# Analysis Notes

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# Chapter 1

# Abstract Integration

Through the notes,

$$\int f d\mu := \int_X f d\mu.$$

And unless stated otherwise, $\mu$ is a positive measure.

## 1.1 Non-negative Lebesgue Integrals

### 1.1.1 Lebesgue integral definition

Let $s : X \to [0, \infty]$ be a measurable simple function of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where $\alpha_i$ are the values of $s(x)$ if $x \in A_i$. For measurable $E$, we define

$$\int_E s \, d\mu = \sum \alpha_i \mu(A_i \cap E)$$

. There is no problem with this definition, as measurable $s$ implies measurable $A_i$ sets.

For measurable $f : X \to [0, \infty]$ we define

$$\int_E f d\mu = \sup \int s \, d\mu,$$

where the supreme is taken from all $s$ such that $0 \leq s \leq f$. It can be proven that there is always a monotonically increasing sequence of simple $s_n$ functions such that $\lim s_n = f$. So the supreme of non-negative $s$ simple functions can approximate $f$ well, so that the integral makes sense.

### 1.1.2 Theorem “change of variables”

Suppose $f : X \to [0, \infty]$ is measurable. Let

$$\varphi(E) = \int_E f d\mu$$

for measurable $E$, then $\varphi$ is a measure on $X$. Furthermore if $g : X \to [0, \infty]$ is measurable,

$$\int g d\varphi = \int f d\mu.$$

## 1.2 Real and complex integrals

### 1.2.1 Definition

If $f$ is complex measurable in $X$, we say that $f \in L^1(X)$ if

$$\int |f| d\mu < \infty.$$

Such $f$ are called Lebesgue integrable functions, or summable functions.

### 1.2.2 Complex integrals

If $f = u + iv$ is measurable and in $L^1(X)$, we define on measurable $E$,

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \left( \int_E v^+ d\mu - \int_E v^- d\mu \right).$$

### 1.2.3 Extended real integrals

If $f : X \to [-\infty, \infty]$ is measurable, we define

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu,$$

for measurable $E$, when one of the integrals on the right are non-infinite, since $\infty - \infty$ is not defined.

### 1.2.4 Theorem

If $f_n : X \to [0, \infty]$ are measurable and

$$f = \sum f_n$$

then

$$\int f d\mu = \sum \int f_n d\mu.$$

### 1.2.5 Theorem

If $f \in L^1(X)$

$$\left| \int f d\mu \right| = \int |f| d\mu.$$

### 1.2.6 Lebesgue's dominated convergence theorem

If $f_n$ are complex measurable functions on $X$ such that

$$f = \lim f_n$$

converges in $X$, and there exists measurable complex $g$ in $X$ such that

$$|f_n| \leq g.$$

Then $f \in L^1(X)$,

$$\lim \int |f - f_n| d\mu,$$

and

$$\int f d\mu = \lim \int f_n d\mu.$$

### 1.2.7 Definition

We say that $P$ holds almost everywhere (a.e) in $E \subset X$, if $P$ holds in $E - N$ where $\mu(N) = 0$.

For example, we say $f = g$ a.e for measurable $f, g$ on the measure space $X$ if they differ on a set of measure 0. If this holds, for any measurable $E$ we have

$$\int_E f d\mu = \int_E g d\mu.$$

Notice that $f \sim g$ if $f = g$ a.e is an equivalence relation.

### 1.2.8 Definition

We extend the definition of measurable function: If $f$ defined on measurable $E \subset X$, we say $f$ is measurable in $X$, if $\mu(X - E)$ and $f^{-1}(V) \cap E$ is measurable for open $V$.

If we care about integrating this function over $X$, $f$ need not be defined on $X - E$ as $\mu(X - E) = 0$ thus $\int_X f d\mu = \int_E f d\mu$ no matter what $f$ is defined to be in $X - E$.

### 1.2.9 Theorem

Let $f_n$ be complex measurable functions defined a.e in $X$ such that

$$\sum \int |f_n| d\mu < \infty.$$

Then

$$f = \sum f_n,$$

converges a.e in $X$, $f \in L^1(X)$, and

$$\int f d\mu = \sum \int f_n d\mu.$$

PROOF: Let $S = \{x : f_n(x) \text{ is defined } \forall n\}$, so that $\mu(X - S) = 0$. Defining $\varphi : S \to C$ by $\varphi = \sum |f_n|$, then by 1.2.4,

$$\int_S \varphi d\mu = \sum \int_S |f_n| d\mu < \infty. \tag{1.1}$$

*Let $E = \{x \in S : \varphi(x) < \infty\}$, it follows that $\mu(X - E) = 0$ by 1.1.*

Proof: Let $A = X - S$ and $B = X - E$, $B$ is the disjoint union of $A \cap B = A$ and $B - A$, then $\mu(B) = \mu(A) + \mu(B - A)$. As $B - A = (X - E) - (X - S) = S - E$ which are the points $x \in S$ on which $\varphi(x) = \infty$, $\mu(B - A) = 0$ as 1.1 must hold. So we have proven $\mu(X - E) = \mu(X - S) = 0$. //