Module 1 - Ordinary Differential Equations (ODE).

Consider the homogeneous differential eqn/s with constant coefficients $a_0, a_1, a_2, \ldots, a_n$.

$$\cancel{x\frac{d^2y}{dx^2} + x\frac{dy}{dx}} + \cdot y = 0$$

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \quad \text{—(1)}$$

When $n=1$ $\quad a_0\frac{dy}{dx} + a_1 y = 0$ $\div a_0$ $\Rightarrow \frac{dy}{dx} + \frac{a_1}{a_0}y = 0$ (i.e) $\frac{dy}{dx} + ay = 0$ for some a

Multiplying both sides by e^{ax}

$$e^{ax}\frac{dy}{dx} + ae^{ax}y = 0$$

$$\frac{d}{dx}(ye^{ax}) = y \cdot ae^{ax} + e^{ax} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx}(e^{ax}y) = 0$$

$$\int \frac{d}{dx}(ye^{ax})dx = \int 0\,dx$$

$$ye^{ax} = C$$

$$\Rightarrow ye^{ax} = c, \text{ a const}$$

$$\Rightarrow y = ce^{-ax} \longrightarrow \frac{dy}{dx} + ay = 0$$

Let $y = Ce^{mx}$ trial solution

$$\frac{dy}{dx} = cme^{mx}, \quad \frac{d^2y}{dx^2} = cm^2e^{mx} \cdots \quad \frac{d^ny}{dx^n} = cm^ne^{mx}$$

$$a_0cm^ne^{mx} + a_1cm^{n-1}e^{mx} + a_2cm^{n-2}e^{mx} + \cdots + a_{n-1}cm\,e^{mx} + a_n ce^{mx} = 0$$

$$ce^{mx}(a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n) = 0.$$

$D^n \leftrightarrow m^n$ $\quad D = \frac{d}{dx}$ $\quad D \leftrightarrow m$.

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_{n-1} m + a_n = 0$$ ② Auxiliary eqn.

↳ n roots.

$y = c_1 e^{m_1 x}$ & $y = c_2 e^{m_2 x}$

Let them be $m_1, m_2, m_3, \dots m_n$. Let them be distinct

$c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$

$\dots y = c_n e^{m_n x}$ to be the solutions

$$\frac{d^2y}{dx^2} + 4y = \cancel{0}$$

$CF = CS$

$CS = CF + PI$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ will be the solution.

$m_1 = \alpha + i\beta \quad m_2 = \alpha - i\beta$

homo

Complementary fns.

$\parallel$

complete solution

non-homogeneous

$CF + PI$

$\parallel$

CS.

$a_0 m^2 + a_1 m + a_2 = 0$

$m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$.

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$

$= e^{\alpha x}\left(c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right)$

$= e^{\alpha x}\left(\cos\beta x\,\underbrace{(c_1+c_2)}_{A} + i\sin\beta x\,\underbrace{(c_1-c_2)}_{B}\right)$

$= e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$

If 2 of its roots are real and equal $m_1 = m_2 = k$

$$y = (c_1 + c_2 x)e^{kx} + c_3 e^{m_3 x} + c_4 e^{m_4 x} \cdots$$

1. To obtain the Auxiliary equn $D \leftrightarrow m$, $D = \frac{d}{dx}$.
2. If the roots are real and distinct, $m_1, m_2, \ldots m_n$.

   then $CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$.
3. If 3 of its roots are equal say $m_1 = m_2 = m_3 = m$

   $CF = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$
4. If the roots are imaginary say $\alpha \pm i\beta$

   $CF = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ Second order.

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Second order
first deg - linear
coeffts - consts
homogeneous.

A.E: $m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$, $m = 3, 2$

$\frac{d}{dx} \leftrightarrow m$

roots are real and distinct

$CF = c_1 e^{3x} + c_2 e^{2x}$

Because it is homo no P.I

$\therefore$ complete Soln is $y = c_1 e^{3x} + c_2 e^{2x}$

2. $D^2y - 4Dy + 4y = 0.$ $\qquad D = \frac{d}{dx}.$

A.E $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$(Ax + B)$

$\therefore$ CS = CF $= (A + Bx)e^{2x}$

$\sqrt{-9} = \sqrt{-1}\sqrt{9}$

$\pm i3.$

3. $(D^2 + 9)y = 0.$

A.E $m^2 + 9 = 0$

$m^2 = -9$ or $m = \pm 3i$

$\alpha \pm i\beta.$ $\quad \alpha = 0 \quad \beta = 3$

CS=CF+~~PI~~

CF : $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$e^{0x}(A\cos 3x + B\sin 3x)$

$\frac{d^2y}{dx^2} + 4y = \boxed{x^2}$

CF $= (A\cos 3x + B\sin 3x) =$ CS $= y.$

non-hom

$\frac{d^2y}{dx^2} + 4y = 0$

hom