We will assume that f is a monotone increasing boolean function such that the following holds for some absolute constants $0 < \alpha < 1$, $0 < \delta < 1$, and for $\epsilon = o_n(1)$:

1. $$\mathbb{E} f \quad = \quad \alpha,$$

2. $$\Pr_{S, |S| = \frac{n}{4}} \Big\{ \Pr_{B : S \subseteq B} \{ f(B) = 1 \} \le 1 - \delta \Big\} \quad \ge \quad 1 - \epsilon,$$

and try to reach contradiction.

Let $\mathcal{F} = \{B : f(B) = 1\}$. We start with observing that the second assumption has the following corollary:

Lemma 0.1: *Let L_m denote the m'th level of the discrete cube $\{0,1\}^n$. There exists a constant $\delta' \lesssim \delta$ such that for all $\frac{n}{4} \le m \le \frac{5n}{8} - \omega\left(\sqrt{n}\right)$ holds*

$$|\mathcal{F} \cap L_m| \;\le\; \left(1 - \delta'\right) \cdot \binom{n}{m}.$$

Proof: Let $\mathcal{F}_m = \mathcal{F} \cap L_m$. Let $P_m = \Big\{ (S, B) : |S| = \frac{n}{4},\; S \subseteq B, B \in \mathcal{F}_m \Big\}$. We will estimate P_m in two ways.

First, we introduce some more notation. Let S be of cardinality $\frac{n}{4}$ and let $C_S = \{B, S \subseteq B\}$ be the $\frac{3n}{4}$-dimensional subcube containing the supersets of S. By the second assumption above, for $(1 - \epsilon)$ fraction of the sets S holds $|C_S \cap \mathcal{F}| \le (1 - \delta) \cdot 2^{\frac{3n}{4}}$. Let S be one of these sets. The main observation means that for some $\delta' \lesssim \delta$ and for all m which are sufficiently below the height of the middle level of C_S (which is $\frac{5n}{8}$, and the choice $m \le \frac{5n}{8} - \omega\left(\sqrt{n}\right)$ works fine) the set $C_S \cap \mathcal{F}_m$ should be of cardinality at most $(1 - \delta') \cdot \binom{\frac{3n}{4}}{m - \frac{n}{4}}$. Indeed, assume to the contrary that this does not hold. Then for all $m \le t \le n$ we have, as a simple corollary of the monotonicity of $C_S \cap \mathcal{F}$, that

$$|C_S \cap \mathcal{F}_m| \;\ge\; \left(1 - \delta'\right) \cdot \binom{\frac{3n}{4}}{t - \frac{n}{4}},$$

which implies $|C_S \cap \mathcal{F}| > (1 - \delta) \cdot 2^{\frac{3n}{4}}$, for an appropriate choice of δ', contradicting the choice of S.

So, for $(1 - \epsilon)$ fraction of the sets S of cardinality $\frac{n}{4}$ holds $C_S \cap \mathcal{F}_m \le (1 - \delta') \cdot \binom{\frac{3n}{4}}{m - \frac{n}{4}}$. This means

$$|P_m| \;\le\; (1 - \epsilon) \cdot \left(1 - \delta'\right) \cdot \binom{n}{\frac{n}{4}} \binom{\frac{3n}{4}}{m - \frac{n}{4}} + \epsilon \cdot \binom{n}{\frac{n}{4}} \binom{\frac{3n}{4}}{m - \frac{n}{4}} \;\approx\; \left(1 - \delta'\right) \cdot \binom{n}{\frac{n}{4}} \binom{\frac{3n}{4}}{m - \frac{n}{4}}.$$

On the other hand, any $B \in \mathcal{F}_m$ contributes $\binom{m}{\frac{n}{4}}$ pairs to P_m. And hence

$$|\mathcal{F}_m| \lesssim (1-\delta') \cdot \frac{\binom{n}{\frac{n}{4}}\binom{\frac{3n}{4}}{m-\frac{n}{4}}}{\binom{m}{\frac{n}{4}}} = (1-\delta') \cdot \binom{n}{m}.$$

∎

We now introduce some more notation. For $0 < p < 1$ let μ_p denote the product measure which assigns probability p to 1 in each coordinate, independently for different coordinates. Let $M(p) = \mu_p(\mathcal{F})$ and observe that $M\left(\frac{1}{2}\right) = \mathbb{E}\, f = \alpha$. We recall the following facts: For a monotone increasing set $\mathcal{F}$ the function M is increasing, and moreover (Russo's formula, see e.g. [1]. eq. 8.8):

$$\frac{dM}{dp} = \frac{1}{p(1-p)} \sum_{i=1}^{n} I_{p,i}\left(\mathcal{F}\right),$$

where $I_{p,i}\left(\mathcal{F}\right)$ is the i'th influence w.r.t. μ_p of f, that is $I_{i,p}(f) = \mu_p\big(\{x : f(x) \neq f\left(x + e_i\right)\}\big)$.

Note that most of the mass in the μ_p measure is concentrated on sets of weight $pn \pm O_p\left(\sqrt{n}\right)$, and hence it is an immediate corollary of Lemma 0.1 that for any $p < \frac{5}{8}$ holds

$$M(p) = \mu_p\left(\mathcal{F}\right) \lesssim \left(1-\delta'\right) \approx 1 - \delta.$$

Fix such a p, and for simplicity assume (with slight inaccuracy) in the following $p = \frac{5}{8}$.

