Analysis Notes

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Chapter 1

Measures

1.1 Measure space

A pair $(X, \mathcal{M})$ is a measure space if $\mathcal{M} \subset \mathcal{P}(X)$ and

1. $X \in \mathcal{M}$.
2. $X - A \in \mathcal{M}$ if $A \in \mathcal{M}$.
3. $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$ if $A_n \in \mathcal{M}$.

The elements of $\mathcal{M}$ are called measurable sets. While $\mathcal{M}$ itself is called a σ-algebra.

1.1.1 Existence of measurable spaces

If $\mathcal{F} \subset \mathcal{P}(X)$ for a set X, then there exists a smallest σ-algebra $\mathcal{M}$ such that $\mathcal{F} \subset \mathcal{M}$. We say that $\mathcal{M}$ is the σ-algebra generated by $\mathcal{F}$.

1.1.2 Borel spaces

For a topological space X there exists a smallest σ-algebra $\mathcal{B}$ that contains all open sets in X, we call $\mathcal{B}$ the Borel σ-algebra generated by the topological space X, so that X is also a measurable space.

Of importance are the F_σ, and G_δ sets, which are respectively the countable union of closed sets, and the countable intersection of open sets.

1.2 Measurable functions

$f : X \to Y$ is a measurable function if Y is a topological space, X is a measurable space, and $f^{-1}(V)$ is measurable for every open V.

Notice that measurable function don't necessarily have a notion of continuity, since the space X may not be a topological space. Whereas in a Borel space, we can have both continuous and measurable functions.

We sometimes call the measurable functions in the Borel space, a Borel measurable function, a Borel function or a Borel mapping. By the definition of the Borel space, every continuous function is also a Borel measurable function. Since the inverse of an open set is open, and a Borel space has every open set in the topology of the domain.

1.3 Measures

A *positive* measure μ is a function with range in $[0, \infty]$ on measurable spaces X, defined for measurable sets, with the property of being countably additive. That is

$$\mu(\cup A_i) = \sum_i \mu(A_i)$$

for countably many A_i.

A complex measure, is a complex function, defined on measurable sets of a measurable space X, which is countably additive. Real measures, are a subclass of the complex measures. Neither complex or real measures map to infinity.

We call measurable spaces with a measure, a measure space.

1.3.1 Some non-trivial properties

1. $\lim \mu(A_n) = \mu(A)$ when $A = \cup_n A_n$ for countably many measurable sets A_n such that
$$A_1 \subset A_2 \subset \cdots .$$

2. $\lim \mu(A_n) = \mu(A)$ when $A = \cap_n A_n$ for countably many measurable sets A_n such that
$$A_1 \supset A_2 \supset \cdots ,$$
with $\mu(A_1) < \infty$.

Chapter 2

Abstract Integration

Through the notes,

$$\int f d\mu := \int_X f d\mu.$$

And unless stated otherwise, μ is a positive measure.

2.1 Non-negative Lebesgue Integrals

2.1.1 Lebesgue integral definition

Let $s : X \to [0, \infty]$ be a measurable simple function of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where α_i are the values of $s(x)$ if $x \in A_i$. For measurable E, we define

$$\int_E s \; d\mu = \sum \alpha_i \mu(A_i \cap E)$$

. There is no problem with this definition, as measurable s implies measurable A_i sets.

For measurable $f : X \to [0, \infty]$ we define

$$\int_E f d\mu = \sup \int s \; d\mu,$$

where the supreme is taken from all s such that $0 \le s \le f$. It can be proven that there is always a monotonically increasing sequence of simple s_n functions such that $\lim s_n = f$. So the supreme of non-negative s simple functions can approximate f well, so that the integral makes sense.

2.1.2 Theorem "change of variables"

Suppose $f : X \to [0, \infty]$ is measurable. Let

$$\varphi(E) = \int_E f d\mu$$

for measurable E, then φ is a measure on X. Furthermore if $g : X \to [0, \infty]$ is measurable,

$$\int g d\varphi = \int f d\mu.$$

2.2 Real and complex integrals

2.2.1 Definition

If f is complex measurable in X, we say that $f \in L^1(X)$ if

$$\int |f| d\mu < \infty.$$

Such f are called Lebesgue integrable functions, or summable functions.

2.2.2 Complex integrals

If $f = u + iv$ is measurable and in $L^1(X)$, we define on measurable E,

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \left(\int_E v^+ d\mu - \int_E v^- d\mu \right).$$

2.2.3 Extended real integrals

If $f : X \to [-\infty, \infty]$ is measurable, we define

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu,$$

for measurable E, when one of the integrals on the right are non-infinite, since $\infty - \infty$ is not defined.

