Practical :- 1

Topic:- Limits and Continuity

Ex:-1) $\lim\limits_{x\to a}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

$$\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\times\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\times\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right]$$

$$\lim_{x\to a}\frac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$\lim_{x\to a}\frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(3a-3x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$=\frac{1}{3}\lim_{x\to a}\frac{\cancel{(a-x)}(\sqrt{3a+x}+2\sqrt{x})}{\cancel{(a-x)}(\sqrt{a+2x}+\sqrt{3x})}$$

$$=\frac{1}{3}\ \frac{\sqrt{3a+a}+2\sqrt{a}}{\sqrt{a+2a}+\sqrt{3a}}$$

$$=\frac{1}{3}\times\frac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$$

$$=\frac{1}{3}\times\frac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$$

$$=\frac{1}{3}\times\frac{\cancel{4}^{2}\cancel{\sqrt{a}}}{\cancel{2}\sqrt{3}\cancel{\sqrt{a}}}$$

$$=\frac{2}{3\sqrt{3}}$$

2. $\lim\limits_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$$\lim_{y\to 0}\left(\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\times\frac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right)$$

$$\lim_{y\to 0}\frac{a+y-a}{y\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}$$

$$\lim_{y\to 0}\frac{\cancel{y}}{\cancel{y}\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}$$

$$=\frac{1}{\sqrt{a+0}\,(\sqrt{a+0}+\sqrt{a})}$$

$$=\frac{1}{\sqrt{a}\,(\sqrt{a}+\sqrt{a})}$$

$$=\frac{1}{\sqrt{a}\,(2\sqrt{a})}$$

$$=\frac{1}{2a}$$

3. $\lim\limits_{x\to \pi/6}\dfrac{\cos x-\sqrt{3}\sin x}{\pi-6x}$

By substituting $x-\frac{\pi}{6}=h$

$x = h+\pi/6$

where $h\to 0$

$$\lim_{h\to 0}\frac{\cos(h+\pi/6)-\sqrt{3}\sin(h+\pi/6)}{\pi-6(h+\pi/6)}$$

No.	
1.	Li
2.	D
3.	Ap
4.	Ap
	N
5.	In
6.	A
	N
7.	D
8.	E
9.	Li
10.	D

$$\lim_{h\to 0} \frac{\cosh.\cos\frac{\pi}{6} - \sin h.\sin\frac{\pi}{6} - \sqrt{3}\sin h.\cos\frac{\pi}{6} + \cos h.\sin\frac{\pi}{6}}{\pi - 6\left(\frac{6h+\pi}{6}\right)}$$

$\cos\frac{\pi}{6} = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2}$

$$\lim_{h\to 0} \frac{\cosh.\frac{\sqrt{3}}{2} - \sin h\frac{1}{\sqrt{2}} - \sqrt{3}\left(\sin h.\frac{\sqrt{3}}{2} + \cosh.\frac{1}{\sqrt{2}}\right)}{\pi - 6h + \pi}$$

$$\lim_{h\to 0} \frac{\cancel{\cos h\frac{\sqrt{3}}{2}}\, h - \sin\frac{h}{2} - \sin\frac{3h}{2} - \cancel{\cos\frac{\sqrt{3}}{2}h}}{-6h}$$

$$\lim_{h\to 0} \frac{\cancel{-}\sin\frac{4h}{2}}{\cancel{-}6h}$$

$$\lim_{h\to 0} \frac{\sin 4h}{3\cancel{12}h}$$

$$\frac{1}{3}\lim_{h\to 0} \frac{\sin h}{h}$$

$$= \frac{1}{3}\times 1 = \frac{1}{3}$$

4) $$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}}\right]$$

By rationalizing numerator & Denominator both

$$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}} \times \frac{\sqrt{x^2+5}+\sqrt{x^2-3}}{\sqrt{x^2+5}+\sqrt{x^2-3}} \times \frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2+1}}\right]$$

$$\lim_{x\to\infty}\frac{(x^2+5-x^2+3)(\sqrt{x^2+3}-\sqrt{x^2+1})}{(x^2+3-x^2-1)(\sqrt{x^2+5}+\sqrt{x^2-3})}$$

$$\lim_{x\to\infty}\frac{{}^{4}\cancel{8}(\sqrt{x^2+3}+\sqrt{x^2+1})}{\cancel{2}(\sqrt{x^2+5}+\sqrt{x^2-3})}$$

$$4\lim_{x\to\infty}\frac{\sqrt{x^2(1+\frac{3}{x^2})}+\sqrt{x^2(1+\frac{1}{x^2})}}{\sqrt{x^2(1+\frac{5}{x^2})}+\sqrt{x^2(1-\frac{3}{x^2})}}$$

After appling limit
we get,
$= 4$

5) i) $f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x \le \frac{\pi}{2}$
$= \frac{\cos x}{\pi - 2x}$, for $\frac{\pi}{2} < x < \pi$ } at $x = \frac{\pi}{2}$

$$f(\tfrac{\pi}{2}) = \frac{\sin 2(\frac{\pi}{2})}{\sqrt{1-\cos 2(\frac{\pi}{2})}} \quad \therefore f(\tfrac{\pi}{2}) = 0.$$

f at $x = \frac{\pi}{2}$ define.

a) $$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\pi - 2x}$$

By substituting method
$x - \frac{\pi}{2} = h$
$x = h + \frac{\pi}{2}$
where $h \to 0$

$$\lim_{h\to 0}\frac{\cos(h+\frac{\pi}{2})}{\pi - 2(h+\frac{\pi}{2})}$$

$$\lim_{h\to 0}\frac{\cos(h+\frac{\pi}{2})}{\pi - 2\left(\frac{2h+\pi}{2}\right)}$$

$$\lim_{h\to 0}\frac{\cos(h+\frac{\pi}{2})}{-2h}$$ using $(\cos(A+B) = \cos A.\cos B - \sin A\sin B$

$$\lim_{h\to 0}\frac{\cosh.\cos\frac{\pi}{2}.\sin h.\sin\frac{\pi}{2}}{-2h}$$

No.	
1.	Li
2.	D
3.	Ap
4.	Ap
	N
5.	I
6.	A
	N
7.	I
8.	E
9.	U
10.	I

$\lim_{h \to 0} \frac{\cos h \cdot 0 - \sin h}{-2h}$

$\lim_{h \to 0} \frac{\cancel{-}\sin h}{\cancel{-}2h}$

$\frac{1}{2} \lim_{h \to 0} \frac{\sin h}{h}$

$= \frac{1}{2}$

b) $\lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2^-} \frac{\sin 2x}{\sqrt{1 - \cos 2x}}$

using $\sin 2x = 2 \sin x \cdot \cos x$

$\lim_{x \to \pi/2^-} \frac{2 \sin x \cdot \cos x}{\sqrt{2 \sin^2 x}}$

$\lim_{x \to \pi/2^-} \frac{2 \cancel{\sin x} \cdot \cos x}{\sqrt{2}\, \cancel{\sin x}}$

$\lim_{x \to \pi/2^-} \frac{2 \cos x}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cos x$

$\therefore$ L.H.L $\neq$ R.H.S

$\therefore f$ is not continuous at $x = \pi/2$

5] iii) $f(x) = \frac{x^2 - 9}{x - 3} \quad 0 < x < 3$

$= x + 3 \quad 3 \le x \le 6$

$= \frac{x^2 - 9}{x + 3} \quad 6 \le x < 9$

$\Big\}$ at $x = 3$ & $x = 6$

at $x = 3$

i) $f(x) = \frac{x^2 - 9}{x - 3} = 0$

f at $x = 3$ define

(ii) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x + 3$

$f(3) = x + 3 = 3 + 3 = 6$

f is define at $x = 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x + 3) = 6$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

$\therefore$ L.H.L = R.H.L

f is continuous at $x = 3$

for $x > 6$

$f(6) = \frac{x^2 - 9}{x + 3} = \frac{36 - 9}{6 + 3} = \frac{\cancel{27}^{3}}{\cancel{9}} = 3$

2) $\lim_{x \to 6^+} \frac{x^2 - 9}{x + 3}$

$\lim_{x \to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$

$\lim_{x \to 6^+} (x - 3) = 6 - 3 = 3$

$\lim_{x \to 6^-} x + 3) = 3 + 6 = 9$

$\therefore$ L.H.S $\neq$ R.H.S

function is not continuous.

