BMAT 102L - DIFFERENTIAL EQUATIONS AND TRANSFORMS

Module - 2 (Partial Differential Equations) - Tutorial sheet 1

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- Form the PDE of the following by eliminating arbitrary const

(1) $(x-a)^2 + (y-b)^2 = z^2 \cot^2\alpha$ — (1)

Differentiating w.r.t x

$2(x-a) = 2zp\cot^2\alpha$ — (2)

$x-a = zp\cot^2\alpha$ — (2)

Differentiating w.r.t y :

$2(y-b) = 2zq\cot^2\alpha$

$(y-b) = zq\cot^2\alpha$ — (3)

substitute (2) and (3) in (1)

$z^2p^2\cot^4\alpha + z^2q^2\cot^4\alpha = z^2\cot^2\alpha$

$z^2\cot^4\alpha\,[p^2+q^2] = z^2\cot^2\alpha$

$p^2+q^2 = \frac{1}{\cos^2\alpha} \Rightarrow \underline{\underline{p^2+q^2 = \tan^2\alpha}}$

(2) $z = (x^2+a^2)(y^2+b^2)$ — (1)

Differentiating w.r.t x

$p = 2x(y^2+b^2)$ — (2)

$y^2+b^2 = \frac{p}{2x} \rightarrow$ (2)

Differentiating w.r.t y

$q = 2y(x^2+a^2)$

$$x^2 + a^2 = \frac{q}{2y} \quad - (3)$$

Substitute the (3) and (2) in (1)

$$z = \frac{p}{2x} \frac{q}{2y}$$

$\underline{\underline{pq = 4xyz}}$ is the reqd PDE

(3) $$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Differentiating w.r.t x

$$2p = \frac{2x}{a^2}. \quad \Rightarrow a^2 = \frac{2x}{2p} = \frac{x}{p} \quad - (2)$$

Differentiating w.r.t y.

$$2q = \frac{2y}{b^2} \quad \Rightarrow \quad b^2 = \frac{2y}{2q} = \frac{y}{q} \quad - (3)$$

Put (3) and (2) on (1)

$$2z = \frac{x^2}{x/p} + \frac{y^2}{y/q}$$

$$\Rightarrow 2z = x^2 \cdot \frac{p}{x} + y^2 \cdot \frac{q}{y}$$

$\Rightarrow 2z = xp + yq$ is the reqd PDE.

II Find the PDE

(1) of all planes passing through the origin.

Solution)

$lx + my + nz = a$ — (1)

and $l^2 + m^2 + n^2 = 1$ — (2)

$n = \sqrt{1 - l^2 - m^2}$

Substitute (2) in (1)

$lx + my + \sqrt{1 - l^2 - m^2}\, z = a$ — (3)

Differentiate eq (3) w.r.t x

$l + m \times 0 + \sqrt{1 - l^2 - m^2}\, p = 0$

$l = -\sqrt{1 - l^2 - m^2}\, p.$ — (4)

Differentiate eq (3) w.r.t y

$0 + m + \sqrt{1 - l^2 - m^2}\, q = 0$

$m = -\sqrt{1 - l^2 - m^2}\, q$ — (5)

Substitute n, (4) and (5) in eq (1)

$x - \sqrt{1 - l^2 - m^2}\, p - \sqrt{1 - l^2 - m^2}\, q y + \sqrt{1 - l^2 - m^2}\, z = a$

$-px - qy + z = \dfrac{a}{\sqrt{1 - l - m^2}}$ — (6)

we know

$l^2 + m^2 + n^2 = 1$

$(1 - l^2 - m^2) p^2 + (1 - l^2 - m^2) q^2 + (1 - l^2 - m^2) = 1$

$p^2 + q^2 + 1 = \dfrac{1}{1 - l^2 - m^2}$

eq (6), $-px - qy + z = a(p^2 + q^2 + 1)^{1/2}$

$\therefore z = px + qy + a(p^2 + q^2 + 1)^{1/2}$ reqd PDE

$px - qy - z = 0$ //

② PDE of all the planes having equal ... y axis.

