Module 1 - Ordinary Differential Equations (ODE).

Consider the homogeneous differential eqn/: with constant coefficients $a_0, a_1, a_2, \ldots, a_n$.

$$x\frac{d^2y}{dx^2} + \cancel{x\frac{dy}{dx}} + \cdot y = 0$$

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \quad \text{--- (1)}$$

When $n=1$ $\quad a_0\frac{dy}{dx} + a_1y = 0$. $\div a_0$ $\Rightarrow \frac{dy}{dx} + \frac{a_1}{a_0}y = 0$ (i.e) $\frac{dy}{dx} + ay = 0$ for some a

Multiplying both sides by e^{ax}

$$\underline{e^{ax}\frac{dy}{dx} + ae^{ax}y = 0}$$

$$\frac{d}{dx}(ye^{ax}) = y.ae^{ax} + e^{ax}.\frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx}(e^{ax}y) = 0$$

$$\int \frac{d}{dx}(ye^{ax})dx = \int 0\,dx$$

$$\Rightarrow ye^{ax} = C, \text{ a const}$$

$$ye^{ax} = C$$

$$\Rightarrow \underline{y = Ce^{-ax}} \longrightarrow \frac{dy}{dx} + ay = 0$$

Let $y = Ce^{mx}$ trial solution

$$\frac{dy}{dx} = cme^{mx}, \quad \frac{d^2y}{dx^2} = cm^2e^{mx} \ldots \quad \frac{d^ny}{dx^n} = cm^ne^{mx}$$

$$a_0cm^ne^{mx} + a_1cm^{n-1}e^{mx} + a_2cm^{n-2}e^{mx} + \cdots + a_{n-1}cm\,e^{mx} + a_n ce^{mx} = 0$$

$$ce^{mx}(a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots + a_{n-1}m + a_n) = 0.$$

$D^n \leftrightarrow m^n \quad D = \frac{d}{dx} \quad D \leftrightarrow m.$

$$a_0 m^{n} + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_{n-1} m + a_n = 0$$ Auxiliary eqn.

n roots.

$y = c_1 e^{m_1 x}$ & $y = c_2 e^{m_2 x}$

Let them be $m_1, m_2, m_3, \dots, m_n$. Let them be distinct

$c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$

$\dots$ $y = c_n e^{m_n x}$ to be the solutions

$\frac{d^2y}{dx^2} + 4y = x^2$

CF = CS

CS = CF + PI

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ will be the solution.

$m_1 = \alpha + i\beta \quad m_2 = \alpha - i\beta$

homo

Complementary fns.

$\|$

Complete solution

non-homogeneous

CF + PI.

$\|$

CS.

$a_0 m^2 + a_1 m + a_2 = 0$

$m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left(c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right)$$

$$= e^{\alpha x}\left(\cos\beta x\,\underbrace{(c_1 + c_2)}_{A} + \underbrace{i\sin\beta x\,(c_1 - c_2)}_{B}\right)$$

$$= e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

2 of its $m_1 = m_2 = k$
If ~~the~~ roots are real and equal

$$y = (c_1 + c_2 x)e^{kx} + c_3 e^{m_3 x} + c_4 e^{m_4 x} \dots$$

1. To obtain the Auxiliary eqn $D \leftrightarrow m$, $D = \frac{d}{dx}$.
2. If the roots are real and distinct, $m_1, m_2, \dots m_n$.
   then $CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$.
3. If 3 of its roots are equal say $m_1 = m_2 = m_3 = m$
   $CF = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
4. If the roots are imaginary say $\alpha \pm i\beta$
   $CF = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ Second order.

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

second order
first deg - linear
coeffts - consts
homogeneous.

$\frac{d}{dx} \leftrightarrow m$

A.E: $m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$, $m = 3, 2$

roots are real and distinct

$CF = c_1 e^{3x} + c_2 e^{2x}$

Because it is homo no P.I

$\therefore$ complete Soln is $y = c_1 e^{3x} + c_2 e^{2x}$

2. $D^2y - 4Dy + 4y = 0$. $D = \frac{d}{dx}$.

A.E $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$(Ax + B)$

$\therefore CS = CF = (A + Bx)e^{2x}$

$\sqrt{-9} = \sqrt{-1}\sqrt{9}$

$\pm i3$.

3. $(D^2 + 9)y = 0$.

A.E $m^2 + 9 = 0$

$m^2 = -9$ or $m = \pm 3i$

$\alpha \pm i\beta$. $\alpha = 0$ $\beta = 3$

$CS = CF + \cancel{PI}$

CF : $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$e^{0x}(A\cos 3x + B\sin 3x)$

$CF = (A\cos 3x + B\sin 3x) = CS = y$.

$\frac{d^2y}{dx^2} + 4y = \boxed{x^2}$

non-hom

$\boxed{\frac{d^2y}{dx^2} + 4y = 0}$

hom

16.02.2022.

Second order linear non-homogeneous differential equn/: with const/: coeffts/:

1. $y'' - 4y = \boxed{8x^2 - 2x}$ — (1)

$m_1 = 2$
$m_2 = -2$

Auxiliary equn/: $m^2 - 4 = 0$

$m^2 = 4$, $m = \pm 2$.

The roots are real and distinct.

$8x^2 - 2x$

$16x - 2$

16

complementary fn/: : $c_1 e^{2x} + c_2 e^{-2x}$

md of Undetermined coeffts :

Let the initial Solution be $y_p = ax^2 + bx + c$

$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x)$

$y_p' = 2ax + b$

$y_p'' = 2a$

$2a - 4(ax^2 + bx + c) = 8x^2 - 2x$

$-4ax^2 - 4bx + (2a - 4c) = 8x^2 - 2x$

coeffts of x^2: $-4a = 8 \Rightarrow a = -2$

Coeffts of x: $-4b = -2 \Rightarrow b = 1/2$

const/. terms : $2a - 4c = 0$

$2a = 4c$ (or) $-4 = 4c \Rightarrow c = -1$.

$y_p = -2x^2 + \frac{1}{2}x - 1$

$\times \begin{cases} y_p' = -4x + 1/2 \\ y_p'' = -4 \end{cases}$

$y_p'' - 4y_p = -4 - 4(-2x^2 + 1/2 x - 1)$

$= 8x^2 - 2x$

Complete solution = comp fn + PI

$$y = \overbrace{c_1 e^{2x} + c_2 e^{-2x}} \underbrace{- 2x^2 + \tfrac{1}{2} - 1}.$$

2. $$y = y_h + y_{PI}.$$

Solve $y'' + 2y' - 3y = 4e^{2x}$ — (1).

Auxiliary eqn, $m^2 + 2m - 3 = 0$. ← To find y_h.

$(m+3)(m-1) = 0$

$m = -3, 1$. The roots are real & distinct.

$\therefore \quad y_h = c_1 e^{-3x} + c_2 e^{x}$.

To find y_p: Let the initial soln be $y_p = ce^{2x}$

$y_p' = 2ce^{2x} \qquad y_p'' = 4ce^{2x}$.

Sub in (1). $4ce^{2x} + 2(2ce^{2x}) - 3ce^{2x} = 4e^{2x}$.

$5ce^{2x} = 4e^{2x}$

$\Rightarrow c = 4/5$

Thus $y_p = \frac{4}{5} e^{2x}$

GS = CF + PI

$y = y_h + y_p = c_1 e^{-3x} + c_2 e^{x} + \frac{4}{5} e^{2x}$.

$m^2+2m-3=0$

$(m+3)(m-1)=0$

$m=-3, +1$

$c_1e^{-3x}+c_2e^{x}$

3. $y''+2y'-3y=8e^x$

$y_p = ce^x$

$y_p' = ce^x$

$y_p'' = ce^x$

$ce^x+2ce^x-3ce^x = 0$

Let the initial soln. be $cxe^x = y_p$

$y_p' = c(xe^x + e^x.1)$

$= ce^x(x+1)$

$y_p'' = c\left[e^x(1) + (x+1)e^x\right]$

$= ce^x + cxe^x + ce^x$

$= 2ce^x + cxe^x = ce^x(x+2).$

$(x+2)ce^x + 2ce^x(x+1) - 3cxe^x = 8e^x$

$ce^x\left[x+2+2x+2-3x\right] = 8e^x$

$\Rightarrow ce^x[4] = 8e^x \Rightarrow 4c=8 \Rightarrow c=2.$

$\therefore y_p = 2xe^x$

$y = y_h + y_p = c_1e^{-3x} + c_2e^{+x} + 2xe^x.$

23.02.2022.

1. Solve $y'' - 6y' + 9y = 5e^{3x}$ (1)

$y_{PI} = ce^{3x}$ $y'' = 9ce^{3x}$ $y' = 3ce^{3x}$

$(9c - 18c + 9c)e^{3x} = 0 \neq 5e^{3x}$

$y_{PI} = cxe^{3x}$ on substitution LHS $= 0 \neq$ RHS

A.E $m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3$ is a double root.

C.F or hom solution $y_h = (c_1 + c_2 x)e^{3x}$.

To find the P.I

Let the proposed solution be $y_{PI} = cx^2 e^{3x}$

$y' = c\left[x^2 \cdot 3e^{3x} + e^{3x} \cdot 2x\right]$

$= c(3x^2 + 2x)e^{3x}$

$y'' = c\left[(3x^2 + 2x) \cdot 3e^{3x} + e^{3x}(6x+2)\right]$

LHS of (1) $= c\left[(3x^2 + 2x)3e^{3x} + e^{3x}(6x+2)\right] - 6\left[c(3x^2 + 2x)e^{3x}\right] + 9\left[cx^2 e^{3x}\right]$

$= ce^{3x}\left[9x^2 + 6x + 6x + 2 - 18x^2 - 12x + 9x^2\right]$

$= 2ce^{3x}$.

RHS (1) $= 5e^{3x}$. Equating $2ce^{3x} = 5e^{3x}$

$c = 5/2$

$\therefore y_c = y_h + y_{PI}$

$= (c_1 + c_2 x)e^{3x} + \frac{5}{2}x^2 e^{3x}$.

2. Solve $y'' + 9y = -4x \sin 3x$.

$y_h = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$ | If roots are imaginary

A.E: $m^2 + 9 = 0 \Rightarrow m^2 = -9 \Rightarrow m = \pm 3i = \alpha \pm i\beta$ (say)

$\alpha = 0,\ \beta = 3$

$$y_h = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$= c_1 \cos 3x + c_2 \sin 3x.$$

To find PI: Let the proposed solution be

$$y_p = (ax+b)\cos 3x + (cx+d)\sin 3x$$

$$y_p' = (ax+b)(-3\sin 3x) + \cos 3x\,(a) + (cx+d)\,3\cos 3x + \sin 3x\,(c).$$

$$= \sin 3x(-3ax - 3b + c) + \cos 3x(a + 3cx + 3d).$$

$$y_p'' = (-3ax - 3b + c)(3\cos 3x) + \sin 3x(-3a) + (a + 3cx + 3d)(-3\sin 3x) + \cos 3x\,(3c).$$

$$= \cos 3x(-9ax - 9b + 3c + 3c) + \sin 3x(-3a - 3a - 9cx - 9d)$$

$$= \cos 3x(-9ax - 9b + 6c) + \sin 3x(-6a - 9cx - 9d).$$

LHS of (1) $= \cos 3x(-9ax - 9b + 6c) + \sin 3x(-6a - 9cx - 9d)$

$$+ 9\cos 3x\,(ax+b) + 9\sin 3x\,(cx+d).$$

$$= \cos 3x(6c) + \sin 3x(-6a).$$

RHS of (1) $= -4x \sin 3x$ | Equating

$a = 0 \quad c = 0 \quad \therefore y_p = ?$ ~~X~~.

✓ $y_p = (ax+b)x\cos 3x + (cx+d)x\sin 3x = (ax^2+bx)\cos 3x + (cx^2+dx)\sin 3x$

$y_p' = (ax^2+bx)(-\sin 3x(3)) + \cos 3x(2ax+b)$ ✓
$+ (cx^2+dx)(3\cos 3x) + \sin 3x(2cx+d)$ ✓
$= \sin 3x(-3ax^2-3bx+2cx+d) + \cos 3x(2ax+b+3cx^2+3dx)$.

$y_p'' =$

$\cos 3x(2a+6c+3d-9ax^2-9bx+6x+3d)$
$+ \sin 3x(-6ax-3b+2c-6ax-3b-9cx^2-9dx)$

On substitution

$\cos 3x(2a+6\cancel{cx}+6d+6cx) + \sin 3x(-12ax-6b+2c) = -4x\sin 3x$

$x\cos 3x$ terms: $12cx\cos 3x = 0$.
$\Rightarrow c = 0$.

$x\sin 3x$ terms: $-12a\cancel{x\sin 3x} = -4\cancel{x\sin 3x}$
$a = 1/3$

$\cos 3x$ terms: $2a+6d=0$
$a = 1/3$
So $2/3 + 6d = 0$
$6d = -2/3$
$d = \frac{-2}{18} = -1/9$.

$\sin 3x$ terms: $-6b+2c=0$
$b=0$ as $c=0$

$y_p = \frac{1}{3}x^2\cos 3x - 1/9\, x\sin 3x$.

$y_c = y_h + y_p$.

3. Solve: $y'' + 4y = x + 2e^{-2x}$

RHS	Proposed Solution
1. $P(x)$	$Q(x)$
2. ce^{ax}	de^{ax} if a is not a root of the auxiliary eqn.
3. ce^{ax}	dxe^{ax} if a is one of the roots of the auxiliary eqn.
4. ce^{ax}	dx^2e^{ax} if a is a double root of the auxiliary eqn.
To be checked 5. $a\cos bx$ (or) $b\sin bx$	$c\cos bx + d\sin bx$ if b is not a root of the a. eqn.
6. $P(x)\cos bx / P(x)\sin bx$	$Q(x)\cos bx + R(x)\sin bx$
7. $P(x)e^{ax}\cos bx / P(x)e^{ax}\sin bx$	$Q(x)e^{ax}\cos bx + R(x)e^{ax}\sin bx$

23/02/22.

Tutorial sheet - 1.

3(ii). $2y'' - 5y = x^2 + 5e^{-4x}$

A.E $2m^2 - 5 = 0 \Rightarrow m^2 = 5/2 \Rightarrow m = \pm\sqrt{5/2}$.

$\therefore$ CF or $y_h = c_1 e^{\sqrt{5/2}\,x} + c_2 e^{-\sqrt{5/2}\,x}$.

To find PI$_1$:

Let the proposed soln. be $ax^2 + bx + c = y_{p_1}$.

$y_{p_1}' = 2ax + b \qquad y_{p_1}'' = 2a$.

LHS: $2(2a) - 5(ax^2 + bx + c) = x^2 = $ RHS.

$= -5ax^2 - 5bx + (4a - 5c) = x^2 \Rightarrow -5a = 1 \Rightarrow a = -1/5$

Coeff of x: $b = 0$

Const: $4a - 5c = 0 \Rightarrow 5c = 4a$

$\Rightarrow c = 4/5\, a = 4/5(-1/5)$

$= -4/25$

$\therefore y_{p_1} = -1/5\, x^2 - 4/25$

To find PI$_2$: Let the proposed solution be $ce^{-4x} = y_{p_2}$

$y_{p_2}' = -4ce^{-4x}, \quad y_{p_2}'' = 16ce^{-4x}$.

Sub. $2(16ce^{-4x}) - 5ce^{-4x} = 5e^{-4x}$

$27ce^{-4x} = 5e^{-4x} \Rightarrow c = \dfrac{5}{27}$

$\therefore y_{p_2} = \dfrac{5}{27} e^{-4x}$

$y = y_h + y_{p_1} + y_{p_2}$.

25.02.2022 Method of variation of parameters.

1. Solve $\frac{d^2y}{dx^2} + a^2 y = \tan ax$, by the method of variation of parameters.

$\alpha \pm i\beta$

$\alpha = 0$

$\beta = a$

To find hom soln. :

A.E $m^2 + a^2 = 0 \Rightarrow m = \pm ia$

$\therefore$ CF or $y_h = e^{0x}(c_1 \cos ax + c_2 \sin ax)$ $\qquad Pf_1 + Qf_2$

$= c_1 \cos ax + c_2 \sin ax$

$= c_1 f_1 + c_2 f_2$, $f_1 = \cos ax$, $f_2 = \sin ax$.

$f_1' = -a\sin ax$ $\quad f_2' = a\cos ax$.

Let $X = \tan ax$ = RHS.

$f_1 f_2' - f_2 f_1'$

$= a(\cos^2 ax + \sin^2 ax)$

$= a$

$$P = -\int \underbrace{\frac{f_2 X}{f_1 f_2' - f_2 f_1'}}_{\text{Wronskian}} dx$$

Wronskian $= \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

$= f_1 f_2' - f_2 f_1'$

$$= -\int \frac{\sin ax \tan ax}{a} dx$$

$$= -\int \frac{\sin^2 ax}{a\cos ax} dx.$$

$$= -\int \frac{1-\cos^2 ax}{a\cos ax} dx = -\left\{\int \frac{1}{a\cos ax} dx - \int \frac{\cos^2 ax}{\cos ax} dx\right\}$$

$$= -\frac{1}{a}\left\{\int \sec ax\, dx - \int \cos ax\, dx\right\}$$

$\frac{1}{a}\log(\sec ax + \tan ax)$

$$P = -\frac{1}{a}\left\{\left(\frac{\log(\sec ax + \tan ax)}{a}\right) - \left(\frac{\sin ax}{a}\right)\right\}$$

$$= \frac{\sin ax}{a^2} - \frac{\log(\sec ax + \tan ax)}{a^2}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'} dx$$

$$= \int \frac{\cos ax \tan ax}{a} dx = \frac{1}{a} \int \cancel{\cos ax} \cdot \frac{\sin ax}{\cancel{\cos ax}} dx$$

$$= -\frac{1}{a}\left(\frac{\cos ax}{a}\right) = -\frac{\cos ax}{a^2}$$

$$PI = Pf_1 + Qf_2$$

$$= \left[\frac{\sin ax}{a^2} - \frac{\log(\sec ax + \tan ax)}{a^2}\right] \cos ax + \left[\frac{-\cos ax}{a^2}\right] \sin ax$$

$$y_p = -\left[\frac{\log(\sec ax + \tan ax)}{a^2}\right] \cos ax. \qquad y_h + y_p = y.$$

2. $(2D^2 - D - 3)y = 25e^{-x}$. $\qquad D = \frac{d}{dx}$.

A.E $2m^2 - m - 3 = 0$

$2m^2 - 3m + 2m - 3 = 0$

$m(2m-3) + 1(2m-3) = 0 \Rightarrow (m+1)(2m-3) = 0$

$\Rightarrow m = -1, 3/2$.

CF or $y_h = c_1 e^{-x} + c_2 e^{3/2 x}$

$f_1 = e^{-x}$ $\quad$ $f_2 = e^{3/2 x}$

$f_1' = -e^{-x}$ $\quad$ $f_2' = \frac{3}{2} e^{3/2 x}$

$f_1 f_2' - f_2 f_1' = \frac{3}{2} e^{1/2 x} + e^{1/2 x}$

$= \frac{5}{2} e^{x/2}.$

$X = RHS = 25 e^{-x}$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_2 f_1'} \, dx$$

$$= -\int \frac{e^{3/2 x} \cdot 25 e^{-x}}{5/2 \, e^{x/2}} \, dx$$

$$= -\frac{2}{5} \int 25 \, dx = -10x.$$

$e^{3/2 x - x - x/2}$

$= e^0 = 1$

$e^{3/2 x} e^{-x}$

$e^{3/2 x - x}$

$= e^{1/2 x}$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'} \, dx = \int \frac{e^{-x} \cdot 25 e^{-x}}{5/2 \, e^{x/2}} \, dx$$

$$= \frac{2 \times 25}{5} \int e^{-5x/2} \, dx$$

$$= 10 \left(\frac{e^{-5/2 x}}{-5/2} \right)$$

$$= -4 e^{-5/2 x}.$$

$-2x - x/2$

$$\int e^{-ax} \, dx = \left(\frac{e^{-ax}}{-a} \right)$$

$$PI = P f_1 + Q f_2 = -10x e^{-x} - 4 e^{-5/2 x} e^{+3/2 x}$$

$$= -10x e^{-x} - 4 e^{-x}$$

$y_h + y_p = y.$

3. $\dfrac{d^2y}{dx^2} + y = x\sin x$.

A.E $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m=\pm i$

$y_h = e^{0x}(c_1\cos x + c_2\sin x)$.

$= c_1\cos x + c_2\sin x$.

$f_1=\cos x \quad f_2=\sin x$.

$f_1' = -\sin x \quad f_2' = \cos x$.

$f_1f_2' - f_2f_1' = \cos^2 x + \sin^2 x = 1$.

$X = x\sin x = RHS$

$$P = -\int \frac{f_2 X}{f_1f_2' - f_2f_1'}\,dx$$

$$= -\int \frac{\sin x \cdot x\sin x}{1}\,dx$$

$$= -\int x\sin^2 x\,dx$$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$.

$$= -\int x\left(\frac{1-\cos 2x}{2}\right)dx = -\int \frac{x}{2}\,dx + \int \frac{x\cos 2x}{2}\,dx$$

$u=$
dv

$$\int x\cos 2x\,dx.$$

$$= x\left(\frac{\sin 2x}{2}\right) - 1\left(\frac{-\cos 2x}{4}\right) + 0$$

$$= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}$$

$$= -\frac{1}{2}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]$$

$$P. = \frac{1}{4}\left[x\sin 2x + \frac{\cos 2x}{2} - x^2\right].$$

$$Q = \int \frac{f_1 X}{f_1f_2' - f_2f_1'}\,dx.$$

$\sin 2x = 2\sin x\cos x$

$$= \int \frac{\cos x \cdot x\sin x}{1}\,dx = \int x\left(\frac{\sin 2x}{2}\right)dx$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right.$$

$$= -\frac{x\cos 2x}{4} + \frac{\sin 2x}{8}$$

$$y_p = Pf_1 + Qf_2 = \left[\frac{1}{4}x\sin 2x + \frac{\cos 2x}{8} - \frac{x^2}{4}\right]\cos x$$

$$+ \left[-\frac{x\cos 2x}{4} + \frac{\sin 2x}{8}\right]\sin x.$$

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$(A + Bx)e^{2x}$

$f_1 = e^{2x}$ $f_2 = xe^{2x}$

26.08.2022 Euler's homogeneous linear differential equation (due to Euler)

General form of the homogeneous linear diff^l equ^n with variable coeff^{ts} is

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x).$$

Euler's equation.

$x\frac{dy}{dx}$, $x^2\frac{d^2y}{dx^2}$ } substitute

When $n=2$

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = f(x).$$

Let $z = \log x$ ($x = e^z$.) $\quad \frac{dz}{dx} = \frac{1}{x} \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

~~Let~~ $\cancel{\theta = \frac{d}{dz}}$. Then $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \boxed{\frac{dy}{dz}}$ — (1)

Differentiating (1) once again w.r.to x

$D = \frac{d}{dx}$

$D = \frac{\partial}{\partial x}$ $D' = \frac{\partial}{\partial y}$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 = \frac{d^2y}{dz^2}\cdot \boxed{\frac{dz}{dx}}$$

$\theta = \frac{d}{dz}$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}\frac{d^2y}{dz^2} \quad \times x$$

$D = d/dx$ $\quad \theta = d/dz$ $\quad \theta^2 = \frac{d^2}{dz^2}$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2} \Rightarrow x^2\frac{d^2y}{dx^2} = \boxed{\frac{d^2y}{dz^2}} - x\frac{dy}{dx} = \boxed{\frac{d^2y}{dz^2}} - \boxed{\frac{dy}{dz}} \quad (1)$$

$$= \theta^2 y - \theta y$$

$$= \theta(\theta-1)y$$

Book work

$D = d/dx$ $\quad \theta = d/dz$ $\quad dy/dz = \theta y$ $\quad d^2y/dz^2 = \theta^2 y$

Problems :

$\therefore$ Let $x = e^z$ or $z = \log x$

$x^2y'' = \theta(\theta-1)y$

$xy' = \theta y$.

1. Solve $x^2y'' - 3xy' = x + 11$.

Substituting $\theta(\theta-1)y - 3\theta y = e^z + 11$.

$\Rightarrow (\theta^2 - 4\theta)y = e^z + 11$.

$x = e^z$

$x + 11 = e^z + 11$.

A.E : $m^2 - 4m = 0$

$e^{4z} = (e^z)^4 = x^4$

$m(m-4) = 0 \Rightarrow m = 0, 4$

$y_h = c_1 e^{0z} + c_2 e^{4z} = c_1 + c_2 x^4$

To find PI_1:

$y_{p_1} = ce^z \quad y_{p_1}' = ce^z \quad y_{p_1}'' = ce^z$.

Sub $ce^z - 4ce^z = e^z \Rightarrow -3ce^z = e^z \Rightarrow c = -1/3$.

$y_{p_1} = -\frac{1}{3}e^z$

To find PI_2: $y_{p_2} = k$. $y_{p_2}' = 0 \quad y_{p_2}'' = 0$

Sub. LHS $= 0 \neq 11 =$ RHS $\therefore y_{p_2} = k$ cannot be a proposed solution

$\therefore$ Let $y_{p_2} = kz$ be the proposed soln. $y_{p_2}' = k \quad y_{p_2}'' = 0$.

Substituting $0 - 4k = 11 \quad k = -11/4$

$\therefore y_{p_2} = -\frac{11}{4}z$.

$y_{p_1} + y_{p_2} = -\frac{1}{3}e^z - \frac{11}{4}z = -\frac{1}{3}x - \frac{11}{4}\log x$

$y = c_1 + c_2x^4 - \frac{x}{3} - \frac{11}{4}\log x$.

2. Solve $x^2y'' - xy' + y = x^2$.

$x = e^z \quad z = \log x$.

$x^2y'' \leftrightarrow \theta(\theta-1)y \quad xy' \leftrightarrow \theta y$.

Sub $[\theta(\theta-1) - \theta + 1]y = e^{2z}$. ✓

$\Rightarrow (\theta^2 - 2\theta + 1)y = e^{2z}, \quad \theta = \frac{d}{dz}$

A.E: $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ twice.

$\therefore y_h = (c_1 + c_2 z)e^z = (c_1 + c_2 \log x) \cdot x$.

To find PI: $y_p = ce^{2z}$ ✓ $\quad y_p' = 2ce^{2z}$ ✓ $\quad y_p'' = 4ce^{2z}$ ✓

sub $\cancel{4ce^{2z}} - \cancel{4ce^{2z}} + ce^{2z} = e^{2z}$

$c = 1$.

$\therefore y_p = e^{2z} = x^2 \qquad \therefore y = y_h + y_p = (c_1 + c_2 \log x)x + x^2$.

3. Solve $(x^2D^2 - 3xD + 4)y = x^2\cos(\log x)$

$x = e^z \Rightarrow z = \log x \quad \& \quad xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$(\theta^2 - \theta - 3\theta + 4)y = e^{2z}\cos z$.

$(\theta^2 - 4\theta + 4)y = e^{2z}\cos z \quad \text{---} (1)$.

$(c_1 + c_2 z)e^{2z}$

$(c_1 + c_2 \log x)x^2$

$y_h = (c_1 + \underline{\underline{c_2 \log x}})x^2$ ✓

Let $y_p = e^{2z}(c_1\cos z + c_2 \sin z)$

$$y_p' = e^{2z}(-c_1\sin z + c_2\cos z) + (c_1\cos z + c_2\sin z).2e^{2z}$$

$$= e^{2z}\left((2c_2 - c_1)\sin z + (2c_1 + c_2)\cos z\right).$$

$$y_p'' = e^{2z}\left((2c_2 - c_1)\cos z + (2c_1 + c_2)(-\sin z)\right)$$

$$+\left((2c_2 - c_1)\sin z + (2c_1 + c_2)\cos z\right) 2e^{2z}.$$

Substituting in (1).

$$x = e^{z}, \quad x^2 = e^{2z}, \quad z = \log x$$

$$-e^{2z}\cos z$$

$$y_p = -x^2\cos(\log x).$$

$$y = (c_1 + c_2\log x)x^2 - x^2\cos(\log x).$$

4. Solve $x^2y'' + 4xy' + 2y = 6x$.

$$x = e^{z} \Rightarrow z = \log x.$$

$$x^2y'' \leftrightarrow \theta(\theta-1)y, \quad xy' \leftrightarrow \theta y.$$

Sub. $(\theta^2 - \theta)y + 4\theta y + 2y = 6e^{z}.$

$$(\theta^2 + 3\theta + 2)y = 6e^{z} \quad — (1)$$

A E: $m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -2, -1.$

$$y_h = c_1e^{-2z} + c_2e^{-z} = c_1x^{-2} + c_2x^{-1} = \frac{c_1}{x^2} + \frac{c_2}{x}.$$

To find PI:

$$y_p = ce^{z} \qquad y_p' = ce^{z} = y_p''$$

sub in (1).

$$ce^{z} + 3ce^{z} + 2ce^{z} = 6e^{z} \Rightarrow 6ce^{z} = 6$$

$$\Rightarrow c = 1. \quad \therefore y_p = e^{z} \quad \therefore y = y_h + y_p$$

5. $xy'' - 3y' = x^2$ Ans: $(c_1 + c_2 x^4) - \frac{x^3}{3}$ ✓

6. $(x^2D^2 - xD + 2)y = x\log x$ Ans: $y = c_1\cos(\log x) + c_2\sin(\log x) + x\log x$.