6) i) $f(x) = \dfrac{1-\cos 4x}{x^2}$ $x < 0$; $= k$ $x = 0$ } at $x = 0$

Solⁿ :- f is continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0} \frac{1-\cos 4x}{x^2} = k$$

$$\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = k$$

$$2\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = k$$

$$2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2 = k$$

$$2(2)^2 = k$$

$$\therefore k = 8$$

ii) $f(x) = (\sec^2 x)^{\cot^2 x}$ $x \neq 0$; $= k$ $x = 0$ } at $x = 0$

Solⁿ :- $f(x) = (\sec^2 x)^{\cot^2 x}$

using

$\tan^2 x - \sec^2 x = 1$

$\therefore \sec^2 x = 1 + \tan^2 x$ &

$\cot^2 x = \dfrac{1}{\tan^2 x}$

$$\therefore \lim_{x\to 0} (\sec^2 x)^{\cot^2 x}$$

$$\lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$$

we know that

$$\lim_{x\to 0} (1+px)^{\frac{1}{px}} = e$$

$$\therefore k = e$$

iii) $f(x) = \dfrac{\sqrt{3}-\tan x}{\pi - 3x}$ $x \neq \pi/3$; $= k$ $x = \pi/3$ } at $x = \pi/3$

$x - \pi/3 = h$

$x = h + \pi/3$

where $h \to 0$

$$f(\pi/3 + h) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

$$\lim_{h\to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$$\lim_{h\to 0} \frac{\sqrt{3} - \dfrac{\tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}}{\pi - \pi - 3h}$$

$$\lim_{h\to 0} \frac{\dfrac{\sqrt{3}(1 - \tan\pi/3 \cdot \tan h) - (\tan \pi/3 + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3}\times\sqrt{3} - \tan h) - (\sqrt{3} + \tan h)}{1 - \sqrt{3}\cdot \tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\cancel{\sqrt{3}} - 3\tan h - \cancel{\sqrt{3}} - \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\cancel{-}4\tan h}{\cancel{-}3h(1-\sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{4\tan h}{3h(1-\sqrt{3}\tan h)}$$

$$\frac{4}{3}\lim_{h\to 0}\frac{\tan h}{h} \quad \lim_{h\to 0}\frac{1}{(1-\sqrt{3}\tan h)} \qquad \frac{\tan h}{h}=1$$

$$=\frac{4}{3}\frac{1}{(1-\sqrt{3}(0))}$$

$$=\frac{4}{3}\left(\frac{1}{1}\right)=\frac{4}{3}$$

7) i) $f(x)=\dfrac{1-\cos 3x}{x\tan x}$ $\quad x\neq 0$
$\quad = 9 \quad x=0$ } at $x=0$

$$f(x)=\frac{1-\cos 3x}{x\cdot\tan x}$$

$$\lim_{x\to 0} = \frac{2\sin^2 \frac{3}{2}x}{x\cdot\tan x}$$

$$\lim_{x\to 0} = \frac{\frac{2\sin^2 \frac{3}{2}x}{x^2}\times x^2}{\frac{x\cdot\tan x}{x^2}\times x^2}$$

$$= 2\lim_{x\to 0}\frac{\left(\frac{3}{2}\right)^2}{1}$$

$$=\cancel{2}\times\frac{9}{\cancel{4}2}$$

$$=\frac{9}{2}$$

$\lim_{x\to 0} fx = \frac{9}{2} \qquad 9 = f(0)$

f is not continuous at $x=0$

Redefine function

$f(x)=\begin{cases}\dfrac{1-\cos 3x}{x\cdot\tan x} & x\neq 0\\ \frac{9}{2} & x=0\end{cases}$

Now $\lim_{x\to 0} f(x)=f(0)$

f has removable discontinuity at $x=0$

ii) $f(x)=\dfrac{(e^{3x}-1)\sin x^\circ}{x^2}$ $\quad x\neq 0$
$\quad =\frac{\pi}{6} \quad x=0$ } at $x=0$

$$\lim_{x\to 0}\frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$$

$$\lim_{x\to 0}\frac{e^{3x}-1}{x}\lim_{x\to 0}\frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$\lim_{x\to 0}\frac{3\cdot e^{3x}-1}{3x}\lim_{x\to 0}\frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$\cancel{3}\log e\cdot\frac{\pi}{\cancel{180}_{60}}=\frac{\pi}{60}=f(0)$$

f is continuous at $x=0$

8) $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$ $\quad x \neq 0$

is continuous at $x=0$

$\therefore$ Given, f is continuous at $x=0$

$\lim\limits_{x\to 0} f(x) = f(0)$

$\lim\limits_{x\to 0} \dfrac{e^{x^2} - \cos x}{x^2} = f(0)$

$\lim\limits_{x\to 0} \dfrac{e^{x^2} - \cos x - 1 + 1}{x^2}$

$\lim\limits_{x\to 0} \dfrac{(e^{x^2} - 1) + (1 - \cos x)}{x^2}$

$\lim\limits_{x\to 0} \dfrac{e^{x^2}-1}{x^2} + \lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

$\log e + \lim\limits_{x\to 0} \dfrac{2\sin^2 x/2}{x^2}$

$\log e + 2 \lim\limits_{x\to 0} \left(\dfrac{\sin x/2}{x}\right)^2$

Multiply with 2 in Num & Denominator

$= 1 + 2 \times \frac{1}{2} = \frac{3}{2} = f(0)$

9) $f(x) = \dfrac{\sqrt{2} - \sqrt{1-\sin x}}{\cos^2 x}$ $\quad x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$\lim\limits_{x\to \pi/2} \dfrac{\sqrt{2} - \sqrt{1-\sin x}}{\cos^2 x} \times \dfrac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$

$\lim\limits_{x\to \pi/2} \dfrac{2 - 1 + \sin x}{\cos^2 x\,(\sqrt{2} + \sqrt{1+\sin x})}$

$\lim\limits_{x\to \pi/2} \dfrac{1+\sin x}{1-\sin^2 x\,(\sqrt{2} + \sqrt{1+\sin x})}$

$\lim\limits_{x\to \pi/2} \dfrac{\cancel{1+\sin x}}{(1-\sin x)\cancel{(1+\sin x)}(\sqrt{2} + \sqrt{1+\sin x})}$

$\lim\limits_{x\to \pi/2} \dfrac{1}{(1-\sin x)(\sqrt{2} + \sqrt{1+\sin x})}$

$= \dfrac{1}{2\sqrt{2}}$

$= \dfrac{1}{2(\sqrt{2}+\sqrt{2})}$

$= \dfrac{1}{2(2\sqrt{2})}$

$= \dfrac{1}{4\sqrt{2}}$

$\therefore f(\pi/2) = \dfrac{1}{4\sqrt{2}}$

Practical :- 0?

Topic :- Derivative

Q1) Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable.

i) $\cot x$

$f(x) = \cot x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x - a}$$

$$= \lim_{x \to a} \frac{\tan a - \tan x}{(x-a) \tan x . \tan a}$$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(a+h-a) \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h) \tan a}$$

formula :- $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$

$\tan A - \tan B = \tan(A-B)(1 + \tan A . \tan B)$

$$= \lim_{h \to 0} \frac{\tan(a-a-h) - (1 + \tan a . \tan(a+h))}{h \times \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{-\tan h}{h} \times \frac{1 + \tan a \tan(a+h)}{\tan(a+h) \tan a}$$

$$= -1 \times \frac{1 + \tan^2 a}{\tan^2 a}$$

$$= \frac{-\sec^2 a}{\tan^2 a}$$

$$= \frac{-1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

$\therefore Df(a) = -\cos^2 a$

$\therefore f$ is differentiable $\forall a \in \mathbb{R}$

ii) $\operatorname{cosec} x$

$f(x) = \operatorname{cosec}(x)$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sin a - \sin x}{(x-a) \sin a . \sin x}$$

put $x - a = h$

$x = a + h$

$x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a) \sin a . \sin(a+h)}$$

formula :-

$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) . \sin\left(\frac{C-D}{2}\right)$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right)\cdot \sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \,\sin(a+h)}$$

$$= \lim_{h\to 0} \frac{-\sin \frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a\, \sin(a+h)}$$

$$= -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$$

$$= -\frac{\cos a}{\sin^2 a} = -\cot a \cdot \operatorname{cosec} a$$

iii) $\sec x$

$f(x) = \sec x$

$$Df(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x-a}$$

$$= \lim_{x\to a} \frac{\cos a - \cos x}{(x-a)\cos a \cdot \cos x}$$

put $x - a = h$

$x = a+h$

as $x \to a$, $h \to 0$

$$Df(a) = \lim_{h\to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cdot \cos(a+h)}$$

