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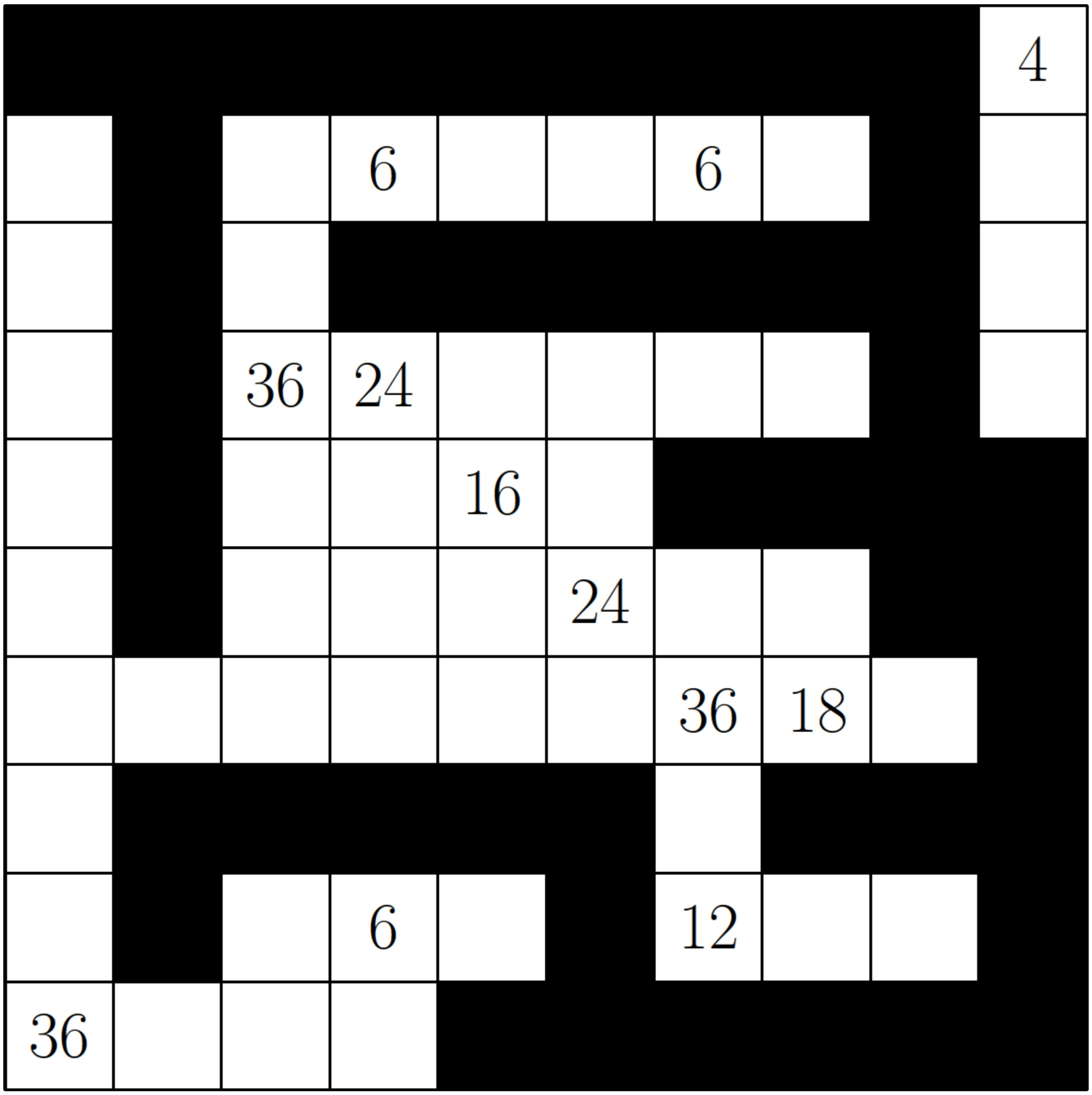
USA Mathematical Talent Search

Yr	Round	Problem
36	3	1

Solution shown below:



The only restriction for the new valid configuration is that the angle between the hour and the minute hands must be equal to the one of the invalid configuration. Let this angle be θ. We seek to find the closest valid time sharing the angle θ between the hour and the minute hands.

Throughout a natural 12-hour cycle, the hour and minute hand will overlap in 11 distinct places, uniformly spaced throughout the clock, due to the constant rate of change of the hour and minute hands. Throughout each rotation, all possible angles in between the minute and hour hand are exhausted, meaning that valid times with our angle θ will also occur 11 times, just as when $\theta = 0°$.

This will lead to valid configurations with angle θ every $\frac{360}{11}^\circ$ around the clock. Observe that the clockwise distance between the invalid configuration and the valid configuration is D, with $D \in (0, \frac{360}{11})$. Since the invalid configuration is chosen randomly, it is also random $\pmod{\frac{360}{11}}$, so we only need to find the average value of D.

We use an integral to compute the expected value:

$$\mathbb{E}[D] = \frac{360}{11} \cdot \int_0^1 D\, dD$$

$$= \frac{360}{11} \cdot \frac{1}{2} = \boxed{\frac{180}{11}^\circ}$$

We first prove statement a. Rewrite x:

$$x = \sqrt{2a - 2\sqrt{a^2 - b}}$$

We can set $b = a^2 - (a-2)^2 = 4a - 4$. Simplifying:

$$x = \sqrt{2a - 2(a-2)}$$

$$x = \sqrt{2a - 2a + 4}$$

$$x = 2$$

Therefore, for all $a \geq 2$, there exists a positive integer $b = 4a - 4$ such that x is a positive integer. $\square$

We prove statement b.

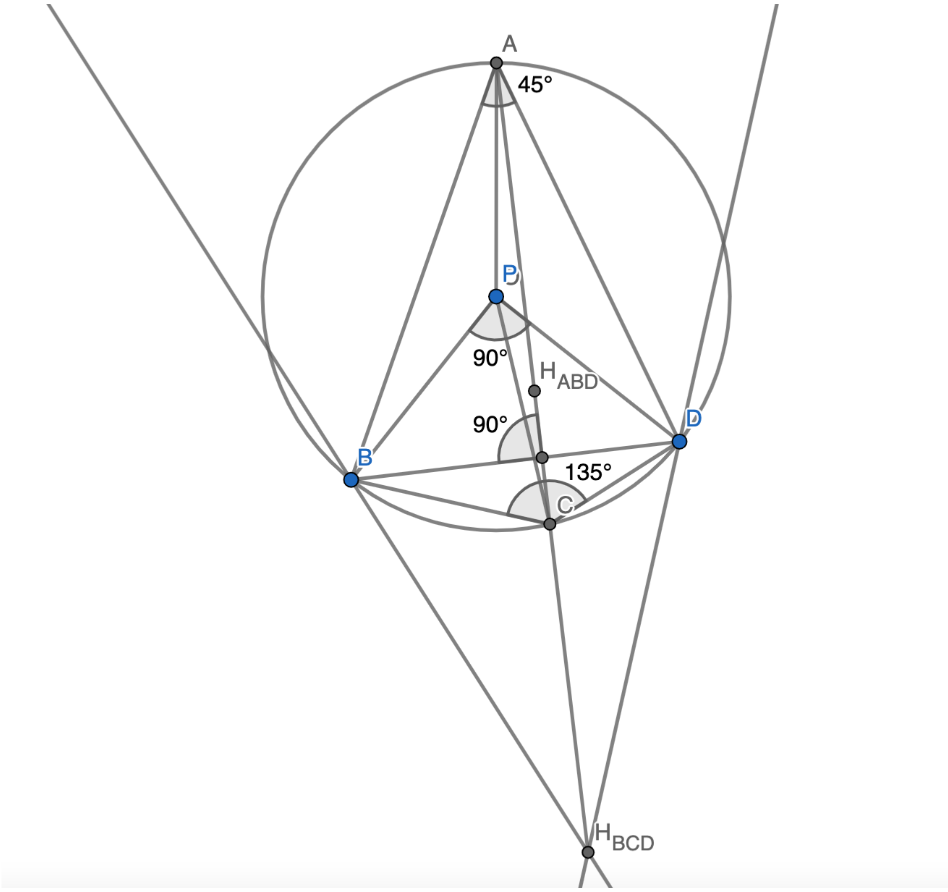
For sufficiently large a, specifically for all $a \geq 8$, we can set $b = a^2 - (a - 8)^2 = 16a - 64$. Simplifying:

$$x = \sqrt{2a - 2(a-8)}$$

$$x = \sqrt{2a - 2a + 64}$$

$$x = 8$$

We have proved that for $a \geq 2$, $b = 4a - 4$ works. For sufficiently large a, both $b = 4a - 4$ and $b = 16a - 64$ work and set x to a positive integer. Therefore, for all sufficiently large a, there are at least two b such that x is a positive integer. $\square$



Direction 1: $\overline{\mathrm{AC}} \perp \overline{\mathrm{BD}} \implies \overline{\mathrm{PB}} \perp \overline{\mathrm{PD}}$

We are given that $m < BAP = m < CAD$.

It follows the the lines AP and AC are the same angular distance from the angle bisector of A. In other words, AP is the reflection of AC over the angle bisector of A, and vice versa.

Note that the orthocenter of ABD lies on AC. By definition of an isogonal conjugate, we know that the isogonal conjugate of H_{ABD} will lie on the line AC reflected over the angle bisector of A. Therefore, it will lie on AP.

We are given that $m < BCP = m < ACD$.

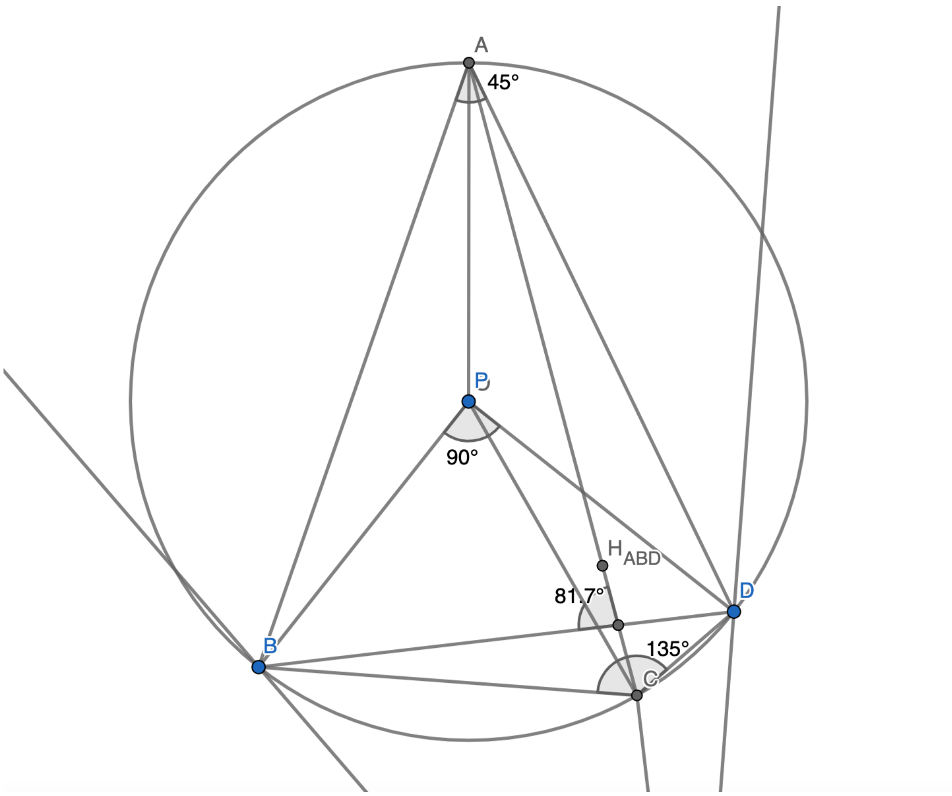
It follows the the lines CP and CA are the same angular distance from the angle bisector of C. In other words, CP is the reflection of CA over the angle bisector of C, and vice versa.

Note that the orthocenter of BCD lies on AC. By definition of an isogonal conjugate, we know that the isogonal conjugate of H_{BCD} will lie on the line CA reflected over the angle bisector of C. Therefore, it will lie on CP.

The is because the isogonal conjugate of the orthocenter of a triangle is the same triangle's circumcenter. Since $ABCD$ is cyclic, the circumcenter of BCD and ABD are equal. Therefore, isogonal conjugate of H_{BCD} and H_{ABD} are both the same, the circumcenter of $ABCD$.

Note that we have shown the isogonal conjugate of H_{BCD} and H_{ABD} are equal, and that this isogonal conjugate lies on both AP and CP. Therefore, this point is P. Combined with our previous results, we conclude that P is the circumcenter of $ABCD$.

Observe that BAD and BPD both subtend the same arc BD. Therefore, by the inscribed angle theorem, $m < BPD = 2 \cdot m < BAD = 2 \cdot 45° = 90°$. Then, it immediately follows that $\overline{\text{PB}} \perp \overline{\text{PD}}$. $\square$



Direction 2: not $(\overline{\mathrm{AC}} \perp \overline{\mathrm{BD}}) \implies$ **not** $(\overline{\mathrm{PB}} \perp \overline{\mathrm{PD}})$

We prove the disjunction of the forward direction, showing that we cannot have both conditions. Assume, for the sake of contradiction, that $\overline{\mathrm{AC}} \not\perp \overline{\mathrm{BD}}$ and $(\overline{\mathrm{PB}} \perp \overline{\mathrm{PD}})$.

Assume P is not the circumcenter. Due to the constraint of the isogonal conjugates of ABD and BCD being equal to the circumcenter, we note that P must be the circumcenter. If P is not the circumcenter, it becomes impossible for both angle equality conditions given in the problem statement and the isogonal conditions to hold.

Then without loss of generality, let P remain unchanged, such that

$m < BAP$ also is unchanged. For a given A and D, and restricting C to arc BD, there is one unique angle $< CAP$ that allows $m < CAD = m < BAP$. Therefore, there is exactly one choice for C such that $m < CAD = m < BAP$, this is easy to see.

However, in the forward direction, we have shown that choosing the unique C such that $AC \perp BD$ causes $m < CAD = m < BAP$.
Therefore, if $AC \not\perp BD$, there is no choice of C such that $m < CAD = m < BAP$. Hence, these coexisting conditions are impossible. $\square$

We first show $a < 3$. This leaves us very few cases to search through.

Let $a \geq 3$, then the equation becomes

$$2^k \cdot 2^3 \cdot 5^b = 3^c + 1$$

With $k, b, c \geq 0$. Take the equation $\pmod 8$:

$$3^c + 1 \equiv 0 \pmod 8$$

$$3^c \equiv 7 \pmod 8$$

$$3^0 \equiv 1, 3^1 \equiv 3, 3^2 \equiv 1 \ldots$$

$3^c \pmod 8$ will alternate between 1 and 3 infinitely, and will never equal 7.

Therefore, there are no solutions (a, b, c) with $a > 2$.

Case 1: a = 0

$$5^b = 3^c + 1$$

Take the equation $\pmod 2$:

$$1 \equiv 1 + 1 \pmod 2$$

This is impossible. There are no solutions in this case.

Case 2: a = 1

$$2 \cdot 5^b = 3^c + 1$$

Consider the equation $\pmod 4$:

$$2 \equiv (-1)^c + 1 \pmod 4$$

This implies c is even.

Let $b \geq 1$. Rewrite the equation as

$$10 \cdot 5^{b-1} = 9^{\frac{c}{2}} + 1$$

Consider the equation $\pmod{10}$:

$$0 \equiv (-1)^{\frac{c}{2}} + 1 \pmod{10}$$

This implies $\frac{c}{2}$ is odd. We return to the original equation with this result.

$$2 \cdot 5^b = 3^c + 1$$

Since $\frac{c}{2}$ is odd, we can rewrite this as:

$$2 \cdot 5^b = 9^{\frac{c}{2}} - (-1)^{\frac{c}{2}}$$

We apply lifting the exponent, taking v_5 of both sides.

$$b = v_5(10) + v_5(\frac{c}{2}) = 1 + v_5(c)$$

$$b - 1 = v_5(c)$$

This implies $c \geq 5^{b-1}$. We can set a strong upper bound on b using this fact. Substitute $c = 5^{b-1}$, the worst-case scenario, which allows b to grow as large as possible:

$$2 \cdot 5^{b-1} \geq 3^{5^{b-1}}$$

$b = 1$ holds. $b = 2$ does not. As b increases, the RHS will grow much faster than the LHS. Therefore, when $b \geq 1$, the solutions occur only when $b = 1$.

Subcase 2.1: $a = 1, b = 1$

$2^1 \cdot 5^1 - 3^c = 1$

$c = 2$

Therefore, the solution derived from this case is $(1, 1, 2)$.

Now consider the case $b = 0$ separately.

Subcase 2.2: $a = 1, b = 0$

$2^1 \cdot 1 - 3^c = 1$

$c = 0$

Therefore, the solution derived from this case is $(1, 0, 0)$.

Case 3: a = 2

$$4 \cdot 5^b = 3^c + 1$$

We show that this cannot hold for any $b > 0$. Let $k, c \geq 0$ then

$$4 \cdot 5 \cdot 5^k = 3^c + 1$$

Taking the equation $\pmod{20}$, this becomes

$$3^c \equiv 19 \pmod{20}$$

We look through the first few values

$$3^0 \equiv 1, 3^1 \equiv 3, 3^2 \equiv 0, 3^3 \equiv 7, 3^4 \equiv 1 \ldots$$

$3^c \pmod{20}$ will cycle with period 4 and never equal 19. Therefore, $b < 1$.

Subcase 3.1: $a = 2, b = 0$

$$2^2 - 3^c = 1$$

$$c = 1$$

The solution derived from this case is $(2, 0, 1)$.

Combining results from all 3 cases, the ordered triples of nonnegative integers (a, b, c) satisfying this equation are

$$\boxed{(1, 1, 2), (1, 0, 0), (2, 0, 1)}$$