Problem 1

$$\int_{x_0}^{x_2} f(x)dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi)$$

Solving for $f(x) = x^n$, when n = 0, 1, and 2. We get 3 equations in 3 unknowns a_0, a_1, and a_2 which can be represented by the matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_0 \\ \frac{1}{2}(x_2^2 - x_0^2) \\ \frac{1}{3}(x_2^3 - x_0^3) \end{bmatrix} \tag{1}$$

Substituting $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$, we get:

$$\begin{bmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2h \\ 2x_0 h + 2h^2 \\ 2x_0^2 h + 4x_0 h^2 + \frac{8}{3}h^3 \end{bmatrix} \tag{2}$$

$R_2 \leftarrow R_2 - x_0 R_1 \qquad R_3 \leftarrow R_3 - x_0^2 R_1 \qquad R_3 \leftarrow R_3 - (2x_0 + h)R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & h & 2h \\ 0 & 0 & 2h^2 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2h \\ 2h^2 \\ \frac{2}{3}h^3 \end{bmatrix} \tag{3}$$

Solving using back-substitution, we get:

$$a_0 = \frac{h}{3} \quad a_1 = \frac{4h}{3} \quad a_2 = \frac{h}{3}$$

as we had expected. Now, when n = 4, i.e., $f(x) = x^4, f^{(4)}(x) = f^{(4)}(\xi(x)) = 4! = 24$. So,

$$\int_{x_0}^{x_2} x^4 dx = \frac{h}{3}[x_0^4 + 4x_1^4 + x_2^4] + 24k \tag{4}$$

Solving for k, we get:

$$\left.\frac{x^5}{5}\right|_{x_0}^{x_0+2h} = \frac{h}{3}[1 \cdot x_0^4 + 4 \cdot (x_0 + h)^4 + 1 \cdot (x_0 + 2h)^4] + 24k \tag{5}$$

We can put $x_0 = 0$ for simplification to get:

$$\frac{(2h)^5}{5} = \frac{h}{3}[0 + 4h^4 + (2h)^4] + 24k$$
$$\frac{32h^5}{5} = \frac{20}{3}h^5 + 24k$$
$$k = -\frac{h^5}{90}$$

Problem 2

Romberg Integration

a) $f(x) = x^{1/3}$

n	Function evaluations	Result	Trapezoidal Result
12	2049	0.749995	0.749989

b) $f(x) = x^2 e^{-x}$

n	Function evaluations	Result	Trapezoidal Result
4	9	0.160602	0.161080

Problem 3

Gaussian Quadrature

a) $f(x) = x^{1/3}$

Results for different values of n:

Function evaluations	n = 2	n = 3	n = 4	n = 5
Result	0.75977	0.75385	0.75194	0.75113
Romberg Result	0.69580	0.73063	0.74250	0.74704
Romberg evaluations	3	5	9	17

b) $f(x) = x^2 e^{-x}$

Results for different values of n:

Function evaluations	n = 2	n = 3	n = 4	n = 5
Result	0.159410	0.160595	0.160602	0.160602
Romberg Result	0.162401	0.160610	0.160602	-
Romberg evaluations	3	5	9	17

Comparing the results, we can see that Gaussian outperforms Romberg in terms of accuracy, and in terms of the number of function evaluations required, it does even better (we could say exponentially better).

Problem 4

$$\frac{dy}{dt} = \frac{1}{t^2} - \frac{y}{t} - y^2 \qquad (1 \leq t \leq 2) \qquad y(1) = -1$$

Euler's method

The different plots for different values of n (step size $\Delta t = 0.1(2^{-n})$) are as follows:

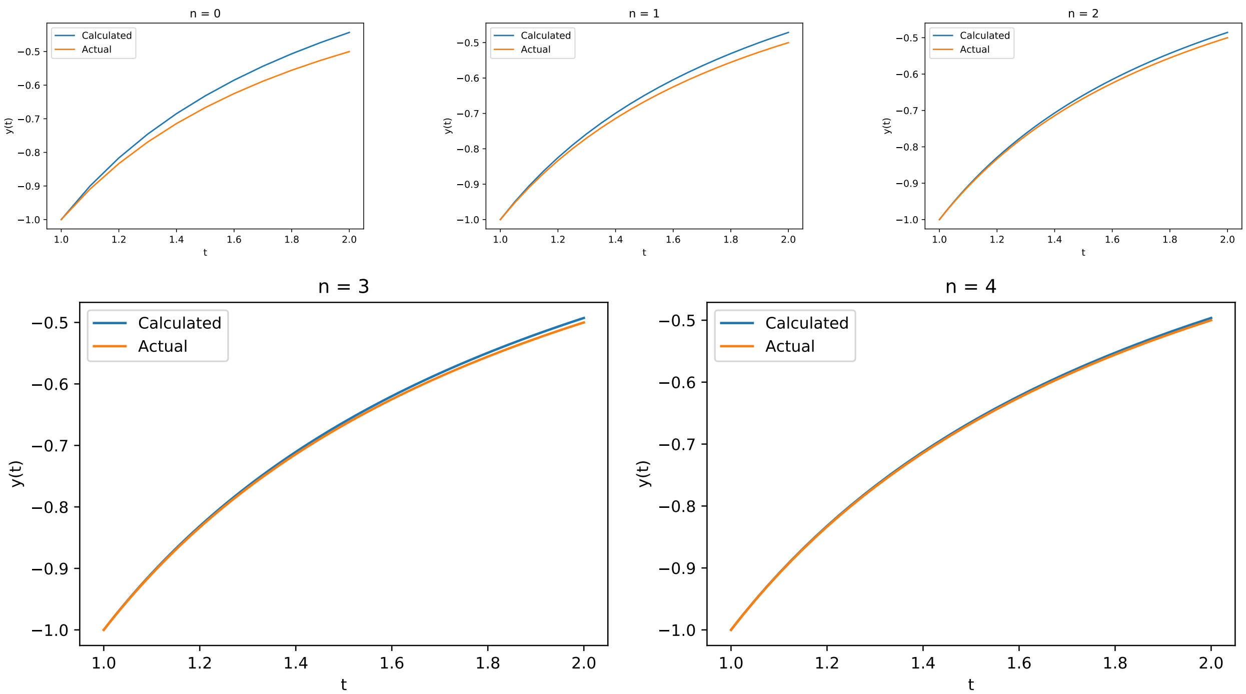

Figure 1: Euler's method for different values of n

The values calculated for different n are given in the table (actual value is -0.5):

n	y(2)	Absolute error
0	-0.4431	0.0569
1	-0.4712	0.0288
2	-0.4855	0.0145
3	-0.4927	0.0073
4	-0.4964	0.0036

Modified Euler's method

The different plots for different values of n (step size $\Delta t = 0.1(2^{-n})$) are as follows:

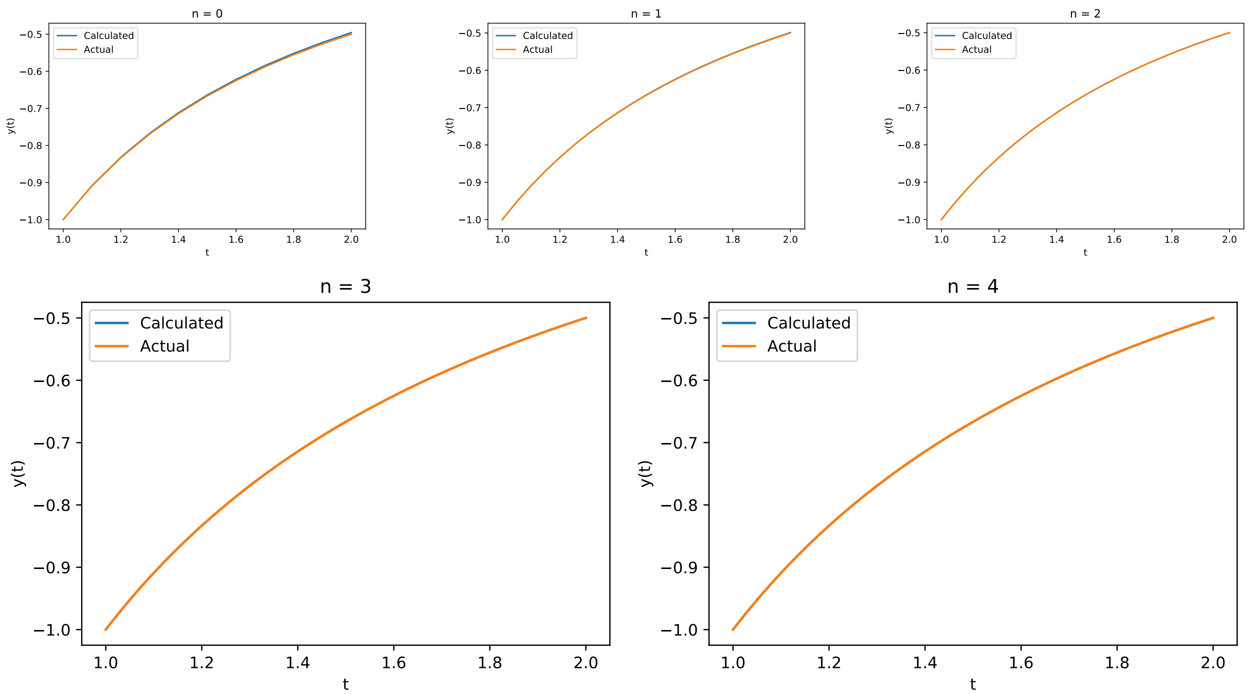

Figure 2: Modified Euler's method for different values of n

The values calculated for different n are given in the table (actual value is -0.5):

n	y(2)	Absolute error
0	-0.49555	0.00445
1	-0.49885	0.00115
2	-0.49971	0.00029
3	-0.49992	0.00008
4	-0.49998	0.00002

Let's compare the errors side by side

n	Error(Euler's)	Error(Modified Euler's)
0	0.0569	0.00445
1	0.0288	0.00115
2	0.0145	0.00029
3	0.0073	0.00008
4	0.0036	0.00002

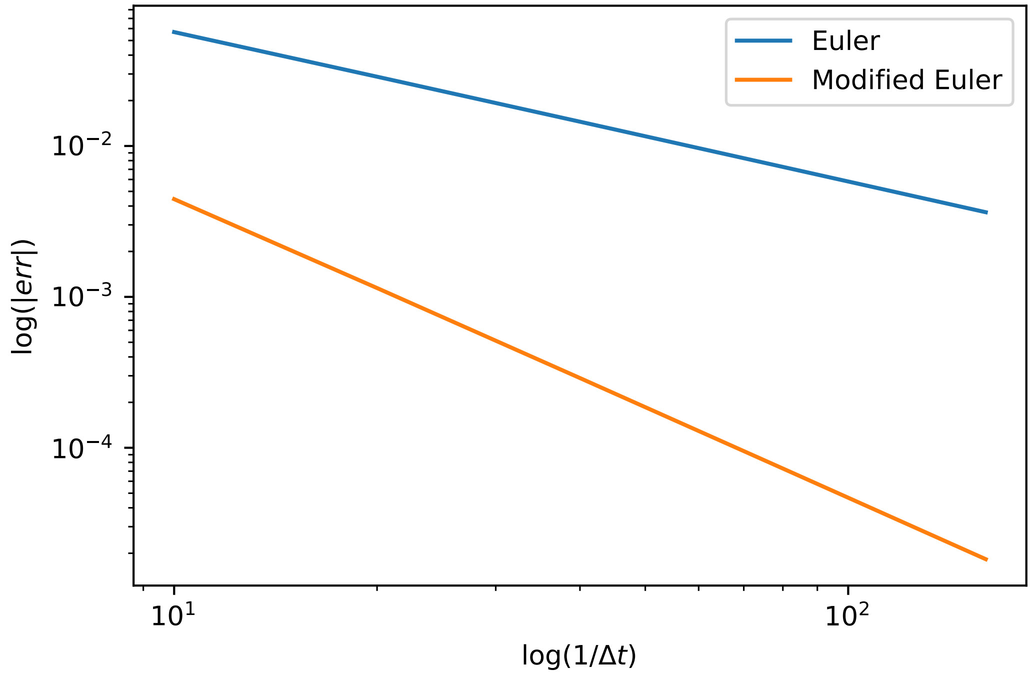

Figure 3: loglog plot of $1/\Delta t$ vs Absolute error for Euler and Modified Euler's methods

As we had expected, the error in both methods decreases with decreasing step size.

But, Modified Euler's performs better as it not only gives better approximation but also has a faster rate of decrease of error.