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Homework-6

Problem 1

I have found a time step size of 1/16 years to be appropriate for this problem. The plot is as follows:

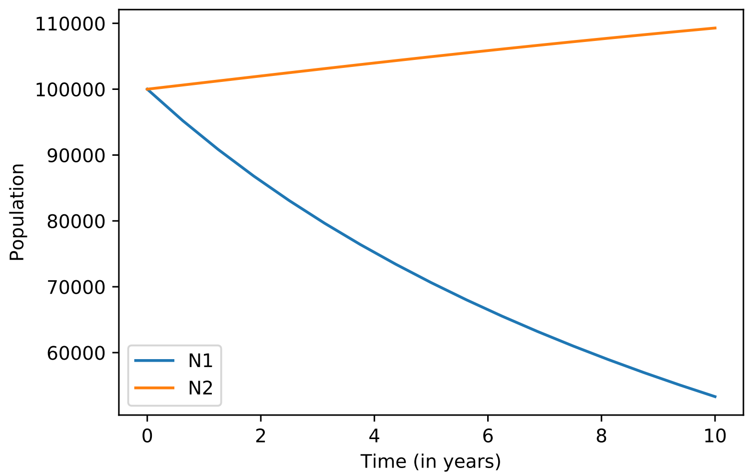

Figure 1: RK-4 for h = 1/16 years

Increasing the time-step size by a factor of 2, 4, 8, and 16, we get the following plot for the relative errors at N(10) vs h^{-1}

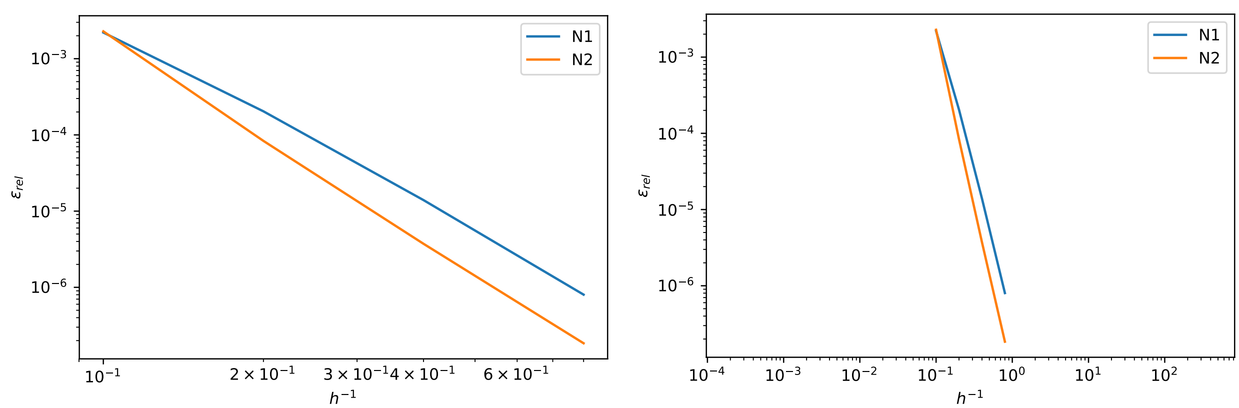

Figure 2: Loglog plot of Relative error vs h^{-1}

Plotting this with an equal aspect ratio, we can see that the graph is a straight line with slope (magnitude) approximately = 4, which is what we'd expect with the RK-4 method.

Problem 2

The maximum allowable step sizes for the different methods are given in the table ranked in decreasing order of h(most to least stable):

Method	h
Adams Moulton	2.0
Adams Predictor/Corrector	0.8
Midpoint rule	0.66
Adams Bashforth	0.33

For the predictor corrector method, the plots for different values of h are as follows:

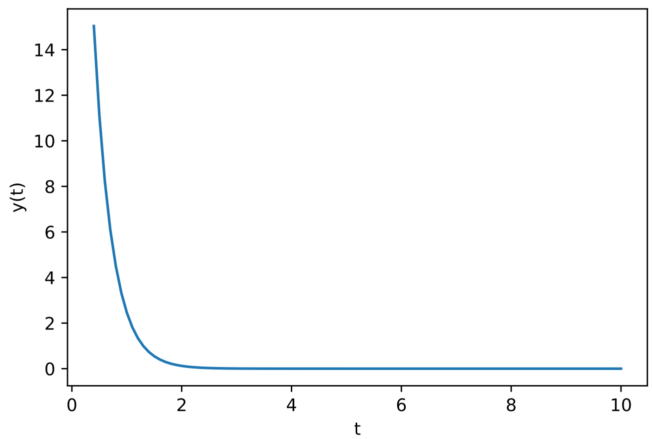

Figure 3: h = 0.1 (Stable)

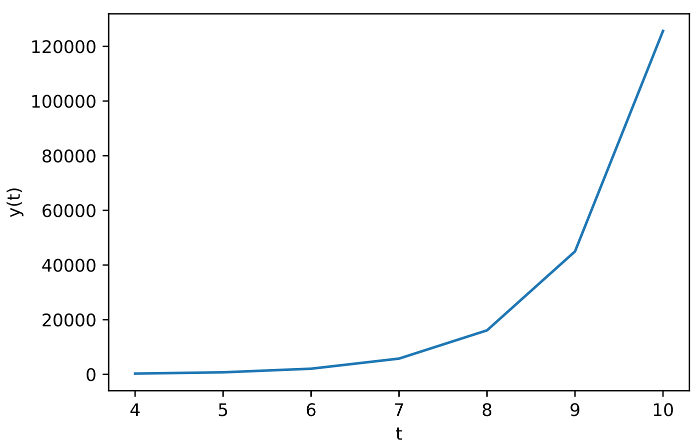

Figure 4: h = 1 (Unstable)

Another plot for h=0.3 is one which is stable but not very accurate.

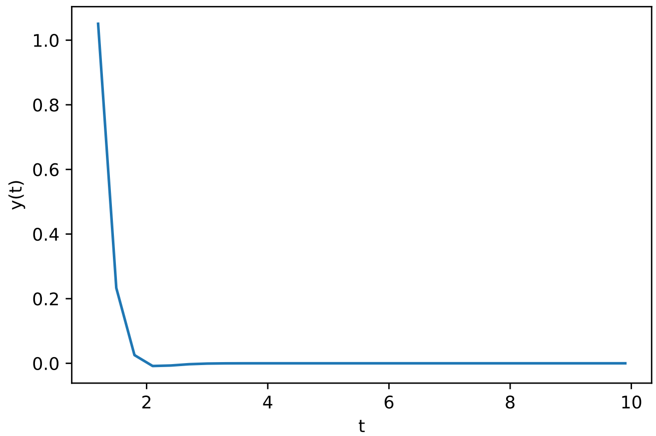

Figure 5: h = 0.3 (Stable but not very accurate)

The stability analysis is done by hand which I am attaching here as images:

2) $\frac{dy(t)}{dt} = -3y(t)$

i) 2nd order Runge-Kutta method :

$$y_{n+1} = y_n + \frac{k_1}{2} + \frac{k_2}{2} .$$

where $k_1 = hf(y_n, t_n)$ and $k_2 = hf\left(y_n + \frac{k_1}{2}, t_n + \frac{h}{2}\right)$

$$k_1 = -3hy_n \;; \quad k_2 = -3h\left(y_n - \frac{3}{2}hy_n\right) = -3hy_n\left(1 - \frac{3}{2}h\right)$$

$$\therefore \quad y_{n+1} = y_n + \frac{1}{2}(-3hy_n) + (-3hy_n)\left(1 - \frac{3}{2}h\right)$$

$$= y_n\left(1 - 3h + \frac{9h^2}{2}\right)$$

For stability, $\left\|1 - 3h + \frac{9h^2}{2}\right\| \leq 1 \implies \boxed{h \leq \frac{2}{3}}$

ii) 2-step Adam's Bashforth

$$y_{n+1} = y_n + \frac{3h}{2}f(y_n, t_n) - \frac{h}{2}f(y_{n-1}, t_{n-1})$$

$$y_{n+1} - \left(1 + \frac{3(-3)h}{2}\right)y_n - \frac{3h}{2}y_{n-1} = 0$$

Putting $y_n = \lambda^n$,

$$\lambda^{n+1} - \left(1 - \frac{9h}{2}\right)\lambda^n - \frac{3h}{2}\lambda^{n-1} = 0$$

$$\lambda^2 - \left(1 - \frac{9h}{2}\right)\lambda - \frac{3h}{2} = 0$$

$$\implies \lambda = \frac{1 - 9h/2 \pm \sqrt{1 + \frac{81h^2}{4} - 9h + 6h}}{2}$$

$$\lambda = \frac{1}{2}\left[1 - \frac{9h}{2} \pm \sqrt{1 - 3h + \frac{81h^2}{4}}\right]$$

$$\sqrt{+1 - 3h + \frac{81h^2}{4}} = \left(1 + 3h - \frac{81h^2}{4}\right)^{1/2} = 1 + \frac{1}{2}\left(-3h + \frac{81}{4}h^2\right) - \frac{9h^2}{8}$$

$$= 1 + \frac{(-3h)}{2} + 9h^2$$

$$\lambda = \frac{1}{2}\left[1 - \frac{9h}{2} \pm \left(1 - \frac{3h}{2} + 9h^2\right)\right]$$

$$\lambda_1 = 1 - 3h + 9h^2 \quad , \quad \lambda_2 = -\frac{3h}{2} - 9h^2$$

For stability, $|\lambda_1| \le 1$ and $|\lambda_2| \le 1$

i.e.,

$$-1 \le 1 - 3h + 9h^2 \le 1 \quad \text{and} \quad -1 \le -\frac{3h}{2} - 9h^2 \le 1$$

$$-\frac{1}{4} \le \left(3h - \frac{1}{2}\right)^2 \le \frac{1}{4} \qquad -\frac{17}{16} \le \left(3h + \frac{1}{4}\right)^2 \le \frac{17}{16}$$

$$\cancel{-\frac{1}{2}} \le \left(3h - \frac{1}{2}\right) \le \frac{1}{2} \qquad 3h + \frac{1}{4} \le \frac{\sqrt{17}}{4}$$

$$h \le \frac{1}{3} \quad \text{and} \quad h \le \frac{\sqrt{17} - 1}{12}$$

so, $\boxed{h \le \frac{1}{3}}$

iii) 2-step Adams Moulton

$$y_{i+1} = y_i + \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right]$$

$$= y_i + \frac{h}{12}\left[5(-3)y_{i+1} + 8(-3)y_i - (-3)y_{i-1}\right]$$

$$\lambda^{n+1} = \lambda^n + \frac{h}{12}\left[-15\lambda^{n+1} - 24\lambda^n + 3\lambda^{n-1}\right]$$

$$12\lambda^2 = 12\lambda + \left[-15h\lambda^2 - 24h\lambda + 3h\right]$$

$$\lambda^2\left[1 + 5/4h\right] + \lambda\left[2h - 1\right] - \frac{h}{4} = 0$$

$$\lambda = \frac{(1-2h) \pm \sqrt{4h^2 + 1 - 4h - h^2 \times \frac{5}{4}}}{2(1 + 5/4h)}$$

$$\lambda = \frac{(1-2h) \pm \left(1 - \frac{3h}{2} + \frac{3h^2}{2}\right)}{2 + 5/2h}$$

For stability, $|\lambda| \leq 1$

i.e.

$$-1 \leq \frac{2 - 2h - \frac{3}{2}h + \frac{3h^2}{2}}{2 + 5/2h} \leq 1 \quad \text{and} \quad -1 \leq \frac{-2h + \frac{3h}{2} - \frac{3h^2}{2}}{2 + 5/2h} \leq 1$$

$$-1 \leq \frac{\frac{3h^2}{2} - \frac{7}{2}h + 2}{2 + 5/2h} \leq 1 \quad \text{and} \quad -1 \leq \frac{-\frac{3h^2}{2} - \frac{h}{2}}{2 + 5/2h} \leq 1$$

$$\frac{3h^2}{2} - \frac{7h}{2} + \cancel{2} \leq \cancel{2} + \frac{5}{2}h \qquad \frac{h}{2} + \frac{3}{2}h^2 \leq 2 + \frac{5}{2}h^2$$

$$\frac{3h^2}{2} - 6h \leq 0 \qquad 3h^2 - 4h - 4 \leq 0$$

$$h \leq 4 \qquad h \leq 2$$

$$\therefore \quad \boxed{h \leq 2}$$

iv) 2-step Predictor Corrector Method

Predictor : $w_{i+1} = w_i + \frac{h}{2}[3f_i - f_{i-1}]$

$f_i = -3w_i$

$w_{i+1} = w_i + \frac{h}{2}[-9w_i + 3w_{i-1}]$

Corrector : $w_{i+1} = w_i + \frac{h}{12}[5f_{i+1} + 8f_i - f_{i-1}]$

$= w_i + \frac{h}{12}[-15w_{i+1} - 24w_i + 3w_{i-1}]$

$= w_i + \frac{h}{12}\left[-15\left(w_i + \frac{h}{2}(-9w_i + 3w_{i-1})\right) - 24w_i + 3w_{i-1}\right]$

$w_{i+1} = w_i + \frac{h}{4}\left[-5w_i - \frac{5h}{2}(-9w_i + 3w_{i-1}) - 9w_i + w_{i-1}\right]$

$= w_i\left(1 - \frac{13}{4}h + \frac{45}{8}h^2\right) + w_{i-1}\left(\frac{h}{4} - \frac{15}{8}h^2\right)$

$8w_{i+1} = w_i(45h^2 - 26h + 8) + w_{i-1}(2h - 15h^2)$

$\therefore \quad 8\lambda^2 - \lambda(45h^2 - 26h + 8) + 15h^2 - 2h = 0$

For stability, $|\lambda| \leq 1$

i.e., $45h^2 - 26h + 8 \pm \sqrt{(45h^2 - 26h + 8)^2 + 32(2h - 15h^2)} \leq 16$

Solving this eq^n, we get :

$\boxed{h \leq 0.8}$