Formula :- $-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \cos a\, \cos(a+h)}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin \frac{-h}{2} \times \frac{-1}{2}}{\cos a \cos(a+h) \times -\frac{h}{2}}$$

$$= -\frac{1}{2} \times \frac{-2 \sin\left(\frac{2a+0}{2}\right)}{\cos a \cdot \cos(a+0)}$$

$$= -\frac{1}{2} \times -2 \frac{\sin a}{\cos a \times \cos a}$$

$= \tan a \cdot \sec a$

Q.2) If $f(x) = 4x+1$, $x \leq 2$

$= x^2+5$ $x > 0$, at $x = 2$, then

find function is differentiable or not

Solⁿ :-

L.H.D :-

$$Df(2^-) = \lim_{x\to 2^-} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to 2^-} \frac{4x+1-(4\times 2+1)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x\to 2^-} \frac{4(x-2)}{(x-2)}$$

$Df(2^-) = 4$

R.H.D :-

$$Df(2^+) = \lim_{x\to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x\to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 2+2$$

$$Df(2^+) = 4$$

R.H.D = L.H.D

$\therefore f$ is differentiable at $x=2$

Q3) If $f(x) = 4x+7$, $x<3$
$= x^2+3x+1$, $x \geq 3$ at $x=3$, then
find f is differentiable or not?

Solⁿ :-

RHD :- $Df(3^+) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3}$

$$= \lim_{x\to 3^+} \frac{x^2+3x+1-(3^2+3\times 3+1)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+3x+1-19}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+3x-18}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= 3+6$$

$$Df(3^+) = 9$$

L.H.D :-

$$Df(3^-) = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4x+7-19}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$Df(3^-) = 4$$

$\therefore$ RHD $\neq$ LHD

$\therefore f$ is not differentiable at $x=3$

Q.4 If $f(x) = 8x-5$, $x \leq 2$
$= 3x^2-4x+7$, $x>2$ at $x=2$, then
find f is differentiable or not?

Solⁿ :-

$$f(2) = 8\times 2-5 = 16-5 = 11$$

R.H.D :-

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 3 \times 2 + 2$$

$$Df(2^+) = 8$$

L.H.D :-

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 8$$

$$Df(2^-) = 8$$

$\therefore$ L.H.D = R.H.D

$\therefore$ f is differentiable at $x = 3$

No.
1.
2.
3.
4.
5.
6.
7.
8.
9.
10.

Practical :- 3

Topic :- Application of Derivative

(1) Find the intervals in which function is increasing or decreasing

i) $f(x) = x^3 - 5x + 11$

ii) $f(x) = x^2 - 4x$

iii) $f(x) = 2x^3 + x^2 - 20x + 4$

iv) $f(x) = x^3 - 27x + 5$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

(2) Find the intervals in which function is concave upwards or concave downwards.

i) $y = 3x^2 - 2x^3$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

iii) $y = x^3 - 27x + 5$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

v) $y = 2x^3 + x^2 - 20x + 4$

Q.1

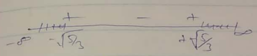
i) To find increasing,

$f'(x) > 0$

$f'(x) = 3x^2 - 5 > 0$

$= 3x^2 > 5$

$\therefore x = \pm\sqrt{5/3}$



$x \in (-\infty, -\sqrt{5/3}) \cup (+\sqrt{5/3}, +\infty)$

To find decreasing $f'(x) < 0$

$f'(x) < 0$

$= 3x^2 - 5 < 0$

$3x^2 < 5$

$x = \pm\sqrt{5/3}$

$x \in (-\sqrt{5/3}, +\sqrt{5/3})$

(ii) $f(x) = x^2 - 4x$

To find increasing,

$f'(x) > 0$

$f'(x) = 2x - 4 > 0$

$= 2(x-2) > 0$

$x - 2 > 0$

$x = 2$

$x \in (2, \infty)$

To find decreasing

$f'(x) < 0$

$f'(x) = 2x - 4 < 0$

$x - 2 < 0$

$\therefore x = 2$

$x \in (-2, -\infty)$

iii) $f(x) = 2x^3 - x^2 - 20x + 4$

$f(x) > 0$ for increasing

$f'(x) = 6x^2 - 2x - 20 > 0$

$= 6x^2 - 12x + 10x - 20 > 0$

$= 6x(x-2) + 10(x-2) > 0$

$= 6x + 10 > 0 \ , \ x - 2 > 0$

$x = -5/3 \ , \ x = 2$

$x \in (-5/3, -\infty) \cup (2, +\infty)$

$f'(x) < 0$ for decreasing

$f'(x) = 6x^2 - 2x - 20 < 0$

$= 6x(x-2) + 10(x-2) < 0$

$= 6x + 10 < 0, \ x - 2 < 0$

$\therefore x = -5/3 \ , \ x = 2$

$x \in (-5/3, 2)$

iv) $f(x) = x^3 - 27x + 5$

$f'(x) > 0$ for increasing

$f'(x) \cancel{>0} = 3x^2 - 27 > 0$

$= x^2 - 9 > 0$

$= (x-3)(x+3) > 0$

$= x = \pm 3$

$x \in (-3, -\infty) \cup (+3, +\infty)$

$f'(x) < 0$ for decreasing

$f'(x) = 3x^2 - 27 > 0$

$= x^2 - 9 > 0$

$= (x-3)(x+3) > 0$

$x = \pm 3$

$x \in (-3, +3)$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

$f'(x) > 0$ for increasing

$f'(x) = 6x^2 - 18x - 24 > 0$

$= x^2 - 3x - 4 > 0$

$= x^2 - 4x + x - 4 > 0$

$= x(x-4) + 1(x-4) > 0$

$(x+1) > 0 \ , \ x - 4 > 0$

$x \in (-1, -\infty) \cup (4, +\infty)$

$f'(x) < 0$ for decreasing

$f'(x) = 6x^2 - 18x - 24 < 0$

$= x^2 - 3x - 4 < 0$

$= x^2 - 4x + x - 4 < 0$

$= x(x-4) + 1(x-4) < 0$

$x + 1 < 0 \ , \ x - 4 < 0$

$x \in (-1, 4)$

Q.2.

i) $y = 3x^2 - 2x^3$

case 1 :- f^x is concave upwards iff $f''(y) > 0$,

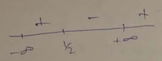
$f'(y) = 6x - 6x^2$

$f''(y) = 6 - 12x$

$f''(y) > 0$

$6 - 12x > 0$

$x = 1/2$



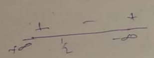
$x \in (-\infty, 1/2)$

case II :- f^x is concave downward iff $f''(y) < 0$

$f''(y) < 0$

$6 - 12x < 0$

$x = 1/2$



$x \in (1/2, +\infty)$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

caseI :- f^x is concave upwards iff $f''(xy) > 0$

$f'(y) = 4x^3 - 18x^2 + 24x + 5$

$f''(y) = 12x^2 - 36x + 24$

$\therefore f''(y) > 0$

$12x^2 - 36x + 24 > 0$

$x^2 - 3x + 2 > 0$

$x(x-2) - 1(x-2) > 0$

$(x-1)(x-2) > 0$

$x = 1, \; x = 2$

$x \in (1, -\infty) \cup (2, +\infty)$

case II :- $f''(y) < 0$, then it is concave downwards

$f''(y) < 0$

$= 12x^2 - 36x + 24 < 0$

$x^2 - 3x + 2 < 0$

$x^2 - 2x - x + 2 < 0$

$x(x-2) - 1(x-2) < 0$

$(x-1) \cdot (x-2) < 0$

$x = 1, \; x = 2$

$\therefore x \in (1, 2)$

iii) $y = x^3 - 27x + 5$

$f''(y) > 0$ for concaving upward.

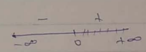
$f'(y) = 3x^2 - 27$

$f''(y) = 6x$

$f''(y) > 0$

$6x > 0$

$\therefore x > 0$



$x \in (0, +\infty)$

$f''(y) < 0$ for concave downward.

$f''(y) = 6x < 0$

$\therefore x < 0$

$\therefore x \in (0, -\infty)$

- A

iv) $y = 69 - 24x - 9x^2 + 2x^3$

$f''(y) > 0$ for concave upwards.

$f'(y) = -24 - 18x + 6x^2$

$f''(y) = 12x - 18$

$f''(y) > 0$

$12x - 18 > 0$

$12x > 18$

$x = 18/12$

$x = 3/2$

$x \in (3/2, +\infty)$

$f''(y) < 0$ for concave downwards.

$f''(y) < 0$

$12x - 18 < 0$

$12x < 0$

$x = 3/2$

$\therefore x \in (3/2, -\infty)$

v) $y = \cancel{2x^3} + x^2 - 20x + 4$

$f''(y) > 0$ for concave upwards

$f'(y) = 6x^2 + 2x - 20$

$f''(y) = 12x + 2$

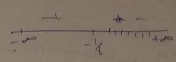
$\therefore f''(y) > 0$

$\therefore 12x + 2 > 0$

$12x = -2$

$x = -2/12$

$= -1/6$



$\therefore x \in (-1/6, +\infty)$

$f''(y) < 0$ for concave downward

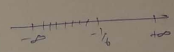
$f''(y) < 0$

$12x + 2 < 0$

$12x = -2$

$x = -2/12$

$x = -1/6$



~~$\therefore x \in (-1/6, -\infty)$~~

$x \in (-\infty, -1/6)$

18/12/19

Practical :- 4

Topic :- Application of Derivative & Newton's Method.

Q.1. find maximum & minimum value of following functions.

i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now consider $f'(x) = 0$

$2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum values at $x = 2$

$\therefore f(2) = 2^2 + \frac{16}{2^2}$

$= 4 + \frac{16}{4}$

$= 4 + 4$

$= 8$

$f''(-2) = 2 + \frac{96}{-2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f is minimum value at $x = -2$

$\therefore$ function reaches minimum value at $x = 2$ and $x = -2$

ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

Now consider $f'(x) = 0$

$\therefore -15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$\therefore f''(1) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minimum value at $x = 1$

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0$

$\therefore$ f has maximum value at $x = -1$

$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 5$

$\therefore$ f has ~~minimum~~ maximum value ~~at~~ 5 at $x = -1$ and has ~~maximum~~ minimum value 1 at $x = 1$

iii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$

Consider $f'(x) = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$3x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore f$ has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1$

$= 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore f$ has minimum value at $x = 2$

$f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore f$ has maximum value 1 at $x = 0$ and has minimum value -3 at $x = 2$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

Consider: $f'(x) = 0$

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore f$ has minimum value at $x = 2$

$\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$\therefore f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore f$ has maximum ~~at~~ value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1$

$= 8$

$\therefore f$ has maximum value 8 at $x = -1$ and has minimum value -19 at $x = 2$

Q2) find the root of following equation by Newton method (Take 4 iteration only) correct upto 4 deci...

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = 0 + \frac{9.5}{55}$$

$$\therefore x_1 = 0.1727$$

$$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$$
$$= 0.0051 - 0.0895 - 9.4985 + 9.5$$
$$= -0.0829$$

$$f'(x_1) = 3(0.1727)^3 - 6(0.1727) - 55$$
$$= 0.0895 - 1.0362 - 55$$
$$= -55.9467$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{0.0829}{55.9467}$$

$$= 0.1712$$

$$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$$
$$= 0.0050 - 0.0879 - 9.416 + 9.5$$
$$= 0.0011$$

$$f'(x) = 3(0.1712)^2 - 6(0.1712) - 55$$
$$= 0.0879 - 1.0272 - 55$$
$$= -55.9393$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.1712 + \frac{0.0011}{55.9393}$$

$$= 0.1712$$

$\therefore$ The root of equation is 0.1712

ii) $f(x) = x^3 - 4x - 9$ $[2, 3]$

$f'(x) = 3x^2 - 4$

$$f(2) = 2^3 - 4(2) - 9$$
$$= 8 - 8 - 9$$
$$= -9$$

$$f(3) = 3^3 - 4(3) - 9$$
$$= 27 - 12 - 9$$
$$= 6$$

let $x_0 = 3$ be the initial approximation,

$\therefore$ By Newton's Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{6}{23}$$

$$= 2.7392$$

$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$
$= 20.5528 - 10.9568 - 9$
$= 0.596$

$f'(x_1) = (2.7392)^2 - 4$
$= 22.5096 - 4$
$= 18.5096$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 2.7392 - \dfrac{0.596}{18.5096}$

$= 2.7071$

$f(x_2) = (2.7071)^3 - 4(2.7071) - 9$
$= 19.8386 - 10.8284 - 9$
$= 0.0102$

$f'(x_2) = 3(2.7071)^2 - 4$
$= 21.9851 - 4$
$= 17.9851$

$\therefore x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 2.7071 - \dfrac{0.0102}{17.9857}$

$= 2.7071 - 0.0056$
$= 2.7015$

$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$
$= 19.7158 - 10.806 - 9$
$= -0.0901$

$f'(x_3) = 3(2.7015)^2 - 4$
$= 21.8943 - 4$
$= 17.8943$

$\therefore x_4 = 2.7015 + \dfrac{0.0901}{17.8943}$
$= 2.7015 + 0.0050$
$= 2.7065$

$\therefore$ The root of equation is 2.7065

iii) $F(x) = x^3 - 1.8x^2 - 10x + 17$ $[1, 2]$
$f'(x) = 3x^2 - 3.6x - 10$
$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$
$= 1 - 1.8 - 10 + 17$
$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$
$= 8 - 87.2 - 20 + 17$
$= -2.2$

Let $x_0 = 2$ be initial approximation
By Newton's Method,
$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 2 - \dfrac{2.2}{5.2}$
$= 2 - 0.4230$
$= 1.577$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$
$= 3.9219 - 4.4764 - 15.77 + 17$
$= 0.6755$

$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$
$= 7.4608 - 5.6772 - 10$
$= -8.2164$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$
$= 1.577 + \dfrac{0.6755}{8.2164}$
$= 1.577 + 0.0822$
$= 1.6592$

$$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$$
$$= 4.5677 - 4.9653 - 16.592 + 17$$
$$= 0.0204$$

$$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$$
$$= 8.2588 - 5.97312 - 10$$
$$= -7.7143$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 1.6592 + \frac{0.0204}{7.7143}$$
$$= 1.6592 + 0.0026$$
$$= 1.6618$$

$$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$$
$$= 4.5892 - 4.9708 - 16.618 + 17$$
$$= 0.0004$$

$$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$$
$$= 8.2847 - 5.9824 - 10$$
$$= -7.6977$$

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 1.6618 + \frac{0.0004}{7.6977}$$
$$= 1.6618$$

$\therefore$ The root of equation is 1.6618

Practical :- 5

Integration

Q.1) Solve the following Integration.

i) $$I = \int \frac{1}{\sqrt{x^2+2x-3}}\,dx$$

$$I = \int \frac{1}{\sqrt{x^2+2x-3}}\,dx$$

$$I = \int \frac{1}{\sqrt{x^2+2x+1-4}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2-4}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2-(2)^2}}\,dx$$

$\because a^2+2ab+b^2 = (a+b)^2$

Put $x+1 = t$

$dx = dt$

where $t = 1$

$t = x+1$

$$\int \frac{1}{\sqrt{t^2-4}}\,dt$$

$$= \log|t+\sqrt{t^2-4}| + c \quad \left[\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log|x+\sqrt{x^2-a^2}|\right]$$

$t = x+1$

$$= \log|x+1+\sqrt{(x+1)^2-4}| + c$$

$$= \log|x+1+\sqrt{x^2+2x-3}| + c$$

ii) $\int (4e^{3x}+1)\,dx$

$I = \int (4e^{3x}+1)\,dx$

$= \int 4e^{3x}\,dx + \int 1\,dx$

$= 4\int e^{3x}\,dx + \int dx$

$= \frac{4e^{3x}}{3} + x + c$ $\quad [\because \int e^{ax}\,dx = \frac{1}{a}e^{ax}]$

$= \frac{4e^{3x}}{3} + x + c$

iii) $\int 2x^2 - 3\sin(x) + 5\sqrt{x}\,dx$

$I = \int 2x^2 - 3\sin x + 5\sqrt{x}\,dx$

$= \int 2x^2 - 3\sin x + 5x^{1/2}\,dx$

$= 2\int x^2\,dx - 3\int \sin x + 5\int x^{1/2}\,dx$

$= \frac{2}{3}x^3 + 3\cos x + \frac{10\,{}^3\sqrt{x}}{3} + c$

$[\because \int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$\because \int \sin x\,dx = -\cos x$

$= \frac{2x^3 + 10\,{}^3\sqrt{x}}{3} + 3\cos x + c$

iv) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$I = \int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$= \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{4}{\sqrt{x}}\right)dx$

$= \int \frac{x^3}{x^{1/2}}\,dx + \int \frac{3x}{x^{1/2}}\,dx + \int \frac{4}{x^{1/2}}\,dx$

$= \int x^{5/2}\,dx + 3\int x^{1/2}\,dx + 4\int x^{-1/2}\,dx$

$= \frac{x^{5/2+1}}{5/2+1} + 3\frac{x^{1/2+1}}{1/2+1} + 4\frac{x^{-1/2+1}}{-1/2+1}$

$= \frac{x^{7/2}}{7/2} + 3\frac{x^{3/2}}{3/2} + 4\frac{x^{1/2}}{1/2}$

$= \frac{2x^{7/2}}{7} + 2x^{3/2} + 8\sqrt{x} + c$

v) $\int t^7 \times \sin(2t^4)\,dt$

put $u = 2t^4$

$du = 2\times 4t^3\,dt$

$= \int t^7 \times \sin(2t^4) \times \frac{1}{2\times 4t^3}\,du$

$= \int t^4 \sin(2t^4) \times \frac{1}{2\times 4}\,du$

$= \int t^4 \sin 2t^4 \times \frac{1}{8}\,du$

$= \int \frac{t^4 \times \sin 2t^4}{8}\,du$

Substitute t^4 with $\frac{u}{2}$

$= \int \frac{u}{2} \times \sin(2\frac{u}{2})\,du$

$$= \int \frac{\frac{u}{2} \times \sin\frac{(u)}{2} \, du}{8}$$

$$= \frac{1}{16} \int u + \sin(u) \, du$$

$$= \frac{1}{16} (u \times (-\cos(u)) - \int -\cos(u) \, du$$

$[\because \int u \, dv = uv - \int v \, du$

where $u = v$

$dv = \sin(u) \times du$

$du = 1 \, du \quad v = -\cos(u)$

$$= \frac{1}{16} \times (u \times (-\cos v)) + \int \cos(v) \, du$$

$$= \frac{1}{16} \times (u \times (-\cos(u)) + \sin(u)) \quad [\because \int \cos x \, dx = \sin x \, dx$$

Resubstituting $u = 2t^4$

$$= \frac{1}{16} \times (2t^4 \times (-\cos(2t^4)) + \sin(2t^4))$$

$$= \frac{-t^4 \times \cos(2t^4)}{8} + \frac{\sin(2t^4)}{16} + C$$

vi) $\int \sqrt{x} \, (x^2 - 1) \, dx$

$$I = \int \sqrt{x} \, (x^2 - 1) \, dx$$

$$= \int x^{1/2} (x^2 - 1) \, dx$$

$$= \int x^{5/2} - x^{1/2} \, dx$$

$$= \int x^{5/2} \, dx - \int x^{1/2} \, dx$$

$$= \frac{x^{5/2 + 1}}{5/2 + 1} - \frac{x^{1/2 + 1}}{1/2 + 1}$$

$$= \frac{x^{7/2}}{7/2} - \frac{x^{3/2}}{3/2} = \frac{2x^{7/2}}{7} - \frac{2x^{3/2}}{3} + C$$

vii) $\int \frac{\cos x}{\sqrt[3]{\sin(x)^2}} \, dx$

$$I = \int \frac{\cos x}{\sqrt[3]{\sin(x)^2}} \, dx$$

$$= \int \frac{\cos x}{\sin x^{3/2}} \, dx$$

put $t = \sin x$

$dt = \cos x \, dx$

$$= \int \frac{dt}{t^{2/3}}$$

$$= \int t^{-2/3} \, dt$$

$$= \frac{t^{-2/3 + 1}}{-2/3 + 1}$$

$$= \frac{t^{1/3}}{1/3} + C$$

$$= 3 \sqrt[3]{t} + C$$

$$= 3 \sqrt[3]{\sin x} + C$$

viii) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$$I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$$

$$I = \int \frac{1}{x^3} \sin \frac{1}{x^2} \, dx$$

Let, $t = \frac{1}{x^2}$

$\therefore t = x^{-2} \quad \therefore dt = \frac{-2}{x^3} \, dx$

$$I = -\frac{1}{2}\int \frac{-2}{x^3} \sin \frac{1}{x^2}\, dx$$

$$= -\frac{1}{2} \int \sin t\, dt$$

$$= -\frac{1}{2} [-\cos t] + c$$

$$= \frac{1}{2} \cos t + c$$

Resubstituting $t = \frac{1}{x^2}$

$$\therefore I = \frac{1}{2} \cos\left(\frac{1}{x^2}\right) + c$$

ix) $\int e^{\cos^2 x} \sin 2x\, dx$

$$I = \int e^{\cos^2 x} \sin 2x\, dx$$

let $\cos^2 x = t$

$-2\cos x \sin x\, dx = dt$

$-2\sin 2x\, dx = dt$

$$I = \int -\sin 2x\, e^{\cos^2 x}\, dx$$

$$= -\int e^t\, dt$$

$$= -e^t + c$$

Resubstituting $t = \cos^2 x$

$$\therefore I = -e^{\cos^2 x} + c$$

x) $\int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$

$$I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$$

Let, $t = x^3 - 3x^2 + 1$

$dt = (3x^2 - 6x)\, dx$

$dt = 3(x^2 - 2x)\, dx$

$\frac{dt}{3} = (x^2 - 2x)\, dx$

$$I = \int \frac{1}{t} \frac{dt}{3}$$

$$= \frac{1}{3} \int \frac{dt}{t}$$

$$= \frac{1}{3} \log t + c$$

Resubstituting $t = x^3 - 3x^2 + 1$

$$\therefore I = \frac{1}{3} \log (x^3 - 3x^2 + 1) + c$$

Practical :- 6

Topic :- Application of integration and Numerical integration

Q.1) Find the length of following curve

1) $x = t - \sin t,\ y = 1 - \cos t,\quad t \in [0, 2\pi]$

$\rightarrow L = \int_0^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$

$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$

$= \int_0^{2\pi} 2\,|\sin \frac{t}{2}|\, dt$

$= \int_0^{4\pi} 2 \sin \frac{t}{2}\, dt$

$= \left(-4\cos\left(\frac{t}{2}\right)\right)_0^{2\pi}$

$= (-4\cos\pi) - (-4\cos 0)$

$= 4 + 4$

$= 8$

2) $y = \sqrt{4 - x^2}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}} \times (-2x)$

$= \frac{x}{\sqrt{4-x^2}}$

$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_{-2}^{2} \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$

$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$

$= 2\int_{-2}^{2} \frac{1}{\sqrt{2^2 - x^2}}\, dx$

$= 2\left(\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = 2\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$

$= 2\left(\sin^{-1}(1) - \sin^{-1}(-1)\right)$

$\therefore L = 2\pi$

3. $y = x^{3/2} \qquad x \in [0, 4]$

$\rightarrow \dfrac{dy}{dx} = \dfrac{3}{2} x^{1/2}$

$$L = \int_0^1 \sqrt{1} + \left(\frac{dy}{dx}\right)^2 dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4} x} \; dx$$

$$= \frac{1}{2} \int_0^4 \sqrt{4 + 9x} \, dx$$

$$= \frac{1}{2} \left[\frac{(4 + 9x)^{3/2}}{3/2} \times \frac{1}{9} \right]$$

$$= \frac{1}{27} \left[(4 + 9x)^{3/2} \right]$$

$$= -\frac{1}{27} \left[(4+0)^{3/2} - (4+36)^{3/2} \right]$$

$$= -\frac{1}{27} \left(40^{3/8} - 8 \right) \text{ unit.}$$

4. $x = 3x \sin t \qquad y = 3 \cos t$,

$\dfrac{dx}{dt} = 3\cos t \qquad \dfrac{dy}{dt} = -3 \sin t$

$$L = \int_0^{2x} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

$$= \int_0^{2x} \sqrt{(3\cos t)^2 + (3 \sin t)^2} \, dt$$

$$= \int_0^{2x} \sqrt{9 \sin^2 + 9 \cos^2} \, dt$$

$$= \int_0^{2x} 3\sqrt{1} \, dt$$

$$= 3 \int_0^{2x} dt$$

$$= 3 [x]^{2x}$$

$$= 3(2x - 0)$$

$\therefore L = 6x$ units.

