Exercice 3:

Let's talk a bit about X_1:

- its density function: $f(x) = \begin{cases} 1 & \text{if } x \in [a; a+1] \\ 0 & \text{elsewhere} \end{cases}$
- $E(X_1) = a + \frac{1}{2}$ (This is intuitive but we could have calculated it with $\int_a^{a+1} x \cdot 1 \, dx$.)
- $E(X_1^2) = \int_a^{a+1} x^2 \, dx = \frac{(a+1)^3 - a^3}{3} = a^2 + a + \frac{1}{3}$.
- $V(X_1) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ (OMG!)

As usual, we will do many tests, if a lot of X_i are greater than $\frac{1}{2}$ then a must not be zero!

Formally, $\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{\text{probability}} a + \frac{1}{2}$.

(weak law of large number)

$$P\left(\left| \frac{1}{n} \sum_{i=1}^{n} X_i - \left(a + \frac{1}{2}\right) \right| \geq \varepsilon \right) \leq \frac{V(X_1)}{n \varepsilon^2} = \frac{1}{\underset{\substack{\uparrow \\ \text{OMG!}}}{12} n \varepsilon^2}$$

So, $P\left(\frac{1}{n} \sum_{i=1}^{n} X_i \notin \left[\left(a + \frac{1}{2}\right) - \varepsilon ; a + \frac{1}{2} + \varepsilon \right] \right) \leq \frac{1}{12 n \varepsilon^2}$

We perform several test (n) and compute $\sum_{i=1}^{n} \frac{x_i}{n}$.

If, let's say for $\varepsilon = \frac{1}{\sqrt{12 n \beta}}$, we find out that our empirical mean $\frac{1}{n} \sum_{i=1}^{n} x_i$ is close to a value μ (which must be $a + \frac{1}{2}$) then, if $\left|\mu - \frac{1}{2}\right| < \xi$ we answer "$a = 0$"

if not, "$a \neq 0$".

The probability to be false will be less than

$$\frac{1}{12\, n\, \xi^2} = \beta.$$

$\rightarrow$ If $\frac{1}{n}\sum_{i=1}^{n} x_i \notin [\mu - \xi ; \mu + \xi]$ then ξ is to small and we must change but in that case we cannot guarantee an error probability less than β. (I mean I don't know how to do ...)

NB: With this method we do not need to distinguish the kind of error!