Discrete Math Assigment 5

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November 15, 2013

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1. *A fair (unbalanced) coin is tossed repeatedly until the first time we see the combination HEAD TAIL (in this order, but not necessarily consecutively). Calculate the distribution of the number of tosses, i.e. give a closed form expression for the probability that their number in this experiment is equal to k for any integer $k \geq 2$.*

   Given a fair coin, $p(H) = p(T) = 0.5$. By the definition of geometric distribution, the probability that we will see H after n tosses is

   $$p(T_1 \ldots T_n H) = (1 - p(H))^n \cdot p(H)$$
   $$= (1 - p(T))^n \cdot p(T) = p(H_1 \ldots H_n T)$$

   Since tossing some number n TAILs and a HEAD will have no effect on our probability of tossing m HEADs, $p(T_1 \ldots T_n H)$ and $p(H_1 \ldots H_m T)$ are independent. By the definition of independence, we know that

   $$p((T_1 \ldots T_n H) \wedge (H_1 \ldots H_m T)) = p(T_1 \ldots T_n H) \cdot p(H_1 \ldots H_m T)$$

   We want to find the expression for the probability of that event given k number of tosses total. We must take into account all cases $p(T_1 \ldots T_i H) \cdot p(H_1 \ldots H_j T)$ where $i + j + 2 = k$, since any positive values of i and j are valid as long as we get a total of k tosses. Therefore in k coinflips, where $k \geq 2$ and $i + j + 2 = k$,

   $$p((T_1 \ldots T_i H) \wedge (H_1 \ldots H_j T)) = \sum_{i=0}^{k-2} (1 - p(H))^i \cdot p(H) \cdot (1 - p(T))^{k-2-i} \cdot p(T)$$
   $$= \sum_{i=0}^{k-2} 0.25 \cdot (0.5)^i \cdot (0.5)^{k-2-i}$$
   $$= (k - 1) \cdot (0.5)^k$$

2. *You roll three fair dice. Compute the expectation of the maximum of the three results, and the expectation of their minimum.*

   We will use the expectation theorem:

   $$E(X) = \sum_{r \in X(S)} p(X = r) r$$

   Given a three dice rolls represented as an ordered triplet (i, j, k) and X_{max} such that $X_{max}((i, j, k)) = max(i, j, k)$, to use the expectation formula, we need $p(X = r)$ for all $1 \leq r \leq 6$.

   To find $p(X_{max} = r)$: given (i, j, k) let E be the event that $i = r$ and $j, k \leq r$, F be the event that $j = r$ and $i, k \leq r$, and G be the event that $k = r$ and $i, j \leq r$. We want to find $p(E \cup F \cup G)$. Using the inclusion-exclusion principle,

   $$p(E \cup F \cup G) = p(E) + p(F) + p(G) - p(E \cap F) - p(E \cap G) - p(G \cap F) + p(E \cap F \cap G)$$

   The denominator in our probability calculations is the total number of unique ordered triplets, $6 \cdot 6 \cdot 6$. The numerator will be the number of ordered triplets which satisfy our event expression.
   $p(E) = p(F) = p(G) = r^2/6^3$, since in the numerator, we have a fixed value for one variable, and r choices

for the other two, $1 \cdot r \cdot r$.

$p(E \cap F) = p(E \cap G) = p(G \cap F) = r/6^3$, since in the numerator we have a fixed value for two variables and r for the last one, $1 \cdot 1 \cdot r$.

$p(E \cap F \cap G) = 1/6^3$, since in the numerator, there is only triplet (r,r,r).

Plugging in these values,

$$p(E \cup F \cup G) = \frac{3r^2 - 3r + 1}{6^3} = p(X_{max} = r)$$

Plugging this probability into the expectation equations, we get that the expectation of the maximum is

$$E(X_{max}) = \sum_{r \in X(S)} \frac{(3r^2 - 3r + 1) \cdot r}{6^3} = \frac{119}{24} \approx 4.96$$

To find the expected value of the minimum, we do essentially the same calucations, except that the probabilities are different — every place where we previously used an r, we now want to use $(7 - r)$ because where we previously wanted to find values less than or equal to r, now we want values greater than or equal to r. For example, in finding $p(i = r, j \geq r, k \geq r)$, we count $(7 - r)^2$ valid triplets since we have a fixed value for i and $7 - r$ choices for each j, k.