By the intermediate value theorem, for some $\frac{1}{2} < p < \frac{5}{8}$, and for the appropriate absolute constant $K \leq 8p(1-p)(1-\alpha)$ holds

$$\sum_{i=1}^{n} I_{p,i}\left(\mathcal{F}\right) = p(1-p)\frac{dM}{dp} = p(1-p)\frac{M\left(\frac{5}{8}\right) - M\left(\frac{1}{2}\right)}{\frac{1}{8}} := K.$$

Remark 0.2: In fact, note that for this claim (that sum of influences is bounded for some $\frac{1}{2} < p < \frac{5}{8}$ we don't need the first lemma. It follows trivially from the fact that $M\left(\frac{5}{8}\right) \leq M(1) \leq 1$. We need the first lemma and its implications below, in the last paragraph, when we restrict to $\mathcal{C}$. ∎

From now on we fix this p. Note that $\mu_p\left(\mathcal{F}\right) \geq \mathbb{E}\, f = \alpha$. Next, we choose a small constant γ, so that $\gamma \ll \alpha$ and $\gamma \ll \delta$, and recall that by a result of Friedgut, since $\sum_{i=1}^{n} I_{p,i}\left(\mathcal{F}\right) \leq K$, for some absolute constant c, there a subset I of cardinality at most $c_1 = c^{\frac{K}{\gamma}}$ coordinates, and a boolean function g ("junta") depending only on coordinates in I such that $\mu_p\left(\{f \neq g\}\right) \leq \gamma$. Moreover, if f is monotone, so is g.

Let now $\mathcal{C}$ be the c_1-codimensional subcube in which all coordinates in I are 1. From now on we restrict our discussion to $\mathcal{C}$, in particular we restrict $\mathcal{F}$ and μ_p to $\mathcal{C}$, denoting them with superscript $\mathcal{C}$.

Lemma 0.3:

$$\mu_p^{\mathcal{C}}\left(\mathcal{F}^{\mathcal{C}}\right) \geq 1 - \frac{\gamma}{\alpha - \gamma}.$$

Proof: First note that $\mu_p(\{f \neq g\}) \leq \gamma$ implies $\mathbb{E}\, g \geq \mathbb{E}\, f - \gamma \geq \alpha - \gamma$. In particular, since $\gamma < \alpha$ we have that $\mathbb{E}\, g > 0$, and $g \neq 0$. Since g is monotone and depends only on coordinates in I this implies that on $\mathcal{C}$ the function g equals 1.

Next, write a vector $x \in \{0,1\}^n$ as $x = (yz)$, where, in the obvious notation, $y \in \{0,1\}^I$ and $z \in \{0,1\}^{n \setminus I}$. For $y \in \{0,1\}^I$ define $F(y) = \mathbb{E}_{z \in \{0,1\}^{n \setminus I}} f(yz)$ and note that F is monotone increasing on $\{0,1\}^I$. Hence $\mu_p^{\mathcal{C}}\left(\mathcal{F}^{\mathcal{C}}\right) = F(1)$ is the maximal value $F(y)$ attains.

Hence we have, recalling that $g(yz)$ depends only on y, and hence we can write it as $g(y)$,

$$\gamma \;\geq\; \mathop{\mathbb{E}}_{x}(f(x) - g(x)) \;=\; \sum_{y \in \{0,1\}^I} \mu_p(y) \cdot \mathop{\mathbb{E}}_{z \in \{0,1\}^{n \setminus I}} \big(f(yz) - g(yz)\big) \;\geq$$

$$\sum_{y \in \{0,1\}^I : g(y)=1} \mu_p(y) \cdot \mathop{\mathbb{E}}_{z \in \{0,1\}^{n \setminus I}} \big(f(yz) - g(yz)\big) \;=\; \sum_{y \in \{0,1\}^I : g(y)=1} \mu_p(y) \cdot (1 - F(y)) \;\geq$$

$$\sum_{y \in \{0,1\}^I : g(y)=1} \mu_p(y) \cdot (1 - F(1)) \;=\; \mathbb{E}\, g \cdot (1 - F(1)) \;\geq\; (\alpha - \gamma) \cdot (1 - F(1)).$$

Rearranging, we get the claim of the lemma. ∎

Next, we claim that, by the total expectation formula, the second assumption above holds for $\mathcal{C}$ and for $\mathcal{F}^{\mathcal{C}}$ with ϵ replaced by $\epsilon_1 \approx 4^{c_1} \cdot \epsilon$ (since a random subset S of $[n]$ of cardinality $\frac{n}{4}$ contains I with probability about $\left(\frac{1}{4}\right)^{c_1}$). Hence, reproving Lemma 0.1 for $\mathcal{C}$ shows that for any $q < \frac{5}{8}$ holds $\mu_q^{\mathcal{C}}\left(\mathcal{F}^{\mathcal{C}}\right) \lesssim 1 - \delta$. But since $p < \frac{5}{8}$ we can choose $q > p$ and hence, by the above

$$\mu_q^{\mathcal{C}}\left(\mathcal{F}^{\mathcal{C}}\right) \;\geq\; \mu_p^{\mathcal{C}}\left(\mathcal{F}^{\mathcal{C}}\right) \;\geq\; 1 - \frac{\gamma}{\alpha - \gamma},$$

reaching contradiction, by our assumptions on γ. ∎

References

[1] R. O'Donnel, **Analysis of Boolean functions**, Cambridge University Press, 2014.