Rémi
Pellerin
M1IF

EPR 2015-2016

Exercice 1:

1) Not at all. For instance, if B and A are independent, then A and $\overline{B}$ are independent (R) and so

$P(A) = P(A|\overline{B}) = P(A|B)$: no information on $P(B)$!

(R) $P(A) = P((A \cap B) \uplus (A \cap \overline{B}))$
$= P(A \cap B) + P(A \cap \overline{B})$ (σ-additivity)

If A, B are independent then,
$P(A) = P(A)P(B) + P(A \cap \overline{B})$
so $P(A)[1 - P(B)] = P(A)P(\overline{B}) = P(A \cap \overline{B})$.

2) Weak law of large numbers: Let $(X_n)_{n \in \mathbb{N}^*}$ be a sequence of independent RV having the same finite variance $V(X)$ and the same mean $E(X)$.
Under those hypothesis, $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ converges in probability to (the constant RV $E(X)$) ie

$\forall \varepsilon > 0, \quad P(|Y_n - E(X)| > \varepsilon) \xrightarrow[n \to +\infty]{} 0.$

Strong law of large numbers: Let $(X_n)_{n \in \mathbb{N}^*}$ be a sequence of iid RV such that $\forall n \in \mathbb{N}^*$, $E(|X_n|) < \infty$ (integrability). Then $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ converges almost surely to $E(X)$ ($= E(X_{12})$ for example).

Central-limit theorem: Same thing but this time the X_i's must have a finite variance σ^2 (and so a finite mean μ).

$$Y_n = \frac{\sum_{i=1}^{n} X_i - \mu}{\frac{\sigma}{\sqrt{n}}}$$ converges in law to Y

such that $Y \sim \mathcal{N}(0,1)$. $\left(\forall x \in \mathbb{R},\ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right)$

4) Exponential law of parameter λ (>0).

density function: $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

$$F_X(x) = P(X \geq x) = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

$\rightarrow F_X(x) \in [0; 1[$!

If $x \geq 0$, $F_X(x) = 1 - e^{-\lambda x} \iff x = \frac{-\ln(1 - F_X(x))}{\lambda}$

Algo: $y \leftarrow$ Random ( )

If $y \leq \frac{1}{2}$ then
 return 0
Else if $y < 1$ then
 return $-\frac{1}{\lambda}(\ln(1-y))$
Else
 fail

Notice that the last case "almost surely never occurs"!

Remark: We could have done $-\frac{1}{\lambda}\ln(y)$ instead of $-\frac{1}{\lambda}\ln(1-y)$.

5/ Let's denote by N_t the number of requests arriving during time t. For each t, N_t is a real R.V. We will express t in minutes.

* $$P(N_{20}=2 \mid N_{60}=2) \underset{\text{Bayes}}{=} \frac{P(N_{60}=2 \mid N_{20}=2)\, P(N_{20}=2)}{P(N_{60}=2)}$$

Recall: $P(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ $(\lambda t > 0 \ldots)$

$$P(N_{20}=2 / N_{60}=2) = \frac{P(N_{40}=0)}{P(N_{60}=2)} \quad P(N_{20}=2)$$

because $(N_{40}=0) = (N_{60}=2 / N_{20}=2)$

$$P(N_{20}=2 / N_{60}=2) = \frac{e^{-40\lambda}}{\frac{(60\lambda)^2}{2!}e^{-60\lambda}} \quad \frac{(20\lambda)^2}{2!}e^{-20\lambda}$$

$$= \frac{1}{9}$$

Remark: This result doesn't depend on how we "discretise" the time! So it isn't useful to specify which time unit (here the minute) we choose!

* $P(N_{20} \geq 1 \mid N_{60}=2) = 1 - P(N_{20}=0 \mid N_{60}=2)$

Indeed, if $P(A)>0$, $P(\cdot \mid A)$ is a probability measure on Ω and $\forall k \geq 3$, $P(N_{20}=3 / N_{60}=2) = 0$.

$$P(N_{20}=0 \mid N_{60}=2) = \frac{P(N_{60}=2 \mid N_{20}=0)\, P(N_{20}=0)}{P(N_{60}=2)}$$

$$= \frac{P(N_{40}=2)\, P(N_{20}=0)}{P(N_{60}=2)}$$

$$= \frac{\frac{(40\lambda)^2}{2!} e^{-40\lambda} e^{-20\lambda}}{\frac{(60\lambda)^2}{2!} e^{-60\lambda}}$$

$$= \frac{4}{9}$$

Once again, this does not depend on the time unit we consider!

6) This markov chain is irreducible (the graph is strongly connex), aperiodic ($0 < p, q < 1$), positive recurrent (because the number of states is finite). Hence, the system converges to a unique stationnary distribution.

$$P = (p_{ij}) = \begin{array}{cc} \text{idle} & \text{busy} \\ \end{array} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \begin{array}{c} \text{idle} \\ \text{busy} \end{array}$$

$$\pi P = \pi \iff \begin{cases} \pi_1 (1-p) + q\, \pi_2 = \pi_1 \\ p\, \pi_1 + (1-q)\, \pi_2 = \pi_2 \end{cases}$$

$$\implies \pi_1 = \frac{q}{p} \pi_2$$

$$\pi = \left[1 - \frac{1}{1+\frac{q}{p}} ; \frac{1}{1+\frac{q}{p}} \right]$$

$f_n(w) \xrightarrow[n \to \infty]{} \frac{1}{1+\frac{q}{p}}$ whatever trajectory we look at (ergodicity!)