soln) $\frac{x}{a}+\frac{y}{a}+\frac{z}{c}=0$ - ①

Differentiating w.r.t x

$\frac{1}{a}+0+P/c=0$

$\frac{P}{c}=-\frac{1}{a}$ - ②

Differentiating w.r.t y

$0+\frac{1}{a}+\frac{q}{c}=0$

$\frac{q}{c}=\frac{-1}{a}$ => ③

divide eq ② and ③

$\frac{P/\cancel{c}}{q/\cancel{c}}=\frac{-1/\cancel{a}}{-1/\cancel{a}}$ => $p=q$ ∴ $\underline{\underline{p-q=0}}$ reqd PDE.

III Form the PDE of the following by eliminating arbitrary functions.

① $z=f_1(x)f_2(y)$

Diff w.r.t x -②

$p=f_1'(x)$ - ②

Diff w.r.t y.

$q=f_2'(y)$ - ③

② ÷ ③ $\frac{p}{q}=\frac{f_1'(x)}{f_2'(y)}$.

Ans) ~~z=q~~ $z\frac{d^2z}{dx\,dy}=\frac{dz}{dx}\frac{dz}{dx}$.

② $z=f(xy/z)$

Diff w.r.t x

$p=\frac{dz}{dx}=f'\left(\frac{xy}{z}\right)\left(\frac{zy-xyp}{z^2}\right)$ - ①

Diff w.r.t y

$$q = \frac{dz}{dy} = f'\left(\frac{xy}{z}\right)\left(\frac{zx - xyq}{z^2}\right) \quad - (2)$$

divide (2) by (3)

$$\frac{p}{q} = \frac{\frac{zy - xyp}{\cancel{z^2}}}{\frac{zx - xyq}{\cancel{z^2}}} = \frac{zy - xyp}{zx - xyq}$$

$$\cancel{z}xp - \cancel{xyqp} = \cancel{z}yq - \cancel{xypq}$$

$$xp = qy$$

(3) $z = f(x^2+y^2)$

diff w.r.t x

$$p = f'(x^2+y^2)\,2x \longrightarrow (1)$$

diff w.r.t y

$$q = f'(x^2+y^2)\,2y \longrightarrow (2)$$

(1) ÷ (2), $\frac{p}{q} = \frac{f'\cancel{(x^2+y^2)}\,\cancel{2}x}{f'\cancel{(x^2+y^2)}\,\cancel{2}y}$

$\Rightarrow$ $Py = qx$ reqd PDE.

(4) $\phi\left[z^2 - xy, \frac{x}{z}\right] = 0$

Let $u = z^2 - xy$

$v = \frac{x}{z}$

using determinants $\Rightarrow$ $\begin{vmatrix} \frac{du}{dx} & \frac{dv}{dx} \\ \frac{du}{dy} & \frac{dv}{dy} \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 2pz - y & \dfrac{z - xp}{z^2} \\ 2pq - x & \dfrac{0 - yq}{z^2} \end{vmatrix}$$

(with x written above $0 - yq$ in the lower-right entry)

$$\Rightarrow (2pz - y)\left(-\frac{xq}{z^2}\right) - \left(\frac{z - xp}{z^2}\right)(2pq - x)$$

$\Rightarrow x^2 p + (2z^2 - xy)q = xz$ reqd PDE

⑤ $f(x^2 + y^2, z - xy) = 0$

$u = x^2 + y^2$

$v = z - xy$

$$\begin{vmatrix} \dfrac{du}{dx} & \dfrac{dv}{dx} \\ \dfrac{du}{dy} & \dfrac{dv}{dy} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2x & p - y \\ 2y & q - x \end{vmatrix}$$

