph and F be a set of faces of planar drawing of G. Then

$$|V| - |E| + |F| = 2$$

Ex:



1 2

$6 - 8 + 2 = 0$.

→ Non-planar.

taking and placing d out:

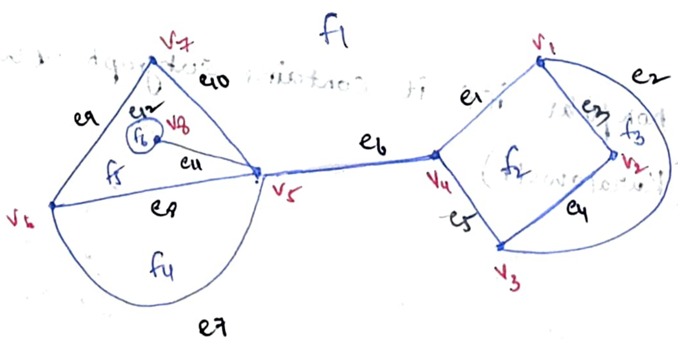
DUAL OF A PLANAR GRAPH:

Let G be a planar graph

→ $F(G)$ - set of faces

→ $\phi(F)$ - no. of faces

* Each planar has exactly one unbounded face.
* G is a connected planar graph $|V| - |E| + |F| = 2$.



$V = \{v_1, v_2, \dots v_8\}$

$E = \{e_1, e_2, \dots e_{12}\}$

$F = \{f_1, f_2, \dots f_6\}$

Note: 1) G is a planar graph and f is a face.

2) $b(f)$ - boundary of a face f.

$b(f_2) = v_1 e_1 v_4 e_5 v_3 e_4 v_2 v_3 v_1$

$b(f_5) = v_7 e_{10} v_5 e_{12} v_8 e_{11} v_5 e_8 v_6 e_9 v_7$

3) A face f is said to be incident with every vertices and edges in its boundary.

) If e is a cut edge in a planar graph, just one face is incident with e. Otherwise there are two faces incident with e.

5) the degree, $d_G(f)$, of a face f is the number of edges with which edge it is incident.

$d(f_2) = 4$

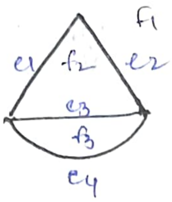
$d(f_5) = 6$

Note: Number of edges in $\underline{b(f)}$.

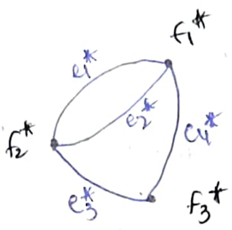
Dual graph G^* of a planar graph G:-

1) For each face f in G, there is vertex f^* in G^*
2) For each edge e in G, there is an edge e^* in G^*.
3) Two vertices f^* and g^* in G^* are joint by an edge e^* in G^* iff their corresponding faces f and g in G are seperated by an edge e in G.

Then G^* is called Dual.

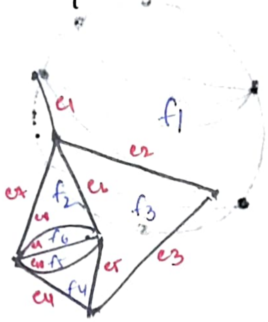


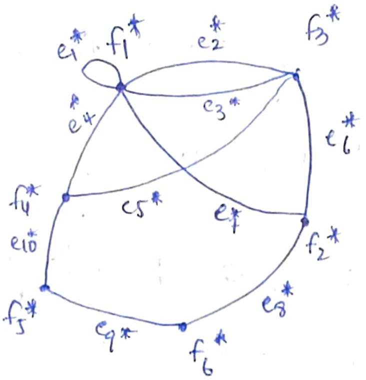
G



G^*

Ex:

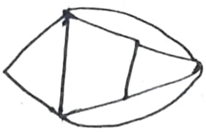




G-connect planar $G^{**} \cong G$

Ex:

G_1



G_2

$\therefore G_1 \cong G_2$

Check if $G_1^* \cong G_2^*$ $\rightarrow$ Not isomorphic.

graph:

Grinberg's theorem:

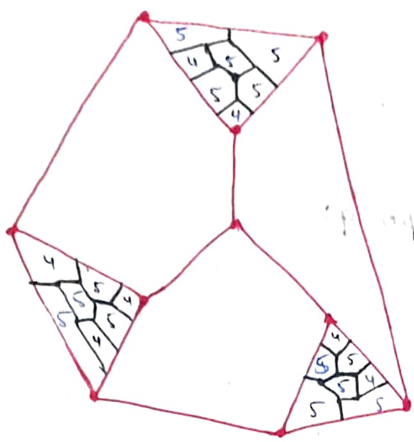
St

22/11

3-Connected: if we remove 3 edge graph → disconnect

* 3-regular, 3-connected non hamilton planar graph:

E



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Grinbergs Theorem: Let G be a loopless plane graph with ham Cycle C. Then

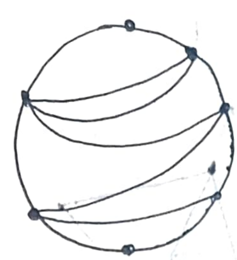
$$\sum_{i=1}^{v}(i-2)(\phi_i' - \phi_i'') = 0.$$

* where ϕ_i' and ϕ_i'' are number of faces of degree i Contained in int C and ext C, respectively.

Proof: E' subset of $E(G)\setminus E(C)$ Contained in C and $\varepsilon' = |E'|$

Then C Contains exactly $(\varepsilon'+1)$ faces.

$$\sum_{i=1}^{v}\phi_i' = \varepsilon' + 1 \rightarrow (1)$$

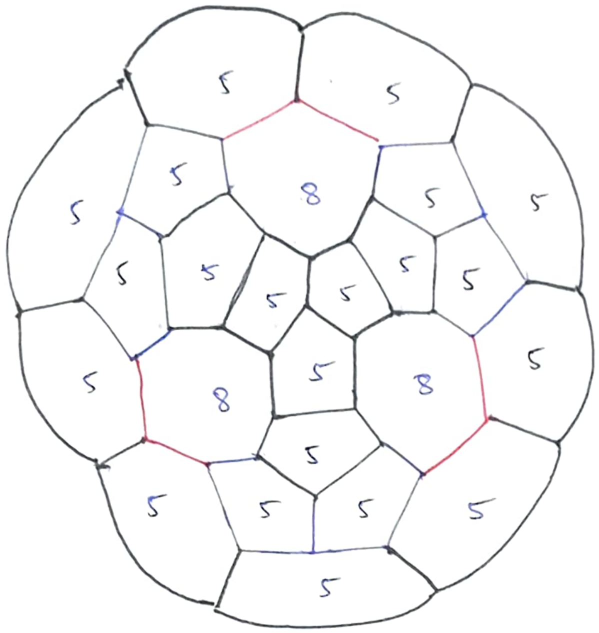


Now each edge in E' is on the boundary of int C. and each edge of C is on the boundary of exactly one face in int C.

$$\sum_{i=1}^{v} i\phi_i' = 2\varepsilon' + v \rightarrow (2)$$

from (1) and (2) → $v - 2 = \sum_{i=1}^{v}(i-2)\phi_i'$

Grinbergs Graph :



faces of degree 5, 8, 9

$(5-2)(\phi_5' - \phi_5'') +$

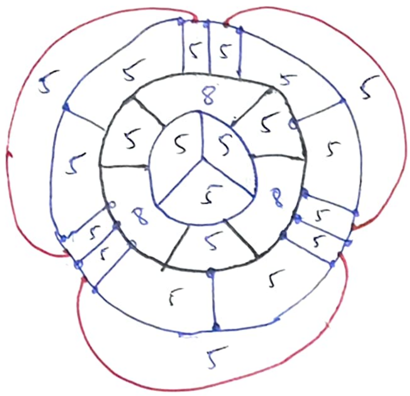
$(8-2)(\phi_8' - \phi_8'') +$

$(9-2)(\phi_9' - \phi_9'')$

$= 0.$

Note:

Cor: Only one face having degree not 2 and 3, others have 2 and 3 then G is non hamiltonian.



H/W: Show that no hamilton cycle contain both the edges e and e' in the following graph: