511 assignment

1Q

(a)

The binary relation Eq_k is first-order definable in $\mathbb{N}$ using the equality relation $=$.

Step 1: Define $\varphi_{\mathbf{mod}}(x, d, y)$

Define the well-formed formula (WFF) $\varphi_{\mathrm{mod}}(x, d, y)$ for the modulo operation:

$$\varphi_{\mathrm{mod}}(x, d, y) \equiv \exists q(x = q \cdot d + y \ \wedge \ 0 \leq y < d)$$

This WFF states that for any x, d, y, $x \mod d = y$ if and only if there exists some q such that $x = qd + y$ and $0 \leq y < d$.

Step 2: Define $\mathbf{Eq}_k(x_1, x_2)$

Using φ_{mod}, define $\mathrm{Eq}_k(x_1, x_2)$:

$$\mathrm{Eq}_k(x_1, x_2) \equiv \exists y_1 \exists y_2 (\varphi_{\mathrm{mod}}(x_1, 2^k, y_1) \wedge \varphi_{\mathrm{mod}}(x_2, 2^k, y_2) \wedge y_1 = y_2)$$

This formula asserts that x_1 and x_2 are related by Eq_k if their remainders when divided by 2^k are equal.

Conclusion

By defining φ_{mod} and using it to express Eq_k, we have shown that Eq_k is first-order definable in $\mathbb{N}$ using the equality relation.

(b)

Part 1: Prove that $\mathbf{Eq}_k$ is a Congruence Relation

To show that Eq_k is a congruence relation, we need to demonstrate that it is an equivalence relation and compatible with the operations in the structure N.

Equivalence Relation:

- Reflexivity: For any $i \in \mathbb{N}$, $\mathrm{Eq}_k(i, i)$ holds as $i \mod 2^k = i \mod 2^k$.
- Symmetry: If $\mathrm{Eq}_k(i, j)$, then $i \mod 2^k = j \mod 2^k$ which implies $j \mod 2^k = i \mod 2^k$, hence $\mathrm{Eq}_k(j, i)$.
- Transitivity: If $\mathrm{Eq}_k(i, j)$ and $\mathrm{Eq}_k(j, k)$, then $i \mod 2^k = k \mod 2^k$, so $\mathrm{Eq}_k(i, k)$.

Compatibility with Operations:

- Addition: For all $a, b, c \in \mathbb{N}$, if $\mathrm{Eq}_k(a, b)$, then $(a + c) \mod 2^k = (b + c) \mod 2^k$.
- Multiplication: Similarly, if $\mathrm{Eq}_k(a, b)$, then $(a \times c) \mod 2^k = (b \times c) \mod 2^k$.

Part 2: Eq_{k+1} is a Finer Congruence Than Eq_k

Every Eq_k-congruence class is the union of two disjoint Eq_{k+1}-congruence classes. Hence, Eq_{k+1} is a finer congruence than Eq_k.

Part 3: Conditions Under Which Eq_k Coincides with Equality

For Eq_k to coincide with equality, we restrict the range of variables such that their differences are less than 2^k. Specifically, if $i, j < 2^k$, then $\mathrm{Eq}_k(i, j)$ holds if and only if $i = j$. Therefore, in any well-formed formula ϕ involving $\approx$ (used here to represent ''), we can replace $\approx$ with Eq_k without changing the truth value of the formula, provided we restrict the domain of the quantified variables to be less than 2^k.

2Q

1. Wff's for {0}, {1}, and {2}

- $\phi_{\{0\}}(x)$ defines $\{0\}$ as the additive identity:

$$\phi_{\{0\}}(x) = \forall y(x + y = y)$$

- $\phi_{\{1\}}(x)$ defines $\{1\}$ as the multiplicative identity (excluding 0):

$$\phi_{\{1\}}(x) = \forall y(x \cdot y = y) \wedge \neg\phi_{\{0\}}(x)$$

- $\phi_{\{2\}}(x)$ defines $\{2\}$ as the sum of two 1s:

$$\phi_{\{2\}}(x) = \phi_{\{1\}}(x + x)$$

2. Wff for the Usual Ordering "¡"

$\phi_<(x, y)$ defines the usual ordering "less than" using the existence of a positive number:

$$\phi_<(x, y) = \exists c(\phi_{\{0\}}(c) \wedge x + c = y)$$

3. Polynomials of Odd Degree

The set Γ contains sentences asserting the existence of a real root for every polynomial of odd degree. Each sentence in Γ for a polynomial of degree n (odd) is:

$$\forall a_0 \forall a_1 \ldots \forall a_n \exists x (a_0 + a_1 \cdot x + a_2 \cdot x \cdot x + \ldots + a_n \cdot x^n = 0)$$

4. Defining Positive Integers in $R_{\mathbf{sin}}$

To define positive integers, we use the sine function's property that $\sin(a) = 0$ if and only if a is an integer multiple of π.

- $\phi_{\{\pi\}}(x)$ uniquely defines π:

$$\phi_{\{\pi\}}(x) = \forall y(\sin(y) = 0 \rightarrow (x \cdot y = y \vee \phi_{\{0\}}(y)))$$

- $\phi_{\text{pos}}(x)$ defines positive integers as those for which $\sin(x \cdot \pi) = 0$ and $x \neq 0$:

$$\phi_{\text{pos}}(x) = \exists y(\phi_{\{\pi\}}(y) \wedge \sin(x \cdot y) = 0 \wedge \neg\phi_{\{0\}}(x))$$

3Q

a

Use first-order resolution to show that $\Gamma \models \varphi$, where

$$\Gamma = \forall x \forall y \forall z\, (P(x,y) \wedge P(y,z) \rightarrow P(x,z)),\ \forall x \forall y\, (P(x,y) \rightarrow P(y,x))$$

and

$$\varphi = \forall x \forall y \forall z\, (P(x,y) \wedge P(z,y) \rightarrow P(x,z)).$$

Conversion to Clausal Form:
The clausal form of Γ and $\neg\varphi$ is given as:
1. From Γ:

- Clause 1: $\{\neg P(x,y), \neg P(y,z), P(x,z)\}$
- Clause 2: $\{\neg P(x,y), P(y,x)\}$

2. From $\neg\varphi$:

- Clause 3: $\{P(x,y), P(z,y), \neg P(x,z)\}$

Applying Resolution:
We attempt to find a resolution between these clauses:
- Resolve Clause 1 and Clause 3 on $P(x,z)$:

- New clause: $\{\neg P(x,y), \neg P(y,z), P(x,y), P(z,y)\}$

- Simplification: $\{\neg P(y,z), P(z,y)\}$ (since $P(x,y)$ and $\neg P(x,y)$ cancel each other)

- Resolve this new clause with Clause 2 on $P(y,x)$:

- New clause: $\{\neg P(y,z), \neg P(x,y)\}$
- This does not directly lead to a contradiction.

- Other combinations do not seem to directly lead to an empty clause or a contradiction.

Conclusion:

Based on this analysis, we cannot conclude that Γ and $\neg\varphi$ are unsatisfiable together, and thus we cannot conclude that $\Gamma \vDash \varphi$. This does not necessarily mean that Γ does not imply φ; it just means that the resolution method did not find a contradiction in this specific approach. Further investigation or alternative strategies might be required.

4Q and 5Q

`https://github.com/rainbowfly29/hwsolutions511/blob/main/hw11.lean`