$\Rightarrow (2x)(q - x) - (p - y)(2y)$

$\Rightarrow 2xq - 2x^2 - 2yp + 2y^2$

$\Rightarrow xq - yp = x^2 - y^2$ reqd PDE

⑥ $xyz = \phi(x^2 + y^2 - z^2)$ — ①

Diff w.r.t x

$y[xp + z] = f'(x^2 + y^2 - z^2) \cdot (2x - 2zp)$ — ②

Diff w.r.t y

$x[yq + z] = f'(x^2 + y^2 - z^2) \cdot (2y - 2zq)$ — ③

② ÷ ③ $\Rightarrow \dfrac{y(xp + z)}{x(yq + z)} = \dfrac{2x - 2zp}{2y - 2zq}$

$\Rightarrow \cancel{\frac{y}{x}} = \cancel{\frac{2x - 2zp}{2y - 2zq}} \Rightarrow \cancel{y^2 - yzq = xy - zpx}$

$\cancel{xq - yp = x^2 - y^2}$

$(yz + xyp)(2y - 2zp) = (xz + xyq)(2x - 2zp)$

reqd PDE

(7) $xy + yz + zx = f\left(\frac{z}{x+y}\right)$ — (1)

Diff w.r.t x

$y + 0 + xp + z = f'\left(\frac{z}{(x+y)}\right)\left(\frac{(x+y)p - z}{(x+y)^2}\right) \to$ (2)

Diff w.r.t y

$q + xq + z = f'\left(\frac{z}{(x+y)}\right)\left(\frac{(x+y)q - z}{(x+y)^2}\right) \to$ (3)

(2) ÷ (3) $\frac{y + xp + z}{q + xq + z} = \frac{xp + yp - z}{xq + yq - z}$

$(y + xp + z)(xq + yq - z) = (xp + yp - z)(q + xq + z)$

$p(x+y)(x+2z) - q(x+y)(y+2z) = z(x+y)$ is the

reqd PDE

(8) $z = (x+y) f(x^2 - y^2)$

Diff w.r.t x

$$P = f(x^2-y^2) + (x+y) \cdot 2x \, f'(x^2-y^2) \quad -(2)$$

Diff w.r.t y

$$q = f(x^2-y^2) + (x+y) \cdot (-2y) f'(x^2-y^2) - (3)$$

(2) ÷ (3) $\quad \frac{P}{q} = \frac{\cancel{(x+y)} \, 2x}{\cancel{(x+y)} (-2y)}$

$py + qx = z$ reqd PDE

(IV) Solve the following equations

(1) $p + q = pq$

can also be written as

$$p + q - pq = 0 \quad -(1)$$

Assume the soln) $\quad z = ax + by - c \quad -(2)$

$$\frac{dz}{dx} = a \qquad \frac{dz}{dy} = b$$

Let $p = a$ and $q = b$ $\quad -(3)$

put (3) in (1)

$$a + b - ab = 0 \quad -(4)$$

$$b = \frac{-a}{1-a} \quad -(5)$$

substitute (5) in (2)

$$z = ax + \frac{+a}{1-a} by + c$$

$\therefore z = ax + \left(\frac{a}{1-a}\right) y + c$ is the reqd solution

(2) $\sqrt{p} + \sqrt{q} = 1$ — (1)

Let the solution be $ax + by + c = z$

$z = ax + by + c$ — (2) $\frac{dz}{dx} = a$ and $\frac{dz}{dy} = b$

$p = a$ $\quad q = b$ — (3)

put (3) in (1)

$\sqrt{a} + \sqrt{b} = 1$ and $\sqrt{b} = 1 - \sqrt{a}$

$b = (1 - \sqrt{a})^2$ — (5)

Put (5) in (2)

$z = ax + (1 - \sqrt{a})^2 + c$

$\therefore z = ax + (1 - \sqrt{a})^2 + c$ is the reqd solution.

(3) $z^2 = 1 + p^2 + q^2$ — (1)

Let solution be $ax + by + c = z$

$z = ax + by + c$ — (2) $\quad \frac{dz}{dx} = a$ and $\frac{dz}{dy} = b$

$p = a$ $\quad q = b$ $\Big\}$ → (3)

Put (3) in (1)

$z^2 = 1 + a^2 + b^2$, $\quad b = \sqrt{z^2 - 1 - a^2}$ — (4)

Put (4) in (2)

$z^2 = 1 + a^2 + \sqrt{z^2 - 1 - a^2}$

$z^2 = 1 + a^2 + \sqrt{z^2 - 1 - a^2}$

$z = 1 + a^4 + z^2 - 1 - a^2$

$z = \cosh\left\{\frac{x + ay}{\sqrt{1 + a^2}} + b\right\}$ // reqd PDE

④ $3p^2 - 2q^2 = 4pq$

$f(p, q) = 3p^2 - 2q^2 - 4pq = 0$

Let the solution be $z = ax + by + c$

where $F(a, b) = 0$

$\Rightarrow 3a^2 \mp 2b^2 - 4ab$

$\Rightarrow 2b^2 - 4ab - 3a^2 = 0$

using Quadratic Equations:

$$b = \frac{-4a \pm \sqrt{16a^2 + 24a^2}}{4} = -a \pm \frac{\sqrt{10}}{2} a$$

$$= a\left[-1 \pm \frac{\sqrt{10}}{2}\right]$$

∴ the complete integral

$$z = ax + \left[\pm \frac{\sqrt{10}}{2}\right] y + c$$

⑤ $p^2 + q^2 - 4pq = 0$

Let the solution be $z = ax + by + c$

$\frac{dz}{dx} = p = a \quad \frac{dz}{dy} = q = b$

using Quadratic Equation,

$$a = \frac{4b + \sqrt{4^2 b^2 - 4b^2}}{2} \Rightarrow a = 2 \pm \sqrt{3}$$

Substitute

$(2 \pm \sqrt{3})x \pm by + c$ read solution putting $z = f(x), x = ay$

$$\frac{dz}{dy} = a = \frac{dz}{dx} a \quad \text{--- (1)}$$

$$\frac{dz}{dx}\left(1 + \frac{dz}{dx} a\right) = \frac{dz}{dx} az$$

$$\frac{dz}{dx} + \left(\frac{dz}{dx}\right)^2 a = \frac{dz}{dx} az$$

$$1 + \frac{dz}{dx} a = az \Rightarrow \frac{dz}{dx} = \frac{z-1}{a}$$

$$\Rightarrow \int \frac{dz}{z-1} = \int \frac{dx}{a} \quad \text{Integrating}$$

put $z - 1 = u$

$dz = du$

$$\Rightarrow \log|z-1| = \frac{x}{a} + c \Rightarrow \log|z-1| = \frac{x+ay}{a} + c.$$

$$\Rightarrow \log(az-1) = x + ay + b$$

(7) $z = px + qy + p^2q^2$ --- (1)

put $P = a$ and $q = b$ into eq (1) gives complete integral

$z = ax + by + a^2b^2$ --- (2)

singular integral : partially differentiate (2) w.r.t a and b

$$\frac{dz}{da} = x + 2ab^2 \qquad \frac{dz}{db} = y + 2ba^2$$

Eq (1) is in clairaut's form

$x = -2ab^2$ (3) and $y = -2a^2b$ (4) then $xy = 4a^3b^3$ (5)

From (3) $\frac{x}{b} = -2ab$. From (4) $\frac{y}{a} = -2ab$

From eq (2) $\Rightarrow z = ab\left[\frac{x}{b} + \frac{y}{a} + ab\right] = ab(-2ab - 2ab + ab)$

$\Rightarrow z = -3a^2b^2$, Now $z^3 = -27a^6b^6 = -27(a^3b^3)^2$

using (5) $z^3 = -27\left(\frac{xy}{4}\right)^2 \Rightarrow z^3 = -\frac{27}{16}x^2y^2$ //

[illegible]

Let $z = \phi(u)$ where $u = x^2 + 2y$

$\frac{\partial z}{\partial x} = 2x \frac{dz}{du}$ $\quad \frac{\partial z}{\partial y} = 2y \frac{dz}{du}$ [illegible]

Sub these into (1)

$\left(\frac{dz}{du}\right)^2 x^2 = \left(\frac{dz}{du}\right)^2 z^2 \left(1 - \left(\frac{dz}{du}\right)^2\right)$

$a^2 = z^2 - z^2 \left(\frac{dz}{du}\right)^2 \Rightarrow \frac{a^2 - z^2}{-z^2} = \left(\frac{dz}{du}\right)^2$

$\Rightarrow \frac{dz}{du} = \frac{\sqrt{z^2 - a^2}}{z} = \frac{\sqrt{a^2 - z^2}}{-z}$

$= \int \frac{-z\,dz}{\sqrt{a^2 - z^2}} = \int du$

$\Rightarrow a^2 - z^2 = (u + c)^2$ is required the complete integral

(ii) $p - y^2 = q + x^2$

Let the eq (ii) be of the form [illegible] $f(x, p) = F(y, q)$: $p - x^2 = q + y^2$

$\Rightarrow p - x^2 = a \Rightarrow p = a + x^2$

$\Rightarrow q + y^2 = a \Rightarrow q = a - y^2$

The complete integral of eq (ii)

$dz = p\,dx + q\,dy \Rightarrow (a + x^2)\,dx + (a - y^2)\,dy$

Integrating $\Rightarrow z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + c$

(iii) $\left(\frac{p}{x} + x\right)^2 = -\left(\frac{q}{y} + y\right)^2$ — (iii)

$\Rightarrow \left(\frac{p}{x} + x\right)^2 = a$ and $-\left(\frac{q}{y} + y\right)^2 = a$

$\Rightarrow \frac{p}{x} + x = \sqrt{a}$ and $-\left(\frac{q}{y} + y\right) = \sqrt{a}$

$\Rightarrow p = x(\sqrt{a} - x)$ $\quad q = -y(\sqrt{a} + y)$

The complete integral of eq (1)

$dz = p\,dx + q\,dy$

$= 2(\sqrt{a} - x)dx + -2(\sqrt{a} + y)dy$

on integrating

$z = [-(x-a)^2 + \frac{1}{2}[\sqrt{1-k^2}\, y - \frac{y^2}{2}] + c]$

(12) $p^2 + q^2 = x + y$

writing the above equation as $\Rightarrow p^2 - x = y - q^2 = a$ (say).

a is arbitrary constant,

$\Rightarrow p^2 - x = a \Rightarrow p = (x+a)^{1/2}$

$\Rightarrow y - q^2 = a \Rightarrow q = (y-a)^{1/2}$

The complete integral of Eq (1)

$dz = p\,dx + q\,dy \Rightarrow (x+a)^{1/2}dx + (y-a)^{1/2}dy$

Integrating $= \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b$

(13) $z = px + qy + \sqrt{1+p^2+q^2}$ - (1)

Put $p = a$ and $q = b$ into eq (1) gives complete integral.

$z = ax + by + \sqrt{1+a^2+b^2}$ - (2)

Singular integral : partially diff (2) w.r.t a and b

$\frac{dz}{da} = x + \frac{1}{\cancel{2}\sqrt{1+a^2+b^2}} \cdot \cancel{2}a \qquad \frac{dz}{db} = y + \frac{\cancel{2}b}{\cancel{2}\sqrt{1+a^2+b^2}}$

Eq (1) is in Clairaut's form,

$x = \frac{-a}{\sqrt{1+a^2+b^2}} \quad \to (3) \qquad y = \frac{-b}{\sqrt{1+a^2+b^2}} \quad \to (4)$

from (3) and (4), we get $x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$

Subtract 1 from both sides, $1 - x^2 - y^2 = 1 - \frac{a^2+b^2}{1+a^2+b^2}$

$1 - x^2 - y^2 = \frac{1}{1+a^2+b^2} \Rightarrow \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$ - (5)

from (5) and (6) we get, $a = \frac{-x}{\sqrt{1-x^2-y^2}}$ and $b = \frac{-y}{\sqrt{1-x^2-y^2}}$
$\to$ (6) $\qquad \to$ (7)

Substituting (5), (6) and (7) in eq (2)

we get, $z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$

$z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2$

$\therefore \boxed{x^2+y^2+z^2=1}$ which is the singular integral.

(14) $z = px + qy + p^2 + pq + q^2$

The given equation is in clairaut's form,

The complete integral $\Rightarrow z = ax + by + a^2 + ab + b^2$ — (1)

Singular integral : partially diff eq (1) w.r.t a and b.

$\frac{dz}{da} = x + 2a + b \qquad \frac{dz}{db} = y + a + 2b$

$a = \frac{1}{3}(y-2x) \qquad b = \frac{1}{3}(x-2y)$

Substitute values a and b in eq (1)

$z = \frac{1}{3}(y-2x)x + \frac{1}{3}(x-2y)y + \frac{1}{9}(y-2x)^2 + \frac{1}{9}(x-2y)(y-2x)$
$+ \frac{1}{9}(x-2y)^2$

Solving we get,

$z = xy - x^2 - y^2$ which is the singular integral.

(15) $z = px + qy + p^2 - q^2$

The given equation in clairaut's form,

The complete integral $\Rightarrow$, $z = ax + by + a^2 - b^2$ — (1)

Singular integral : partially diff (1) w.r.t a and b

$\frac{dz}{da} = x + 2a \qquad \frac{dz}{db} = y - 2b$

$a = -x/2 \qquad b = y/2$

Substitute in eq (1), $z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$

$z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4} \Rightarrow 4z = y^2 - x^2$

is the singular integral

(16) $z = px + qy + p^2q^2$ (same as question (7)).

(17) $pz = 1 + q^2$ — (1)

put $z = f(X)$ where $X = x + ay$

$\frac{dz}{dX} = p = \frac{dz}{dX}$ and $\frac{dz}{dy} = q = \frac{dz}{dX}(a)$.

put them in eq (1)

$\left(\frac{dz}{dX}\right)z = 1 + \left(\frac{dz}{dX}\right)^2 a^2$

Ans) $\frac{1}{4a^2}\left[\frac{z^2}{2} - \frac{z\sqrt{z^2-4a^2}}{2} + \frac{4a^2}{2}\cosh^{-1}\left(\frac{z}{2a}\right)\right] = \frac{x+ay}{2a^2} + b$

(18) $p^3 + q^3 = 8z$

(11) $q = px + p^2$

soln) This is of the type $F_1(x, p) = F_2(q, y)$.

let $q = px + p^2 = k^2/4$ (constant)

(here assuming the constant as $k^2/4$ is only for convenience)

$q = \frac{k^2}{4}$; $p^2 + px - \frac{k^2}{4} = 0$ (which is a quadratic equation in p)

$q = \frac{k^2}{4}$ — (1) $\quad p = \frac{-x \pm \sqrt{x^2+k^2}}{2}$ (2)

we know that, $dz = p\,dx + q\,dy$ — (3)

sub (1), (2) and (3) $\Rightarrow dz = \left(\frac{-x \pm \sqrt{x^2+k^2}}{2}\right)dx + \frac{k^2}{4}dy$

Integrating we get, $\int dz = \int \frac{1}{2}\left(-x \pm \sqrt{x^2+k^2}\right)dx + \int \left(\frac{k^2}{4}\right)dy$

$z = \frac{-x^2}{4} \pm \frac{1}{2}\left\{\frac{x}{2}\sqrt{x^2+k^2} + \frac{k^2}{2}\sin^{-1}\left(\frac{x}{k}\right)\right\} + \frac{k^2y}{4} + b$