Homework-2

September 5, 2019

0.1 Problem 1

0.1.1 i) Newton's Method

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        import seaborn
```

```
In [2]: def my_func_1(x):
            return x + np.exp(-np.square(x))*np.cos(x)

        def my_func_2(x):
            return np.square(my_func_1(x))


        def der_my_func_1(x):
            return 1 - np.exp(-np.square(x))*(np.sin(x) + 2*x*np.cos(x))

        def der_my_func_2(x):
            return 2*my_func_1(x)*der_my_func_1(x)


        def double_der_my_func_1(x):
            return 2 * np.exp(-np.square(x)) * (np.cos(x) * (2*np.square(x) - 1) + np.sin(x)*(

        def double_der_my_func_2(x):
            return 2 * (my_func_1(x) * double_der_my_func_1(x) + np.square(der_my_func_1(x)))
```

```
In [3]: def Newton(func=my_func_1, der_func=der_my_func_1, p_0 = 0, tol = 1e-6, max_iter=1000)
            """returns (p_n, no. of iterations, approximations)"""
            p_n = p_0; i = 0; approximations = [p_n]
            while(i < max_iter):
                derivative = der_func(p_n)
                if abs(derivative) < np.finfo(float).eps:
                    print("Derivative vanished. Stopping iteration")
                    return (p_n, i, approximations)
                i += 1

                p_nplus1 = p_n - func(p_n)/derivative
```

```
            approximations.append(p_nplus1)
            if abs(p_nplus1 - p_n) < tol:
                break
            p_n = p_nplus1

        return (p_nplus1, i, approximations)
```

a) $f(x) = x + e^{-x^2} cos(x)$

```
In [4]: (p_n, iterations, approximations) = Newton()
        print("p_n = ", p_n)
        print("iterations = ", iterations)
        #print("approximations = ", approximations)
```

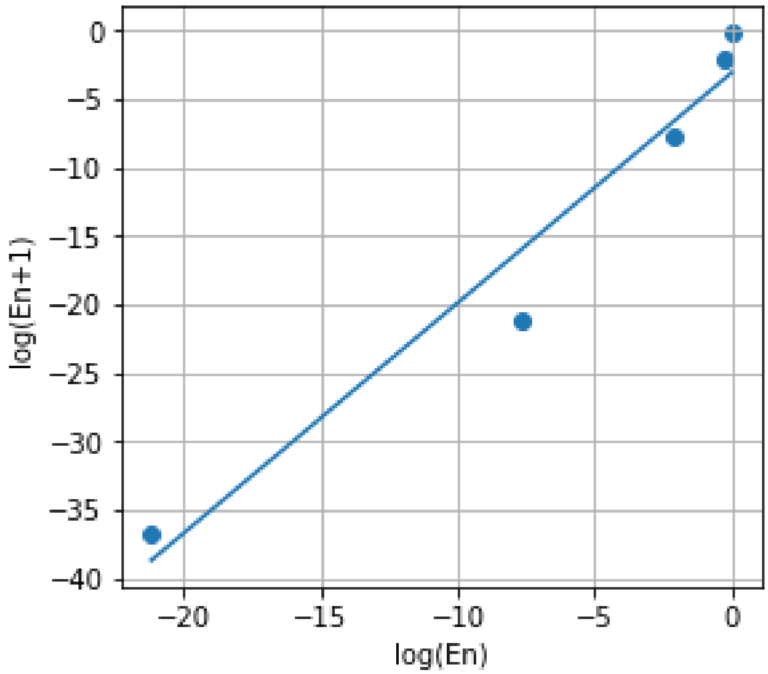
```
p_n =  -0.5884017765009963
iterations =  5
```

```
In [5]: errors = [abs(my_func_1(i)) for i in approximations]

        e_nplus1 = errors[1:]
        e_n = errors[:-1]

        x = np.log(e_n)
        y = np.log(e_nplus1);
```

```
In [6]: fig,ax = plt.subplots()
        plt.xlabel('log(En)')
        plt.ylabel('log(En+1)')
        ax.scatter(x,y)
        ax.plot(x, np.poly1d(np.polyfit(x,y,1))(x))
        ax.grid(True)
        ax.set_aspect(0.5) # = 1/2.0 since 2.0 is the convergence rate of Newton's method
```



So we're getting a curve which can be best fit with a straight line of slope nearly 2 (as expected since Newton's converges quadratically).

Root found = -0.5884017765009963

No. of iterations taken = 5 Let's also use the function value of our approximation (its difference from 0) as our error metric (since we don't know the exact value of the root)

```
In [7]: print("absolute error = ", abs(my_func_1(p_n)))
```

```
absolute error =  1.1102230246251565e-16
```

b) $f(x) = (x + e^{-x^2} cos(x))^2$

```
In [8]: (p_n, iterations, approximations) = Newton(func=my_func_2, der_func=der_my_func_2)
```

```
In [9]: print("p_n = ", p_n)
        print("iterations = ", iterations)
        #print("approximations = ", approximations)
```

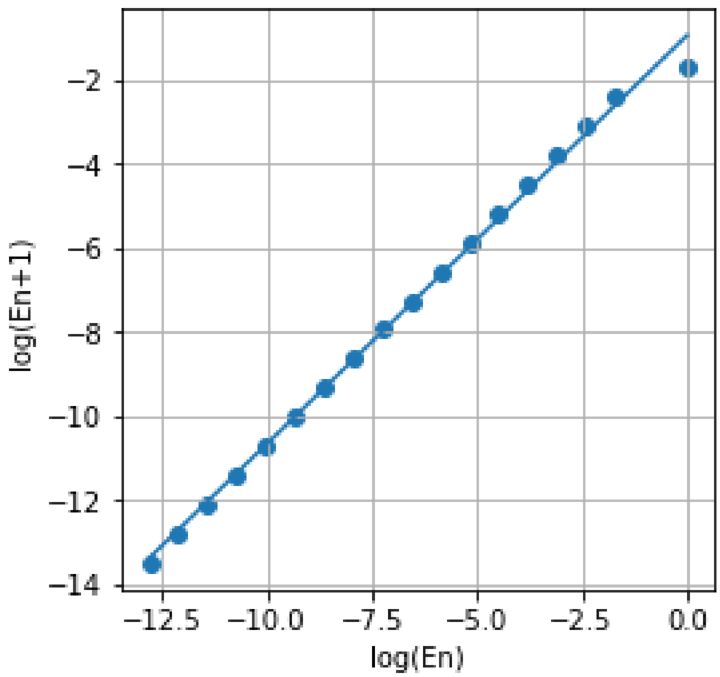
```
p_n =  -0.5884011102599648
iterations =  18
```

```
In [10]: errors = [abs(my_func_1(i)) for i in approximations]
```

```
In [11]: e_nplus1 = errors[1:]
         e_n = errors[:-1]

         x = np.log(e_n)
         y = np.log(e_nplus1);
```

```
In [12]: fig,ax = plt.subplots()
         plt.xlabel('log(En)')
         plt.ylabel('log(En+1)')
         ax.scatter(x,y)
         ax.plot(x, np.poly1d(np.polyfit(x,y,1))(x))
         ax.grid(True)
         ax.set_aspect(1) # = 1/1.0
```



So we're getting a straight line of slope nearly 1 (as expected since Newton's converges linearly when the derivative at the root is 0).

Root found = -0.5884011102599648

No. of iterations taken = 18

```
In [13]: print("absolute error = ", abs(my_func_2(p_n)))
```

```
absolute error =  1.929708636477569e-12
```

0.1.2 ii) Secant Method

In [14]:
```
def Secant(func=my_func_1, p_0 = 0, p_1 = 1, tol = 1e-6, max_iter=1000):
    """returns (p_n, no. of iterations, approximations)"""
    p_n = p_1; p_nminus1 = p_0; i = 0; approximations = [p_nminus1, p_n]
    while(i < max_iter):
        derivative = (func(p_n) - func(p_nminus1))/(p_n - p_nminus1)
        if abs(derivative) < np.finfo(float).eps:
            print("Derivative vanished. Stopping iteration")
            return (p_n, i, approximations)
        i += 1

        p_nplus1 = p_n - func(p_n)/derivative
        approximations.append(p_nplus1)
        if abs(p_nplus1 - p_n) < tol:
            break
        p_nminus1 = p_n
        p_n = p_nplus1


    return (p_nplus1, i, approximations)
```

In [15]:
```
(p_n, iterations, approximations) = Secant()
print("p_n = ", p_n)
print("iterations = ", iterations)
#print("approximations = ", approximations)
```

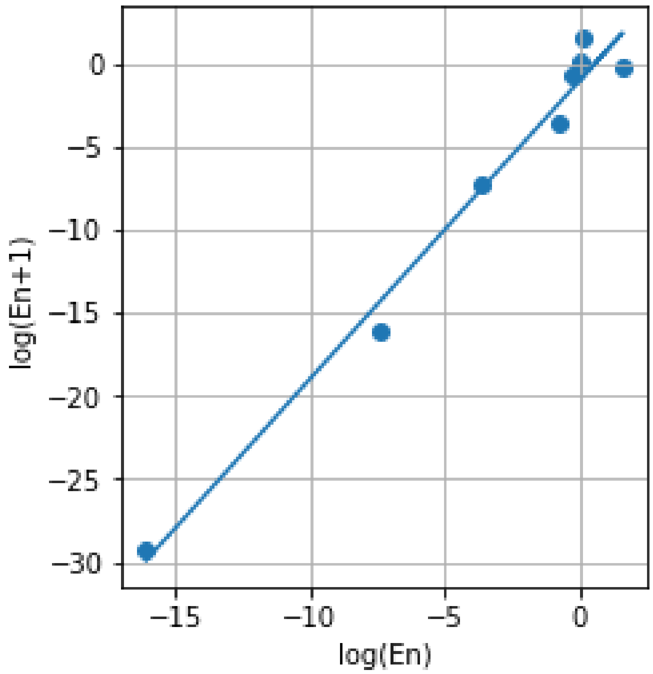
```
p_n =  -0.5884017765009013
iterations =  7
```

In [16]:
```
errors = [abs(my_func_1(i)) for i in approximations]

e_nplus1 = errors[1:]
e_n = errors[:-1]

x = np.log(e_n)
y = np.log(e_nplus1);
```

In [17]:
```
fig,ax = plt.subplots()
plt.xlabel('log(En)')
plt.ylabel('log(En+1)')
ax.scatter(x,y)
ax.plot(x, np.poly1d(np.polyfit(x,y,1))(x))
ax.grid(True)
ax.set_aspect(0.617) # = 1/1.62
```



So we're getting an almost straight line of slope nearly 1.62 (as expected since Secant's has that convergence rate).

Root found = -0.5884017765009013

No. of iterations taken = 7

```
In [18]: print("absolute error = ", abs(my_func_1(p_n)))
```

```
absolute error =  1.9806378759312793e-13
```

b) $f(x) = (x + e^{-x^2} cos(x))^2$

```
In [19]: (p_n, iterations, approximations) = Secant(func=my_func_2)
         print("p_n = ", p_n)
         print("iterations = ", iterations)
         #print("approximations = ", approximations)
```

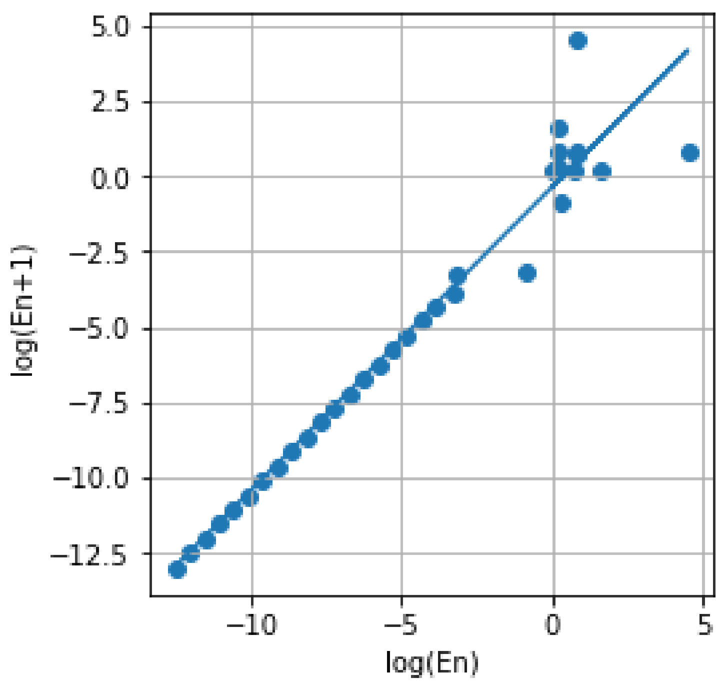
```
p_n =  -0.5884006785832132
iterations =  33
```

```
In [20]: errors = [abs(my_func_1(i)) for i in approximations]
```

```
In [21]: e_nplus1 = errors[1:]
         e_n = errors[:-1]

         x = np.log(e_n)
         y = np.log(e_nplus1);
```

```
In [22]: fig,ax = plt.subplots()
         plt.xlabel('log(En)')
         plt.ylabel('log(En+1)')
         ax.scatter(x,y)
         ax.plot(x, np.poly1d(np.polyfit(x,y,1))(x))
         ax.grid(True)
         ax.set_aspect(1) # = 1/1.0
```



So we're getting an almost straight line of slope nearly 1 (as expected since Secant's has that convergence rate when the multiplicity is 2).

Root found = -0.5884006785832132

No. of iterations taken = 33

```
In [23]: print("absolute error = ", abs(my_func_2(p_n)))
```

```
absolute error =  5.240450526912252e-12
```

0.1.3 iii) Modified Newton's Method

```
In [24]: def numerator_term_func_1(x):
             return my_func_1(x)*der_my_func_1(x)

         def denominator_term_func_1(x):
             return np.square(der_my_func_1(x)) - my_func_1(x)*double_der_my_func_1(x)

         def numerator_term_func_2(x):
             return my_func_2(x)*der_my_func_2(x)

         def denominator_term_func_2(x):
             return np.square(der_my_func_2(x)) - my_func_2(x)*double_der_my_func_2(x)
```

```
In [25]: def ModNewton(numerator_term=numerator_term_func_1, denominator_term = denominator_te
                       p_0 = 0, tol = 1e-6, max_iter=1000):
             """returns (p_n, no. of iterations, approximations)"""
             p_n = p_0; i = 0; approximations = [p_n]
             while(i < max_iter):
                 derivative = denominator_term(p_n)
                 if abs(derivative) < np.finfo(float).eps:
                     printf("Derivative vanished. Stopping iteration")
                     return (p_n, i, approximations)
                 i += 1

                 p_nplus1 = p_n - numerator_term(p_n)/derivative
                 approximations.append(p_nplus1)
                 if abs(p_nplus1 - p_n) < tol:
                     break
                 p_n = p_nplus1

             return (p_nplus1, i, approximations)
```

a) $f(x) = x + e^{-x^2}cos(x)$

```
In [26]: (p_n, iterations, approximations) = ModNewton()
         print("p_n = ", p_n)
         print("iterations = ", iterations)
         #print("approximations = ", approximations)
```

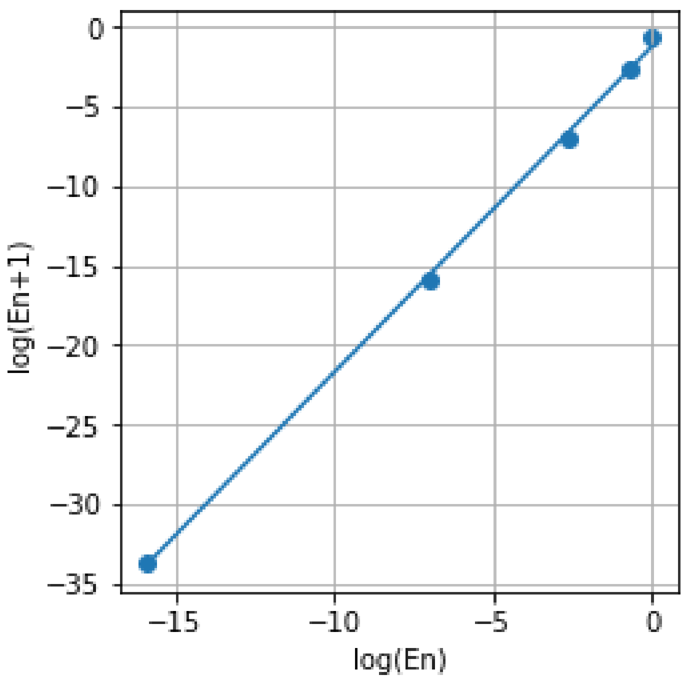
```
p_n =  -0.5884017765009951
iterations =  5
```

```
In [27]: errors = [abs(my_func_1(i)) for i in approximations]

         e_nplus1 = errors[1:]
         e_n = errors[:-1]
```

```
x = np.log(e_n)
y = np.log(e_nplus1);
```

```
In [28]: fig,ax = plt.subplots()
         plt.xlabel('log(En)')
         plt.ylabel('log(En+1)')
         ax.scatter(x,y)
         ax.plot(x, np.poly1d(np.polyfit(x,y,1))(x))
         ax.grid(True)
         ax.set_aspect(0.5) # = 1/2.0
```



Straight line with slope 2, as expected (Modified Newton's convergence rate)

Root found = -0.5884017765009951

No. of iterations taken = 5

```
In [29]: print("absolute error = ", abs(my_func_1(p_n)))

absolute error =  2.4424906541753444e-15
```

b) $f(x) = (x + e^{-x^2} cos(x))^2$

In [30]:
```
(p_n, iterations, approximations) = ModNewton(numerator_term=numerator_term_func_2,
                                              denominator_term = denominator_term_func
print("p_n = ", p_n)
print("iterations = ", iterations)
#print("approximations = ", approximations)
```

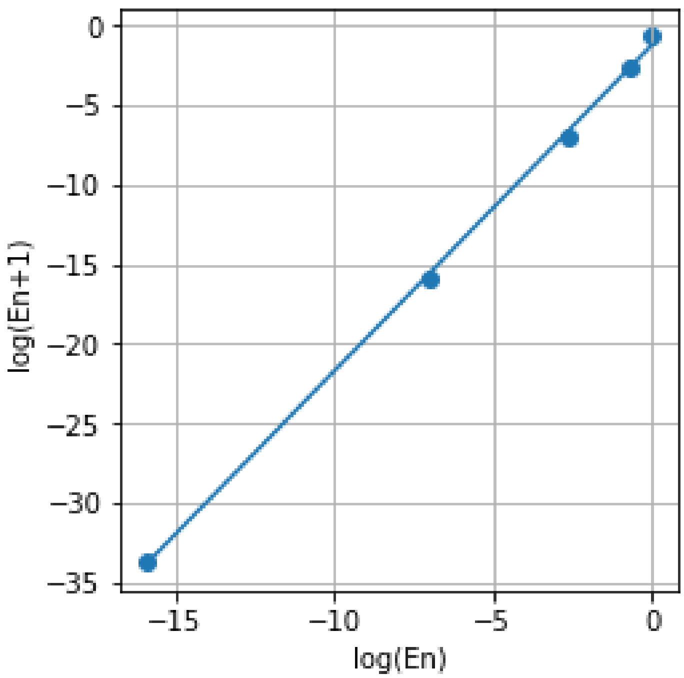
```
p_n =  -0.5884017765009951
iterations =  5
```

In [31]:
```
errors = [abs(my_func_1(i)) for i in approximations]

e_nplus1 = errors[1:]
e_n = errors[:-1]

x = np.log(e_n)
y = np.log(e_nplus1);
```

In [32]:
```
fig,ax = plt.subplots()
plt.xlabel('log(En)')
plt.ylabel('log(En+1)')
ax.scatter(x,y)
ax.plot(x, np.poly1d(np.polyfit(x,y,1))(x))
ax.grid(True)
ax.set_aspect(0.5) # = 1/2.0
```



Straight line with slope 2, as expected (Modified Newton's convergence rate even when multpicity is 2)

Root found = -0.5884017765009951

No. of iterations taken = 5

(We actually get the exact same equations (numerator and denominator terms) and hence the same answer for both problems when using Modified Newton's)

```
In [33]: print("absolute error = ", abs(my_func_2(p_n)))
```

```
absolute error =  5.965760595733902e-30
```

0.2 Problem 2

$$g(x) = x - \phi(x)f(x) - \psi(x)f^2(x)$$

For cubic convergence, g'(p) = g"(p) = 0.
So, differentiating,

$$g'(x) = 1 - (\phi'(x)f(x) + \phi(x)f'(x)) - (\psi'(x)f^2(x) + 2\psi(x)f(x)f'(x))$$

Since p is a root, g(p) = p and f(p) = 0.
Putting x = p gives:

$$0 = 1 - \phi(p)f'(p)$$
$$\phi(p) = \frac{1}{f'(p)}$$

i.e.,

$$\phi(x) = \frac{1}{f'(x)}$$

$\therefore$

$$\phi'(x) = \frac{-f''(x)}{f'(x)^2}$$

We'll simplify it for ease to

$$\phi' = \frac{-f''}{f'^2}$$

So, we can write g'(x) as:

$$
\begin{aligned}
g' &= 1 - (\phi' f + \phi f') - (\psi' f^2 + 2\psi f f') \\
&= 1 - (\frac{-ff''}{f'^2} + 1) - (\psi' f^2 + 2\psi f f') \\
&= \frac{ff''}{f'^2} - (\psi' f^2 + 2\psi f f')
\end{aligned}
$$

Differentiating once more, we get:

$$
g'' = \frac{ff'f''' + f'^2 f'' - 2ff''^2}{f'^3} - (f^2\psi'' + 4ff'\psi' + 2ff''\psi + 2f'^2\psi)
$$

Now, putting x = p gives:

$$
\begin{aligned}
g''(p) = 0 &= \frac{f''}{f'} - 2f'^2\psi \\
\psi &= \frac{f''}{2f'^3}
\end{aligned}
$$

Therefore,

$$
\phi(x) = \frac{1}{f'(x)}
$$

and

$$
\psi(x) = \frac{f''(x)}{2f'(x)^3}
$$

Hence,

$$
g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)f^2(x)}{2f'(x)^3}
$$

Now, since the convergence is cubic, its rate is given by:

$$
\begin{aligned}
\epsilon_{n+1} &= \frac{g'''(p)}{3!}\epsilon_n^3 \\
\frac{\epsilon_{n+1}}{\epsilon_n^3} &= \frac{g'''(p)}{6}
\end{aligned}
$$

Using Wolfram Alpha, we can calculate the triple derivative of g(x)
https://www.wolframalpha.com/input/?i=derivative+of+%28double+derivative+of+x+-+f%5Cf%27+-+%28f%27%27+f%5E2%29%2F%282f%27%5E3%29%29
But the expression is too complicated, even after simplifying it by putting x = p in it.

So, going by the method used on https://en.wikipedia.org/wiki/Halley%27s_method#Cubic_convergence , we can write

$$\frac{\epsilon_{n+1}}{\epsilon_n^3} = \frac{2f'(p)f'''(p) - 3f''(p)^2}{12f'(p)^2}$$

Therefore,

$$\alpha = 3$$

and

$$\lambda = \frac{2f'(p)f'''(p) - 3f''(p)^2}{12f'(p)^2}$$

0.3 Problem 3

In [34]:
```
theta2_0 = 30*np.pi/180;
theta3_0 = 0;
theta4 = 220 * np.pi/180

r1 = 10; r2 = 6; r3 = 8; r4 = 4;

theta2_n = theta2_0;
theta3_n = theta3_0;

Theta = np.array([
    [theta2_n], [theta3_n]
], np.float64);
```

In [35]:
```
def Function(Theta):
    return np.array([
        [r2*np.cos(Theta[0][0]) + r3*np.cos(Theta[1][0]) + r4*np.cos(theta4) - r1],
        [r2*np.sin(Theta[0][0]) + r3*np.sin(Theta[1][0]) + r4*np.sin(theta4)]
    ])
```

In [36]:
```
def Jacobian(Theta):
    return np.array([
        [-r2*np.sin(Theta[0][0]), -r3*np.sin(Theta[1][0])],
        [r2*np.cos(Theta[0][0]), r3*np.cos(Theta[1][0])]
    ])
```

In [37]:
```
def determinant(matrix):
    return matrix[0][0]*matrix[1][1] - matrix[0][1]*matrix[1][0]
```

In [38]:
```
def inverse(matrix):
    return np.array([
        [matrix[1][1], -matrix[0][1]],
        [-matrix[1][0], matrix[0][0]]
    ])/determinant(matrix)
```

I am defining error as the max of errors ($|\theta_2^{n+1} - \theta_2^n|$, $|\theta_3^{n+1} - \theta_3^n|$)

```
In [39]: def CoupledNewton(Theta_n=Theta, tol = 1e-4, max_iter=1000):
             """returns (p_n, no. of iterations, approximations)"""
             i = 0; approximations = [Theta_n]
             while(i < max_iter):
                 Jac = Jacobian(Theta_n)
                 if abs(determinant(Jac)) < np.finfo(float).eps:
                     print("Derivative (Jacobian) vanished. Stopping iteration")
                     return (Theta_n, i, approximations)
                 i += 1
                 Theta_nplus1 = Theta_n - np.matmul(inverse(Jac), Function(Theta_n))
                 approximations.append(Theta_nplus1)
                 if max(abs(Theta_nplus1 - Theta_n)) < tol:
                     break
                 Theta_n = Theta_nplus1

             return (Theta_nplus1, i, approximations)
```

```
In [40]: (Theta, iterations, approximations) = CoupledNewton()
```

```
In [41]: print("theta2 = ", Theta[0][0] * 180/ np.pi)
         print("theta3 = ", Theta[1][0] * 180/ np.pi)
         #print("approximations = ", approximations)
```

```
theta2 =  32.01518099634485
theta3 =  -4.3709878918079985
```

So, the angles in degrees are (after doing the calculations with all degrees in radians):

θ_2 = 32.015

θ_3 = -4.37 = 355.63

No. of iterations = 3

The 2x2 Jacobian was inverted in the process.