7. $x^2y'' + xy' + y = 4\sin(\log x)$ Ans: $y = c_1\cos\log x + c_2\sin\log x - 2\log x\cos(\log x)$

28.02.2022 Legendre's Equation.

$$(ax+b)^n\frac{d^ny}{dx^n} + A_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + A_{n-1}\frac{dy}{dx} + A_ny = f(x).$$

$b=0$

$x = e^z$

$A_1, A_2, \ldots$ are constants

Let $(ax+b) = e^z \Rightarrow z = \log(ax+b)$

$$\frac{dz}{dx} = \frac{1}{(ax+b)}\,a$$

$n=2$ ✓

$$(ax+b)^2\frac{d^2y}{dx^2} + A_1(ax+b)\frac{dy}{dx} + A_2y = f(x)$$

$$\frac{dz}{dx} = \frac{a}{(ax+b)}.$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}.$$

$$\frac{dy}{dx} = \frac{a}{ax+b}\frac{dy}{dz} \Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = a\theta y,\quad \theta = \frac{d}{dz}$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2\theta(\theta-1)y,\quad \theta = \frac{d}{dz}$$

1. Solve $\underset{ax+b}{(2x+5)^2}\frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 0$.—(1)

Let $e^z = (2x+5) \Rightarrow z = \log(2x+5)$.

$$(2x+5)\frac{dy}{dx} = 2\theta y$$

$$(2x+5)^2\frac{d^2y}{dx^2} = 4\theta(\theta-1)y$$

$$(ax+b)\frac{dy}{dx} = a\theta y$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2\theta(\theta-1)y$$

$a=2 \quad b=5$

Sub in (1) $\left(4(\theta^2-\theta) - 12\theta + 8\right)y = 0$ ✓

$$(4\theta^2-4\theta-12\theta+8)y=0 \Rightarrow (4\theta^2-16\theta+8)y=0 \div 4$$

$$(\theta^2-4\theta+2)y=0.$$

A.E. $m^2-4m+2=0$.

$$m=\frac{4\pm\sqrt{16-8}}{2}$$

$$=\frac{4\pm2\sqrt{2}}{2}=2\pm\sqrt{2}.$$

CF: (or) $y_h=c_1e^{(2+\sqrt{2})z}+c_2e^{(2-\sqrt{2})z}$.

$$=c_1e^{(2+\sqrt{2})\log(2x+5)}+c_2e^{(2-\sqrt{2})\log(2x+5)}$$

$$=c_1\left(e^{\log(2x+5)}\right)^{2+\sqrt{2}}+c_2\left(e^{\log(2x+5)}\right)^{2-\sqrt{2}}.$$

$$=c_1(2x+5)^{2+\sqrt{2}}+c_2(2x+5)^{2-\sqrt{2}}$$

$n\log a=\log a^n$

$e^{\log x}=x$

$()\frac{d^2y}{dx^2}+()\frac{dy}{dx}-y=$

$(2x+5)$

RHS $=0$.

$$\therefore\ y=c_1(2x+5)^{2+\sqrt{2}}+c_2(2x+5)^{2-\sqrt{2}}.$$

2. $(2x-1)^2\frac{d^2y}{dx^2}-4(2x-1)\frac{dy}{dx}+8y=\boxed{8x}. \longleftarrow (1).$

Let $(2x-1)=e^z$

$\Rightarrow 2x=e^z+1$

$\Rightarrow x=\frac{e^z+1}{2}$

$(2x-1)\frac{dy}{dx}=2\theta y$

$(2x-1)^2\frac{d^2y}{dx^2}=4\theta(\theta-1)y$

Sub in (1).

$$4\theta(\theta-1)y-8\theta y+8y=\overset{4}{\cancel{8}}\left(\frac{e^z+1}{\cancel{2}}\right)$$

$$(4\theta^2-4\theta-8\theta+8)y=4(e^z+1)$$

$$(4\theta^2-12\theta+8)y = 4(e^z+1) \quad \div 4$$

$$(\theta^2-3\theta+2)y = e^z+1. \quad \text{---(1)}$$

$(2x-1)=e^z$ ✓
$z=\log(2x-1)$

A.E: $m^2-3m+2=0 \Rightarrow (m-2)(m-1)=0$
$\Rightarrow m=1,2$

$$\therefore y_h = c_1e^z + c_2e^{2z} = c_1(2x-1)+c_2(2x-1)^2 \checkmark$$

To find y_p: Proposed soln: $y_{p_1} = \underline{\underline{cze^z}}$

$$y_{p_1}' = c(ze^z+e^z\cdot 1) = c(z+1)e^z.$$

$$y_{p_1}'' = c\left[(z+1)e^z+e^z\cdot 1\right]$$

$$= ce^z(z+2)$$

sub in (1)

$$ce^z(z+2)-3c(z+1)e^z+2cze^z = e^z.$$

$y_{p_1} = cze^z$
$= \cancel{0} - ze^z$

$$c[z+2-3z-3+2z] = 1$$

$$\Rightarrow -1c = 1 \Rightarrow c=-1 \quad \therefore y_{p_1} = \underline{\underline{-1e^z z}} = \underline{\underline{-(2x-1)\log(2x-1)}}$$

To find y_{p_2}: $\quad y_{p_2} = c \quad y_{p_2}' = 0 \quad y_{p_2}'' = 0.$

$$2c = 1 \Rightarrow c = 1/2.$$

$$y = c_1(2x-1)+c_2(2x-1)^2-(2x-1)\underline{\underline{\log(2x-1)}}+1/2$$

3. $(3x+2)^2y'' + 3(3x+2)y' - 36y = 3x^2+4x+1 \longleftarrow (1)$

Ans : $c_1(3x+2)^2 + c_2\dfrac{1}{(3x+2)^2} + \dfrac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$

Let $(3x+2) = e^z \Rightarrow z = \log(3x+2)$

$(3x+2)^2y'' \longleftrightarrow 9\theta(\theta-1)y$

$(3x+2)y' \longleftrightarrow 3\theta y$

$3x+2 = e^z$.

$3x = e^z - 2$

$x = \dfrac{e^z-2}{3}$

$x^2 = \left(\dfrac{e^z-2}{3}\right)^2$

$3x^2 = 3\left(\dfrac{e^z-2}{3}\right)^2 = \dfrac{(e^z-2)^2}{3}$.

Sub in (1), $9\theta(\theta-1)y + 3(3\theta y) - 36y = \left(\dfrac{e^z-2}{3}\right)^2 + 4\left(\dfrac{e^z-2}{3}\right)$

$(9\theta^2 \cancel{-9\theta} \cancel{+9\theta} - 36)y = \dfrac{e^{2z}+4\cancel{-4e^z}\cancel{+4e^z}-8+3}{3}$

$(9\theta^2-36)y = \dfrac{e^{2z}-1}{3}$.

$(\theta^2-4)y = \dfrac{e^{2z}-1}{3\times 9} = \dfrac{e^{2z}-1}{27}$

A.E $m^2-4=0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$.

$\therefore y_h = Ae^{2z} + Be^{-2z} = A(3x+2)^2 + \dfrac{B}{(3x+2)^2}$

To find PI : Proposed Soln/:

$y_{p_1} = cze^{2z}$ $\quad y_{p_1}' = c(z.2e^{2z} + e^{2z}.1)$.

$= c(2z+1)e^{2z}$.

$$y_{p_1}'' = c\left[(2x+1)e^{2x}(2) + e^{2x}(2)\right] = c\left[4e^{2x}(x+1)\right]$$

$$4ce^{2x}(x+1) - 4\left(cxe^{2x}\right) = \frac{e^{2x}}{27}.$$

$$\Rightarrow c\left[\cancel{4x} + 4 - \cancel{4x}\right] = \frac{1}{27} \Rightarrow c = \frac{1}{108} \qquad y_{p_1} = \frac{1}{108}xe^{2x}$$

$$\Rightarrow y_{p_1} = \frac{(3x+2)^2 \log(3x+2)}{108}$$

To find PI_2: Proposed soln. is $k = y_{p_2}$ $\quad y_{p_2}' = 0 = y_{p_2}''$

Sub

$$0 - 4k = \frac{-1}{27} \Rightarrow k = \frac{1}{108}$$

$$y = A(3x+2)^2 + \frac{B}{(3x+2)} + \frac{1}{108}\left((3x+2)^2 \log(3x+2) + 1\right).$$

4. $$(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$$

$$(1+2x) = e^x \Rightarrow x = \log(1+2x).$$

$$(1+2x)^2\frac{d^2y}{dx^2} = 2^2\theta(\theta-1)y = 4(\theta^2-\theta)y$$

$$(1+2x)\frac{dy}{dx} = 2\theta y.$$

02.08.'22.

Applications - ODE.

Physical interpretation of the linear second order ordinary diff. eqn.: $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$

Vibrating Spring/mass system:

Hooke's law: Suppose a flexible string is suspended vertically from a rigid support and then a mass m is attached to its free end.

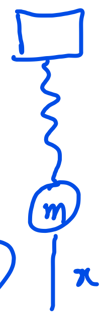
By Hooke's law the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation $\boxed{s}$. $F \propto s$ $F = ks$

(i.e) $F = ks$, k - const. of proportionality called spring const.

Newton's second law:

After a mass m is attached to the spring, it stretches the spring by an amount s and attains a position of equilibrium at which its wt. W is balanced by the restoring force ks

w.k.t $W = mg$



Condition of equilibrium : $mg = ks$ (or) $mg - ks = 0$ — (1)

Assuming that there are no retarding forces (free motion) and if the mass is displaced by an amount x from its equilibrium position, the restoring force = $k(s+x)$

Resultant force (Newton's second law)

$v = \frac{dx}{dt}$

$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg = \underbrace{mg - ks}_{0} - kx \quad \text{by (1)}.$$

rest force

opposite dir

$$\therefore m\frac{d^2x}{dt^2} = -kx.$$

Thus the diff. eqn. of free undamped motion is

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x, \quad \omega^2 = \frac{k}{m}$$

$y = f(x)$

$x = f(t)$.

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \leftarrow \text{S.H.M}$$

$\alpha \pm i\beta$.

A.E $\quad m^2 + \omega^2 = 0 \Rightarrow m^2 = -\omega^2 \Rightarrow m = \pm i\omega$. $\quad \alpha = 0, \beta = \omega$.

CF $\quad x_h = e^{0t}(c_1\cos\omega t + c_2\sin\omega t)$

$$x = c_1\cos\omega t + c_2\sin\omega t$$

$x(t) = A\cos(\omega t + \delta)$

2. A spring with a mass of 2kgs has natural length 0.5m. A force of 25.6 N is required to maintain its stretched length of 0.7m. If it is released with initial velocity 0, find the position of the mass at any time t.

$x = 0.2$

From Hooke's law: $F = ks$

$F = kx$

$= k(0.2)$

$$25.6 = k(0.2)$$

$$k = \frac{25.6}{0.2} = 128$$

$$m\frac{d^2x}{dt^2} + kx = 0.$$

$$2\frac{d^2x}{dt^2} + 128x = 0 \quad \div 2$$

$$\frac{d^2x}{dt^2} + 64x = 0$$

$x(0) = 0.2$.
$x'(0) = 0$.

$m^2 + 64 = 0$

$m = \pm 8i$

$x_h = e^{0x}(c_1 \cos 8t + c_2 \sin 8t)$

$x = c_1 \cos 8t + c_2 \sin 8t$

$x(0) = 0.2 \Rightarrow c_1 = 0.2$

$\therefore x' = -c_1 8 \sin 8t + c_2 8 \cos 8t \Rightarrow 0 = c_2 \quad \therefore x(t) = 0.2\cos 8t$

1 Suppose the spring is subjected to a frictional force or a damping force

$$\text{damping force} = c\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force}.$$

$$= -kx - c\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

A.E

$$mr^2 + cr + k = 0.$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

r_1, r_2

case i : $c^2 - 4mk > 0$ (over damping).

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

case ii $c^2 - 4mk = 0$ (critical damping)

$$r_1 = r_2 = -\frac{c}{2m}.$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = (c_1 + c_2 t)\, e^{-c/2m\, t}$$

case iii $c^2 - 4mk < 0$ (under damping).

$$\boxed{-\frac{c}{2m}} \pm \omega i \quad , \quad \omega = \frac{\sqrt{4mk - c^2}}{2m}.$$

$$x = e^{-c/2m\, t} \left(c_1 \cos\omega t + c_2 \sin\omega t \right)$$

2. $x = c_1 e^{-4t} + c_2 e^{-16t}$.

$x(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$,

$x'(0) = 6$

$$x'(t) = -4c_1 e^{-4t} - 16 c_2 e^{-16t}$$

$x'(0) = 0.6$

$$0.6 = -4c_1 - 16c_2.$$

$$= -4c_1 + 16c_1 \Rightarrow 12c_1 = 0.6$$

$$c_1 = \frac{0.6}{12} = 0.05$$

$$x(t) = 0.05 e^{-4t} - 0.05 e^{-16t}$$

$$= 0.05\left(e^{-4t} - e^{-16t}\right).$$

case iii) external force $F(t)$.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

2. Suppose the spring in ex(1) is immersed in a fluid with damping const, $c = 40$. Find the position of the mass at any time t if it starts from equilibrium position & given a push to start with an initial velocity of 0.6m/s

$$m = 2 \qquad k = \frac{F}{s} = \frac{25.6}{0.2} = 128 \qquad c = 40.$$

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0.$$

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0.$$

A.E $m^2 + 20m + 64 = 0$

$$(m+4)(m+16) = 0$$

$$m = -4, -16.$$

$$\therefore x = c_1 e^{-4t} + c_2 e^{-16t}.$$

$$x(0) = 0 \qquad x'(0) = 0.6.$$

$$0 = c_1 + c_2 \Rightarrow c_2 = -c_1$$

Sub $x = c_1 e^{-4t} - c_1 e^{-16t}$

$$x'(t) = -4c_1 e^{-4t} + 16c_1 e^{-16t}$$

$$x'(0) = -4c_1 + 16c_1 = 0.6$$

$$12c_1 = 0.6 \qquad c_1 = \frac{0.6}{12} = 0.05.$$

$$x(t) = 0.05e^{-4t} - 0.05e^{-16t} = 0.05(e^{-4t} - e^{-16t}).$$

(i) $$m\frac{d^2x}{dt^2} = -kx$$

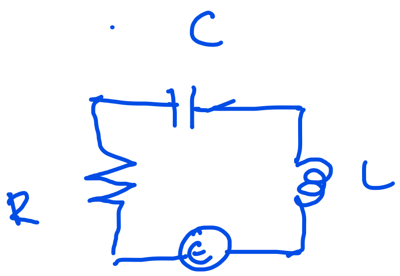
(ii) $$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

(iii) $$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t).$$

$$\therefore m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t).$$

Electric circuits:

03.03.2022. Electric circuits.



Let a source of emf be connected in series with a capacitor, a coil and a resistor as shown in figure.

I – current

By Ohm's law, the voltage drop across the resistance R is IR.

By the Law of Faraday, the voltage drop across the inductance L is $L\frac{dI}{dt}$.

If V is the voltage across the capacitor and E(t) is the voltage across the source, then the voltage eqn is

$$L\frac{dI}{dt} + RI + V = E(t) \quad \text{—(i)}.$$

If C is the capacitance, it is known that the charge Q on a capacitor plate satisfies the relation $Q = CV$ (or) $\frac{Q}{C} = V$.

$$\text{(i)} \longrightarrow \quad L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t) \quad \text{—(ii)}$$

But by definition $I = \frac{dQ}{dt}$

$$\therefore L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$$

If $E(t)=0$, electrical vibrations of the circuit are free.

Auxiliary eqn. is $Lm^2+Rm+\frac{1}{C}=0$

$$m=\frac{-R\pm\sqrt{R^2-4L/C}}{2L}$$

(i) Over damped if $R^2-\frac{4L}{C}>0$

(ii) Critically damped if $R^2-\frac{4L}{C}=0$

(iii) under damped if $R^2-\frac{4L}{C}<0$.

1. Find the charge $q(t)$ on the capacitor in an LCR circuit when $L=0.25$ henry (h), $R=10$ ohms (Ω), $C=0.001$ farad, $E(t)=0$, $q(0)=q_0$ coulombs (C), $i(0)=0$.

$C=0.001 \Rightarrow \frac{1}{C}=1000$

Eqn.: $\frac{1}{4}q''+10q'+1000q=0 \Rightarrow q''+40q'+4000q=0$.

$m^2+40m+4000=0$ $\quad m=-20\pm 60i$

$$q=e^{-20t}(c_1\cos 60t+c_2\sin 60t) \longrightarrow (1).$$

when $t=0$, $q=q_0$ sub

$$q_0=e^0(c_1\underbrace{\cos 0}_{1}+c_2\underbrace{\cancel{\sin 0}}_{0}) \Rightarrow c_1=q_0.$$

$i=0$ when $t=0$ (i.e) $\frac{dq}{dt}=0$ when $t=0$.

Diff (1).

$$\frac{dq}{dr} = e^{-20t}\left(-60c_1 \sin 60t + 60 c_2 \cos 60t\right) + \left(c_1 \cos 60t + c_2 \sin 60t\right)\left(-20e^{-20t}\right).$$

$i(0)=0$

$i(t)=0\big|_{t=0}$

$$\left(\frac{dq}{dt}\right)_{t=0} = 0.$$

$$0 = 1(60)c_2 + c_1 . (-20)(1).$$

$$\Rightarrow +20c_1 = 60c_2 \Rightarrow 20q_0 = 60c_2 \Rightarrow c_2 = \tfrac{1}{3}q_0$$

$$\therefore q(t) = e^{-20t}\left(q_0 \cos 60t + \tfrac{1}{3} q_0 \sin 60t\right)$$

$$= q_0 e^{-20t}\left(\cos 60t + \tfrac{1}{3}\sin 60t\right).$$