Functions Solutions

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Exercises

1. Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.

(a) Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.

Proof. Note that if A_0 is empty, then $A_0 \subseteq f^{-1}(f(A_0))$ trivially. Thus we assume that A_0 is non-empty. Let $x \in A_0$. Then, $f(x) \in f(A_0)$ by definition. Since $f(x) \in f(A_0)$, we have that $x \in f^{-1}(f(A_0))$. Therefore, $A_0 \subseteq f^{-1}(f(A_0))$.
Let $x \in f^{-1}(f(A_0))$. By definition, $f(x) \in f(A_0)$. In particular, this implies that $f(x) = b$ such that $b \in f(A_0)$. Since it may be that f is not injective, we cannot conclude that $x \in A_0$. This is because there may exist some $y \in A_0$ such that $f(y) = f(x) \in f(A_0)$ and $y \neq x$.
If, however, f is injective, then it must be that $y = x$ since $f(y) = f(x)$. In particular, this would imply that $x \in A_0$ must be true since $f(x) \in f(A_0)$. Therefore, we've shown that equality holds if f is injective. $\square$

(b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

Proof. If $f(f^{-1}(B_0))$ is empty, then it is trivially a subset of B_0. Thus we will assume that $f(f^{-1}B_0)$ is non-empty. Let $x \in f(f^{-1}B_0)$. By definition, $x = f(a)$ for at least one $a \in f^{-1}(B_0)$. Since $a \in f^{-1}(B_0)$, it must be that $f(a) \in B_0$. Therefore, $x \in B_0$ and we can conclude that $f(f^{-1}B_0) \subseteq B_0$.
Let $x \in B_0$. Since f is not necessarily surjective, it might be that there exists no $a \in A$ such that $f(a) = x$. As such, $f^{-1}(x) = \emptyset$, and therefore, $f^{-1}(x) \notin f^{-1}(B_0)$. Furthermore, since $f^{-1}(x) \notin f^{-1}(B_0)$, it is also true that $x \notin f(f^{-1}(B_0))$.
Thus, assume that f is surjective. Then, there exists some $a \in A$ such that $f(a) = x$. In particular, since $x \in B_0$, $a \in f^{-1}(B_0)$. Furthermore, we conclude that $f(f^{-1}(B_0))$ contains x. Therefore, $B_0 \subseteq f(f^{-1}(B_0))$ and thus equality holds if f is surjective. $\square$

2. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets:

(a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.

Proof. Let $B_0 \subseteq B_1$. If B_0 is the empty set, then note that $f^{-1}(B_0)$ is also the empty set, since f is a rule of assignment, and as such every element of A must be mapped to at least one element in B. Therefore, it is trivial to note that $f^{-1}(B_0) \subseteq f^{-1}(B_1)$. We arrive at the same conclusion if B_1 is the empty set. Thus, we will assume that B_0 and B_1 are non-empty.

If $f^{-1}(B_0)$ is empty, then it is trivial to note that $f^{-1}(B_0) \subseteq f^{-1}(B_1)$. Similarly if $f^{-1}(B_1)$ is empty. Thus assume that $f^{-1}(B_0)$ is non-empty. Let $x \in f^{-1}(B_0)$ and note that this implies that $f(x) \in B_0$ by definition. Since $B_0 \subseteq B_1$, we have that $f(x) \in B_1$. In particular, this implies that $x \in f^{-1}(B_1)$ by definition. Therefore, we conclude that $f^{-1}(B_0) \subseteq f^{-1}(B_1)$. □

(b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.

Proof. □

(c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

(d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

Show that f preserves inclusions and unions only:

(e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.

(f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that equality holds if f is injective.

(h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; show that equality holds if f is injective.

3. Show that (b), (c), (f), and (g) of Exercise 2 hold for arbitrary unions and intersections.

4. Let $f : A \to B$ and $g : B \to C$.

(a) If $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$.

(b) If f and g are injective, show that $g \circ f$ is injective.

(c) If $g \circ f$ is injective, what can you say about the injectivity of f and g?

(d) If f and g are surjective, show that $g \circ f$ is surjective.

(e) If $g \circ f$ is surjective, what can you say about the surjectivity of f and g?

(f) Summarize your answers to (b)-(c) in the form of a theorem.

5. In general, let us denote the ***identity function*** for a set C by i_C. That is, define $i_C : C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, we say that $h : B \to A$ is a ***right inverse*** for f if $f \circ h = i_B$.

(a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.

(b) Give an example of a function that has a left inverse but no right inverse.

(c) Give an example of a function that has a right inverse but no left inverse.

(d) Can a function have more than one left inverse? More than one right inverse?

(e) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^1$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijective function g. Draw the graphs of g and g^{-1}. (There are several possible choices for g.)