02.08.'22.

Applications - ODE.

Physical interpretation of the linear second order ordinary diffl: equn: $a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + cy = f(t)$

Vibrating Spring/mass system:

Hooke's law: Suppose a flexible string is suspended vertically from a rigid support and then a mass m is attached to its free end.

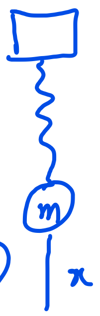
By Hooke's law the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation $\boxed{s}$. $F \propto s$ $F = ks$

(i.e) $F = ks$, k - const: of proportionality called spring const:

Newton's second law:

After a mass m is attached to the spring, it stretches the spring by an amount s and attains a position of equilibrium at which its wt: W is balanced by the restoring force ks

w.k.t $W = mg$



Condition of equilibrium : $mg = ks$ (or) $mg - ks = 0$ — (1)

Assuming that there are no retarding forces (free motion) and if the mass is displaced by an amount x from its equilibrium position, the restoring force $= k(s+x)$

Resultant force (Newton's second law)

$v = \frac{dx}{dt}$

$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg = \underbrace{mg - ks}_{0} - kx \quad \text{by (1)}.$$

rest force opposite dim

$$\therefore m\frac{d^2x}{dt^2} = -kx.$$

Thus the diff. eqn. of free undamped motion is

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x, \quad \omega^2 = \frac{k}{m}$$

$y = f(x)$
$x = f(t)$.

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \leftarrow \text{S.H.M}$$

$\alpha \pm i\beta$.

A.E $\quad m^2 + \omega^2 = 0 \Rightarrow m^2 = -\omega^2 \Rightarrow m = \pm i\omega$. $\quad \alpha = 0, \beta = \omega$.

CF $\quad x_h = e^{0t}(c_1\cos\omega t + c_2\sin\omega t)$

$$x = c_1\cos\omega t + c_2\sin\omega t$$

$x(t) = A\cos(\omega t + \delta)$

2. A spring with a mass of 2kgs has natural length 0.5m. A force of 25.6 N is required to maintain its stretched length of 0.7 m. If it is released with initial velocity 0, find the position of the mass at any time t.

$x = 0.2$

From Hooke's law: $F = ks$

$F = kx$
$= k(0.2)$

$$25.6 = k(0.2)$$

$$k = \frac{25.6}{0.2} = 128$$

$$m\frac{d^2x}{dt^2}+kx=0.$$

$$2\frac{d^2x}{dt^2}+128x=0 \quad \div 2$$

$$\frac{d^2x}{dt^2}+64x=0$$

$x(0)=0.2$.
$x'(0)=0$.

$m^2+64=0$
$m=\pm 8i$

$x_h=e^{0x}(c_1\cos 8t+c_2\sin 8t)$

$x = c_1\cos 8t + c_2\sin 8t$ ✓

$x(0)=0.2 \Rightarrow c_1=0.2$

$\therefore x'=-c_1 8\sin 8t+c_2 8\cos 8t \Rightarrow 0=c_2 \quad \therefore x(t)=0.2\cos 8t$

1 Suppose the spring is subjected to a frictional force or a damping force

$$\text{damping force} = c\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force}.$$

$$= -kx - c\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

A.E $\quad mr^2+cr+k=0.$

$$r=\frac{-c\pm\sqrt{c^2-4mk}}{2m}. \checkmark$$

r_1, r_2

case i : $c^2-4mk>0$ (over damping).

$$x = c_1e^{r_1t}+c_2e^{r_2t} \checkmark$$

case ii $\quad c^2 - 4mk = 0$ (critical damping)

$$r_1 = r_2 = -\frac{c}{2m}.$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = (c_1 + c_2 t)\, e^{-c/2m\, t}$$

Case iii $\quad c^2 - 4mk < 0$ (under damping).

$$\boxed{-\frac{c}{2m}} \pm \omega i \quad , \quad \omega = \frac{\sqrt{4mk - c^2}}{2m}.$$

$$x = e^{-c/2m\, t}\,(c_1 \cos\omega t + c_2 \sin\omega t)$$

2. $\quad x = c_1 e^{-4t} + c_2 e^{-16t}$.

$x(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$,

$x'(0) = 6$

$$x'(t) = -4c_1 e^{-4t} - 16 c_2 e^{-16t}$$

$x'(0) = 0.6$

$$0.6 = -4c_1 - 16c_2.$$

$$= -4c_1 + 16c_1 \Rightarrow 12c_1 = 0.6$$

$$c_1 = \frac{0.6}{12} = 0.05$$

$$x(t) = 0.05e^{-4t} - 0.05e^{-16t}$$

$$= 0.05(e^{-4t} - e^{-16t}).$$

case iii) external force $F(t)$.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

2. Suppose the spring in ex(1) is immersed in a fluid with damping const $c=40$. Find the position of the mass at any time t if it starts from equilibrium position & given a push to start with an initial velocity of 0.6m/s

$$m=2 \qquad k=\frac{F}{s}=\frac{25.6}{0.2}=128 \qquad c=40.$$

$$2\frac{d^2x}{dt^2}+40\frac{dx}{dt}+128x=0.$$

$$\frac{d^2x}{dt^2}+20\frac{dx}{dt}+64x=0.$$

A.E $m^2+20m+64=0$

$$(m+4)(m+16)=0$$

$$m=-4,-16.$$

$$\therefore x=c_1e^{-4t}+c_2e^{-16t}.$$

$$x(0)=0 \qquad x'(0)=0.6.$$

$$0=c_1+c_2 \Rightarrow c_2=-c_1$$

Sub $x=c_1e^{-4t}-c_1e^{-16t}$

$$x'(t)=-4c_1e^{-4t}+16c_1e^{-16t}$$

$$x'(0)=-4c_1+16c_1=0.6$$

$$12c_1=0.6 \qquad c_1=\frac{0.6}{12}=0.05.$$

$$x(t)=0.05e^{-4t}-0.05e^{-16t}=0.05(e^{-4t}-e^{-16t}).$$

(i) $$m\frac{d^2x}{dt^2} = -kx$$

(ii) $$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

(iii) $$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t).$$

$$\therefore m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t).$$

Electric circuits: