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Dichotomies for Linear Difference Equations

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1. Introduction

The analogy between differential equations and difference equations has been recognized and exploited very frequently. Thus, shortly after the publication of PERRON's paper on stability [5], there appeared a corresponding paper by TA LI [3], a student of PERRON's, with an almost identical title, in which similar methods were used to obtain analogous results for difference equations. In both papers a central concern is the relationship, for linear equations, between the condition that the non-homogeneous equation have some bounded solution for every bounded "second member" on the one hand, and a certain form of behaviour of the solutions of the homogeneous equation on the other. PERRON's results for linear differential equations are the starting point of extensive later developments; some recent ones are found in a series of papers by MASSERA and SCHÄFFER, later continued by SCHÄFFER, and brought together in a monograph [4] (see also [1], Chs. 12 and 13). The purpose of the present paper is to provide for linear difference equations the analogues of the most central results for linear differential equations in that theory.

We consider the difference equation (in vector form)

$$x(n) + A(n)\,x(n-1) = f(n) \qquad n = 1, 2, \ldots,$$

and the corresponding homogeneous equation, for a given operator-valued function A. If the values of A were assumed to be invertible, we should obtain a theory that is a pallid and essentially trivialized copy of the theory for differential equations in [4]; our main concern is, therefore, to develop our analysis without any assumption on the invertibility of the "transition operators" $A(n)$. This means that solutions cannot always be continued to the left, or continued uniquely if they can be continued at all; in more precise language, it becomes necessary to consider also solutions of the equation restricted to a set of the form $\{m, m+1, \ldots\}$, $m > 0$. The theory is thus essentially asymmetric. The fact that we deal with functions and solutions with values in a general Banach space rather than the more usual finite-dimensional one is, on the other hand, the source of only minor technical complications and we believe it well worth the effort.

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The selection of topics and the development of the theory is very much conditioned by our knowledge and experience of differential equations, but we have made the paper self-contained in its mathematical content for the convenience of the reader. This may help explain the somewhat extensive Sections 2 and 3, concerning sequences and sequence spaces, which are closely related to the theory of function spaces developed in [6]. In Section 4 we establish our terminology for difference equations and their solutions. Section 5 introduces the concept of a covariant sequence of linear manifolds, use of which replaces the technique of following a linear manifold along the solutions of a differential equation; this device allows us to overcome the difficulty inherent in the non-invertibility of the transition operators. Section 6 refers to solutions belonging to certain sequence spaces, and Section 7 introduces the interesting types of behaviour of the solutions of the homogeneous equation, the so-called "dichotomies"; they are a kind of uniform conditional (simple or exponential) stability conditions. Section 8 is devoted to the relationship, for the non-homogeneous equation, between "test functions" f in a certain sequence space and solutions in another given sequence space. Sections 9 and 10 contain the main theorems connecting the aspects introduced in the previous sections. Section 11 is in the nature of an appendix.

The results, in spite of the complications alluded to above, turn out to be similar to those for differential equations, and sometimes they are actually neater, as may be expected, due to the absence of problems involving the local behaviour of functions (see, e.g., [4], Section 62). There are many possible additional topics we have not touched upon, as, e.g., equations on the set of all integers, or dependence on A of the properties of the solutions. It might turn out to be interesting to consider also difference equations with a real variable and a continuity assumption on the solutions; they may help approach the much more essential topic of functional-differential equations.

We conclude this introduction with a few technical comments.

If E is a locally convex linear topological space and Y is a normed space that is algebraically contained as a linear manifold in E, Y is said to be *stronger than* E if the norm topology of Y is stronger than the topology induced on Y by that of E; that is, if the (solid) unit sphere $\Sigma(Y)$ of Y is E-bounded. In particular, if E is also a normed space, Y is stronger than E — E is said to be *weaker than* Y — if and only if there exists a number $\varrho > 0$ such that $\|y\|_E \leqq \varrho \|y\|_Y$ for all $y \in Y$. If Y is both stronger and weaker than the normed space E, it follows that they coincide as linear topological spaces and their norms are equivalent; they are *norm-equivalent*. A useful device for verifying the completeness of a normed space is the following lemma, of which we include a proof in order to make the exposition self-contained.

1.1. Lemma. *Assume that the normed space Y is stronger than the Fréchet space E (i.e., a metrizable complete locally convex space). Then Y is complete if and only if the E-limit of every Y-Cauchy sequence in $\Sigma(Y)$ belongs to $\Sigma(Y)$.*

Proof. (Cf. [4], 21.B. Instead of assuming that E is metrizable and complete, it is enough to assume that it is a complete, or even only sequentially complete, Hausdorff space.) Every Y-Cauchy sequence is an E-Cauchy sequence and has an E-limit; if a Y-limit exists, it coincides with the E-limit. The "only if" part is then trivial. Let (y_n) be any Y-Cauchy sequence, u its E-limit. For given $\varepsilon > 0$ there exists m such that $\|y_n - y_m\|_Y \leqq \varepsilon$ for $n \geqq m$. The Y-Cauchy sequence $(y_n - y_m)$ in $\varepsilon\Sigma(Y)$ has its E-limit $u - y_m$ in $\varepsilon\Sigma(Y)$ by assumption; i.e., $u \in Y$ and $\|u - y_m\|_Y \leqq \varepsilon$. It follows that (y_n) has u as its Y-limit. Therefore Y is complete.

Some comments on the geometry of a Banach space X over the field F of real or complex numbers, with norm $\|\cdot\|$, follow. (The field of reals is called R.) $d(\cdot,\cdot)$ denotes distance between points and sets (and between sets and sets). A *subspace* is a closed linear manifold. $\tilde{X}$ denotes the Banach algebra of operators on X. In order to measure the degree of "angular apartness" of an element z of X from a subspace Y, we have found it most convenient to use a condition of the form $\|z\| \leqq \lambda d(Y, z)$, where $\lambda > 1$; the cases $z = 0$, $Y = \{0\}$ are thus not excluded. If X is a Hilbert space and $Y \neq \{0\}$, $z \neq 0$, the condition expresses the fact the sine of the angle z forms with Y is $\geqq \lambda^{-1}$.

In order to replace the (bounded) projection along a given subspace onto a complementary subspace, which need not always exist, we use a non-linear analogue. For a given subspace Y and a number $\lambda > 1$, a mapping $q: X \to X$ is a (Y, λ)-*splitting* of X if it satisfies:

(1): $x - q(x) \in Y$ for all $x \in X$;
(2): if $x_1 - x_2 \in Y$, then $q(x_1) = q(x_2)$;
(3): $\|q(x)\| \leqq \lambda \|x\|$ for all $x \in X$.

The following lemma summarizes the information we require concerning splittings. Functional composition is denoted by $\circ$. The proof is trivial and is omitted.

1.2. Lemma. *For given Y and $\lambda > 1$, there always exists at least one (Y, λ)-splitting. A mapping $q: X \to X$ is a (Y, λ)-splitting if and only if $q = \bar{q} \circ \Omega$, where $\Omega: X \to X/Y$ is the canonical epimorphism onto the quotient space, and the mapping $\bar{q}: X/Y \to X$, uniquely determined by q, satisfies $\Omega \circ \bar{q} = I$ (identity on X/Y) and $\|\bar{q}(\xi)\| \leqq \lambda \|\xi\|$ for all $\xi \in X/Y$; we have $q \circ \bar{q} = \bar{q}$, whence $q \circ q = q$. A (Y, λ)-splitting is linear if and only if it is a projection P with null-space Y and $\|P\| \leqq \lambda$.*

Remark. For given Y and $\lambda > 1$, there always exists a *continuous* (Y, λ)-splitting (see [4], 11.E).

2. Sequences

We denote by ω the set $\{0, 1, 2, \ldots\}$ of non-negative integers. If $m \in \omega$, we set $\omega_{[m]} = \{n \in \omega : n \geqq m\} = \{m, m+1, \ldots\}$; thus $\omega_{[0]} = \omega$. We shall arrange the notation in such a way that the subscript $[m]$ will always denote the restriction or extension of some object (sequence, sequence space) to $\omega_{[m]}$.

We are mainly concerned with functions defined on ω or some $\omega_{[m]}$ with values in some Banach space X (i.e., sequences in X, although we usually avoid this term in order to prevent confusion). Such functions will usually be denoted by letters of the same type as those used for the elements of their range space, e.g., f, g for a general Banach space X, but φ, ψ for the scalar field F. If $\lambda \in F$ and φ, f, g, U, V are functions on $\omega_{[m]}$, say, with values in $F, X, X, \tilde{X}, \tilde{X}$, respectively, the functions $\operatorname{Re}\varphi$, $\bar{\varphi}$, $|\varphi|$, $\|f\|$, $\operatorname{sgn} f$, λf, φf, $f+g$, Uf, VU are defined by the corresponding pointwise operations on their values. Obviously, $\varphi = |\varphi| \operatorname{sgn}\varphi$, $f = \|f\| \operatorname{sgn} f$. The *support* of a function $f: \omega_{[m]} \to X$ is the set $\{n \in \omega_{[m]} : f(n) \neq 0\}$. If the support of f is finite, its greatest element is denoted by $s(f)$.

For any function $f: \omega_{[m]} \to X$ and any $p \in \omega$ we define $f_{[p]}: \omega_{[p]} \to X$ by $f_{[p]}(n) = f(n)$ if $n \geqq m$, $f_{[p]}(n) = 0$ otherwise, for all $n \in \omega_{[p]}$; thus, if $p \geqq m$, $f_{[p]}$ is the restriction of f to $\omega_{[p]}$. A constant function defined on ω is in general denoted by the same symbol as its value; correspondingly, the function defined on $\omega_{[m]}$ with the constant value u, say, is denoted by $u_{[m]}$.

Special real-valued functions on ω are the "basis elements" χ^k, $k = 0, 1, \ldots$, defined by $\chi^k(n) = \delta^k_n$ (Kronecker symbol).

We further require a *truncation operator* for functions; it is notationally convenient to define it for all integral indices. If k is any integer and $f: \omega_{[m]} \to X$, say, is given, then $\Theta_k f: \omega_{[m]} \to X$ is defined by $\Theta_k f(n) = f(n)$ if $n \geqq k$, $\Theta_k f(n) = 0$ otherwise, for all $n \in \omega_{[m]}$. (Strictly speaking, the domain should be specified, but no confusion will result from its omission.) We remark that

$$\text{(2.1)} \qquad \text{for } f: \omega_{[m]} \to X, \quad \Theta_k f = f \text{ if } k \leqq m, \quad \Theta_k f = f_{[k][m]} \text{ if } k \geqq 0,$$

which could serve as an alternative definition of Θ_k.

Finally, we introduce *translation operators* for functions defined on ω. If $f: \omega \to X$, then $T^+ f: \omega \to X$ is defined by $T^+ f(0) = 0$, $T^+ f(n) = f(n-1)$ for $n \geqq 1$; and $T^- f: \omega \to X$ is defined by $T^- f(n) = f(n+1)$ for $n \in \omega$. Let k be any integer; we define T^k to be the identity, $(T^+)^k$, or $(T^-)^{-k}$, according as $k = 0$, $k > 0$, or $k < 0$, respectively. A straightforward verification yields

$$\text{(2.2)} \qquad T^{k'} T^{k''} = \Theta_{k'} T^{k'+k''} = T^{k'+k''} \Theta_{-k''}$$

for all integers k', k'' (the truncation operators are redundant in (2.2) unless $k'' < 0 < k'$).

Spaces of functions on ω or $\omega_{[m]}$ will be denoted by bold-face letters; if such a space is normed, thick bars ($|\cdot|$) with an appropriate subscript will be used for the norm.

For a given Banach space X and $m \in \omega$, $\mathbf{s}_{[m]}(X)$ denotes the vector space of all functions $f: \omega_{[m]} \to X$, provided with the topology of pointwise convergence. Since $\mathbf{s}_{[m]}(X)$ is the cartesian product $\bigtimes_{n \geqq m} X$ provided with the co-ordinatewise vector-space structure and the product topology, it is a Fréchet space (i.e., locally convex, metrizable, complete), and a defining set of seminorms is given by $\{f \to \|f(n)\| : n \geqq m\}$. In particular, we set $\mathbf{s}_{[0]}(X) = \mathbf{s}(X)$, $\mathbf{s}_{[m]}(R) = \mathbf{s}_{[m]}$, $\mathbf{s}(R) = \mathbf{s}_{[0]} = \mathbf{s}$. The mapping $f \to f_{[m]}: \mathbf{s}(X) \to \mathbf{s}_{[m]}(X)$ is linear, continuous, and surjective; the mapping $f \to \|f\|: \mathbf{s}_{[m]}(X) \to \mathbf{s}_{[m]}$ is continuous.

Each $\mathbf{s}_{[m]}$ (a space of real-valued functions) is, in addition, a conditionally complete vector lattice under the ordering $\varphi \leqq \psi$, defined pointwise (i.e., $\varphi(n) \leqq \psi(n)$ for all $n \in \omega_{[m]}$), the lattice operations being the pointwise supremum and infimum, denoted by sup, inf. $\varphi \in s_{[m]}$ is *positive* if $\varphi \geqq 0_{[m]}$; a sequence (φ_j) in $s_{[m]}$ is *increasing* if $\varphi_{j+1} - \varphi_j \geqq 0_{[m]}$ for all j. If $\varphi_1, \varphi_2 \in s_{[m]}$, $\varphi_1 \leqq \varphi_2$, then $[\varphi_1, \varphi_2] = \{\psi \in s_{[m]} : \varphi_1 \leqq \psi \leqq \varphi_2\}$ is an *order-interval*. The surjective continuous linear mapping $\varphi \to \varphi_{[m]} : s \to s_{[m]}$ is positive (i.e., positivity-preserving or, equivalently, order-preserving).

3. Sequence spaces

We next consider certain normed spaces of functions on ω and $\omega_{[m]}$. We begin with spaces of real-valued functions on ω.

The class $\not{f}$ consists of all normed spaces $\boldsymbol{f}$, with norm $|\cdot|_{\boldsymbol{f}}$, of real-valued functions on ω (any such space is, algebraically, a linear manifold in $\boldsymbol{s}$) satisfying the following condition:

(f): if $\varphi \in \boldsymbol{f}$ and $\psi \in \boldsymbol{s}$ with $|\psi| \leqq |\varphi|$, then $\psi \in \boldsymbol{f}$ and $|\psi|_{\boldsymbol{f}} \leqq |\varphi|_{\boldsymbol{f}}$; equivalently, if $\varphi \in \Sigma(\boldsymbol{f})$, then $[-|\varphi|, |\varphi|] \subset \Sigma(\boldsymbol{f})$.

3.1. Lemma. *Assume that $\boldsymbol{f} \in \not{f}$. For each $m \in \omega$, either $\varphi(m) = 0$ for all $\varphi \in \boldsymbol{f}$, or $\chi^m \in \boldsymbol{f}$ and $|\varphi(m)| \leqq |\chi^m|_{\boldsymbol{f}}^{-1} |\varphi|_{\boldsymbol{f}}$ for all $\varphi \in \boldsymbol{f}$. Therefore $\boldsymbol{f}$ is stronger than $\boldsymbol{s}$.*

Proof. For fixed $m \in \omega$ and every $\varphi \in \boldsymbol{f}$, we have $|\varphi(m)| \chi^m \leqq |\varphi|$; by (f), $|\varphi(m)| \chi^m \in \boldsymbol{f}$. If $\varphi(m) \neq 0$, we conclude that $\chi^m \in \boldsymbol{f}$, $|\varphi(m)|\, |\chi^m|_{\boldsymbol{f}} \leqq |\varphi|_{\boldsymbol{f}}$. The final statement now follows, since $\sup\{|\varphi(m)| : \varphi \in \Sigma(\boldsymbol{f})\} = 0$ or $|\chi^m|_{\boldsymbol{f}}^{-1}$ for each m.

We set $\operatorname{supp}(\boldsymbol{f}) = \{m \in \omega : \chi^m \in \boldsymbol{f}\}$; every $\varphi \in \boldsymbol{f}$ vanishes outside $\operatorname{supp}(\boldsymbol{f})$. We further set $\not{f}_K = \{\boldsymbol{f} \in \not{f} : \operatorname{supp}(\boldsymbol{f}) = \omega\}$.

We denote by $\mathrm{b}\not{f}$ the subclass of $\not{f}$ consisting of complete spaces, i.e., Banach spaces; and we set $\mathrm{b}\not{f}_K = \mathrm{b}\not{f} \cap \not{f}_K$.

3.2. Lemma. *If $\boldsymbol{f} \in \mathrm{b}\not{f}$, $\boldsymbol{g} \in \not{f}$, then $\boldsymbol{f}$ is stronger than $\boldsymbol{g}$ if (and, of course, only if) $\boldsymbol{f}$ is algebraically contained in $\boldsymbol{g}$.*

Proof. Assume that $\boldsymbol{f}$ is algebraically contained in $\boldsymbol{g}$. If $\boldsymbol{f}$ were not stronger than $\boldsymbol{g}$, there would exist a sequence (φ_n) in $\boldsymbol{f}$ (hence in $\boldsymbol{g}$) such that $\sum_1^\infty |\varphi_n|_{\boldsymbol{f}} < \infty$ but $|\varphi_n|_{\boldsymbol{g}} \geqq n$, $n = 1, 2, \ldots$. The sequence $\left(\sum_1^n |\varphi_j|\right)$ would be an $\boldsymbol{f}$-Cauchy sequence, hence converge to, say, $\varphi \in \boldsymbol{f}$; since the sequence is increasing and the $\boldsymbol{f}$-limit is the $\boldsymbol{s}$-limit (by Lemma 3.1), $0 \leqq |\varphi_n| \leqq \varphi$ for all n; but then, in $\boldsymbol{g}$, $n \leqq |\varphi_n|_{\boldsymbol{g}} \leqq |\varphi|_{\boldsymbol{g}} < \infty$ for all n, a contradiction. (We observe that $\boldsymbol{g}$ need not be complete and that the Open-Mapping Theorem is not used.)

If $\boldsymbol{f} \in \not{f}$, we define $\mathrm{k}_0\boldsymbol{f}$ as the linear manifold of those $\varphi \in \boldsymbol{f}$ that have finite support, provided with the norm of $\boldsymbol{f}$; and $\mathrm{k}\boldsymbol{f}$ as the closure in $\boldsymbol{f}$ of $\mathrm{k}_0\boldsymbol{f}$, provided with the norm of $\boldsymbol{f}$. It is clear that $\varphi \in \mathrm{k}\boldsymbol{f}$ if and only if $\varphi \in \boldsymbol{f}$ and $\lim_{m\to\infty} |\Theta_m \varphi|_{\boldsymbol{f}} = 0$. A space $\boldsymbol{f} \in \not{f}$ such that $\mathrm{k}\boldsymbol{f} = \boldsymbol{f}$ is called *lean*. Finally, for given $m \in \omega$, $\Theta_m \boldsymbol{f}$ is the linear manifold $\{\varphi \in \boldsymbol{f} : \varphi(n) = 0, n = 0, \ldots, m-1\} = \{\Theta_m \varphi : \varphi \in \boldsymbol{f}\}$ — a subspace of $\boldsymbol{f}$ by (f) and Lemma 3.1 — provided with the norm of $\boldsymbol{f}$.

3.3. Lemma. *If $\boldsymbol{f} \in \not{f}$, then $\mathrm{k}_0\boldsymbol{f}$, $\mathrm{k}\boldsymbol{f}$, $\Theta_m \boldsymbol{f} \in \not{f}$; if $\boldsymbol{f} \in \not{f}_K$, then $\mathrm{k}_0\boldsymbol{f}$, $\mathrm{k}\boldsymbol{f} \in \not{f}_K$; if $\boldsymbol{f} \in \mathrm{b}\not{f}$, then $\mathrm{k}\boldsymbol{f}$, $\Theta_m \boldsymbol{f} \in \mathrm{b}\not{f}$.*

Proof. Trivial.

We require one more operation to make new $\not{f}$-spaces out of old. For given $\boldsymbol{f} \in \not{f}$, its *local closure* $\mathrm{lc}\boldsymbol{f}$ is the space of real-valued functions on ω defined as follows: for every $\varphi \in s$, $\varphi \in \mathrm{lc}f$ if and only if $\varphi_m = \varphi \sum_0^m \chi^j = \sum_0^m \varphi(j)\chi^j \in f$ for all $m \in \omega$ and $\sup_{m\in\omega} |\varphi_m|_f < \infty$, with $|\varphi|_{\mathrm{lc}f}$ equal to this supremum (actually, since $(|\varphi_m|)$ is an increasing sequence, (f) implies that $(|\varphi_m|_f)$ is increasing, and the supremum is in fact $\lim_{m\to\infty} |\varphi_m|_f$). It is obvious that we have indeed defined a normed space, and that $\mathrm{lc}f \in \not{f}$. The terminology is justified by the first part of the following lemma.

3.4. Lemma. *If* $f \in \not{f}$, *then* $\mathrm{supp}(\mathrm{lc}f) = \mathrm{supp}(f)$, *and* $\Sigma(\mathrm{lc}f)$ *is the s-closure of* $\Sigma(f)$; $\mathrm{lc}f \in \mathrm{b}\not{f}$; $\mathrm{lck}_0 f = \mathrm{lck}f = \mathrm{lc}f$; $\mathrm{k}_0\,\mathrm{lc}f = \mathrm{k}_0 f$.

Proof. The equality $\mathrm{supp}(\mathrm{lc}f) = \mathrm{supp}(f)$ is immediate from the definition. Assume that $\varphi \in \Sigma(\mathrm{lc}f)$; if φ_m is defined as in the preceding paragraph, then $\varphi_m \in \Sigma(f)$ for all $m \in \omega$; but obviously $\varphi_m \to \varphi$ in s, so that $\varphi \in \mathrm{cl}_s \Sigma(f)$, the s-closure of $\Sigma(f)$. Thus $\Sigma(\mathrm{lc}f) \subset \mathrm{cl}_s \Sigma(f)$. Assume conversely that $\varphi \in \mathrm{cl}_s \Sigma(f)$, and let m be fixed. Since $\psi(n) = 0$ for all $n \notin \mathrm{supp}(f)$ and $\psi \in \Sigma(f)$, we must have $\varphi(n) = 0$ for all $n \notin \mathrm{supp}(f)$. Therefore $\varphi_m = \sum_0^{m}{}' \varphi(j)\chi^j \in f$, where the prime indicates a summation restricted to $\mathrm{supp}(f)$. For every $\varepsilon > 0$ there exists $\psi_\varepsilon \in \Sigma(f)$ such that $|\varphi(n) - \psi_\varepsilon(n)| \leqq \varepsilon$ for all $n \in \mathrm{supp}(f)$, $0 \leqq n \leqq m$. By (f) we may assume without loss that $\psi_\varepsilon(n) = 0$ for $n > m$. Then $|\varphi_m - \psi_\varepsilon| = \sum_0^{m}{}' |\varphi(j) - \psi_\varepsilon(j)|\,\chi^j \leqq \varepsilon \sum_0^{m}{}' \chi^j$, whence $|\varphi_m|_f \leqq |\psi_\varepsilon|_f + |\varphi_m - \psi_\varepsilon|_f \leqq 1 + \varepsilon \left|\sum_0^{m}{}' \chi^j\right|_f$. Since $\varepsilon > 0$ was arbitrary, $\varphi_m \in \Sigma(f)$; since this holds for all m, $\varphi \in \Sigma(\mathrm{lc}f)$. Thus $\mathrm{cl}_s \Sigma(f) \subset \Sigma(\mathrm{lc}f)$, and equality holds.

The fact that $\mathrm{lc}f$ is complete now follows from Lemma 1.1, since $\Sigma(\mathrm{lc}f)$ is closed and bounded in the Fréchet space s.

The final statement of the lemma is again immediate from the definition.

A space $f \in \not{f}$ such that $\mathrm{lc}f = f$, i.e., such that $\Sigma(f)$ is s-closed, is called *locally closed*. For every $f \in \not{f}$, $\mathrm{lc}f$ is locally closed.

We now consider subclasses of $\not{f}$ consisting of spaces with certain properties of translation invariance. Thus $\not{t}$ is the class of all $f \in \not{f}$ that satisfy in addition the following conditions:

(z): $f \neq \{0\}$;
(t): if $\varphi \in f$, then $T^-\varphi$, $T^+\varphi \in f$ and $|T^-\varphi|_f, |T^+\varphi|_f \leqq |\varphi|_f$.

Obviously, if $f \in \not{t}$ and $\varphi \in f$, then $T^k\varphi \in f$, $|T^k\varphi|_f \leqq |\varphi|_f$ for all integers k. On account of (2.2), $T^+T^- = \Theta_1$, $T^-T^+ =$ identity; therefore (t) is equivalent, in the presence of (f), to the assumption that $\varphi \in f$ implies $T^-\varphi \in f$, together with

($\mathrm{t}^\to$): if $\varphi \in f$, then $T^+\varphi \in f$ and $|T^+\varphi|_f = |\varphi|_f$.

We also consider the class $\not{t}^\to$ of all $f \in \not{f}$ that satisfy (z) and ($\mathrm{t}^\to$); thus $\not{t} \subset \not{t}^\to \subset \not{f}$. The respective subclasses of complete spaces are denoted by $\mathrm{b}\not{t}$ and $\mathrm{b}\not{t}^\to$.

3.5. Lemma. *If* $f \in \not{t}$, *then* $\mathrm{supp}(f) = \omega$, *i.e.*, $\not{t} \subset \not{f}_K$. *If* $f \in \not{t}^\to$, *then* $\mathrm{supp}(f) = \omega_{[s_0]}$, *where* $s_0 = s_0(f) = \inf \mathrm{supp}(f)$.

Proof. Trivial from (t), ($\mathrm{t}^\to$), since $\mathrm{supp}(f) \neq \emptyset$ by (z).

If $f \in \ell^{\rightarrow}$ and p is a positive integer, $\sum_{m}^{m+p-1} \chi^j \in f$ for all $m \geqq s_0(f)$, and by ($\mathrm{t}^{\rightarrow}$) we may define the function $\beta(f;p) = \left|\sum_{s_0}^{s_0+p-1} \chi^j\right|_f = \left|\sum_{m}^{m+p-1} \chi^j\right|_f$ for all $m \geqq s_0$, where $s_0 = s_0(f)$; $\beta(f;p)$ is non-decreasing with p. In particular, we set $\beta(f) = \beta(f;1) = |\chi^m|_f$ for all $m \geqq s_0$. If $f \in \ell$, all this holds of course with $s_0 = 0$.

3.6. Lemma. *If $f \in \ell^{\rightarrow}$, then $\mathrm{k}_0 f$, $\mathrm{k}f$, $\Theta_m f$, $\mathrm{lc}f \in \ell^{\rightarrow}$; if $f \in \ell$, then $\mathrm{k}_0 f$, $\mathrm{k}f$, $\mathrm{lc}f \in \ell$.*

Proof. Immediate from the definitions.

Important particular instances of spaces in ℓ are the spaces l^p, $1 \leqq p \leqq \infty$. We abbreviate $|\cdot|_{l^p}$ to $|\cdot|_p$, $1 \leqq p < \infty$, and $|\cdot|_{l^\infty}$ to $|\cdot|$. All these spaces are obviously locally closed, and all except l^∞ (sometimes called $\boldsymbol{m}$ in the literature) are lean; we write $l_0^\infty = \mathrm{k}\, l^\infty$, the space (often called $\boldsymbol{c}_0$) of functions φ with $\lim_{n\to\infty} \varphi(n) = 0$, with the supremum norm. The spaces l^1, l^∞, l_0^∞ occupy particularly important positions in the structure of the classes ℓ, $\ell^{\rightarrow}$, as we now show.

3.7. Lemma. *If $f \in \ell$ or $\ell^{\rightarrow}$, f is stronger than l^∞; if $f \in \ell$ and $\Theta_m 1 \in f$ for some $m \in \omega$, then f is weaker than, hence norm-equivalent to, l^∞. If $f \in \mathrm{b}\ell$, f is weaker than l^1; if $f \in \mathrm{b}\ell^{\rightarrow}$, f is weaker than $\Theta_{s_0(f)} l^1$. If $f \in \mathrm{b}\ell$, then f is weaker than l_0^∞ if and only if $\beta(f;p)$ is bounded, i.e., $\lim_{p\to\infty} \beta(f;p) < \infty$.*

Proof. We set $s_0 = s_0(f)$. Assume $f \in \ell^{\rightarrow}$ (in particular, $f \in \ell$), and let $\varphi \in f$ be given. For any $m \geqq s_0$, i.e., any $m \in \operatorname{supp}(f)$, we have, by Lemma 3.1, $|\varphi(m)| \leqq \beta^{-1}(f)\, |\varphi|_f$. Therefore $\varphi \in l^\infty$, $|\varphi| \leqq \beta^{-1}(f)\, |\varphi|_f$, and thus f is stronger than l^∞. Assume $f \in \ell$, $\Theta_m 1 \in f$; then $1 = \Theta_m 1 + \sum_{0}^{m-1} \chi^n \in f$. For any $\varphi \in l^\infty$ we have $|\varphi| \leqq |\varphi|$; thus $\varphi \in f$ and $|\varphi|_f \leqq |\varphi|\, |1|_f$, so that f is weaker than l^∞.

Assume that $f \in \mathrm{b}\ell^{\rightarrow}$, and let $\varphi \in \Theta_{s_0} l^1$ be given. Then $\left(\sum_{0}^{n} \varphi(j)\, \chi^j\right)$ is an f-Cauchy sequence, since $\sum_{0}^{\infty} |\varphi(n)|\, |\chi^n|_f = \beta(f)\, |\varphi|_1 < \infty$. Its f-limit exists, and is therefore its s-limit (by Lemma 3.1); but this s-limit is of course the l^1-limit φ. The preceding computation yields $|\varphi|_f \leqq \beta(f)\, |\varphi|_1$, so that f is weaker than $\Theta_{s_0} l^1$. In particular, if $f \in \mathrm{b}\ell$, so that $s_0 = 0$, then f is weaker than $\Theta_0 l^1 = l^1$.

Assume that $f \in \mathrm{b}\ell$. If f is weaker than l_0^∞, there exists $\varrho > 0$ such that $\beta(f;p) \leqq \varrho\beta(l_0^\infty;p) = \varrho < \infty$, $p = 1, 2, \ldots$. Assume, conversely, that $\beta(f;p) \leqq \varrho < \infty$, $p = 1, 2, \ldots$, and let $\varphi \in l_0^\infty$ be given. There exists an increasing sequence (m_k) in ω, with $m_0 = 0$ and such that $|\varphi(n)| \leqq 2^{-k} |\varphi|$ for $n \geqq m_k$, $k = 0, 1, \ldots$. Then $\left(\sum_{i=0}^{k} \sum_{j=m_i}^{m_{i+1}-1} \varphi(j)\, \chi^j\right)$ is an f-Cauchy sequence, since $\sum_{i=0}^{\infty} \left|\sum_{j=m_i}^{m_{i+1}-1} \varphi(j)\, \chi^j\right|_f \leqq$ $\leqq \sum_{i=0}^{\infty} 2^{-i} |\varphi|\, \beta(f; m_{i+1} - m_i) \leqq 2\varrho|\varphi| < \infty$. As in the preceding argument, the f-limit exists and coincides with the l_0^∞-limit φ; the computation yields $|\varphi|_f \leqq 2\varrho|\varphi|$, and f is indeed weaker than l_0^∞.

We must now introduce spaces of functions defined on a given $\omega_{[m]}$ and with values in a given Banach space X. The fundamental definition is as follows:

if $\boldsymbol{f}\in\not{f}$, $m\in\omega$, and X is a given Banach space, $\boldsymbol{f}_{[m]}(X)$ is the class of all functions $f\colon\omega_{[m]}\to X$ such that $\|f_{[0]}\|=\|f\|_{[0]}\in\boldsymbol{f}$, with the norm $|f|_{\boldsymbol{f}}=|\|f_{[0]}\||_{\boldsymbol{f}}$ (where we have abbreviated the subscript $\boldsymbol{f}_{[m]}(X)$ to $\boldsymbol{f}$ without danger of confusion). This is indeed obviously a normed space (algebraically a linear manifold in $\boldsymbol{s}_{[m]}(X)$). According to our usual procedure, we write $\boldsymbol{f}(X)=\boldsymbol{f}_{[0]}(X)$, $\boldsymbol{f}_{[m]}=\boldsymbol{f}_{[m]}(R)$, and we indeed find $\boldsymbol{f}_{[0]}(R)=\boldsymbol{f}$, so that this notation is consistent. Thus, $f\in\boldsymbol{f}_{[m]}(X)$ if and only if $f_{[0]}\in\boldsymbol{f}(X)$ — and $|f|_{\boldsymbol{f}}=|f_{[0]}|_{\boldsymbol{f}}$ — and if and only if $\|f\|\in\boldsymbol{f}_{[m]}$ — and $|f|_{\boldsymbol{f}}=|\|f\||_{\boldsymbol{f}}$.

With the definitions of $\mathrm{k}_0(\boldsymbol{f}_{[m]}(X))$, $\mathrm{k}(\boldsymbol{f}_{[m]}(X))$, $\Theta_p(\boldsymbol{f}_{[m]}(X))$ (for $p\in\omega$), $\mathrm{lc}(\boldsymbol{f}_{[m]}(X))$ that follow by obvious analogy from the corresponding ones for $\not{f}$-spaces, it is immediate that the spaces thus defined are respectively identical with $(\mathrm{k}_0\boldsymbol{f})_{[m]}(X)$, $(\mathrm{k}\boldsymbol{f})_{[m]}(X)$, $(\Theta_p\boldsymbol{f})_{[m]}(X)$, $(\mathrm{lc}\boldsymbol{f})_{[m]}(X)$, and that parentheses may therefore be omitted.

3.8. Lemma. *Let $\boldsymbol{f}\in\not{f}$, $m\in\omega$, and the Banach space X be given. Then $\boldsymbol{f}_{[m]}(X)=\{f_{[m]}\colon f\in\boldsymbol{f}(X)\}$, and the mapping $f\to f_{[m]}\colon\boldsymbol{f}(X)\to\boldsymbol{f}_{[m]}(X)$ is a norm-diminishing epimorphism whose restriction to the subspace $\Theta_m\boldsymbol{f}(X)$ is an isometrical isomorphism of $\Theta_m\boldsymbol{f}(X)$ onto $\boldsymbol{f}_{[m]}(X)$. $\boldsymbol{f}_{[m]}(X)$ is stronger than $\boldsymbol{s}_{[m]}(X)$. If $\boldsymbol{f}$ is complete, i.e., if $\boldsymbol{f}\in\mathrm{b}\not{f}$, then $\boldsymbol{f}_{[m]}(X)$ is complete.*

Proof. All but the last statement follows directly from the definitions, from (f), and from Lemma 3.1. To prove the last statement, it is sufficient to prove that $\boldsymbol{f}(X)$ is complete, since $\boldsymbol{f}_{[m]}(X)$ is isometrically isomorphic to the subspace $\Theta_m\boldsymbol{f}(X)$ of $\boldsymbol{f}(X)$. Let then (f_n) be an $\boldsymbol{f}(X)$-Cauchy sequence in $\Sigma(\boldsymbol{f}(X))$, and f its $\boldsymbol{s}(X)$-limit (which exists, since $\boldsymbol{s}(X)$ is a Fréchet space and $\boldsymbol{f}(X)$ is stronger). Since $g\to\|g\|\colon\boldsymbol{s}(X)\to\boldsymbol{s}$ is continuous, $\|f_n\|\to\|f\|$ as $n\to\infty$ in $\boldsymbol{s}$; on the other hand, $(\|f_n\|)$ is an $\boldsymbol{f}$-Cauchy sequence in $\Sigma(\boldsymbol{f})$; hence its $\boldsymbol{f}$-limit exists and must coincide with $\|f\|$, so that $\|f\|\in\Sigma(\boldsymbol{f})$. Hence $f\in\Sigma(\boldsymbol{f}(X))$; by Lemma 1.1, $\boldsymbol{f}(X)$ is complete.

3.9. Lemma. *Let $\boldsymbol{f}\in\not{f}_K$, $m, m'\in\omega$, $m'>m$, and the Banach space X be given. For any function $f\colon\omega_{[m]}\to X$, $f\in\boldsymbol{f}_{[m]}(X)$ if and only if $f_{[m']}\in\boldsymbol{f}_{[m']}(X)$; consequently, for any two functions $f, g\colon\omega_{[m]}\to X$ that coincide for all $n\geqq m'$, either one is in $\boldsymbol{f}_{[m]}(X)$ if and only if the other is.*

Proof. If $f\in\boldsymbol{f}_{[m]}(X)$, then $f_{[0]}\in\boldsymbol{f}(X)$ and $f_{[m']}=f_{[0][m']}\in\boldsymbol{f}_{[m']}(X)$. If, conversely, $f_{[m']}\in\boldsymbol{f}_{[m']}(X)$, we have $\Theta_{m'}f=f_{[0][m]}\in\boldsymbol{f}_{[m]}(X)$; but $\chi^n_{[m]}\in\boldsymbol{f}_{[m]}$, $n\in\omega$, since $\boldsymbol{f}\in\not{f}_K$; therefore $\|f\|=\|\Theta_{m'}f\|+\sum_{n=m}^{m'-1}\|f(n)\|\,\chi^n_{[m]}\in\boldsymbol{f}_{[m]}$, and hence $f\in\boldsymbol{f}_{[m]}(X)$.

This completes the account of the results on sequence spaces that we shall actually require. However, to make the nature of the spaces $\boldsymbol{f}_{[m]}(X)$ quite clear to the reader and to convince him that they are natural objects in their own right we state a few facts in the form of two lemmas. The proofs are quite elementary and are substantial simplifications of the proofs for the corresponding results about function spaces given in [6].

3.10. Lemma. *Let $m\in\omega$ and the Banach space X be given. A normed space $\boldsymbol{F}$ of functions $f\colon\omega_{[m]}\to X$ satisfies the condition*

($\mathrm{f}_{[m]}(X)$): *if* $f \in \boldsymbol{F}$ *and* $g \in s_{[m]}(X)$ *with* $\|g\| \leqq \|f\|$, *then* $g \in \boldsymbol{F}$ *and* $|g|_{\boldsymbol{F}} \leqq |f|_{\boldsymbol{F}}$

if and only if $\boldsymbol{F} = \boldsymbol{f}_{[m]}(X)$ *for some* $\boldsymbol{f} \in \mathscr{f}$. *A normed space* $\boldsymbol{F}$ *of functions* $f: \omega \to X$ *satisfies the conditions* ($\mathrm{f}(X)$) *and*

($\mathrm{z}(X)$): $\boldsymbol{F} \neq \{0\}$;

($\mathrm{t}(X)$): *if* $f \in \boldsymbol{F}$, *then* $T^- f$, $T^+ f \in \boldsymbol{F}$ *and* $|T^- f|_{\boldsymbol{F}}$, $|T^+ f|_{\boldsymbol{F}} \leqq |f|_{\boldsymbol{F}}$,

or ($\mathrm{f}(X)$), ($\mathrm{z}(X)$), *and*

($\mathrm{t}^{\to}(X)$): *if* $f \in \boldsymbol{F}$, *then* $T^+ f \in \boldsymbol{F}$ *and* $|T^+ f|_{\boldsymbol{F}} = |f|_{\boldsymbol{F}}$,

if and only if $\boldsymbol{F} = \boldsymbol{f}(X)$ *for some* $\boldsymbol{f} \in \mathscr{t}$, *or* $\boldsymbol{f} \in \mathscr{t}^{\to}$, *respectively.*

3.11. Lemma. *For every* $\boldsymbol{f} \in \mathscr{f}$, *every* $m \in \omega$, *and every Banach space* X, $\Sigma(\mathrm{lc}\boldsymbol{f}_{[m]}(X))$ *is the* $s_{[m]}(X)$-*closure of* $\Sigma(\boldsymbol{f}_{[m]}(X))$.

4. Equations and solutions

We shall be mainly concerned with the following equations in a Banach space X *(we exclude once and for all the trivial case* $X = \{0\}$*)*:

$$\text{(I)}\quad x(n) + A(n)\,x(n-1) = 0, \qquad n = 1, 2, \ldots,$$

$$\text{(II)}\quad x(n) + A(n)\,x(n-1) = f(n), \qquad n = 1, 2, \ldots;$$

here $A \in s_{[1]}(\tilde{X})$, $f \in s_{[1]}(X)$; and a *solution* is a function $x \in s(X)$ that satisfies (I) or (II), respectively. We could of course write (I) and (II) in an argumentless form, using translation operators, but this is awkward and does not provide much insight. For given $A \in s_{[1]}(\tilde{X})$, $f \in s_{[1]}(X)$, and $m \in \omega$, a function $x \in s_{[m]}(X)$ is a *solution of* ($\mathrm{I}_{[m]}$) or *of* ($\mathrm{II}_{[m]}$) (the equation (I) or (II) "restricted to $\omega_{[m]}$") if it satisfies (I) or (II), respectively, for $n = m+1$, $m+2$, $\ldots$. If x is a solution of ($\mathrm{I}_{[m]}$) or ($\mathrm{II}_{[m]}$) and $m' \geqq m$, then $x_{[m']}$ is a solution of ($\mathrm{I}_{[m']}$) or ($\mathrm{II}_{[m']}$), respectively.

Assume $A \in s_{[1]}(\tilde{X})$ given. We define the products

$$\text{(4.1)}\quad \begin{aligned} &U(n,n) = I, \quad n \geqq 0, \\ &U(n,n_0) = (-1)^{n-n_0} A(n) \ldots A(n_0+1) = -A(n)\,U(n-1,n_0) \in \tilde{X}, \\ &\hspace{22em} n > n_0 \geqq 0. \end{aligned}$$

These operators satisfy

$$\text{(4.2)}\quad \begin{aligned} &U(n,n-1) = -A(n), \qquad n \geqq 1, \\ &U(n,n_0) = U(n,n_1)\,U(n_1,n_0), \qquad n \geqq n_1 \geqq n_0 \geqq 0. \end{aligned}$$

It is a matter of immediate verification that $x \in s_{[m]}(X)$ is a solution of ($\mathrm{I}_{[m]}$), or of ($\mathrm{II}_{[m]}$) for a given f, if and only if it satisfies, respectively:

$$\text{(4.3)}\qquad x(n) = U(n,m)\,x(m), \qquad n \in \omega_{[m]},$$

$$\text{(4.4)}\qquad x(n) = U(n,m)\,x(m) + \sum_{i=m+1}^{n} U(n,i)\,f(i), \qquad n \in \omega_{[m]};$$

these formulae provide the unique solution of the "initial-value problem". These solutions of course actually satisfy, respectively,

$$\text{(4.5)}\qquad x(n) = U(n,n_0)\,x(n_0), \qquad n \geqq n_0 \geqq m,$$

$$\text{(4.6)}\qquad x(n) = U(n,n_0)\,x(n_0) + \sum_{i=n_0+1}^{n} U(n,i)\,f(i), \qquad n \geqq n_0 \geqq m.$$

5. Covariant sequences

We assume the Banach space X and the function $A \in s_{[1]}(\tilde{X})$ given.

A sequence Y of linear manifolds in X, more precisely, a function Y defined on ω whose values are linear manifolds in X, is a *covariant sequence (for A)* if

$$(5.1) \qquad A(n)\,Y(n-1) \subset Y(n), \quad A(n)\,(X \backslash Y(n-1)) \subset X \backslash Y(n), \quad n = 1, 2, \ldots .$$

It follows from the definition that if Y is a covariant sequence then

$$(5.2) \qquad U(n, n_0)\,Y(n_0) \subset Y(n), \quad U(n, n_0)\,(X \backslash Y(n_0)) \subset X \backslash Y(n), \quad n \geqq n_0 \geqq 0 .$$

In view of (4.5), if x is a solution of $(\mathrm{I}_{[m]})$ it follows from (5.2) that $x(n) \in Y(n)$ for some $n \in \omega_{[m]}$ if and only if this is the case for all $n \in \omega_{[m]}$; such a solution is said to *lie in* $Y_{[m]}$ (or *in* Y, if $m=0$). Similarly, using (4.6), if $f \in s_{[1]}(X)$ has finite support and x is a solution of $(\mathrm{II}_{[m]})$, then $x(n) \in Y(n)$ for some $n \in \omega_{[m]}$, $n \geqq s(f)$, if and only if this is the case for all such n; the solution x is then said to *lie eventually in* $Y_{[m]}$, or *in* Y.

If Y, Y' are covariant sequences, (5.2) implies that if $Y(n_0) \subset Y'(n_0)$ for some $n_0 \in \omega$ then $Y(n) \subset Y'(n)$ for all $n \leqq n_0$; if this is the case for all $n \in \omega$ we write $Y \subset Y'$. In particular, every covariant sequence Y satisfies $Y_0 \subset Y \subset X$, where X is the constant sequence and Y_0 is the covariant sequence defined by

$$Y_0(n) = \bigcup_{i=n}^{\infty} U^{-1}(i, n)\,(\{0\}) = \{x(n) : x \text{ a solution of } (\mathrm{I}_{[n]}),\ x(p) = 0$$
$$\text{for some (equivalently, for all sufficiently large) } p \in \omega_{[n]}\} .$$

A covariant sequence whose values are subspaces is a *closed covariant sequence*. Let Y be such a sequence; we denote by $\Omega_Y(n)$ the canonical epimorphism of X onto the quotient space $X/Y(n)$, $n \in \omega$.

5.1. Lemma. *Let Y be a closed covariant sequence. For $n \geqq n_0 \geqq 0$ there exists a unique monomorphism $V_Y(n, n_0) : X/Y(n_0) \to X/Y(n)$ such that*

$$(5.3) \qquad \Omega_Y(n)\,U(n, n_0) = V_Y(n, n_0)\,\Omega_Y(n_0) .$$

These mappings satisfy

$$(5.4) \qquad \|V_Y(n, n_0)\| \leqq \|U(n, n_0)\| ,$$

$$(5.5) \qquad V_Y(n, n_0) = V_Y(n, n_1)\,V_Y(n_1, n_0), \qquad n \geqq n_1 \geqq n_0 \geqq 0.$$

Proof. If $x, x' \in X$, (5.2) implies that $U(n, n_0)\,(x' - x) \in Y(n)$ if and only if $x' - x \in Y(n_0)$. Thus, for every $\xi \in X/Y(n_0)$, $\Omega_Y(n)\,U(n, n_0)x$ for all $x \in \xi$ is one and the same well-defined element of $X/Y(n)$, which we call $V_Y(n, n_0)\xi$, and the (obviously linear) mapping $V_Y(n, n_0)$ is injective and satisfies (5.3); it is obviously unique in this respect, and $\|V_Y(n, n_0)\xi\| \leqq \inf\{\|U(n, n_0)x\| : x \in \xi\} \leqq$ $\leqq \|U(n, n_0)\|\,\|\xi\|$, so that (5.4) holds. (5.5) follows from (5.3) and (4.2) and the uniqueness of V_Y.

An important question concerns the surjectivity (hence invertibility) of $V_Y(n, n_0)$. An answer is provided by the following lemma.

5.2. Lemma. *For a given $m \in \omega_{[1]}$, the following statements are equivalent:*
(a): *$V_Y(m, 0)$ is surjective, hence, equivalently, an isomorphism;*
(b): *$U(m, 0)\, X + Y(m) = X$;*
(c): *for every $x \in X$, (II) with $f = \chi^m_{[1]} x$ has a solution that lies eventually in Y;*
(d): *$V_Y(n, n_0)$ is surjective, hence an isomorphism, if $0 \leqq n_0 \leqq n \leqq m$;*
(e): *$U(n, n_0)\, X + Y(n) = X$ if $0 \leqq n_0 \leqq n \leqq m$;*
(f): *$A(n)\, X + Y(n) = X$ if $1 \leqq n \leqq m$.*

Proof. The fact that surjectivity of the monomorphism $V_Y(n, n_0)$ implies invertibility, i.e., its being an isomorphism, is a consequence of the Open-Mapping Theorem, since $X/Y(n_0)$, $X/Y(n)$ are Banach spaces.

The proof will be carried out according to the diagram (a)→(d)→(e)→→(f)→(b)→(c)→(a).

(a) *implies* (d): By (5.5), $V_Y(m, n)\, V_Y(n, 0) = V_Y(m, 0)$. If $V_Y(m, 0)$ is invertible, $V_Y(m, n)$ must be surjective, hence invertible, and $V_Y(n, 0)$ must be invertible; again, $V_Y(n, n_0)\, V_Y(n_0, 0) = V_Y(n, 0)$, hence $V_Y(n, n_0)$ is surjective.

(d) *implies* (e): If $x \in X$ we may choose $y \in X$ such that $V_Y(n, n_0)\, \Omega_Y(n_0)\, y = \Omega_Y(n) x$. By (5.3), $\Omega_Y(n)\; (x - U(n, n_0)\, y) = \Omega_Y(n) x - V_Y(n, n_0)\, \Omega_Y(n_0)\, y = 0$, so that $x - U(n, n_0)\, y \in Y(n)$.

(e) *implies* (f): Trivial, by taking $n_0 = n - 1$ in (e).

(f) *implies* (b): We prove $U(m, m-i)\, X + Y(m) = X$ by induction on $i = 0, \ldots, m$. For $i = 0$ it is trivial; assume that it holds for some $i < m$; then, using (f) and (4.1) and (5.2),

$$X = U(m, m-i)\, X + Y(m) = U(m, m-i)\, (A(m-i)\, X + Y(m-i)) + Y(m)$$
$$\subset U(m, m-i-1)\, X + Y(m) + Y(m),$$

and the statement holds with $i + 1$ instead of i. Therefore it holds for m instead of i, and (b) is satisfied.

(b) *implies* (c): On account of (b) we have $x = y - U(m, 0) z_0$ for some $z_0 \in X$, $y \in Y(m)$. We let z be the solution of (II) with $f = \chi^m_{[1]} x$ that satisfies $z(0) = z_0$. By (4.4), $z(m) = U(m, 0)\, z(0) + f(m) = y \in Y(m)$. Since $s(f) = m$ (if we exclude the trivial case $x = 0$), z lies eventually in Y.

(c) *implies* (a): For given $x \in X$, $x \neq 0$, let z be a solution of (II) with $f = \chi^m_{[1]} x$ that lies eventually in Y. Since $s(f) = m$, this means, by (4.4), that $z(m) = U(m, 0)\, z(0) + x \in Y(m)$, or $V_Y(m, 0)\, \Omega_Y(0)\, z(0) = \Omega_Y(m)\, U(m, 0)\, z(0) = -\Omega_Y(m) x$. Since x was arbitrary, $V_Y(m, 0)$ is surjective.

Remark. Even if the covariant sequence Y is not closed, the conclusion of Lemma 5.2, as well as its proof, are valid in a purely algebraic sense.

A closed covariant sequence Y that satisfies the equivalent conditions of Lemma 5.2 for every $m \in \omega_{[1]}$ is a *regular covariant sequence.*

5.3. Lemma. *If Y is a closed covariant sequence, it is regular if and only if condition* (a) *or* (b) *or* (c) *of Lemma 5.2 holds for each m in an infinite subset of ω. If Y' is another closed covariant sequence and $Y \subset Y'$, then Y' is regular if Y is.*

Proof. Lemma 5.2.

5.4. Lemma. *Let Y, Y' be closed covariant sequences, $Y \subset Y'$. If Y is regular and $Y(m)$ has finite co-dimension with respect to $Y'(m)$ for some m, then $Y(n)$*

has the same finite co-dimension with respect to $Y'(n)$ for every $n \in \omega$. Conversely, if Y' is regular and $Y(n)$ has one and the same finite co-dimension with respect to $Y'(n)$ for every $n \in \omega$, then Y is regular. In particular, if Y is a closed covariant sequence and $Y(m)$ has finite co-dimension (with respect to X) for some m, Y is regular if and only if $Y(n)$ has the same finite co-dimension for every $n \in \omega$. More in particular, if X is finite-dimensional, a covariant sequence Y is regular if and only if the dimension of its terms is constant.

Proof. The co-dimension of $Y(n)$ with respect to $Y'(n)$ is the dimension of $\Omega_Y(n)\, Y'(n)$, $n \in \omega$. Further, by (5.2) applied to Y' and by (5.3), we have, for all $n \geqq n_0 \geqq 0$,

$$\begin{aligned} &V_Y(n, n_0)\, \Omega_Y(n_0)\, Y'(n_0) \subset \Omega_Y(n)\, Y'(n), \\ (5.6)\quad &V_Y(n, n_0)\,((X/Y(n_0)) \backslash \Omega_Y(n_0)\, Y'(n_0)) = V_Y(n, n_0)\, \Omega_Y(n_0)\,(X \backslash Y'(n_0)) \subset \\ &\subset \Omega_Y(n)\,(X \backslash Y'(n)) = (X/Y(n)) \backslash \Omega_Y(n)\, Y'(n)\,. \end{aligned}$$

If Y is regular, $V_Y(n, n_0)$ is an isomorphism, and therefore its restriction to $\Omega_Y(n_0)\, Y'(n_0)$ is an isomorphism of this space onto $\Omega_Y(n)\, Y'(n)$; if either of these spaces has finite dimension, the other has the same finite dimension.

Assume conversely that Y' is regular and that $\Omega_Y(m)\, Y'(m)$ has the same finite dimension as $\Omega_Y(0)\, Y'(0)$ for every $m \in \omega_{[1]}$. Since $V_Y(m, 0)$ is injective, it must map $\Omega_Y(0)\, Y'(0)$ *onto* $\Omega_Y(m)\, Y'(m)$, by (5.6) and the dimensionality assumption. Therefore

$$\Omega_Y(m)\, U(m, 0)\, Y'(0) = V_Y(m, 0)\, \Omega_Y(0)\, Y'(0) = \Omega_Y(m)\, Y'(m)\,,$$

whence

$$U(m, 0)\, X + Y(m) = U(m, 0)\, X + (U(m, 0)\, Y'(0) + Y(m)) = U(m, 0)\, X + Y'(m) = X,$$

since Y' is regular. Since m was arbitrary, Y is also regular.

The particular cases follow using the fact that the constant sequence X is regular covariant.

6. *d*-solutions

Let $A \in s_{[1]}(\tilde{X})$ and a space $\boldsymbol{d} \in \mathfrak{b}\not{f}_K$ be given. A solution x of $(\mathrm{I}_{[m]})$, or of $(\mathrm{II}_{[m]})$ for a given f, is a *$\boldsymbol{d}$-solution* if $x \in \boldsymbol{d}_{[m]}(X)$.

6.1. Lemma. *Let $m' \geqq m \geqq 0$ be given. A solution x of $(\mathrm{I}_{[m]})$, or of $(\mathrm{II}_{[m]})$ for a given f, is a $\boldsymbol{d}$-solution if and only if $x_{[m']}$ is a $\boldsymbol{d}$-solution of $(\mathrm{I}_{[m']})$ or $(\mathrm{II}_{[m']})$, respectively.*

Proof. We know that $x_{[m']}$ is a solution. The conclusion follows from Lemma 3.9.

We define $X_{0\boldsymbol{d}}(m) = \{x(m) : x$ is a $\boldsymbol{d}$-solution of $(\mathrm{I}_{[m]})\}$, and let $X_{0\boldsymbol{d}}$ be the sequence formed by these linear manifolds. In particular, we set $X_0 = X_{0 l^\infty}$, $X_{00} = X_{0 l_0^\infty}$.

6.2. Lemma. *$X_{0\boldsymbol{d}}$ is a covariant sequence.*

Proof. For any $x_0 \in X$ and $n \geqq 1$, $x_0 \in X_{0\boldsymbol{d}}(n-1)$ if and only if $x_0 = x(n-1)$ for some $\boldsymbol{d}$-solution x of $(\mathrm{I}_{[n-1]})$; but by Lemma 6.1 this is the case if and only if $-A(n)\, x_0 = x(n) = x_{[n]}(n) \in X_{0\boldsymbol{d}}(n)$.

6.3. Lemma. *For each $m \in \omega$, the set $\boldsymbol{X}_{\boldsymbol{d}[m]}$ of $\boldsymbol{d}$-solutions of* $(\mathrm{I}_{[m]})$ *is a subspace of $\boldsymbol{d}_{[m]}(X)$. The linear mapping $\Pi(m): x \to x(m): \boldsymbol{X}_{\boldsymbol{d}[m]} \to X_{0\boldsymbol{d}}(m)$ is bounded and bijective.*

Proof. The set of all solutions of $(\mathrm{I}_{[m]})$ is a subspace of $\boldsymbol{s}_{[m]}(X)$, since the topology in this space is that of pointwise convergence. $\boldsymbol{X}_{\boldsymbol{d}[m]}$ is the intersection of this set with the space $\boldsymbol{d}_{[m]}(X)$, and is therefore closed in this space, which is stronger than $\boldsymbol{s}_{[m]}(X)$ (Lemma 3.8). $\Pi(m)$ is obviously linear; it is bijective on account of (4.3) and the definition of $X_{0\boldsymbol{d}}(m)$, and bounded since $\|\Pi(m)\,x\| \leqq \leqq |x|_{\boldsymbol{d}}\,|\chi^m|_{\boldsymbol{d}}^{-1}$.

6.4. Lemma. *A linear manifold $Y \subset X_{0\boldsymbol{d}}(m)$ has* $\mathrm{cl}\,Y \subset X_{0\boldsymbol{d}}(m)$ *if and only if there exists a number $S_{Y,m} \geqq 0$ such that every solution x of* $(\mathrm{I}_{[m]})$ *with $x(m) \in Y$ satisfies $|x|_{\boldsymbol{d}} \leqq S_{Y,m}\|x(m)\|$.*

Proof. The condition is necessary: If $\mathrm{cl}\,Y \subset X_{0\boldsymbol{d}}(m)$, $\Pi(m)^{-1}(\mathrm{cl}\,Y)$ is a subspace of $\boldsymbol{X}_{\boldsymbol{d}[m]}$, and $\Pi(m)$ restricted to this subspace is a bounded bijective linear mapping of this subspace, a Banach space, onto the Banach space $\mathrm{cl}\,Y$, hence an isomorphism; the conclusion holds with $S_{Y,m}$ the norm of the inverse isomorphism.

The condition is sufficient: If $S_{Y,m}$ exists, $\Pi(m)^{-1}$ restricted to Y is bounded and therefore has a bounded extension $\Psi: \mathrm{cl}\,Y \to \boldsymbol{X}_{\boldsymbol{d}[m]}$, since $\boldsymbol{X}_{\boldsymbol{d}[m]}$ is a Banach space. Both the trivial injection and $\Pi(m)\,\Psi$ map $\mathrm{cl}\,Y$ into X, and their restrictions to the dense set Y coincide; therefore $\Pi(m)\,\Psi$ is the trivial injection, and $\mathrm{cl}\,Y = \Pi(m)\,\Psi(\mathrm{cl}\,Y) \subset \Pi(m)\,\boldsymbol{X}_{\boldsymbol{d}[m]} = X_{0\boldsymbol{d}}(m)$.

The proof breaks down if $Y = \{0\}$, but then the conclusion is trivially satisfied.

6.5. Lemma. *$X_{0\boldsymbol{d}}(m)$ is closed if and only if there exists a number $S_m \geqq 0$ such that every $\boldsymbol{d}$-solution x of* $(\mathrm{I}_{[m]})$ *satisfies $|x|_{\boldsymbol{d}} \leqq S_m\|x(m)\|$.*

Proof. Lemma 6.4.

The minimum values of $S_{Y,m}$ and S_m in the preceding lemmas are henceforth denoted by these symbols or, in full, by $S_{Y\boldsymbol{d},m}(A)$, $S_{\boldsymbol{d},m}(A)$, respectively. If $m = 0$, the index 0 is omitted.

7. Dichotomies

Let $A \in \boldsymbol{s}_{[1]}(\tilde{X})$ be given.

A regular covariant sequence Y *induces a dichotomy [an exponential dichotomy] for A* if there exist a number $N > 0$ [numbers ν, ν', $N > 0$] and, for every $\lambda > 1$, numbers $N' = N'(\lambda) > 0$, $\lambda_0 = \lambda_0(\lambda) > 1$ such that for each $m \in \omega$ and any pair of solutions y, z of $(\mathrm{I}_{[m]})$ with $y(m) \in Y(m)$ (i.e., y lying in $Y_{[m]}$) and $\|z(m)\| \leqq \lambda d(Y(m), z(m))$ we have:

(Di) [(Ei)]: $\|y(n)\| \leqq N\|y(n_0)\|$ $[\|y(n)\| \leqq N e^{-\nu(n-n_0)}\|y(n_0)\|]$

for all $n \geqq n_0 \geqq m$;

(Dii) [(Eii)]: $\|z(n)\| \geqq N'^{-1}\|z(n_0)\|$ $[\|z(n)\| \geqq N'^{-1} e^{\nu'(n-n_0)}\|z(n_0)\|]$

for all $n \geqq n_0 \geqq m$;

(Diii): $\|z(n)\| \leqq \lambda_0 d(Y(n), z(n))$ for all $n \geqq m$.

A dichotomy is often described as an *ordinary* dichotomy, to distinguish it from an exponential dichotomy.

The fact that the covariant sequence Y is regular allows us to show that the statement that it induces a dichotomy, ordinary or exponential, is equivalent to an apparently much weaker statement; this is included in the following theorem. Mention of various possible intermediate statements has been omitted.

7.1. Theorem. *Let Y be a regular covariant sequence. The following statements are equivalent:*

(a): *Y induces an ordinary [exponential] dichotomy for A.*

(b): *There exists a number $N>0$ [numbers $\nu, \nu', N>0$] and, for every $\lambda>1$, numbers $N'_0=N'_0(\lambda)>0$, $\lambda_0=\lambda_0(\lambda)>1$ such that for each $m\in\omega$ and any pair of solutions y, z of* $(\mathrm{I}_{[m]})$ *with $y(m)\in Y(m)$ (i.e., y lying in $Y_{[m]}$) and $\|z(m)\|\leqq \lambda d(Y(m), y(m))$ we have* (Diii) *and*

$$(\mathrm{Di}_0)\ [(\mathrm{Ei}_0)]:\quad \|y(n)\|\leqq N\|y(m)\| \quad [\|y(n)\|\leqq N e^{-\nu(n-m)}\|y(m)\|]$$

for all $n\geqq m$;

$$(\mathrm{Dii}_0)\ [(\mathrm{Eii}_0)]:\quad \|z(n)\|\geqq N_0'^{-1}\|z(m)\| \quad [\|z(n)\|\geqq N_0'^{-1} e^{\nu'(n-m)}\|z(m)\|]$$

for all $n\geqq m$.

(c): *There exist: a number $N>0$ [numbers ν, $N>0$] such that* (Di_0) [(Ei_0)] *holds for each $m\in\omega$ and any solution y of* $(\mathrm{I}_{[m]})$ *lying in $Y_{[m]}$; and a $(Y(0),\lambda')$-splitting q of X for some $\lambda'>1$, and numbers $N''>0$ [numbers ν', $N''>0$], $\lambda'_0>1$, such that any solution z' of* (I) *with $q(z'(0))=z'(0)$ satisfies*

$$(\mathrm{Dii}')\ [(\mathrm{Eii}')]:\quad \|z'(n)\|\geqq N''^{-1}\|z'(n_0)\| \quad [\|z'(n)\|\geqq N''^{-1}e^{\nu'(n-n_0)}\|z'(n_0)\|]$$

for all $n\geqq n_0\geqq 0$;

$$(\mathrm{Diii}'):\quad \|z'(n)\|\leqq \lambda'_0 d(Y(n), z'(n)) \qquad \text{for all } n\geqq 0.$$

Proof. We give the proof for the exponential dichotomy only; the proof for the ordinary dichotomy is the same with the exponentials deleted.

(a) *implies* (c): Since for any $\lambda'>1$ and any $(Y(0),\lambda')$-splitting q the assumption $q(z'(0))=z'(0)$ yields $\|z'(0)\|\leqq \lambda' d(Y(0), z'(0))$, this implication is trivial, and (c) holds with the same ν, ν', N, and $N''=N'(\lambda')$, $\lambda'_0=\lambda_0(\lambda')$.

(c) *implies* (b): Let $m\in\omega$ and $\lambda>1$ be given. Since (Ei_0) is satisfied by assumption, it remains to verify (Eii_0), (Diii) for an arbitrary solution z of $(\mathrm{I}_{[m]})$ with $\|z(m)\|\leqq \lambda d(Y(m), z(m))$. Since the covariant sequence Y is regular, we may define the solution z' of (I) by specifying $z'(0)=\bar q(V_Y^{-1}(m,0)\,\Omega_Y(m)\,z(m))$, where $\bar q$ is obtained from q as in Lemma 1.2. Thus indeed $q(z'(0))=z'(0)$. Now

$$\begin{aligned}\Omega_Y(m)\,z'(m)&=\Omega_Y(m)\,U(m,0)\,z'(0)=V(m,0)\,\Omega_Y(0)\,\bar q(V_Y^{-1}(m,0)\,\Omega_Y(m)\,z(m))\\&=\Omega_Y(m)\,z(m),\end{aligned}$$

by Lemma 1.2 and (5.3); thus $z(m)-z'(m)\in Y(m)$, and therefore

$$\|z(m)\|\leqq \lambda d(Y(m), z(m))=\lambda d(Y(m), z'(m))\leqq \lambda\|z'(m)\|. \tag{7.1}$$

We set $y=z-z'_{[m]}$, a solution of $(\mathrm{I}_{[m]})$ lying in $Y_{[m]}$.

For any $n \geqq m$ we have $d(Y(n), z'(n)) = d(Y(n), z(n)) \leqq \|z(n)\|$; and (7.1), (Eii′), (Diii′) imply

$$\|z(m)\| \leqq \lambda \|z'(m)\| \leqq \lambda N'' e^{-\nu'(n-m)} \|z'(n)\| \leqq \lambda \lambda'_0 N'' e^{-\nu'(n-m)} d(Y(n), z'(n)) \leqq \leqq \lambda \lambda'_0 N'' e^{-\nu'(n-m)} \|z(n)\|,$$

so that (Eii_0) holds with $N'_0 = N'_0(\lambda) = \lambda \lambda'_0 N''$.

Again for any $n \geqq m$ we have, applying (Ei_0) to y and using (Diii′), (Eii′), (7.1),

$$\|z(n)\| \leqq \|z'(n)\| + \|y(n)\| \leqq \|z'(n)\| + N \|z(m) - z'(m)\| \leqq \leqq \|z'(n)\| + (1+\lambda) N \|z'(m)\| \leqq (1 + (1+\lambda) N N'') \|z'(n)\| \leqq \leqq \lambda'_0 (1 + (1+\lambda) N N'') d(Y(n), z'(n)) = \lambda_0 d(Y(n), z(n)),$$

where $\lambda_0 = \lambda_0(\lambda) = \lambda'_0(1 + (1+\lambda) N N'')$, and (Diii) is satisfied with this λ_0.

(b) *implies* (a): Let $m \in \omega$ and $\lambda > 1$ be given and let y, z be as in the definition of an exponential dichotomy. If $n \geqq n_0 \geqq m$, we apply (Ei_0) to $y_{[n_0]}$, with n_0 instead of m, since $y_{[n_0]}(n_0) = y(n_0) \in Y(n_0)$; and we find (Ei) with the same ν, N. Again, if $n \geqq n_0 \geqq m$, (Diii) implies $\|z_{[n_0]}(n_0)\| = \|z(n_0)\| \leqq \lambda_0 d(Y(n_0), z_{[n_0]}(n_0))$, so that we may apply ($\mathrm{Eii}_0$) to $z_{[n_0]}$, with n_0 instead of m and λ_0 instead of λ, and we find (Eii) with the same ν' and with $N' = N'(\lambda) = N'_0(\lambda_0(\lambda))$. (Diii) of course holds by assumption with the given $\lambda_0 = \lambda_0(\lambda)$.

In an important special case, the "apartness" condition is redundant in the definition of an exponential (but not of an ordinary) dichotomy.

7.2. Lemma. *Assume that* $A \in l^{\infty}_{[1]}(\tilde{X})$. *Then condition* (Diii) *is redundant in the definition of an exponential dichotomy, and so is* (Diii′) *in the corresponding statement* (c) *of Theorem 7.1.*

Proof. Let $n_0 \geqq m \geqq 0$ be given and let z be any solution of ($\mathrm{I}_{[m]}$) satisfying (Eii) for some given ν', N'. Let $\varrho > 1$ be given and let y be a solution of ($\mathrm{I}_{[n_0]}$) lying in $Y_{[n_0]}$ and such that $\|z(n_0) - y(n_0)\| \leqq \varrho d(Y(n_0), z(n_0))$. If y satisfies (Ei) for given ν, N (with n_0 instead of m), there exists a positive integer p such that $N'^{-1} e^{\nu' p} - N e^{-\nu p} \geqq 1$. We then find:

$$\|z(n_0+p) - y(n_0+p)\| \leqq \|U(n_0+p, n_0)\| \, \|z(n_0) - y(n_0)\| \leqq \varrho |A|^p d(Y(n_0), z(n_0)),$$

$$\|z(n_0+p) - y(n_0+p)\| \geqq \|z(n_0+p)\| - \|y(n_0+p)\| \geqq \geqq N'^{-1} e^{\nu' p} \|z(n_0)\| - N e^{-\nu p} (\|z(n_0)\| + \|z(n_0) - y(n_0)\|) \geqq \geqq \|z(n_0)\| - \varrho N e^{-\nu p} d(Y(n_0), z(n_0)),$$

and, since $\varrho > 1$ was arbitrary, $\|z(n_0)\| \leqq (|A|^p + N e^{-\nu p}) d(Y(n_0), z(n_0))$, so that (Diii) holds, with $\lambda_0 = |A|^p + N e^{-\nu p}$. Since p depends only on ν, ν', N, N', the conclusion holds (for the last part, replace z by z', N' by N'', (Eii) by (Eii′), (Diii) by (Diii′), λ_0 by λ'_0).

Remark. A similar statement does not seem to hold with respect to the omission of condition (Diii) in statement (b) of Theorem 7.1.

We next state a few results, some to be proved later, concerning the relationship between a covariant sequence inducing a dichotomy and the sequences $X_{0\boldsymbol{d}}$ for spaces $\boldsymbol{d} \in \mathrm{b}\ell$.

7.3. Theorem. *Assume that the regular covariant sequence Y induces a dichotomy for A. Then $X_{00} \subset Y \subset X_0$, and X_{00} is closed. For every $\boldsymbol{d} \in \mathrm{b}\ell$ that is not norm-equivalent to $\boldsymbol{l}^\infty$, $X_{0\boldsymbol{d}} \subset X_{00}$.*

Proof. Let $m \in \omega$ be given. The inclusions $X_{00}(m) \subset Y(m) \subset X_0(m)$ are trivial by conditions (Di), (Dii), since for any $z_0 \in X \backslash Y(m)$ there exists $\lambda > 1$ such that $\|z_0\| \leqq \lambda d(Y(m), z_0)$. The fact that $X_{00}(m)$ is closed follows from Lemma 6.5 with $S_m = N$, on account of the preceding inclusion and (Di). Let $\boldsymbol{d} \in \mathrm{b}\ell$ be given, and let x be a $\boldsymbol{d}$-solution of $(\mathrm{I}_{[m]})$ with $x(m) \notin X_{00}(m)$. If $x(m) \notin Y(m)$, there exists $N' > 0$ such that $\|x(n)\| \geqq N'^{-1} \|x(m)\|$ for all $n \geqq m$, and therefore $1_{[m]} \leqq N' \|x(m)\|^{-1} \|x\| \in \boldsymbol{d}_{[m]}$. If, on the other hand, $x(m) \in Y(m) \backslash X_{00}(m)$, (Di) yields $\|x(n_0)\| \geqq N^{-1} \limsup_{n\to\infty} \|x(n)\| = \eta > 0$ for each $n_0 \geqq m$, and again $1_{[m]} \leqq \eta^{-1} \|x\| \in \boldsymbol{d}_{[m]}$. In either case, $1_{[m]} \in \boldsymbol{d}_{[m]}$, i.e., $\Theta_m 1 \in \boldsymbol{d}$, whence $\boldsymbol{d}$ is norm-equivalent to $\boldsymbol{l}^\infty$ (Lemma 3.7). The conclusion follows.

7.4. Theorem. *Assume that the regular covariant sequence Y induces a dichotomy for A. Then every closed covariant sequence Y' such that $Y \subset Y' \subset X_0$ is regular and induces a dichotomy for A.*

Proof. See Section 9 (after Theorem 9.3).

7.5. Theorem. *Assume that the regular covariant sequence Y induces a dichotomy for A. If the (closed) covariant sequence X_{00} is regular and the co-dimension of $X_{00}(0)$ with respect to $Y(0)$ is finite (equivalently, by Lemma 5.4, if $X_{00}(n)$ has the same finite co-dimension with respect to $Y(n)$ for each $n \in \omega$), then X_{00} induces a dichotomy for A.*

Proof. There is a complicated geometrical proof, in the spirit of [4], Theorem 41. F (Proof I). We give an indirect proof in Section 9 (after Theorem 9.3).

For exponential dichotomies the corresponding result is much simpler:

7.6. Theorem. *Assume that the regular covariant sequence Y induces an exponential dichotomy for A. For every $\boldsymbol{d} \in \mathrm{b}\ell$, $X_{0\boldsymbol{d}} = Y$.*

Proof. Let $m \in \omega$ be given. If y is a solution of $(\mathrm{I}_{[m]})$ with $y(m) \in Y(m)$, (Ei) implies $y \in \boldsymbol{l}^1_{[m]}(X)$, hence $y \in \boldsymbol{d}_{[m]}(X)$ (Lemma 3.7); thus $Y(m) \subset X_{0\boldsymbol{d}}(m)$. If z is a solution of $(\mathrm{I}_{[m]})$ with $z(m) \notin Y(m)$, then (Eii) holds with v' and some N', hence $z \notin \boldsymbol{l}^\infty_{[m]}(X)$, a fortiori $z \notin \boldsymbol{d}_{[m]}(X)$ (Lemma 3.7); thus $X_{0\boldsymbol{d}}(m) \subset Y(m)$, and equality holds.

8. Admissibility and $(\boldsymbol{b}, \boldsymbol{d})$-sequences

We are concerned now with equation (II) for a given $A \in s_{[1]}(\tilde{X})$.

An $\not{\ell}$-*pair* is a pair $(\boldsymbol{b}, \boldsymbol{d})$ of sequence spaces, where $\boldsymbol{b} \in \mathrm{b}\not{\ell}$, $\boldsymbol{d} \in \mathrm{b}\not{\ell}_K$. If $(\boldsymbol{b}_1, \boldsymbol{d}_1)$ is another $\not{\ell}$-pair, it is *stronger than* $(\boldsymbol{b}, \boldsymbol{d})$, and the latter pair is *weaker than* the former, if $\boldsymbol{b}_1$ is weaker than $\boldsymbol{b}$ and $\boldsymbol{d}_1$ is stronger than $\boldsymbol{d}$. An $\not{\ell}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ is a ℓ-*pair* or a $\ell^{\to}$-pair if $\boldsymbol{b} \in \mathrm{b}\ell$ or $\boldsymbol{b} \in \mathrm{b}\ell^{\to}$, respectively, and $\boldsymbol{d} \in \mathrm{b}\ell$ in both cases.

An $\not{\ell}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ is *admissible (for A)* if (II) has a $\boldsymbol{d}$-solution for every $f \in \boldsymbol{b}_{[1]}(X)$. (Strictly speaking, we should say that the pair $(\boldsymbol{b}_{[1]}(X), \boldsymbol{d}(X))$ is admissible, and the definition is meaningful for arbitrary pairs of classes of functions, but we use the abbreviated notation since we only have to do with

$\not{f}$-pairs.) If $(\boldsymbol{b}, \boldsymbol{d})$ is an admissible $\not{f}$-pair and $(\boldsymbol{b}_1, \boldsymbol{d}_1)$ is an $\not{f}$-pair that is weaker than $(\boldsymbol{b}, \boldsymbol{d})$, it is obviously also admissible.

An $\not{f}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ is *regularly admissible (for A)* if it is admissible, and the covariant sequence $X_{0\boldsymbol{d}}$ (see Lemma 6.2) is closed. ($X_{0\boldsymbol{d}}$ actually is a regular covariant sequence in the important cases; see Lemma 8.6 and Theorem 8.8.)

The fundamental result concerning admissibility is a boundedness theorem.

8.1. Theorem. *If the $\not{f}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ is admissible for A, there exists a number $K \geqq 0$ such that for any $f \in \boldsymbol{b}_{[1]}(X)$ and any $\varrho > 1$ there exists a $\boldsymbol{d}$-solution x of* (II) *with $|x|_{\boldsymbol{d}} \leqq \varrho K |f|_{\boldsymbol{b}}$.*

Proof. Let $Y \subset \boldsymbol{d}(X)$ be the linear manifold of all $\boldsymbol{d}$-solutions of all equations (II) with $f \in \boldsymbol{b}_{[1]}(X)$. The linear mapping $x \to x_{[1]} + A(T^+ x)_{[1]}: Y \to \boldsymbol{b}_{[1]}(X)$, which maps each solution onto the corresponding second member of (II), is surjective by assumption and its graph is closed in $\boldsymbol{s}(X) \times \boldsymbol{s}_{[1]}(X)$, by (4.4), a fortiori in the stronger topology of $\boldsymbol{d}(X) \times \boldsymbol{b}_{[1]}(X)$ (Lemma 3.8). By the Open-Mapping Theorem there exists $k > 0$ such that for each $f \in \boldsymbol{b}_{[1]}(X)$ there is an element $x \in Y$ that is mapped into f and satisfies $|x|_{\boldsymbol{d}} \leqq k |f|_{\boldsymbol{b}}$ (cf. [2], Theorem 2.12.1). The conclusion holds with K the infimum of the possible values of k.

The least value of K satisfying the conclusion of Theorem 8.1 is denoted in full by $K_{\boldsymbol{b},\boldsymbol{d}}(A)$, but subscripts and/or argument are omitted if confusion is unlikely.

8.2. Lemma. *Let the $\not{f}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ be given, and assume that there exists a linear manifold $\boldsymbol{f}$, dense in $\boldsymbol{b}_{[1]}(X)$, and a number $k > 0$, such that for every $f \in \boldsymbol{f}$ there exists a $\boldsymbol{d}$-solution x of* (II) *with $|x|_{\boldsymbol{d}} \leqq k |f|_{\boldsymbol{b}}$. Then $(\boldsymbol{b}, \boldsymbol{d})$ is admissible.*

Proof. Let $f \in \boldsymbol{b}_{[1]}(X)$ be given. There exists a sequence (f_k) in $\boldsymbol{f}$ such that $\lim_{k\to\infty} |f - f_k|_{\boldsymbol{b}} = 0$. Adding $f_0 = 0$ and taking a subsequence, if necessary, we may assume that $\sum_0^\infty |f_{k+1} - f_k|_{\boldsymbol{b}} < \infty$. By the assumption there exist $\boldsymbol{d}$-solutions y_k of $y_k(n) + A(n)\, y_k(n-1) = f_{k+1}(n) - f_k(n)$ such that $|y_k|_{\boldsymbol{d}} \leqq k |f_{k+1} - f_k|_{\boldsymbol{b}}$. Set $x_k = \sum_{i=0}^{k-1} y_i$, $k = 1, 2, \ldots$; then (x_k) is a $\boldsymbol{d}(X)$-Cauchy sequence; let x be its $\boldsymbol{d}(X)$-limit, hence its pointwise limit. But x_k is a solution of $x_k(n) + A(n)\, x_k(n-1) = f_k(n)$, and (f_k) has the $\boldsymbol{b}_{[1]}(X)$-limit f, which is therefore also the pointwise limit. Therefore x is a $\boldsymbol{d}$-solution of (II). A slightly more careful proof would have shown that $K_{\boldsymbol{b},\boldsymbol{d}} \leqq k$.

We omit the proof of the following theorem (cf. [4], Theorem 51.G).

8.3. Theorem. *If the $\not{f}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ is admissible for A, and if there exists a subspace Y of X, $Y \subset X_{0\boldsymbol{d}}(0)$, such that X/Y is reflexive (in particular, if X is reflexive or $X_{0\boldsymbol{d}}(0) = X$), then* $(\mathrm{lc}\boldsymbol{b}, \mathrm{lc}\boldsymbol{d})$ *is also admissible.*

We next introduce a concept that, in view of Theorem 8.1, generalizes the properties of the sequence $X_{0\boldsymbol{d}}$ in the presence of admissibility (cf. Theorem 8.8).

If $(\boldsymbol{b}, \boldsymbol{d})$ is an $\not{f}$-pair, a covariant sequence Y is a $(\boldsymbol{b}, \boldsymbol{d})$*-sequence (for A)* if $Y \subset X_{0\boldsymbol{d}}$ and if there exists a number $K_Y \geqq 0$ such that for every $f \in \mathrm{k}_0\boldsymbol{b}_{[1]}(X)$ and every $\varrho > 1$ there is a solution x of (II) that lies eventually in Y (hence is a $\boldsymbol{d}$-solution, by Lemma 3.9) and satisfies $|x|_{\boldsymbol{d}} \leqq \varrho K_Y |f|_{\boldsymbol{b}}$. The least value of K_Y with this property is denoted in full by $K_{Y\boldsymbol{b},\boldsymbol{d}}(A)$, with the usual omissions if confusion is not to be feared. If the covariant sequence Y is, in addition, closed, it is a *closed* $(\boldsymbol{b}, \boldsymbol{d})$*-sequence.*

8.4. Lemma. *Assume that Y is a $(\boldsymbol{b}, \boldsymbol{d})$-sequence for A. Then:*

(a): *Y is an $(\mathrm{lc}\boldsymbol{b}, \boldsymbol{d})$-sequence with the same K_Y, and a $(\boldsymbol{b}_1, \boldsymbol{d}_1)$-sequence for any $\not\ell$-pair $(\boldsymbol{b}_1, \boldsymbol{d}_1)$ weaker than $(\mathrm{lc}\boldsymbol{b}, \boldsymbol{d})$.*

(b): *Every covariant sequence Y' with $Y \subset Y' \subset X_{0\boldsymbol{d}}$ is also a $(\boldsymbol{b}, \boldsymbol{d})$-sequence (in particular, $X_{0\boldsymbol{d}}$ itself is one).*

(c): *If $Y \subset X_{0\mathrm{k}\boldsymbol{d}}$, then Y is a $(\boldsymbol{b}, \mathrm{k}\boldsymbol{d})$-sequence.*

Proof. We recall that $\mathrm{k}_0 \mathrm{lc}\boldsymbol{b} = \mathrm{k}_0 \boldsymbol{b}$ (Lemma 3.4). The proof is then trivial in view of the definitions.

8.5. Lemma. *Assume that $(\boldsymbol{b}, \boldsymbol{d})$ is an $\not\ell$-pair, $\boldsymbol{b} \neq \{0\}$, and that Y is a $(\Theta_m \boldsymbol{b}, \boldsymbol{d})$-sequence for some $m \in \omega$, $0 \leqq m \leqq \sup \mathrm{supp}(\boldsymbol{b})$. Then Y is a $(\boldsymbol{b}, \boldsymbol{d})$-sequence for A.*

Proof. We may assume $m \geqq 1$, and $m \in \mathrm{supp}(\boldsymbol{b})$, since otherwise $\Theta_m \boldsymbol{b} = \Theta_{m'} \boldsymbol{b}$ for some $m' > m$, $m' \in \mathrm{supp}(\boldsymbol{b})$. Set $K_Y = K_{Y\Theta_m \boldsymbol{b}, \boldsymbol{d}}$.

Let $f \in \mathrm{k}_0 \boldsymbol{b}_{[1]}(X)$ and $\varrho > 1$ be given, and set

$$g = \Theta_m f + \chi_{[1]}^m \sum_{j=1}^{m-1} U(m,j) f(j) \in \mathrm{k}_0 \Theta_m \boldsymbol{b}_{[1]}(X)$$

with

$$|g|_{\boldsymbol{b}} \leqq |f|_{\boldsymbol{b}} + |\chi^m|_{\boldsymbol{b}} \sum_{j=1}^{m-1} \|U(m,j)\| \, \|f(j)\| \leqq$$
$$\leqq \left(1 + |\chi^m|_{\boldsymbol{b}} \sum_{j=1}^{m-1}{}' \|U(m,j)\| \, |\chi^j|_{\boldsymbol{b}}^{-1}\right) |f|_{\boldsymbol{b}},$$

were here and throughout the proof the prime indicates a summation restricted to indices in $\mathrm{supp}(\boldsymbol{b})$. By assumption, there exists a solution y of $y(n) + A(n)\, y(n-1) = g(n)$, $n = 1, 2, \ldots$, such that y lies eventually in Y and $|y|_{\boldsymbol{d}} \leqq \varrho K_Y |g|_{\boldsymbol{b}}$. Let x be the solution of (II) with $x(0) = y(0)$. By (4.4) we have

$$x(n) = y(n) + \sum_{i=1}^{n} U(n,i) f(i), \qquad n = 0, \ldots, m-1$$
$$x(m) = U(m,0)\, y(0) + \sum_{i=1}^{m} U(m,i) f(i) = y(m);$$

and since $f(n) = g(n)$ for $n > m$, also $x(n) = y(n)$ for $n > m$. Thus x also lies eventually in Y, and $x - y = \sum_{n=1}^{m-1} \chi^n \sum_{i=1}^{n} U(n,i) f(i)$, so that

$$|x-y|_{\boldsymbol{d}} \leqq \sum_{n=1}^{m-1} \sum_{i=1}^{n}{}' \|U(n,i)\| \, |\chi^n|_{\boldsymbol{d}} |\chi^i|_{\boldsymbol{b}}^{-1} |f|_{\boldsymbol{b}}.$$

Therefore $|x|_{\boldsymbol{d}} \leqq k|f|_{\boldsymbol{b}}$, where

$$k = \varrho K_Y \left(1 + |\chi^m|_{\boldsymbol{b}} \sum_{j=1}^{m-1}{}' \|U(m,j)\| \, |\chi^j|_{\boldsymbol{b}}^{-1}\right) + \sum_{n=1}^{m-1} \sum_{i=1}^{n}{}' \|U(n,i)\| \, |\chi^n|_{\boldsymbol{d}} |\chi^i|_{\boldsymbol{b}}^{-1}.$$

Some special results refer to closed $(\boldsymbol{b}, \boldsymbol{d})$-sequences.

8.6. Lemma. *If* $(\boldsymbol{b}, \boldsymbol{d})$ *is an* $\not{f}$*-pair such that* $\operatorname{supp}(\boldsymbol{b})$ *is an infinite set (in particular, a* $\not{t}$*-pair or* $\not{t}^{\rightarrow}$*-pair), then every closed* $(\boldsymbol{b}, \boldsymbol{d})$*-sequence is regular.*

Proof. For every $x \in X$ and every $m \neq 0$ in $\operatorname{supp}(\boldsymbol{b})$ we have $f = \chi^{m}_{[1]} x \in \mathrm{k}_0 \boldsymbol{b}_{[1]}(X)$, and by assumption there exists, for a given closed $(\boldsymbol{b}, \boldsymbol{d})$-sequence Y, a solution of (II) with this f that lies eventually in Y. The conclusion follows from Lemma 5.3.

8.7. Theorem. *Assume that* Y *is a closed* $(\boldsymbol{b}, \boldsymbol{d})$*-sequence for* A*, and let* q *be a* $(Y(0), \lambda)$*-splitting of* X *for some* $\lambda > 1$*. For every* $f \in \mathrm{k}_0 \boldsymbol{b}_{[1]}(X)$ *there then exists a unique solution* x *of* (II) *that lies eventually in* Y *and satisfies* $q(x(0)) = x(0)$*. This solution satisfies* $|x|_{\boldsymbol{d}} \leqq \lambda K'_Y |f|_{\boldsymbol{b}}$*, where* $K'_Y = K_{Y\boldsymbol{b},\boldsymbol{d}}(1 + 2S_{Y\boldsymbol{d}}|\chi^0|_{\boldsymbol{d}}^{-1})$.

Proof. Let $f \in \mathrm{k}_0 \boldsymbol{b}_{[1]}(X)$ be given, and choose some $\varrho > 1$. By assumption, there exists a solution x' of (II) that lies eventually in Y and satisfies $|x'|_{\boldsymbol{d}} \leqq \varrho K_Y |f|_{\boldsymbol{b}}$. We let x be the solution of (II) with $x(0) = q(x'(0))$; then we indeed have $q(x(0)) = x(0)$; also, $x' - x$ is a solution of (I) with $x'(0) - x(0) \in Y(0)$, so that it lies in Y, and x therefore lies eventually in Y. If y is any solution of (II) lying eventually in Y and with $q(y(0)) = y(0)$, $y - x$ is a solution of (I) lying (eventually) in Y, hence $y(0) - x(0) \in Y(0)$, so that $y(0) = q(y(0)) = q(x(0)) = x(0)$; we conclude that $y = x$, and uniqueness is established.

Further, with x' as above, $|x' - x|_{\boldsymbol{d}} \leqq S_{Y\boldsymbol{d}} \|x'(0) - x(0)\| = S_{Y\boldsymbol{d}} \|x'(0) - q(x'(0))\| \leqq$ $\leqq (1 + \lambda) S_{Y\boldsymbol{d}} \|x'(0)\| \leqq (1 + \lambda) S_{Y\boldsymbol{d}} |\chi^0|_{\boldsymbol{d}}^{-1} |x'|_{\boldsymbol{d}}$. Therefore $|x|_{\boldsymbol{d}} \leqq (1 + (1 + \lambda) S_{Y\boldsymbol{d}} |\chi^0|_{\boldsymbol{d}}^{-1}) \times$ $\times |x'|_{\boldsymbol{d}} \leqq \varrho \lambda K'_Y |f|_{\boldsymbol{b}}$ with K'_Y as in the statement. Since $\varrho > 1$ was arbitrary, the conclusion follows.

The relationship between admissibility of $(\boldsymbol{b}, \boldsymbol{d})$ and $(\boldsymbol{b}, \boldsymbol{d})$-sequences is described in the following theorem.

8.8. Theorem. *If the* $\not{f}$*-pair* $(\boldsymbol{b}, \boldsymbol{d})$ *is [regularly] admissible, then* $X_{0\boldsymbol{d}}$ *is a [closed]* $(\boldsymbol{b}, \boldsymbol{d})$*-sequence for* A*. Conversely, if there exists a* $(\boldsymbol{b}, \boldsymbol{d})$*-sequence for* A*, then* $(\mathrm{k}\boldsymbol{b}, \boldsymbol{d})$*, is admissible.*

Proof. If $f \in \mathrm{k}_0 \boldsymbol{b}_{[1]}(X)$, a solution x of (II) is a $\boldsymbol{d}$-solution if and only if it is eventually in $X_{0\boldsymbol{d}}$. The first part of the conclusion then follows directly from Theorem 8.1. Assume, conversely, that there exists a $(\boldsymbol{b}, \boldsymbol{d})$-sequence. Then the assumption of Lemma 8.2 is satisfied for the $\not{f}$-pair $(\mathrm{k}\boldsymbol{b}, \boldsymbol{d})$ with $\boldsymbol{f} = \mathrm{k}_0 \boldsymbol{b}_{[1]}(X)$, which is dense in $\mathrm{k}\boldsymbol{b}_{[1]}(X)$, and with $k = \varrho K_Y$ for some arbitrary $\varrho > 1$.

8.9. Corollary. *If* $(\boldsymbol{b}, \boldsymbol{d})$ *is an* $\not{f}$*-pair and* $\boldsymbol{b}$ *is lean,* $(\boldsymbol{b}, \boldsymbol{d})$ *is admissible if and only if there exists [if and only if* $X_{0\boldsymbol{d}}$ *is] a* $(\boldsymbol{b}, \boldsymbol{d})$*-sequence for* A*.*

9. The main theorems: ordinary dichotomies

We again assume that $A \in s_{[1]}(\tilde{X})$ is given. We give the main direct and converse theorems connecting $(\boldsymbol{b}, \boldsymbol{d})$-sequences and dichotomies.

9.1. Theorem. *Assume that* $(\boldsymbol{b}, \boldsymbol{d})$ *is a* $\not{t}$*-pair or a* $\not{t}^{\rightarrow}$*-pair. If* Y *is a closed* $(\boldsymbol{b}, \boldsymbol{d})$*-sequence for* A *(in particular, if* $(\boldsymbol{b}, \boldsymbol{d})$ *is regularly admissible and* $Y = X_{0\boldsymbol{d}}$*), then* Y *is regular and induces a dichotomy for* A*.*

Proof. The parenthetical assumption is indeed a particular case, by Theorem 8.8. Since the pair $(\boldsymbol{b}, \boldsymbol{d})$ is stronger than $(\Theta_m \boldsymbol{l}^1, \boldsymbol{l}^\infty)$ with $m = s_0(\boldsymbol{b})$ (Lemma 3.7), Y is a closed $(\Theta_m \boldsymbol{l}^1, \boldsymbol{l}^\infty)$-sequence for A (Lemma 8.4, (a)). By

Lemma 8.5, Y is a closed (l^1, l^∞)-sequence for A; let K'_Y be the corresponding number in Theorem 8.7.

Y is a regular covariant sequence, by Lemma 8.6; we prove that statement (c) of Theorem 7.1 — not in square brackets — holds for Y, thus establishing the conclusion. For this purpose we choose an arbitrary $\lambda' > 1$ and an arbitrary $(Y(0), \lambda')$-splitting q of X.

Let $m \in \omega_{[1]}$ be chosen, and let y be a solution of $(\mathrm{I}_{[m]})$ lying in $Y_{[m]}$ and z' a solution of (I) with $q(z'(0)) = z'(0)$. Define $x \in s(X)$ by setting $x(n) = z'(n)$, $0 \leqq n < m$, and $x(n) = y(n)$, $n \geqq m$. Then x is a solution of (II) with

$$f = \chi^m(y(m) + A(m)\, z'(m-1)) = \chi^m(y(m) - z'(m));$$

also, $x(0) = z'(0)$, so that $q(x(0)) = x(0)$; and, since y lies in $Y_{[m]}$, x lies eventually in Y. Therefore Theorem 8.7 yields

$$|x| \leqq \lambda' K'_Y |f|_1 = \lambda' K'_Y \|y(m) - z'(m)\|. \tag{9.1}$$

To establish (Di_0), assume $z' = 0$ in the preceding argument. For every $n \geqq m$, (9.1) yields $\|y(n)\| = \|x(n)\| \leqq |x| \leqq \lambda' K'_Y \|y(m)\|$. This argument only covers the cases with $m \neq 0$; if y is a solution of (I) lying in Y, the preceding argument applied to $m = 1$ and $y_{[1]}$ shows, for any $n \geqq 1$, $\|y(n)\| \leqq \lambda' K'_Y \|y(1)\| \leqq \leqq \lambda' K'_Y \|A(1)\|\, \|y(0)\|$; since finally $\|y(0)\| = \|y(0)\|$, (Di_0) now holds for all $m \in \omega$ with $N = \max\{1, \lambda' K'_Y, \lambda' K'_Y \|A(1)\|\}$ (actually, the factor $\lambda' > 1$ could be deleted, since its choice was arbitrary and y does not depend on it).

To obtain (Dii′), let z' be as above and take any $n > n_0 \geqq 0$; set $m = n$, $y = 0$. Then (9.1) yields $\|z'(n_0)\| = \|x(n_0)\| \leqq |x| \leqq \lambda' K'_Y \|z'(n)\|$. If we also include the case $n = n_0$, (Dii′) holds with $N'' = \max\{1, \lambda' K'_Y\}$.

Finally, to obtain (Diii′), let z' be as above and take any $n > 0$; set $m = n$ and let y be any solution of $(\mathrm{I}_{[n]})$ lying in $Y_{[n]}$. Then (9.1) yields $\|y(n)\| = \|x(n)\| \leqq \leqq |x| \leqq \lambda' K'_Y \|z'(n) - y(n)\|$, so that

$$\|z'(n)\| \leqq \|y(n)\| + \|z'(n) - y(n)\| \leqq (1 + \lambda' K'_Y)\, \|z'(n) - y(n)\|.$$

Since $y(n)$ is an arbitrary element of $Y(n)$, we conclude $\|z'(n)\| \leqq (1 + \lambda' K'_Y) \times \times d(Y(n), z'(n))$. If we also include $n = 0$, where $\|z'(0)\| = \|q(z'(0))\| \leqq \lambda' d(Y(0), z'(0))$, we see that (Diii′) holds with $\lambda'_0 = \max\{\lambda', 1 + \lambda' K'_Y\}$.

9.2. Theorem. *If the regular covariant sequence Y induces a dichotomy for A, then Y is a (closed) (l^1, l^∞)-sequence for A, and (l^1, l^∞) is admissible for A.*

Proof. By the assumption, statement (c) of Theorem 7.1 — not in square brackets — holds for Y, with certain numbers N, $N'' > 0$, λ', $\lambda'_0 > 1$, and a $(Y(0), \lambda')$-splitting q of X. Let $\bar{q}\colon X/Y(0) \to X$ be the mapping associated with q by Lemma 1.2.

For given $m \in \omega_{[1]}$ and $u \in X$ define $x_{m,u}$ to be the solution of (II) for $f = \chi^m_{[1]} u$ with $x_{m,u}(0) = \bar{q}(-V_Y^{-1}(m, 0)\, \Omega_Y(m) u)$. Let z' be the solution of (I) with $z'(0) = x_{m,u}(0)$ and y the solution of $(\mathrm{I}_{[m]})$ with $y(m) = x_{m,u}(m)$. It is clear that $x_{m,u}(n) = z'(n)$, $0 \leqq n < m$, and $x_{m,u}(n) = y(n)$, $n \geqq m$. Now $q(z'(0)) = q(x_{m,u}(0)) = x_{m,u}(0) = z'(0)$ (since $q \circ \bar{q} = \bar{q}$, by Lemma 1.2). Using (4.4), (5.3), and Lemma 1.2, we find

$$\Omega_Y(m)\,y(m) = \Omega_Y(m)\,x_{m,u}(m) = \Omega_Y(m)\,U(m,0)\bar{q}(-V_Y^{-1}(m,0)\,\Omega_Y(m)u) + \Omega_Y(m)\,f(m)$$
$$= V_Y(m,0)\,\Omega_Y(0)\bar{q}(-V_Y^{-1}(m,0)\,\Omega_Y(m)u) + \Omega_Y(m)u = 0\,,$$

so that $y(m) \in Y(m)$; hence y lies in $Y_{[m]}$, and $x_{m,u}$ lies eventually in Y.

Now (II) with $n=m$ yields $u = y(m) + A(m)\,z'(m-1) = y(m) - z'(m)$. It follows that $d(Y(m), z'(m)) = d(Y(m), u) \leqq \|u\|$. We may now apply ($\mathrm{Di}_0$), (Dii′), (Diii′) and find

$$(9.2)\quad \begin{aligned} &\|x_{m,u}(n)\| = \|z'(n)\| \leqq N''\,\|z'(m)\| \leqq \lambda_0' N''\,d(Y(m), z'(m)) \leqq \lambda_0' N''\,\|u\|\,, \\ &\qquad\qquad 0 \leqq n < m\,, \\ &\|x_{m,u}(n)\| = \|y(n)\| \leqq N\,\|y(m)\| = N\,\|u + z'(m)\| \leqq (1+\lambda_0')\,N\,\|u\|\,, \qquad n \geqq m\,. \end{aligned}$$

Thus $x_{m,u} \in l^\infty(X)$, with $|x_{m,u}| \leqq k\,\|u\|$, where $k = \max\{\lambda_0' N'', (1+\lambda_0')\,N\}$.

