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DISCRETE ADMISSIBILITY AND EXPONENTIAL DICHOTOMY FOR EVOLUTION FAMILIES

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Abstract. In this paper we study the uniform exponential dichotomy property for evolution families using discrete - time admissibility of some suitable pairs of spaces, so-called discrete Schäffer spaces, which are invariant at translations . The obtained result generalize some results published by Coffman, Schäffer, Ben - Artzi, Gohberg, Pinto.

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1 Introduction

The concept of exponential dichotomy of linear differential equations was introduced by O. Perron in 1930 [18], which is concerned with the problem of conditional stability of a system $x' = A(t)x + f(t, x)$ in a finite-dimensional space. After seminal researches of O. Perron, relevant results concerning the extension of Perron's problem in the more general framework of infinite-dimensional Banach spaces were obtained by M. G. Krein, J. L. Daleckij, R. Bellman, J. L. Massera and J. J. Schäffer. In the last three decades a great number of papers about dichotomies and qualitative behavior of evolutionary processes was published. We have different characterization of exponential dichotomy for a strongly continuous, exponentially bounded evolution family in the papers due to N. van Minh [15,16], Y. Latushkin[3,9,10,11], P. Randolph [10,11], P. Preda[14,20], M. Megan[13,14], R. Schnaubelt [11,23], S. Montgomery -Smith[9]. For the case of discrete-time systems analogous results were firstly obtained by Ta Li in 1934 [see 24]. In his paper, we remark the same central concern as in Perron's work, but in another terms. In fact it was proposed that the non-homogeneous equation is responsible in some sense for the asymptotic behaviour of the solutions for the homogeneous equation. In this spirit were established connections between the condition that the non-homogeneous equation has some bounded solution for every bounded "second member" on the one hand and a certain form of conditional stability of the solutions of the homogeneous equation on the other.

This idea was later extensively developed for the discrete-time systems in the infinite-dimensional case by Ch.V. Coffman and J.J. Schäffer in 1967 [4] and D. Henry in 1981[7]. More recently we have the papers due to A. Ben-Artzi[2], I. Gohberg[2], M. Pinto[19], J. P. La Salle[8]. Applications of this "discrete-time theory" to stability theory of linear infinite-dimensional continuous-time systems have been presented by Przyluski and Rolewicz in [21]. The aim of this paper is to give discrete-time criterion for the dichotomy of continuous evolution families. In order to express uniform exponential dichotomy for evolution families, we shall use the admissibility of a pair of Schäffer spaces.This characterization include, as particular cases, many interesting situations among them we note (l^p, l^q)-admissibility, (c_0, c_0)-admissibility and also (l^Φ, l^Φ)-admissibility, where l^Φ is a discrete Orlicz space(for details see the next section bellow). Our methods are different from the methods used frequently where the input space and the output space are the same.

2 Preliminaries

First, let us fix some standard notation. For X a Banach space we will denote by $l^p(X)$ and $l^\infty(X)$ the normed spaces,

$$l^p(X) = \{f : \mathbf{N} \to X : \sum_{n=0}^{\infty} \|f(n)\|^p < \infty\}, \quad p \in [1, \infty),$$

$$l^\infty(X) = \{f : \mathbf{N} \to X : \sup_{n\in\mathbf{N}} \|f(n)\| < \infty\}.$$

We note that $l^p(X), l^\infty(X)$ are Banach spaces endowed with the respectively noms

$$\|f\|_p = (\sum_{n=0}^{\infty} \|f(n)\|^p)^{1/p} \quad ;$$

$$\|f\|_\infty = \sup_{n\in\mathbf{N}} \|f(n)\|.$$

Also, we will put $c(X)$ and $c_0(X)$ for the space of all convergent sequences and respectively for the space of all sequences that tends toward 0. These are closed subspaces of $l^\infty(X)$. For the simplicity of notations we denote by $l^p = l^p(\mathbf{R}), l^\infty = l^\infty(\mathbf{R}), c = c(\mathbf{R}), c_0 = c_0(\mathbf{R})$. At last we consider the linear transformation, $T : \mathbf{R}^\mathbf{N} \to \mathbf{R}^\mathbf{N}$ defined by

$$(Tf)(n) = \begin{cases} 0 & , \quad n = 0 \\ f(n-1) & , \quad n \geq 1. \end{cases}$$

Definition 2.1. *A Banach space E is said to be a discrete Schäffer space if the following conditions are satisfied:*

$s_1)\chi_{\{0\}} \in E$ *where χ_A is the characteristic function of $A \subset \mathbf{N}$;*

$s_2)$ *If $f \in \mathbf{R}^\mathbf{N}, g \in E$ and $|f| \leq |g|$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$;*

s_3) *$f \in E$ if and only if $Tf \in E$ and $\|Tf\|_E = \|f\|_E$, for all $f \in E$.*

Example 2.1. It is easy to check that l^p, l^∞, c_0 are discrete Schäffer spaces and that c is not.

Another remarkable example of discrete Schäffer spaces are the discrete Orlicz spaces. For more convenience we will recall the definition of a discrete Orlicz space. Let $\varphi : I\!R_+ \to I\!R_+$ be a function which is non-decreasing, left-continuous and not identically 0 or ∞ on $(0, \infty)$. Define

$$\Phi(t) = \int_0^t \varphi(s)ds$$

A function Φ of this form is called a Young function. For $f : I\!N \to I\!R$ a real sequence and Φ a discrete Young function we define

$$m^\Phi(f) = \sum_{k=0}^{\infty} \Phi(|f(k)|).$$

The set l^Φ of all f for which there exists a $j > 0$ that $m^\Phi(jf) < \infty$ is easily checked to be a linear space. With the norm

$$\rho^\Phi(f) = \inf\{j > 0 : m^\Phi(\frac{1}{j}f) \le 1\}$$

the space (l^Φ, ρ^Φ) becomes a Banach space which is easy to see that verify the conditions $s_1), s_2), s_3)$.

If E is a discrete Schäffer space we denote by

$$E(X) = \{f : \mathbf{N} \to X : (\|f(n)\|)_{n\in\mathbf{N}} \quad \text{is in} \quad E\}$$

Remark 2.1. $E(X)$ is a Banach space endowed with the norm

$$\|f\|_{E(X)} = \| \, \|f(\cdot)\| \, \|_E$$

Remark 2.2. For any discrete Schäffer space E we have the properties
i) $l^1 \subset E \subset l^\infty$

ii) $\|f\|_\infty \le \frac{1}{\|\chi_{\{0\}}\|_E}\|f\|_E \quad , \quad \text{for all} \quad f \in E$

iii) $\|f\|_E \le \|\chi_{\{0\}}\|_E \|f\|_1 \quad , \quad \text{for all} \quad f \in l^1$.
For the proof of this fact see for instance [17] or [25].

For a discrete Schäffer space E, we denote by $\alpha_E, \beta_E : \mathbf{N} \to \mathbf{R}_+$, the following applications

$$\alpha_E(n) = \inf\{\alpha > 0 : \sum_{k=0}^{n} |f(k)| \le \alpha\|f\|_E \quad , \quad \text{for all} \quad f \in E\}$$

$$\beta_E(n) = \|\chi_{\{0,\dots,n\}}\|_E$$

It is known that α_E, β_E are nondecreasing sequences and moreover

$$\sum_{k=m}^{m+n} |f(k)| \leq \alpha_E(n)\|f\|_E \quad , \quad \text{for all} \quad f \in E \quad \text{and all} \quad m, n \in \mathbf{N}$$

Proposition 2.1. *If E is a discrete Schäffer space then*

$$n + 1 \leq \alpha_E(n)\beta_E(n) \leq 2n + 1, \quad \textit{for all} \quad n \in \mathbf{N}.$$

Proof. If we put $f = \chi_{\{0,\dots,n\}}$ in the inequality

$$\sum_{k=0}^{n} |f(k)| \leq \alpha_E(n)\|f\|_E,$$

we obtain

$$n + 1 \leq \alpha_E(n)\beta_E(n), \quad \text{for all} \quad n \in \mathbf{N}.$$

Let us define $V : E \to E, (Vf)(n) = f(n+1)$. By s_3) it results that V is well defined. Obviously V is linear, and by using the fact that

$$(TVf)(n) = \begin{cases} 0, & n = 0 \\ f(n), & n \in \mathbf{N}^* \end{cases}$$

and s_2), we have

$$||Vf||_E = ||TVf||_E \leq ||f||_E, \quad \text{for all} \quad f \in E.$$

Having in mind that

$$\sum_{k=0}^{n} |f(k)|\chi_{\{0,\dots n\}} = \sum_{k=0}^{n} V^k\Big(|f|\chi_{\{0,\dots n\}}\Big) + \sum_{j=1}^{n} T^j\Big(|f|\chi_{\{0,\dots n-j\}}\Big)$$

and by applying again the property s_2 it follows that

$$\sum_{k=0}^{n} |f(k)|\chi_{\{0,\dots n\}} \leq (2n+1)||f||_E,$$

for all $n \in \mathbf{N}^*$ and every $f \in E$ and hence

$$n + 1 \leq \alpha_E(n)\beta_E(n) \leq 2n + 1, \quad \text{for all} \quad n \in \mathbf{N}^*.$$

If we observe that the above inequality is also true for $n = 0$, the proof is complete.

Example 2.2. By a simple computation we obtain:

$$\alpha_{l^p}(n) = (n+1)^{1-\frac{1}{p}}, \qquad \beta_{l^p}(n) = (n+1)^{\frac{1}{p}} \quad \text{for} \quad p \in [1, \infty)$$

$$\alpha_{l^\infty}(n) = \alpha_{C_0}(n) = n+1, \quad \beta_{l^\infty}(n) = \beta_{C_0}(n) = 1.$$

Definition 2.2. *A family of bounded linear operators acting on X denoted by $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ is called an evolution family if the following properties hold:*

$e_1)$ $U(t,t) = I$ *(where I is the identity operator on X) for all $t \geq 0$;*

$e_2)$ $U(t,s) = U(t,r)U(r,s)$, *for all $t \geq r \geq s \geq 0$;*

$e_3)$ *there exists $M > 0, \omega \geq 0$ such that*

$$\|U(t,s)\| \leq Me^{\omega(t-s)} \quad , \quad \textit{for all} \quad t \geq s \geq 0.$$

Definition 2.3. *An application $P : \mathbf{R}_+ \to B(X)$ is said to be a dichotomy projection family if*

i) $P^2(t) = P(t)$, *for all, $t \geq 0$;*

ii) $P(\cdot)x$ *is bounded for all $x \in X$.*

We also denote by $Q(t) = I - P(t) \quad , \quad t \geq 0$.

Definition 2.4. *An evolution family $\mathcal{U}$ is said to be uniformly exponentially dichotomic (u.e.d.) if there exist P a dichotomy projection family and two constants $N, \nu > 0$ such that the following conditions hold:*

$d_1)$ $U(t,s)P(s) = P(t)U(t,s)$, *for all $t \geq s \geq 0$;*

$d_2)$ $U(t,s) : KerP(s) \to KerP(t)$ *is an isomorphism for all $t \geq s \geq 0$;*

$d_3)$ $\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$, *for all $x \in ImP(s), t \geq s \geq 0$;*

$d_4)$ $\|U(t,s)x\| \geq \frac{1}{N}e^{\nu(t-s)}\|x\|$, *for all $x \in KerP(s), t \geq s \geq 0$.*

In what follows we will consider the evolution families $\mathcal{U}$ for which there exists P, a dichotomy projection family, such that $d_1)$ and $d_2)$ are satisfied.

In this case we will denote by

$$U_1(t,s) = U(t,s)_{|ImP(s)} \quad , \quad U_2(t,s) = U(t,s)_{|KerP(s)}$$

Remark 2.3. The evolution family $\mathcal{U}$ is u.e.d. if and only if there exist the constants $N_1, N_2, \nu_1, \nu_2 > 0$ such that

$$\|U_1(t,s)\| \leq N_1 e^{-\nu_1(t-s)}, \qquad \|U_2^{-1}(t,s)\| \leq N_2 e^{-\nu_2(t-s)} \quad ,$$

for all $t \geq s \geq 0$.

If E, F are two discrete Schäffer spaces we give,

Definition 2.5. *The pair (E, F) is said to be admissible to $\mathcal{U}$ if the following statements hold*

$a_1)$ $\sum\limits_{k=n}^{\infty} \|U_2^{-1}(k,n)Q(k)f(k)\| < \infty$, *for all* $f \in E(X), n \in \mathbf{N}$;

$a_2)$ $x_f : \mathbf{N} \to X, x_f(n) = \sum\limits_{k=0}^{n} U_1(n,k)P(k)f(k) - \sum\limits_{k=n}^{\infty} U_2^{-1}(k,n)Q(k)f(k)$, *lies in* $F(X)$.

Lemma 2.1. *If the pair (E, F) is admissible to $\mathcal{U}$ then there is $K > 0$ such that*

$$\|x_f\|_{F(X)} \leq K\|f\|_{E(X)} \quad , \quad \textit{for all} \quad f \in E(X)$$

Proof. We set now $V_m : E(X) \to l^1(X)$

$$(V_m f)(k) = \begin{cases} U_2^{-1}(k,m)Q(k)f(k) & , \quad k \geq m \\ 0 & , \quad k < m \end{cases}$$

It is obvious that V_m is a linear operator, for all $m \in \mathbf{N}$.

If we consider $m \in \mathbf{N}, \{f_n\}_{n\in\mathbf{N}} \subset E(X), f \in E(X), g \in l^1(X)$ such that

$$f_n \xrightarrow{E(X)} f \quad , \quad V_m f_n \xrightarrow{l^1(X)} g \quad ,$$

then, by Remark 2.2. it results that

$$f_n(k) \to f(k), (V_m f_n)(k) \to g(k) \quad , \quad \text{for all} \quad k \in \mathbf{N} \quad ,$$

and hence $V_m f = g$, which implies that V_m is also bounded for all $m \in \mathbf{N}$.

Let us define the linear operator $W : E(X) \to F(X)$, given by

$$(Wf)(m) = \sum_{k=0}^{m} U_1(m,k)P(k)f(k) - \sum_{k=m}^{\infty} U_2^{-1}(k,m)Q(k)f(k)$$

If $\{g_n\}_{n\in\mathbf{N}} \subset E(X), g \in E(X), h \in F(X)$ such that

$$g_n \xrightarrow{E(X)} g \quad , \quad W g_n \xrightarrow{F(X)} h$$

Then

$$\begin{aligned} &\|(Wg_n)(m) - (Wg)(m)\| \leq \\ \leq \quad &\sum_{k=0}^{m} \|U_1(m,k)P(k)(g_n(k) - g(k))\| + \sum_{k=m}^{\infty} \|U_2^{-1}(k,m)Q(k)(g_n(k) - g(k))\| \\ \leq \quad &(\sum_{k=0}^{m} \|U_1(m,k)P(k)\|) \frac{1}{\|\chi_{\{0\}}\|_E} \|g_n - g\|_{E(X)} + \|V_m(g_n - g)\|_1, \end{aligned}$$

for all $m, n \in \mathbf{N}$.

It follows, using again the Remark 2.2., that $Wg = h$. So we obtain that

$$\|x_f\|_{F(X)} = \|Wf\|_{F(X)} \le \|W\|\|f\|_{E(X)} \quad , \quad \text{for all} \quad f \in E(X) \quad \text{as required.}$$

Lemma 2.2. *Let $g : \{(t,t_0) \in \mathbf{R}^2 : t \ge t_0 \ge 0\} \to \mathbf{R}_+$ be a function such that the following properties hold.*

1) $g(t,t_0) \le g(t,s)g(s,t_0) \quad , \quad$ for all $\quad t \ge s \ge t_0 \ge 0$;

2) $\sup\limits_{0 \le t_0 \le t \le t_0+1} g(t,t_0) < \infty$;

3) there exist $h \in c_0$ and $g(m+n,n) \le h(m) \quad , \quad$ for all $\quad m,n \in \mathbf{N}$.
Then there exist two constants $N, \nu > 0$ such that

$$g(t,t_0) \le Ne^{-\nu(t-t_0)} \quad , \quad \text{for all} \quad t \ge t_0 \ge 0$$

Proof. Let $a = \sup\limits_{0 \le t_0 \le t \le t_0+1} g(t,t_0), m_0 = \min\left\{m \in \mathbf{N}^* : h(m) \le \dfrac{1}{e}\right\}$.

Conditions 1) and 2) imply that $\sup\limits_{0 \le t_0 \le t \le t_0+2m_0} g(t,t_0) \le a^{2m_0}$.

Fix $t_0 \ge 0, t \ge t_0 + 2m_0, m = \left[\frac{t}{m_0}\right], n = \left[\frac{t_0}{m_0}\right]$ where $[s]$ is the largest integer equal or less than $s \in \mathbf{R}$. One can see that $m_0 m \le t < m_0(m+1), m_0 n \le t_0 < m_0(n+1), m \ge n+2$, and so,

$$\begin{aligned} g(t,t_0) &\le g(t,m_0m)g(m_0m,m_0(n+1))g(m_0(n+1),t_0) \le \\ &\le a^{4m_0} \prod_{k=n+2}^{m} g(m_0k, m_0(k-1)) \le a^{4m_0} \prod_{k=n+2}^{m} h(m_0) \\ &\le a^{4m_0} e^{-(m-n-1)} \le a^{4m_0} e^{-\frac{t-t_0}{m_0}+2}. \end{aligned}$$

If we note that

$$g(t,t_0) \le a^{2m_0} \le a^{2m_0} e^2 e^{-\frac{t-t_0}{m_0}} \quad , \quad \text{for all} \quad t_0 \ge 0, t \in [t_0, t_0+2m_0]$$

we obtain easily that

$$g(t,t_0) \le Ne^{-\nu(t-t_0)} \quad , \quad \text{for all} \quad t \ge t_0 \ge 0 \quad , \quad \text{where}$$

$$N = max\{a^{4m_0}e^2, a^{2m_0}e^2\}, \ \nu = \frac{1}{m_0}.$$

3 The Main Result

We start with the following

Lemma 3.1. *The pair (l^1, l^∞) is admissible to $\mathcal{U}$ if and only if there exists $K > 0$ such that*

$$\|U_1(m,n)\| \le K, \quad \|U_2^{-1}(m,n)\| \le K \quad , \quad \textit{for all} \quad (m,n) \in \mathbf{N}^2 \quad \textit{with} \quad m \ge n.$$

Proof. Sufficiency: It is a simple computation.
Necessity: Let $m \in \mathbf{N}, x \in X$, and $f : \mathbf{N} \to X, f = \chi_{\{m\}}x$. It is easy to verify that $f \in l^1(X), \|f\|_1 = \|x\|$ and

$$(x_f)(k) = \sum_{j=0}^{k} U_1(k,j)P(j)f(j) - \sum_{j=k}^{\infty} U_2^{-1}(j,k)Q(j)f(j)$$

$$= \begin{cases} U_1(k,m)P(m)x & , \quad k > m \\ -U_2^{-1}(m,k)Q(m)x & , \quad k < m \end{cases}$$

and so $\|U_1(k,m)P(m)x\| \le \|x_f\|_\infty \le K\|f\|_1 = K\|x\| \quad \text{if} \quad k > m$,

$$\|U_2^{-1}(m,k)Q(m)x\| \le \|x_f\|_\infty \le K\|f\|_1 = K\|x\| \quad \text{if} \quad k < m$$

It is now clear that $\|U_1(m,n)\| \le K$, $\|U_2^{-1}(m,n)\| \le K$ for all $(m,n) \in \mathbf{N}^2$ with $m \ge n$.

Theorem 3.1. *$\mathcal{U}$ is u.e.d. if and only if there exists a pair (E,F) of discrete Schäffer spaces, admissible to $\mathcal{U}$, with $\lim\limits_{n\to\infty} \alpha_E(n)\beta_F(n) = \infty$.*

Proof. Necessity It follows easily from Definition 2.4. that the pair (l^∞, l^∞) is admissible to $\mathcal{U}$.

SufficiencyFirst we observe that if the pair (E,F) is admissible to $\mathcal{U}$, then by Remark 2.2. the pair (l^1, l^∞) is admissible to $\mathcal{U}$ and hence, by Lemma 3.1., there exists $L > 0$ such that

$$\|U_1(m,n)\| \le L, \|U_2^{-1}(m,n)\| \le L \quad , \quad \text{for all} \quad (m,n) \in \mathbf{N}^2 \quad , \quad \text{with} \quad m \ge n.$$

Let $n_0, m \in \mathbf{N}, x \in ImP(n_0), f : \mathbf{N} \to X$, given by

$$f(n) = \begin{cases} U_1(n,n_0)x & , \quad n \in \{n_0, ..., n_0+m\} \\ 0 & , \quad n \notin \{n_0, ..., n_0+m\} \end{cases}$$

Then $f \in E(X), \|f\|_{E(X)} \le L\beta_E(m)\|x\|$ and $f(n) \in ImP(n)$, for all $n \in \mathbf{N}$. It follows that

$$(x_f)(n) = \sum_{k=0}^{n} U_1(n,k)f(k) = \begin{cases} 0 & , \quad n < n_0 \\ (n-n_0+1)U_1(n,n_0)x & , \quad n \in \{n_0, ..., n_0+m\} \\ (m+1)U_1(m,n_0)x & , \quad n \ge n_0+m+1 \end{cases}$$

and so

$$\frac{(m+1)(m+2)}{2}\|U_1(m+n_0,n_0)x\| = \sum_{n=n_0}^{n_0+m}(n-n_0+1)\|U_1(m+n_0,n_0)x\|$$

$$\leq L\sum_{n=n_0}^{n_0+m}(n-n_0+1)\|U_1(n,n_0)x\| = L\sum_{n=n_0}^{n_0+m}\|x_f(n)\| \leq L\alpha_F(m)\|x_f\|_{F(X)}$$

$$\leq KL\alpha_F(m)\|f\|_{E(X)} \leq KL^2\|x\|\alpha_F(m)\beta_E(m) \leq \frac{(2m+1)^2KL^2}{\alpha_E(m)\beta_F(m)}\|x\|.$$

We obtain that

$$\|U_1(m+n_0,m)\| \leq \frac{8KL^2}{\alpha_E(m)\beta_F(m)} \quad , \quad \text{for all} \quad m,n_0 \in \mathbf{N}$$

By Lemma 2.2. it results that there exist two constants $N_1, \nu_1 > 0$ such that

$$\|U_1(t,t_0)\| \leq N_1 e^{-\nu_1(t-t_0)} \quad , \quad \text{for all} \quad t \geq t_0 \geq 0 \quad .$$

Consider again $m, n_0 \in \mathbf{N}, x \in KerP(m+n_0), g : \mathbf{N} \to X$, given by

$$g(n) = \begin{cases} U_2^{-1}(m+n_0,n)x & , \quad n \in \{n_0,...,n_0+m\} \\ 0 & , \quad n \notin \{n_0,...,n_0+m\} \end{cases}$$

Then $g \in E(X), \|g\|_{E(X)} \leq L\beta_E(m)\|x\|$ and $g(n) \in KerP(n)$, for all $n \in \mathbf{N}$. A simple computation shows that

$$\begin{aligned}(x_g)(n) &= -\sum_{k=n}^{n_0+m} U_2^{-1}(k,n)U_2^{-1}(m+n_0,k)x = -\sum_{k=n}^{n_0+m} U_2^{-1}(n_0+m,n)x \\ &= -(n_0+m-n+1)U_2^{-1}(n_0+m,n)x \quad , \quad \text{for all} \quad n \in \{n_0,...,n_0+m\}\end{aligned}$$

and hence

$$\frac{(m+1)(m+2)}{2}\|U_2^{-1}(n_0+m,n_0)x\| = \sum_{n=n_0}^{n_0+m}(n_0+m-n+1)\|U_2^{-1}(n_0+m,n_0)x\|$$

$$\leq L\sum_{n=n_0}^{n_0+m}(n_0+m-n+1)\|U_2^{-1}(n_0+m,n)x\| = L\sum_{n=n_0}^{n_0+m}\|x_g(n)\| \leq L\alpha_F(m)\|x_g\|_{F(X)}$$

$$\leq LK\alpha_F(m)\|g\|_{E(X)} \leq KL^2\alpha_F(m)\beta_E(m)\|x\| \leq \frac{(2m+1)KL^2}{\alpha_E(m)\beta_F(m)}\|x\|.$$

We can state that

$$\|U_2^{-1}(n_0+m,n_0)\| \leq \frac{8KL^2}{\alpha_E(m)\beta_F(m)} \quad , \quad \text{for all} \quad m,n_0 \in \mathbf{N}$$

In order to apply Lemma 2.2. again, we observe that

$$U_2^{-1}(t,t_0) = U_2(t_0,[t_0])U_2^{-1}([t_0]+2,[t_0])U_2([t_0]+2,t) \quad ,$$

for all $0 \leq t_0 \leq t \leq t_0+1$ which implies that $\sup\limits_{0\leq t_0\leq t\leq t_0+1} \|U_2^{-1}(t,t_0)\| \leq M^2 e^{3\omega} L$. Hence there exists two constants $N_2, \nu_2 > 0$ such that

$$\|U_2^{-1}(t,t_0)\| \leq N_2 e^{-\nu_2(t-t_0)} \quad , \quad \text{for all} \quad t \geq t_0 \geq 0$$

By Remark 2.3., it follows that $\mathcal{U}$ is u.e.d.

Applying this result to various pairs of Schäffer spaces we obtain some characterizations for the uniform exponential dichotomy in terms of their admissibility.

Theorem 3.2. *The following assertions are equivalent:*

1) $\mathcal{U}$ is u.e.d.;

2) there exist E a Schäffer space such that the pair (E,E) is admissible to $\mathcal{U}$;

3) there exists $p,q \in [1,\infty], (p,q) \neq (1,\infty)$ such that the pair (l^p, l^q) is admissible to $\mathcal{U}$;

4) there exists E a Schäffer space such that the pair (c_0, E) is admissible to $\mathcal{U}$.

Proof. Follows easily from Theorem 3.1. and Example 2.2

Remark 2.4 From the statement (2) of the Theorem 3.2 and Example 2.1 it follows also that $\mathcal{U}$ is u.e.d. if and only if (l^Φ, l^Φ) is admissible to $\mathcal{U}$, where l^Φ is a discrete Orlicz space.

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5 References

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