To get $p(X_{min} = r)$, we can therefore substitute $(7 - r)$ for r in our previous probability expression. Again using the expectation equation, we get

$$E(X_{min}) = \sum_{r \in X(S)} \frac{(3(7 - r)^2 - 3(7 - r) + 1) \cdot r}{6^3} = \frac{49}{24} \approx 2.041$$

3. *For a function $f : [n] \longrightarrow [n]$, let us define its inverse complexity ic(f) as the number of ordered pairs such that $i < j$ and $f(j) \leq f(i)$. Prove that*

$$p\left(ic(f) \geq \frac{n^2}{3}\right) \leq \frac{3}{4}$$

where f is chosen uniformly at random from the set of all functions.

Let random variable $X = ic(f)$. By the Markov bound theorem we discussed in class, we know that $p(X \geq a) \leq c/a$, where $c = E(X)$ and $a > 0$. So we know that

$$p\left(X \geq \frac{n^2}{3}\right) \leq \frac{3 \cdot E(X)}{n^2}$$

To find $E(X)$, first let us count the number of valid choices of i and j such that $i < j$. There are n valid values of j, and for each of those values, there are $n - 1$ valid choices of j. Then we divide by two to account for the double-counting. So there are

$$\frac{n(n - 1)}{2}$$

ordered pairs (i, j) such that $i < j$.

We can consider X to be the sum of n bernouli variables that indicate whether or not a single ordered pair (i, j), $i < j$ satisfies $f(j) \leq f(i)$. So $X = X_1 + \cdots + X_m$, where $m = \frac{n(n-1)}{2}$. And thus, $E(X) = E(X_1 + \cdots + X_m)$. By linearity of expectation, we also know that $E(X) = E(X_1) + \cdots + E(X_m)$. Finally, because each X_i is a bernoulli variable and the $E(Y) = \sum_{r \in Y(S)} p(Y = r)r$, because all $r \in 0, 1$, we know that $E(X_i) = p(X_i)$. So

$$E(x) = p(X_1) + \cdots + p(X_m)$$

Now we need $p(X_i)$. Now, for each ordered pair X_i, there are $\frac{1}{2}n(n + 1)$ choices for the values of $f(i)$ and $f(j)$ where $f(j) \leq f(i)$, since we are adding the number of choices of $f(j)$ for all $f(i)$ which is $(n + \cdots + 1) = \frac{1}{2}n(n + 1)$. The total number of ordered pairs is of course n^2, n choices for $f(i)$ and n choices

for $f(j)$. So

$$p(X_i) = \frac{n \cdot (n+1)}{2n^2}$$

Therefore we can multiple $p(X_i)$ by m to see that

$$E(X) = \frac{n \cdot (n+1)}{2n^2} \cdot \frac{n(n-1)}{2} = \frac{n^2(n+1)(n-1)}{4n^2} = \frac{n^2-1}{4n^2}$$

Now plugging this value back into the Markov bound, we see that

$$p\left(X \geq \frac{n^2}{3}\right) \leq \frac{3 \cdot (n^2-1)}{4n^4} \leq \frac{3}{4}$$

4. *The mean deviation MD(X) of a random variable X is defined as $E(|X - c|)$, where $c = E(X)$ is the expectation of X (we briefly discussed this notion in class, and demoted it).*

(a) *Prove that for any two random variables X and Y on the same sample space, $MD(X+Y) \leq MD(X) + MD(Y)$.*

Let $c = E(X)$ and $d = E(Y)$. By the definition of MD, then by the linearity of expectation

$$
\begin{aligned}
MD(X+Y) &= E(|(X+Y) - E(X+Y)|) \\
&= E(|(X+Y) - E(X) - E(Y)|) \\
&= E(|(X+Y) - c - d|) \\
&= E(|(X-c) + (Y-d)|)
\end{aligned}
$$

$$
\begin{aligned}
MD(X) + MD(Y) &= E(|X-c|) + E(|Y-d|) \\
&= E(|X-c| + |Y-d|)
\end{aligned}
$$

The triangle inequality states that $|a+b| \leq |a| + |b|$ for all a, b. so we know that

$$
\begin{aligned}
E(|(X-c) - (Y-d)|) &\leq E(|X-c| + |Y-d|) \\
MD(X+Y) &\leq MD(X) + MD(Y)
\end{aligned}
$$

(b) *Prove that if X and Y are additionally known to be independent, then this inequality is always strict, unless one of the variables X, Y is trivial (that is, takes one fixed value with probability 1).*

In order to get a strict inequality, we must show that

$$E(|(X-c) + (Y-d)|) \neq E(|X-c| + |Y-d|)$$

We know that for some a and b, $|a+b| = |a| + |b|$ iff $a, b > 0$ or $a, b < 0$ or $a \cdot b = 0$. Since we are working with non-trivial random variables, we want to show that there exists some values of $(X-c)$ and $(Y-d)$ such that they have different signs.

Since c is the mean value for X, we know that there must be must be values of X both greater than and less than c (since we're using non-trival values). Therefore $p(X < c) > 0$. Similarly, since d is the mean value for Y, we know that there must be must be values of Y both greater than and less than d. So $p(Y > d) > 0$.

If X and Y are independent then from lecture, we know that $p(X < c) \cdot p(Y < d) = p((X < c) \wedge (Y > d))$. Since $p(X < c)$ and $p(Y < d)$ are both non-zero, $p((X < c) \wedge (Y > d)) > 0$. This tells us that there *are* some values of $(X-c)$ and $(Y-d)$ such that they have different signs. Therefore,

$$
\begin{aligned}
E(|(X-c) + (Y-d)|) &\neq E(|X-c| + |Y-d|) \\
MD(X+Y) &< MD(X) + MD(Y)
\end{aligned}
$$

5. *Let f be picked uniformly at random from the set of all functions from $[n]$ to $[n]$. Give a closed form expression for the variance of the random variable $|im(f)|$.*

Let our sample space S be the set of all functions f from $[n]$ to $[n]$. We know that $|S| = n^n$. Let the random variable $|im(f)|$ be called X. for $i \in [n]$, $X_i = 0$ if $i \notin im(f)$ and $X_i = 1$ if $i \in im(f)$.

We know that $E(X_i) = 1 - (1 - \frac{1}{n})^n$, so because X_i si a bernoulli variable, we know that $E(X_i^2) = 1 - (1 - \frac{1}{n})^n$. Therefore,

$$Var(X) = \left(1 - \frac{1}{n}\right)^n \left(1 - \left(1 - \frac{1}{n}\right)^n\right) = \left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n}$$

Since X_i are not mutually independent, we will have to calculate covariances. We know that $Covar(X_i, X_j) = E(X_i, X_j) - E(X_i)E(X_j)$. Once again, because X_i and X_j are bernoulli variables, we just need to find $p(X_i \cap X_j)$ because we already know that $E(X_i) = (1 = \frac{1}{n})^n$. By the thereom from lecture, the variance of $X = E(X^2) - E(X)^2$.

$$\begin{aligned} Covar(X_i, X_j) &= E(X_i, X_j) - E(X_i)E(X_j) \\ &= p(X_i \cap X_j) - p(X_i)p(X_j) \\ &= \left(1 - \frac{2}{n}\right)^n - \left(1 - \left(1 - \frac{1}{n}\right)^n\right)^2 \\ &= 2 \cdot \left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n} + \left(1 - \frac{2}{n}\right)^n \end{aligned}$$

And finally we can conclude that

$$\begin{aligned} Var(X) &= \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Covar(X_i, X_j) \\ &= nVar(X_i) + \frac{n \cdot (n-1)}{2} Covar(X_i, X_j) \\ &= n^2 \left(1 - \frac{1}{n}\right)^n - \frac{n \cdot (n+1)}{2} \left(1 - \frac{1}{n}\right)^{2n} + \frac{n \cdot (n-1)}{2} \left(1 - \frac{2}{n}\right)^n \end{aligned}$$