5) $x = \dfrac{1}{6} y^3 + \dfrac{1}{2y}$

$\rightarrow \therefore \dfrac{dx}{dy} = \dfrac{y^2}{2} - \dfrac{1}{xy^2}$

$$\frac{dx}{dy} = \frac{y^{\cancel{x}} - 1}{xy^2}$$

$$L = \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= \int_0^2 \sqrt{1} + \left(\frac{(y^3 - 1)^2}{4y^2} \right) dy$$

$$= \int_0^2 \sqrt{\frac{(y^6 - 1) + 4xy^3 x 1}{4y}} \, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)}{(2y)^2}} \, dy$$

$$= \int_1^2 \frac{y^4 + 1}{2y^2} \, dy$$

$$= \frac{1}{2} \int_1^2 y^2 \, dy + \frac{1}{2} \int_1^2 y^{-2} \, dy$$

$$= \frac{1}{2}\left[\frac{y}{3} - \frac{y-2}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{17}{6}\right]$$

$$L = \frac{17}{12} \text{ units}$$

Q.2) using Simpson's Rule solve the following :-

1) $\int_0^2 e^x dx$ with $x = 4$

$\rightarrow$ $L = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

x	0	0.5	1	1.5	2
y	1	1.284	2.71	9.58	54.54

$$= \frac{0.5}{3}\left[(1 + (54.54) + 5(1.28 + 9.48) + 2 \times 2.7883)\right]$$

$$= \frac{0.5}{3}\left[55.54 + 43.868 + 5.426\right]$$

$$= \int_0^2 e x^2 dx = 17.3535$$

2) $\int_0^4 x^2 dx$ $n = 4$ $h = \frac{4-0}{4} = 1$

$\rightarrow$

x	0	1	2	3	4
y	0	1	4	9	16
	y_0	y_1	y_2	y_3	y_4

$$\int_0^1 x^2 dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$= \frac{1}{3}(16 + 4(10) + 8)$$

$$= \frac{64}{3}$$

$$= 21.333$$

$$\therefore \int_0^4 x^2 dx = 21.333$$

3) $\int_0^{\pi/3} \sqrt{\sin x}\, dx$

$= 0$

$h = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$

0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$
0	0.416	0.584	0.70	0.875	0.430
y_0	y_1	y_2	y_3	y_4	y_5

$$\int_0^2 \sqrt{\sin x}\, dx = \frac{h}{3}\left(y_0 + y_6 + 4(x_1 + y_3 + y_5) + 2(y_2 + y_6)\right]$$

$$= \frac{\pi/18}{3}\left[0.416 + 0.97 + 4(0.416 + 0.707) + 0.9752 + 2(0.5848 + 0.807)\right]$$

$$= \frac{\pi}{54}\left[1.3473 + 7.996 + 2.773\right]$$

$$= \frac{\pi}{54} \times 12.1163$$

$$\therefore \int_0^{\pi/3} \sqrt{\sin x}\, dx = 0.7049$$

AV

Practical :- 7

Topic :- Differential Equation

Q.1 Solve the following differential equation.

1) $x\frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$p(x) = 1/x \quad Q(x) = e^x/x$

$IF = e\int 1/x \, dx$

$= e^{\int dx}$

$= x$

$y(I.F) = \int Q(x)(IF)\,dx + c$

$= \int \frac{e^x}{x} \times x \, dx + c$

$= \int e^x dx + c$

$\therefore xy = e^x + c$

2) $e^x\frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + 2y = e^{-x}$

$p(x) = 2 \quad Q(x) = e^x$

$I.F = e^{\int p(x)dx}$

$= e^{\int 2dx}$

$= e^{2x}$

$y(I.F) = \int Q(x) + (I.F)\,dx + c$

$y \cdot e^{2x} = \int e^{-x} \times e^{2x} + c$

$= \int e^{-x+2x}\,dx + c$

$= \int e^x dx + c$

$y \cdot e^{2x} = e^x + c$

3) $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\therefore \frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$

$p(x) = 2/x \quad Q(x) = \cos x/x^2$

$I.F = e\int p(x)\,dx$

$= e^{\int 2/x \, dx}$

$= e^{2/x \cdot x}$

$= e^x \cdot x^2$

$I.F = x^2$

$y(I.F) = \int Q(x)(I.F)\,dx + c$

$= \int \frac{\cos x}{x^2} \times x^2\,dx + c$

$= \int \cos x + c$

$x^2 \cdot y = \sin x + c$

2

4) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^2}$

$P(x) = 3/x \qquad Q(x) = \sin x/x^2$

$(I.F) = e^{\int P(x)dx}$

$= e^{\int 3/x\, dx}$

$= e^{3 \log x}$

$= e^{\log x^3}$

$I.F = x^3$

$y(I.F) = \int Q(x)\,(I.F)\,dx + c$

$x^3 y = \int \frac{\sin x}{x^2} \times x^3\, dx + c$

$= \int \sin x\, dx + c$

$x^3 y = -\cos x + c$

5) $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$P(x) = 2 \qquad Q(x) = 2x\, e^{-2x}$

$I.F = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$

$y(I.F) = \int Q(x)\,(I.F)\,dx + c$

$y e^{2x} = \int 2x\, e^{-2x} \times e^{2x}\, dx + c$

$y e^{2x} = \int 2x\, dx + c$

$\therefore y e^{2x} = x^2 + c$

6) $\sec^2 x \cdot \tan y\, dx + \sec^2 y \cdot \tan x\, dy = 0$

$\sec^2 x \cdot \tan y\, dx = \sec^2 y \cdot \tan x\, dy$

$\frac{\sec^2 x\, dx}{\tan x} = -\frac{\sec^2 y\, dy}{\tan y}$

$\therefore \int \frac{\sec^2 x\, dx}{\tan x} : \int \frac{\sec^2 y\, dy}{\tan y}$

$\therefore \log|\tan x| - \log|\tan y| + 1$

$\therefore \log|\tan x - \tan y| = c$

$\tan x \cdot \tan y = e$

7) $\frac{dy}{dx} = \sin^2(x-y+1) - x$

Differentiating both sides

$x - y + 1 = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$1 - \frac{dv}{dx} = \frac{dy}{dx}$

$1 - \frac{dv}{dx} = \sin^2 v$

$$\frac{dy}{dx} = \cos^2 x$$

$$\int \sec^2 v \, dv = \int dx$$

$$\tan v = x + c$$

$$\tan(x+y+1) = x + c$$

8) $$\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$$

put $2x+3y = v$

$$\therefore 2 + \frac{3dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$$

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$$

$$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$$

$$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2} = \frac{3v+3}{v+2}$$

$$\int \frac{v+1}{v+1} dv + \int \frac{1}{v+1} dv = 3x + c$$

$$v + \log(v+1) = 3x + c$$

$$2x + 3y + \log|2x+3y+1| = 3x + c$$

$$3y = x \cdot \log|2x+3y+1| + c$$

Practical :- 8

Topic :- Euler's Method

1) $\frac{dy}{dx} = y + e^x - 2$ $y(0) = 2$ $h = 0.5$ find $y(2) = ?$

sol^n :- $f(x) = y + e^x - 2$ $x_0 = 0$ $y(0) = 2$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$\therefore y(2) = 9.8215$

2) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 1$ $h = 0.2$ find $y(1) = ?$

$x_0 = 0$, $y_0 = 0$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1664	0.6412
3	0.6	0.6412	1.4111	0.9234
4	0.8	0.9234	1.8526	1.2939
5	1	1.2939		

$\therefore y(1) = 1.2939$

3) $\frac{dy}{dx} = \sqrt{\frac{2}{y}}$ $y(0)=1$ $h=0.2$ find $y(1) = ?$

$x_0 = 0$ $y(0) = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2105
3	0.6	1.2105	0.7040	1.3513
4	0.8	1.3513	0.7696	1.5051
5	1	1.5051		

$y(1) = \cancel{7.875}\ 1.5051$

4) $\frac{dy}{dx} = 3x^2+1$ $y(1)=2$ find $y(2)$ $h=0.5$

sol^n:- $y_0 = 2$ $x_0 = 1$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	7.75	7.875
2	2	7.875		

$y(2) = 7.875$

5) $y_0 = 2$, $x_0 = 1$ $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.50	4.4218	59.6569	19.3360
3	1.75	19.3360	1122.6426	2999.9960
4	2	299.9960		

$\therefore y(2) = 299.9960$

Q5) $\frac{dy}{dx} = \sqrt{xy} + 2$ $y(1) = 1$ $h = 0.2$ find $y(1.2) = ?$

$x_0 = 1$ $y_0 = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	3.6
1	1.2	3.6		

$y(1.2) = 3.6$

Practical :- 4

Topic :- Limits & Partial order derivatives

Q1) Evaluate the following limits

1) $\lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

$= \lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$

Apply limit,

$= \frac{(-4)^3-3(-4)+(-1)^2-1}{(-4)(-1)+5} = \frac{-64+12+1-1}{4+5} = -\frac{52}{9}$

2) $\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$

$= \lim_{x,y\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$

Apply limits

$= \frac{(0+1)((2)^2+(0)^2-4(2))}{2+3(0)} = \frac{1(4+0-8)}{2} = \frac{4-8}{2} = \frac{-4}{2} = -2$

3) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$

Apply limit

$= \frac{(1)^2-(1)^2(1)^2}{(1)^3-(1)^2(1)(1)} = \frac{1-1}{1-1} = \frac{0}{0}$

limit does not exist.

Q2)

i) $f(x,y) = xy\,e^{x^2}+y^2$

$f(x) = y(1e^{x^2+y^2}) + xy(e^{x^2+y^2}2x)$

$= ye^{x^2+y^2} + 2x^2y\,e^{x^2+y^2}$

$f_y = x(1\cdot e^{x^2+y^2}) + xy(e^{x^2+y^2}\cdot 2y)$

$= x\cdot e^{x^2+y^2} + 2x^2y\cdot e^{x^2+y^2}$

$f_x = y\,e^{x^2+y^2} + 2x^2y\,e^{x^2+y^2}$

$f_y = x\,e^{x^2+y^2} + 2xy^2\cdot e^{x^2+y^2}$

ii) $f(x,y) = e^x\cos y$

$f(x) = \cos y\,e^x$

$f_y = e^x - \sin y$

$\therefore f_y = -\sin y\,e^x$

iii) $f(x,y) = x^3y^2 - 3x^2y + y^3+1$

$f(x) = y^2 3x^2 - 3y2x + 0 + 0$

$= 3x^2y^2 - 6xy$

$f_y = x^3 2y - 3x^2 + 3y^2$

$= 2x^3y - 3x^2 + 3y^2$

Q3) using definition find value of f_x, f_y at $(0,0)$ for $f(x,y) = \frac{2x}{1+yz}$

$f_x(a,b) \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$

$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$

According to given $(a,b) = (0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h-0}{h} = 2$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\therefore f_x = 2, \quad f_y = 0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h-0}{h} = 2$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\therefore f_x = 2 \quad f_y = 0$

Q4. Find all second order partial derivatives of f. Also verify whether $f_{xy} = f_{yx}$

i) $f(x,y) = \dfrac{y^2 - xy}{x^2}$

$f_{xx} = \dfrac{d^2F}{dx^2}$. $\quad f_{yy} = \dfrac{d^2F}{dy^2}$

Apply q rule.

$$f_x = \frac{x^2(0-y) - (y^2 - xy)\,2x}{x^4}$$

$$= \frac{-x^2y - 2xy^2 + 2x^2y}{x^4}$$

$$\therefore f_x = \frac{x^2y - 2xy^2}{x^4}$$

$$\therefore f_{xx} = \frac{x^4(2xy - 2y^2) - (x^2y - 2xy^2)(4x^3)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - (4x5y - 8x^4y^2)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - (4x^5y - 8x^4y^2)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - 4x^5y + 8x^4y^2}{x^8}$$

$$= \frac{-2x^5y + 6x^4y^2}{x^8}$$

$$= \frac{6x^4y^2 - 2x^5y}{x^8}$$

$$f_{xx} = \frac{6y^2 - 2xy}{x^4}$$

$$F_y = \frac{1}{x^2}(2y - x) \qquad F_y = \frac{2y-x}{x^2}$$

$$\therefore f_{yy} = \frac{1}{x^2}\,2 = \frac{2}{x^2}$$

$$\therefore f_{xy} = \frac{2y - x}{x^2}$$

$$= \frac{x^2(-1) - (2y-x)(2x)}{x^4}$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4}$$

$$= \frac{x^2 - 4x4}{x^4}$$

$$\therefore f_{xy} = \frac{x - 4y}{x^3}$$

$\therefore f_{yx} = \frac{x^2y - 2xy^2}{x^4}$

$= \frac{x^2 - 4xy}{x^4}$

$= \frac{x - 4y}{x^3}$

$= f_{xy}$

$\therefore f_{yx} = f_{xy}$

Hence, verified.

Q.5) find the linearization of $f(x,y)$ at given point.

i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$f_x = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$ $\quad$ $f_y = \frac{2y}{2\sqrt{y^2+x^2}}$

$= \frac{x}{\sqrt{x^2-y^2}}$ $\quad$ $= \frac{y}{\sqrt{x^2+y^2}}$

$f_x(1,1) = \frac{1}{\sqrt{2}}$ $\quad$ $f_y(1,1) = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= \sqrt{2} + \frac{x-1}{\sqrt{2}} + \frac{y-1}{\sqrt{2}}$

$= \sqrt{2} + \frac{x+y-2}{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$

ii) $f(x,y) = 1 - x + y\sin x$ at $\pi/2$

$f(\pi/2, 0) = 1 - \pi/2 + 0 + \sin\pi/2$

$f(\pi/2, 0) = \frac{2-\pi}{2}$

$f_x = -1 + y\cos x$ $\quad$ $f_y = \sin x$

$f_x = (\pi/2, 0) = -1$ $\quad$ $-1 + 0 \cdot \cos\pi/2$

$f_y = (\pi/2, 0) = \sin\pi/2$

$= 1$

$L(x,y) = f(\pi/2, 0) + f_x(\pi/2, 0)(x - \pi/2) + f_y(\pi/2, 0)(y-0)$

$= \frac{2-\pi}{2} + (-1(x - \pi/2) + 1(y))$

$L(x,y) = 1 - x + y$

iii) $f(x,y) = \log x + \log y$

$f(1,1) = \log 1 + \log 1$

$= 0$

$f_x = 1/x$ $\quad$ $f_y = 1/y$

$f_x(1,1) = 1$ $\quad$ $f_y(1,1) = 1$

$f(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 0 + 1(x-1) + 1(y-1)$

$\therefore L(x,y) = x + y - 2$

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Practical :- 10

Q1. Find the directional derivation of the given vector at the given points.

(i) $f(x,y) = x+2y-3$ at $\bar{u} = 3i-j$, $a(1,-1)$

sol^n :- $\bar{u} = 3i-j$

$\therefore \hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{3^2+(-1)^2}} \cdot (3i-j)$

$\therefore \hat{u} = \frac{1}{\sqrt{10}}(3i-j)$

$\hat{u} = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$a = (1,-1)$.

$\therefore f(a) = 1+2(-1)-3$

$= 1+(-2)-3$

$= -4$

$f(a+hu) = f\left((1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)\right)$

$= f\left(\left(1+\frac{3}{\sqrt{10}}h\right), \left(-1-\frac{h}{\sqrt{10}}\right)\right)$

$= 1+\frac{3h}{\sqrt{10}} + 2\left(-1-\frac{h}{\sqrt{10}}\right) - 3$

$= 1+\frac{3h}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$= \frac{h}{\sqrt{10}} - 4$

$\therefore D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h\to 0} \frac{h/\sqrt{10} - 4 - (-4)}{h}$

$= \frac{1}{\sqrt{10}} \lim_{h\to 0} \frac{h}{h}$

$= \frac{1}{\sqrt{10}}$

ii) $f(x,y) = y^2-4x+1$ at $\bar{u} = i+3j$, $a(3,4)$

sol^n :- $f(x,y) = y^2-4x+1$ $\quad a(3,4)$

$\vec{u} = \hat{i} + 5\hat{j}$

$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{i+5j}{\sqrt{1^2+5^2}} = \frac{1}{\sqrt{26}}(i+5j)$

$\therefore \hat{u} = \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(a) = (4)^2 - 4(3) + 1$

$= 16-12+1$

$= 5$

$f(a+hu) = f\left((3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\right)$

$= f\left(\left(3+\frac{h}{\sqrt{26}}\right), \left(4+\frac{5h}{\sqrt{26}}\right)\right)$

$= \left(4+\frac{5h}{\sqrt{26}}\right)^2 - 4\left(3+\frac{h}{\sqrt{26}}\right) + 1$

$= 16 + \frac{40h}{\sqrt{26}} + \frac{25h^2}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} - \frac{36h}{\sqrt{26}} + 5$

$$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{25h^2}{26} - \frac{36h}{\sqrt{26}} + 5 - 5}{h}$$

$$= \lim_{h\to 0} \frac{25h}{26} - \frac{36}{\sqrt{26}}$$

$$D_u f(a) = \frac{36}{\sqrt{26}}$$

iii) $f(x,y) = 2x + 3y$ at $\bar{u} = 3i + 4j$; $a(1,2)$

Sol^n:- $\bar{u} = 3i + 4j$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{3^2+4^2}}(3i+4j)$$

$$= \frac{1}{\sqrt{25}}(3i+4j)$$

$$\hat{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$f(a) = 2(1) + 3(2)$$

$$= 2 + 6$$

$$f(a) = 8$$

$$f(a+hu) = f\left((1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right)$$

$$= f\left(\left(1+\frac{3h}{5}\right), \left(2+\frac{4h}{5}\right)\right)$$

$$= 2\left(1+\frac{3h}{5}\right) + 3\left(2+\frac{4h}{5}\right)$$

$$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$$

$$= \frac{18h}{5} + 8$$

$$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) + f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$$

$$= \lim_{h\to 0} \frac{\frac{18h}{5}}{h}$$

$$D_u f(a) = \frac{18}{5}$$

Q2. Find Greatest vector for the following function at its given point

i) $f(x,y) = x^y + y^x$, $a:(1,1)$

Sol^n:- $f(x,y) = x^y + y^x$

$$f_x = \frac{d}{dx}(x^y + y^x)$$

$$\therefore f_x = yx^{y-1} + y^x \cdot \log y$$

$$f_y = \frac{d}{dy}\, x^y + y^x$$

$$\therefore f_y = xy^{x-1} + x^y \cdot \log x$$

$$\nabla f(x,y) = (f_x, f_y)$$

$$\nabla f(x,y) = (yx^{y-1} + y^x \cdot \log y,\ xy^{x-1} + x^y \log x)$$

$\nabla f(1,1) = (1(1)^{1-1} + 1'\log 1,\ 1(1)^{1-1} + 1'\log 1)$

$\nabla f(1,1) = (1,1)$

ii) $f(x,y) = (\tan^{-1}x)\cdot y^2$. $a = (1,-1)$

sol^n:- $f(x,y) = (\tan^{-1}x)\, y^2$

$f_x = \dfrac{y^2}{1+x^2}$

$f_y = \dfrac{d}{dy}(\tan^{-1}x)\, y^2$

$= 2y \tan^{-1}x$

$\nabla f(x,y) = (f_x, f_y)$

$= \left(\dfrac{y^2}{1+x^2},\ 2y\tan^{-1}x\right)$

$\nabla (f(1,-1) = \left(\dfrac{-1^2}{1+(1)^2},\ 2(-1)\tan^{-1}(1)\right)$

$= \left(\dfrac{1}{1+1},\ -2 \times \pi/4\right)$

$\nabla f(1,-1) = (1/2,\ -\pi/2)$

iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$

sol^n:- $f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= (yz - e^{x+y+z},\ xz - e^{x+y+z},\ xy - e^{x+y+z})$

$\nabla f(1,-1,0) = (-1\times 0 - e^{-1+1+0},\ 1(0) - e^{1-1+0},\ 1(-1) - e^{1-1+0})$

$\therefore \nabla f(1,-1,0) = (-1,-1,-2)$

Q3) Find the eq^n of tangent and normal of each of the following curves.

i) $x^2\cos y + e^{xy} = 2$ at $(1,0)$

sol^n:- $f = x^2\cos y + e^{xy} - 2 = 0$

$f(x) = 2x\cos y + ye^{xy}$

$f(y) = -x^2\sin y + xe^{xy}$

tangent:

$f_x(x - x_0) + f_y(y - y_0) = 0$

$(2x\cos y + ye^{xy})(x-1) + (-x^2\sin y + xe^{xy})(y-0) = 0$

$2x^2\cos y + xye^{xy} - 2x\cos y - ye^{xy} - (x^2\sin y - xe^{xy})y = 0$

ii) $x^2+y^2-2x+3y+2=0$ at $(2,-2)$

sol^n:- $f(x,y)=x^2+y^2-2x+3y+2=0$

$f(x)=2x-2 \qquad f_x(2,-1)=2$

$f(y)=2y+3 \qquad f_y(2,-1)=1$

Tangent:- $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(x-x_0)=0$

$2(x-2)+1(y+2)=0$

$2x-4+y+2=0$

$2x+y-2=0$

Normal:- $x-2y+d=0$

$2-2(-2)+d=0$

$\therefore d=2$

$\therefore x-2y+2=0$

Q4.) Find the equation of tangent and normal line to each of the following surfaces.

i) $x^2-2yz+3y+xz=7$ at $(2,1,0)$

Sol^n:- $f(x,y,z)=x^2-2yz+3y+xz-7$

$f_x=2x+z \qquad f_x(x_0,y_0,z_0)=2(2)+0=4$

$f_y=-2z+3 \qquad f_y(x_0,y_0,z_0)=2(0)+3=3$

$f_z=-2y+x \qquad f_z(x_0,y_0,z_0)=-2(1)+2=0$

Tangent:- $f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0)=0$

$\therefore 4(x-2)+3(y-1)+0(z-0)=0$

$\therefore 4x-8+3y-3=0$

$4x+3y-11=0$

Normal:- $\dfrac{x-x_0}{f_x(x_0,y_0,z_0)}=\dfrac{y-y_0}{f_y(x_0,y_0,z_0)}=\dfrac{z-z_0}{f_z(x_0,y_0,z_0)}$

$\therefore \dfrac{x-2}{4}=\dfrac{y-1}{1}=\dfrac{z-0}{0}$

ii) $3xyz-x-y+z=-4$ at $(1,-1,2)$

sol^n:- $f(x,y,z)=3xyz-x-y+z+4=0$

$f_x=3yz-1 \qquad f_x=3(-1)(2)-1=-7$

$f_y=3xz-1 \qquad f_y=3(1)(2)-1=5$

$f_z=3xy+1 \qquad f_z=3(1)(-1)+1=-2$

Tangent:- $f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0)=0$

$\therefore -7(x-1)+5(y+1)-2(z-2)=0$

$-7x+7+5y+5-2z+4=0$

$-7x+5y-2z+16=0$

$7x-5y+2z-16=0$

Normal:-

$$\frac{x-x_0}{f_x\cancel{(x_0,y_0,z_0)}} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$$

$$\therefore \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

Q5. Find the local maximum & minimum for the following function.

i) $f(x,y) = 3x^2+y^2-3xy+6x-4y$

Sol^n:- $f_x = 6x-3y+6$
$f_y = 2y-3x-4$

$f_x=0$ $f_y=0$

$6x-3y=-6$ $3x-2y=-4$

$6x-4y=-8$

$\therefore y=2$

$\therefore x=0$

$\therefore (x,y)=(0,2)$

$r=f_{xx}=6$

$s=f_{xy}=-3$

$t=f_{yy}=2$

$rt-s^2 = 6(2)-(-3)^2 = 12-9 = 3>0$

$r>0$

$\therefore f^n$ is minimum at $(0,2)$

$f(0,2) = 3(0)^2+(2)^2-3(0)(2)+6(0)-4(2)$
$= 4+0-0+0-8$

$f(0,2) = -4$

ii) $f(x,y) = 2x^4+3x^2y-y^2$

$f_x = 8x^3+6xy$

$f_y = -2y+3x^2$

$f_x=0$ $f_y=0$

$8x^3+6xy=0$ $-2y+3(0)=0$

$x(8x^2+6y)=0$ $\therefore y=0$

$\therefore x=0$

$(x,y)=(0,0)$ is a root

$8x^2+6y=0$

$x^2=-\frac{3}{4}y$

$\therefore f_y = -2y-\frac{3}{4}\times 3y$

$= -4y=0$

$\therefore y=0$

$x^2=0$

$\therefore x=0$

$\therefore (x,y)=(0,0)$ is the only root.