Rémi Pellerin
M1IF

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Exercice 4:

1) First method: "by hands"

$S : \Omega \longrightarrow \mathbb{N}$
$\omega \longmapsto \sum_{i=1}^{Z(\omega)} X_i(\omega)$

$$E(S) = \sum_{n=0}^{\infty} n \, P(S=n)$$

$$= \sum_{n=0}^{\infty} n \sum_{k=0}^{\infty} P\Big(\sum_{i=1}^{k} X_i = n,\ Z=k\Big)$$

↑ because $\Omega = \biguplus_{k\in\mathbb{N}} (Z=k)$

$$= \sum_{k=0}^{\infty} P(Z=k) \sum_{n=0}^{\infty} n \, P\Big(\sum_{i=1}^{k} X_i = n\Big)$$

Z and $\sum X_i$ are independent

↑ Always true in probability (but not always in real life)

$$= \sum_{k=0}^{\infty} P(Z=k) \underbrace{E\Big(\sum_{i=1}^{k} X_i\Big)}_{k\,E(X_1) \text{ (linearity of } E)}$$

OMG!

$$= E(X_1)\,E(Z).$$

Second method: "by feet"

$$E(S) = \sum_{k=0}^{\infty} P(Z=k) \underbrace{E\Big(\sum_{i=1}^{k} X_i \,\Big|\, Z=k\Big)}_{k\,E(X_1)} = E(Z)\,E(X_1)$$

2/ $Z_n = \sum_{i=1}^{Z_{n-1}} X_i$ for $n \geq 1$

$$E(Z_n) = E(Z_{n-1})\, E(X_1)$$
$$= \dots = E(Z_0)\, E(X_1)^n$$

$$E(Z_n) = E(X_1)^n = \mu^n$$

3/ $P(Z_n \neq 0) = P(Z_n \geq 1) \leq \frac{E(Z_n)}{1} \leq \mu^n \xrightarrow[n \to \infty]{} 0$

Markov since Z_n is almost surely positive

4/ <u>Theorem:</u> Let $(X_n)_{n \in \mathbb{N}^*}$ be a family of iid RV which take values in $\mathbb{N}$, N a RV in $\mathbb{N}$ independent from the (X_n).

$S = \sum_{i=1}^{N} X_i$ is a RV and $G_S(t) = G_N(G_{X_1}(t))$ (if defined!)

<u>Lemma:</u> If $X_1, \dots, X_n$ are independent with values in $\mathbb{N}$ then $G_{X_1 + \dots X_n}(t) = \prod_{i=1}^{n} G_{X_i}(t)$ (if defined!)

<u>proof:</u> $S : \Omega \to \mathbb{N}$
$\omega \mapsto \sum_{i=1}^{N(\omega)} X_i(\omega)$

$$G_S(t) = \sum_{n=0}^{\infty} P(S = n)\, t^n$$

⚠ This proof is not "natural" but it's really fast.

$$G_S(t) = E(t^S) = E\left(t^{\sum_{i=1}^{N} X_i}\right)$$

$$= E\left(\prod_{i=1}^{N} t^{X_i}\right)$$

$$= \sum_{n=0}^{\infty} P(N=n)\, E\left(\prod_{i=1}^{n} t^{X_i} \mid N=n\right)$$

$$= \sum_{n=0}^{\infty} P(N=n) \prod_{i=1}^{n} E\left(t^{X_i} \mid N=n\right)$$

Because for any t, (t^{X_i}) are independent!

$$= \sum_{n=0}^{\infty} P(N=n) \prod_{i=1}^{n} E\left(t^{X_i}\right)$$

Because for any t, N and t^{X_i} are independent.

$$\prod_{i=1}^{n} E(t^{X_i}) = \prod_{i=1}^{n} G_{X_1}(t) = G_{X_1}(t)^n \text{ : done!}$$

→ Since $Z_n = \sum_{i=1}^{Z_{n-1}} X_i$, $G_n(x) = G_{Z_n}(x) = G_{Z_{n-1}}(G(x))$

$(n \geq 1)$

5) $G(x) = p + (1-p)t$ (Someone normal would have ~~do~~ $p_0 = 1-p$, $p_1 = p$)

$$G_n(x) = G_{n-1}(p + (1-p)t)$$

$$= G_{n-2}(p + (1-p)[p + (1-p)t])$$

$$= G_{n-2}(p + p(1-p) + (1-p)^2 t)$$

$$G_n(x) = G_{n-3}\left(p + p(1-p) + p(1-p)^2 + (1-p)^3 x\right)$$

$$= G_0\left(\sum_{k=0}^{n-1} p(1-p)^k + (1-p)^n x\right)$$

Simisto will do an induction (# Quentin)

$$G_0(x) = x$$

$$G_n(x) = p\sum_{k=0}^{n-1} (1-p)^k + (1-p)^n x$$

$$G_n(x) = (1-p)^n (x-1) + 1$$

$$= 1 - (1-p)^n + (1-p)^n x$$

(Because G_n is a power serie with a strictly positive radius of convergence ($R \geq 1$)).

$$P(Z_n = 0) = 1 - (1-p)^n \longrightarrow 1$$
$$P(Z_n = 1) = (1-p)^n \longrightarrow 0$$

$$P(T=n) = P(Z_n = 0, Z_{n-1} \neq 0, \dots, Z_0 \neq 1)$$

$$= P(Z_n = 0, Z_{n-1} \geq 1)$$

(markovian process)

$$= \sum_{k=1}^{\infty} P(Z_n = 0, Z_{n-1} = k)$$

$$= P(Z_n = 0 \mid Z_{n-1} = 1)\, P(Z_{n-1} = 1)$$

$$P(T=n) = \begin{cases} p(1-p)^{n-1}, & n \geq 1 \\ 0 \text{ if } n = 0 \text{ (because } Z_0 = 1) \end{cases}$$