2.2.4 Theorem

If $f_n : X \to [0, \infty]$ are measurable and

$$f = \sum f_n$$

then

$$\int f d\mu \leq \sum \int f_n d\mu.$$

2.2.5 Theorem

If $f \in L^1(X)$

$$\left| \int f d\mu \right| = \int |f| d\mu.$$

2.2.6 Lebesgue's dominated convergence theorem

If f_n are complex measurable functions on X such that

$$f = \lim f_n$$

converges in X, and there exists measurable complex g in X such that

$$|f_n| \leq g.$$

Then $f \in L^1(X)$,

$$\lim \int |f - f_n| d\mu,$$

and

$$\int f d\mu = \lim \int f_n d\mu.$$

Chapter 3

Properties almost everywhere

3.1 Definition

We say that P holds almost everywhere (a.e) in $E \subset X$, if P holds in $E - N$ where $\mu(N) = 0$.

For example, we say $f = g$ a.e for measurable f, g on the measure space X if they differ on a set of measure 0. If this holds, for any measurable E we have

$$\int_E f d\mu = \int_E g d\mu.$$

Notice that $f \sim g$ if $f = g$ a.e is an equivalence relation.

3.2 Definition

We extend the definition of measurable function: If f defined on measurable $E \subset X$, we say f is measurable in X, if $\mu(X - E)$ and $f^{-1}(V) \cap E$ is measurable for open V.

If we care about integrating this function over X, f need not be defined on $X - E$ as $\mu(X - E) = 0$ thus $\int_X f d\mu = \int_E f d\mu$ no matter what f is defined to be in $X - E$.

3.3 Theorem

Let f_n be complex measurable functions defined a.e in X such that

$$\sum \int |f_n| d\mu < \infty. \tag{3.1}$$

Then

$$f = \sum f_n, \tag{3.2}$$

converges a.e in X, $f \in L^1(X)$, and

$$\int f d\mu = \sum \int f_n d\mu. \tag{3.3}$$

Proof: Let $S = \{x : f_n(x) \text{ is defined } \forall n\}$, so that $\mu(X - S) = 0$. Defining $\varphi : S \to C$ by $\varphi = \sum |f_n|$, then by 2.2.4,

$$\int \varphi d\mu = \sum \int |f_n| d\mu < \infty. \tag{3.4}$$

Let $E = \{x \in S : \varphi(x) < \infty\}$, it follows that $\mu(X - E) = 0$ by 3.4.

Proof: Let $A = X - S$ and $B = X - E$, B is the disjoint union of $A \cap B = A$ and $B - A$, then $\mu(B) = \mu(A) + \mu(B - A)$. As $B - A = (X - E) - (X - S) = S - E$ which are the points $x \in S$ on which $\varphi(x) = \infty$, $\mu(B - A) = 0$ as 3.4 must hold. So we have proven $\mu(X - E) = \mu(X - S) = 0$. //

Series 3.2 converges absolutely in E by seeing that $\varphi(x) = \sum |f_n(x)| < \infty$ for $x \in E$. So that f restricted to E is well defined, and by 2.2.5, and 3.4,

$$\left| \int f d\mu \right| \leq \int \varphi d\mu < \infty$$

so that $f \in L^1(X)$.

If we set $g_n = \sum_{i=0}^{n} f_i$, we see that on E, $|g_n| \leq \varphi$. Since $\varphi \in L^1(X)$ by 3.4, and $f = \lim g_n$ in E, the dominated convergence theorem says that $f \in L^1(X)$ (since $\mu(X - E) = 0$), and that

$$\lim \int g_n d\mu = \sum \int f_n d\mu = \int f d\mu.$$

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Note: If f_n were defined everywhere in X (so S would be X), when proving that $\mu(X - E) = 0$, we do not imply that $E = X$. Thus the conclusion would still be, that f converges almost everywhere (in all of E).

3.4 Theorem

1. Suppose $f : X \to [0, \infty]$ is measurable and $\mu(E) > 0$. If $\int_E f d\mu = 0$ then $f = 0$ a.e on E.

2. Suppose $f \in L^1(X)$ and $\int_E f d\mu = 0$ for all measurable E. Then $f = 0$ a.e on X.

3. Suppose $f \in L^1(X)$ and

$$\left| \int_X f d\mu \right| = \int_X |f| d\mu.$$

Then there exists $\alpha \in C$ such that $\alpha f = |f|$ a.e in X.

Proof. 1. Let $A_n = \{x \in E : f(x) > \frac{1}{n}\}$. Then

$$\frac{1}{n}\mu(A_n) = \int_{A_n} \frac{1}{n} d\mu \leq \int_{A_n} f d\mu \leq \int_E f d\mu = 0.$$

Thus $\mu(A_n) = 0$, and $\cup A_n = \{x : f(x) > 0\}$ has measure 0 too. ////

3.5 Corollary

Let $f : X \to [0, \infty]$ be measurable, then $f = 0$ a.e in X if and only if $\int_X f dm = 0$.

Proof: If $f = 0$ a.e in X, trivially $\int_X f dm = \int_X 0 dm$. If $\int_X f dm = 0$, by 3.4, $f = 0$ a.e in X. ////

3.6 An averages theorem

Suppose $\mu(X) < \infty$ and that $f \in L^1(X)$, with S closed in the complex plane, and that the averages

$$A_E(f) = \frac{1}{\mu(E)} \int_E f d\mu,$$

lie in S for every measurable E with positive measure. Then $f(x) \in S$ for almost all $x \in X$.

Proof: We want to show that the set where f lies outside of S, specifically the set $f^{-1}(X - S)$, is of measure 0.

For this, it is enough that we show that the open disc Δ contained in $X - S$, is such that $\mu(E) = 0$ where $E = f^{-1}(\Delta)$. This, because as $X - S$ is open, it is the union of countably many open disks. Trivially, E is measurable as f is measurable.

Let Δ be the disk at α of radius $r > 0$. If we had $\mu(E) > 0$, then

$$|A_E(f) - \alpha| = \left| \frac{1}{\mu(E)} \int_E f d\mu - \alpha \right| \tag{3.1}$$

$$= \left| \frac{1}{\mu(E)} \int_E (f - \alpha) d\mu \right| \tag{3.2}$$

$$= \frac{1}{\mu(E)} \left| \int_E (f - \alpha) d\mu \right| \tag{3.3}$$

$$\leq \frac{1}{\mu(E)} \int_E |f - \alpha| d\mu. \tag{3.4}$$

The second equality comes from the fact that the integral of a constant c over a measurable set A, is $c\mu(E)$. The inequality comes from 2.2.5.

Since $f(E) \subset \Delta$, $|f(x) - \alpha| < r$ for x in E. Thus by 3.5,

$$\begin{aligned} & 0 < \int_E (r - |f - \alpha|) d\mu = r\mu(E) - \int_E |f - \alpha| d\mu \\ \Longleftrightarrow & \frac{1}{\mu(E)} \int_E |f - \alpha| d\mu < r, \text{ and then by 3.4} \\ & |A_E(f) - \alpha| < r. \end{aligned}$$

Meaning $A_E(f) \in \Delta \subset X - S$, contradicting the hypothesis that $A_E(f)$ lies in S. Thus $\mu(E) = 0$. ////

3.7 Theorem

Let E_k be a sequence of measurable sets in X, such that

$$\sum \mu(E_k) < \infty. \tag{3.1}$$

Then, almost all x lies in at most finitely many E_k.

Proof: We have to prove that the set where x lies in infinitely many E_k is of measure 0. Consider

$$g = \sum \chi_{E_k}.$$

From its definition $g(x) = \infty$ if and only if x is in infinitely many E_k. By hypothesis E_k are measurable sets, so χ_{E_k} are measurable, then

$$\int g d\mu = \sum \mu(E_k),$$

by 3.3, which is finite by hypothesis. ////

Chapter 4

Positive Borel Measures

4.1 Topological preliminaries

4.1.1 Definitions

X is a Hausdorff space if: For $p \neq q$ in X, p and q have respectively neighborhoods which are disjoint one of another.

X is locally compact if every point of X has a neighborhood whose closure is compact.

Example

Every compact space is locally compact.

By the Heiner-Borel theorem, the compact sets of an euclidian space R^n are precisely those which are close and bounded. Which means, R^n is a locally compact Hausdorff space.

Every metric space is a Hausdorff space.

4.1.2 Theorem

In any topological space, if K is compact and F is closed, then $F \subset K$ implies F is compact.

4.1.3 Theorem

If K is compact in a Hausdorff space X and $p \in X - K$. Then there exists open sets U and W, such that $p \in U$ and $K \subset W$, and $U \cap W = \emptyset$.

Corollaries

1. Compact subsets of Hausdorff spaces are closed.

2. In a Hausdorff space, if F is closed and K is compact, $F \cap K$ is compact.

Point 1 comes from the fact that the theorem implies there are open sets for every point outside of K, so that the union of these open sets is $X - K$, thus K is closed.

Point 2 follows from the fact that K is closed, so that $E \cap K \subset K$ is compact.

4.1.4 Theorem

If $\{K_\alpha\}$ is a collection of compact sets in a Hausdorff space, and the intersection of all K_α, is empty, then for finitely many K_α their intersection is also empty.