COMP3121 ASSIGNMENT1 QUESTION5

5. Determine if $f(n) = O(g(n))$ or $g(n) = O(f(n)$ or both (i.e., $f(n) = \Theta(g(n))$) or neither of the two, for the following pairs of functions

(a) $f(n) = (\log_2(n))^2$; $g(n) = \log_2(n^{\log_2 n})^2$; (6 points)

(b) $f(n) = n^{10}$; $g(n) = 2^{\sqrt[10]{n}}$; (6 points)

(c) $f(n) = n^{1+(-1)^n}$; $g(n) = n$. (8 points)

Answer:

(a) We want to show that $f(n) = \theta\big(g(n)\big)$ we have to show that $f(n) = O\big(g(n)\big)$ and $f(n) = \Omega\big(g(n)\big)$

Use $\log(a^b) = b\log a$.

$$g(n) = \log_2\left(n^{\log_2 n}\right)^2 = \log_2\left(n^{2\log_2 n}\right) = 2\log_2 n \cdot \log_2 n = 2(\log_2 n)^2$$

For $f(n) = \Omega\big(g(n)\big)$ we need to show that $C \cdot g(n) \leq f(n)$ for some positive c and all sufficiently large n. We take c = $\frac{1}{2}$ then is enough to show $\frac{1}{2}(2(log_2 n)^2) \leq (\log_2(n))^2$ for all sufficiently large n.

So $f(n) = \Omega\big(g(n)\big)$.

For $f(n) = O\big(g(n)\big)$ we need to show that $f(n) \leq C \cdot g(n)$ for some positive c and all sufficiently large n. we take c = 1 then is enough to show $(\log_2(n))^2 < 2(\log_2(n))^2$ for all sufficiently large n.

So $f(n) = O\big(g(n)\big)$.

In this case, we can show that $f(n) = \theta\big(g(n)\big)$.

(b) We want to show that f(n) = O(g(n)). We have to show that $n^{10} < c * \left(2^{\sqrt[10]{n}}\right)$ for some positive c and all sufficiently large n. But, since the log function is monotonically increasing, this will hold just in case

$$log\, n^{10} < \log C + \log\left(2^{\sqrt[10]{n}}\right)$$

$$10\log n < \log C + \sqrt[10]{n}$$

We now see that if we take c = 1 then it is enough to show that

$$10\, log\, n < \sqrt[10]{n}$$

$$\frac{10\log n}{\sqrt[10]{n}} < 1$$

We know that L'Hopital's rule can replace $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}$ with $\lim\limits_{x\to\infty}\frac{f'_{(x)}}{g'_{(x)}}$

$$\lim_{n\to\infty}\frac{10\log n}{\sqrt[10]{n}} = \lim_{n\to\infty}\frac{100\sqrt[10]{n^9}}{\ln(2)n} = \lim_{n\to\infty}\frac{90}{\ln(2)\sqrt[10]{n}}$$

For all sufficiently large n, we can know that

$$\lim_{n\to\infty}\frac{90}{\ln(2)\sqrt[10]{n}} = 0$$

In this case for sufficiently large n we have $\frac{10\log n}{\sqrt[10]{n}} < 1$

(c) Just note that 1 + (-1)^n cycles, with one period equal to {0, 2}
Thus, for all n is even number we have 1 + 1 = 2 and for all n is odd number we have 1 -1 = 0. Thus for any fixed constant c > 0 for all n is even number eventually $n^{1+(-1)^n} = n^2 > C \cdot n$ and for all n is odd number we have $n^{1+(-1)^n} = n^0 = 1$ and so $n^{1+(-1)^n} = 1 < C \cdot n$
Thus neither $f(n) = O\big(g(n)\big)$ nor $g(n) = O\big(f(n)\big)$