Let now $f \in \mathrm{k}_0 l^1_{[1]}(X)$ be given, and set $x = \sum_{m=1}^{s(f)} x_{m,f(m)}$. Since $f = \sum_{m=1}^{s(f)} \chi^m_{[1]} f(m)$, x is a solution of (II); it lies eventually in Y, since each summand does. Further, $x \in l^\infty(X)$, and $|x| \leqq \sum_{m=1}^{s(f)} |x_{m,f(m)}| \leqq k \sum_{m=1}^{s(f)} \|f(m)\| = k\,|f|_1$, so that Y is indeed an (l^1, l^∞)-sequence, with $K_Y \leqq k$. The admissibility of (l^1, l^∞) follows from Corollary 8.9, since l^1 is lean.

9.3. Theorem. *A closed covariant sequence Y is regular and induces a dichotomy for A if and only if it is a (closed) (l^1, l^∞)-sequence. If the covariant sequence X_0 is closed, it is regular and induces a dichotomy for A if and only if (l^1, l^∞) is (regularly) admissible.*

Proof. Theorems 9.1 and 9.2.

Proof of Theorem 7.4. Theorem 9.3 and Lemma 8.4, (b) applied to the pair (l^1, l^∞).

Proof of Theorem 7.5. If $X_{00} = Y$, there is nothing to prove. We may therefore assume that Z_0 is a non-trivial finite-dimensional subspace of X such that $X_{00}(0) \cap Z_0 = \{0\}$, $X_{00}(0) + Z_0 = Y(0)$. Consider the real-valued function μ defined on the compact set $S = \{z \in Z_0 \colon \|z\| = 1\}$ by $\mu(z) = \inf_{n\in\omega} \|U(n,0)\,z\|$ (observe that $U(\cdot,0)\,z$ is the solution of (I) with initial value z). Since $S \subset Y(0)$, condition (Di) for the dichotomy induced by Y implies $|\mu(z_2) - \mu(z_1)| \leqq \leqq \sup_{n\in\omega} \|U(n,0)\,(z_2 - z_1)\| \leqq N\,\|z_2 - z_1\|$, so that μ is continuous. Since $X_{00}(0) \cap S = \theta$, another application of (Di) yields $0 < \limsup_{n\to\infty} \|U(n,0)\,z\| \leqq N\,\mu(z)$ for every $z \in S$. Therefore μ has a positive minimum, say $\mu_0 > 0$, on S. We conclude that for every solution z_0 of (I) with $z_0(0) \in Z_0$ we have $\|z_0(n)\| \geqq \mu_0\,\|z_0(0)\|$ for all $n \in \omega$.

Now by Theorem 9.3 Y is an (l^1, l^∞)-sequence; let K_Y be the corresponding constant. We claim that X_{00} is also an (l^1, l^∞)-sequence, and Theorem 9.3 then yields the conclusion.

To establish our claim, let $f \in \mathrm{k}_0 l^1(X)$ and $\varrho > 1$ be given, and set $s = s(f)$. There exists a solution x of (II) that lies eventually in Y (hence $x(s) \in Y(s)$) and satisfies $|x| \leqq \varrho K_Y |f|_1$. Since X_{00} is regular by assumption, we may consider the element $\eta = V_{X_{00}}^{-1}(s, 0)\, \Omega_{X_{00}}(s)\, x(s) \in V_{X_{00}}^{-1}(s, 0)\, \Omega_{X_{00}}(s)\, Y(s) = \Omega_{X_{00}}(0)\, Y(0)$ (cf. proof of Lemma 5.4). Let z_0 be the unique solution of (I) with $z_0(0) \in Z_0$, $\Omega_{X_{00}}(0)\, z_0(0) = \eta$. Then $z = x - z_0$ is a solution of (II). Now

$$\begin{aligned} \Omega_{X_{00}}(s)\, z_0(s) &= \Omega_{X_{00}}(s)\, U(s, 0)\, z_0(0) = V_{X_{00}}(s, 0)\, \Omega_{X_{00}}(0)\, z_0(0) \\ &= V_{X_{00}}(s, 0)\, \eta = \Omega_{X_{00}}(s)\, x(s), \end{aligned}$$

so that $z(s) \in X_{00}(s)$ and z lies eventually in X_{00}. Thus $z \in \boldsymbol{l}_0^\infty(X)$, and

$$|z_0| \leqq N \|z_0(0)\| \leqq N \mu_0^{-1} \limsup_{n \to \infty} \|z_0(n)\| = N \mu_0^{-1} \limsup_{n \to \infty} \|x(n)\| \leqq N \mu_0^{-1} |x|.$$

Therefore $|z| \leqq |x| + |z_0| \leqq (1 + N\mu_0^{-1})\,|x| \leqq (1 + N\mu_0^{-1})\, \varrho K_Y |f|_1$, so that X_{00} is indeed an $(\boldsymbol{l}^1, \boldsymbol{l}^\infty)$-sequence, with $K_{X_{00}} \leqq (1 + N\mu_0^{-1})\, K_Y$.

9.4. Theorem. *If the covariant sequence X_{00} is closed, it is regular and induces a dichotomy for A if and only if it is an $(\boldsymbol{l}^1, \boldsymbol{l}_0^\infty)$-sequence (indeed, the only one); and if and only if $(\boldsymbol{l}^1, \boldsymbol{l}_0^\infty)$ is (regularly) admissible.*

Proof. Theorem 9.3 and Lemma 8.4, (c) — since $\mathrm{k}\,\boldsymbol{l}^\infty = \boldsymbol{l}_0^\infty$ — and, for the last part of the statement, Corollary 8.9.

10. The main theorems: exponential dichotomies

We assume, as before, $A \in \boldsymbol{s}_{[1]}(\tilde{X})$ given. We prove the analogues, for exponential dichotomies, of Theorem 9.1 and 9.2; we require an elementary computation.

10.1. Lemma. *If $\varphi \in \boldsymbol{s}_{[m]}$, $\varphi \geqq 0_{[m]}$, and there exists a positive integer p and a number M such that $\varphi(n_0 + p) \leqq e^{-1} \varphi(n_0)$ and $\varphi(n) \leqq M \varphi(n_0)$, $n_0 \leqq n < n_0 + p$ for every $n_0 \geqq m$, then $\varphi(n) \leqq M e^{1 - p^{-1}(n - n_0)} \varphi(n_0)$ for all $n \geqq n_0 \geqq m$.*

Proof. For given $n \geqq n_0 \geqq m$, let k be the integer such that $n_0 + kp \leqq n < n_0 + (k+1)p$. Then $\varphi(n) \leqq M \varphi(n_0 + kp) \leqq M e^{-k} \varphi(n_0) \leqq M e^{-(p^{-1}(n - n_0) - 1)} \varphi(n_0)$.

10.2. Theorem. *Assume that $(\boldsymbol{b}, \boldsymbol{d})$ is a ℓ-pair or $\ell^{\rightarrow}$-pair that is not weaker than $(\boldsymbol{l}^1, \boldsymbol{l}_0^\infty)$. If Y is a closed $(\boldsymbol{b}, \boldsymbol{d})$-sequence for A (in particular, if $(\boldsymbol{b}, \boldsymbol{d})$ is regularly admissible and $Y = X_{0\boldsymbol{d}}$), then Y is regular and induces an exponential dichotomy for A.*

Proof. The parenthetical assumption is again a particular case, by Theorem 8.8. By Theorem 9.1, Y is a regular covariant sequence and induces an ordinary dichotomy for A; statement (c) of Theorem 7.1 — not in square brackets — is therefore verified for numbers N_0, $N_0'' > 0$ (instead of N, N''), λ', $\lambda_0' > 1$, and a $(Y(0), \lambda')$-splitting q of X. To establish the conclusion it is therefore sufficient, by Theorem 7.1, to show that numbers ν, ν', N, $N'' > 0$ exist such that (Ei$_0$), (Eii$'$) are satisfied for y, z' as in condition (c) of that Theorem, for the same λ' and q.

The assumption on $(\boldsymbol{b}, \boldsymbol{d})$ implies that: *either* (Case I) $\boldsymbol{d}$ is not weaker than $\boldsymbol{l}_0^\infty$, and we replace $(\boldsymbol{b}, \boldsymbol{d})$ without loss by the weaker pair $(\Theta_m \boldsymbol{l}^1, \boldsymbol{d})$ with $m = s_0(\boldsymbol{b})$ (Lemma 3.7), and hence, on account of Lemma 8.5, by $(\boldsymbol{l}^1, \boldsymbol{d})$; *or* (Case II) $\boldsymbol{b}$ is not stronger than (equivalently, norm-equivalent to) $\boldsymbol{l}^1$, and we may replace $(\boldsymbol{b}, \boldsymbol{d})$ by the weaker pair $(\boldsymbol{b}, \boldsymbol{l}^\infty)$ (Lemma 3.7). The two cases are not mutually exclusive, of course.

Case I. Let K'_Y be the number in Theorem 8.7 for Y as an $(\boldsymbol{l}^1, \boldsymbol{d})$-sequence. Choose the positive integer p so large that $\beta(\boldsymbol{d}; p) \geqq e\lambda' K'_Y \max\{N_0^2, N''_0\}$, as we may by Lemma 3.7.

Let y be a solution of $(\mathrm{I}_{[m]})$ lying in $Y_{[m]}$ for a given $m \in \omega$, and let $n_0 \geqq m$ be given. Define $x \in \boldsymbol{s}(X)$ by $x(n) = 0$, $0 \leqq n \leqq n_0$, and $x(n) = y(n)$, $n > n_0$. Then x is a solution of (II) with $f = \chi_{[1]}^{n_0+1} y(n_0+1)$; it lies eventually in Y, and $x(0) = 0 = q(x(0))$. We can therefore apply Theorem 8.7 and find, using (Di_0) for $y_{[n_0]}$,

$$\text{(10.1)} \qquad |x|_{\boldsymbol{d}} \leqq \lambda' K'_Y |f|_1 = \lambda' K'_Y \|y(n_0+1)\| \leqq \lambda' K'_Y N_0 \|y(n_0)\| .$$

Now, using (Di_0) for each $y_{[n]}$, we obtain

$$\|x\| \geqq \sum_{n_0+1}^{n_0+p} \chi^n \|y(n)\| \geqq N_0^{-1} \|y(n_0+p)\| \sum_{n_0+1}^{n_0+p} \chi^n .$$

Taking the $\boldsymbol{d}$-norms and combining with (10.1), we have $\|y(n_0+p)\| \leqq$ $\leqq N_0 \beta^{-1}(\boldsymbol{d}; p) |x|_{\boldsymbol{d}} \leqq e^{-1} \|y(n_0)\|$. We may therefore apply Lemma 10.1 to $\|y\|$ with $M = N_0$, and obtain (Ei), a fortiori (Ei_0), with $\nu = p^{-1}$, $N = eN_0$.

Let z' be a non-trivial solution of (I) with $q(z'(0)) = z'(0)$, and let $n_0 \geqq 0$ be given. Define $x \in \boldsymbol{s}(X)$ by $x(n) = z'(n)$, $0 \leqq n < n_0 + p$, and $x(n) = 0$, $n \geqq n_0 + p$. Then x is a solution of (II) for $f = \chi_{[1]}^{n_0+p} A(n_0+p)\, z'(n_0+p-1) = -\chi_{[1]}^{n_0+p} z'(n_0+p)$; it lies eventually in Y, and $q(x(0)) = x(0)$. By Theorem 8.7 $|x|_{\boldsymbol{d}} \leqq \lambda' K'_Y |f|_1$ $= \lambda' K'_Y \|z'(n_0+p)\|$. Using (Dii′), $\|x\| \geqq \sum_{n_0}^{n_0+p-1} \chi^n \|z'(n)\| \geqq N_0''^{-1} \|z'(n_0)\| \sum_{n_0}^{n_0+p-1} \chi^n$. Taking $\boldsymbol{d}$-norms and combining, we have $\|z'(n_0+p)\| \geqq (\lambda' K'_Y)^{-1} |x|_{\boldsymbol{d}} \geqq$ $\geqq (\lambda' K'_Y N''_0)^{-1} \beta(\boldsymbol{d}; p) \|z'(n_0)\| \geqq e \|z'(n_0)\|$. We may therefore apply Lemma 10.1 to $\|z'\|^{-1}$ and find (Eii′) with $\nu' = p^{-1}$, $N'' = eN''_0$.

Case II. Let K'_Y be the number in Theorem 8.7 for Y as a $(\boldsymbol{b}, \boldsymbol{l}^\infty)$-sequence. There exists $\varphi \in \boldsymbol{b}$, $\varphi \geqq 0$. $|\varphi|_{\boldsymbol{b}} = 1$, such that $\varphi \notin \boldsymbol{l}^1$ (Lemma 3.2); choose the positive integer p so large that $\sigma = \sum_1^p \varphi(n) \geqq e\lambda' K'_Y \max\{N_0, N''_0\}$.

Let y be a solution of $(\mathrm{I}_{[m]})$ lying in $Y_{[m]}$, for some $m \in \omega$, and let $n_0 \geqq m$ be given. Define $x \in \boldsymbol{s}(X)$ by $x(n) = 0$, $0 \leqq n \leqq n_0$; $x(n) = \sigma^{-1} y(n) \sum_{j=n_0+1}^{n} \varphi(j-n_0)$, $n_0 \leqq n \leqq n_0 + p$; and $x(n) = y(n)$, $n \geqq n_0 + p$. Thus x is a solution of (II) with $f = \sigma^{-1} \sum_{n=n_0+1}^{n_0+p} \chi_{[1]}^n \varphi(n-n_0)\, y(n)$. By (Di_0) applied to $y_{[n_0]}$, $\|f\| \leqq \sigma^{-1} N_0 \|y(n_0)\| \times$ $\times (T^{n_0}\varphi)_{[1]}$, so that $|f|_{\boldsymbol{b}} \leqq \sigma^{-1} N_0 \|y(n_0)\|$. Now x lies eventually in Y, and $x(0) = 0 = q(x(0))$. Thus Theorem 8.7 yields $\|y(n_0+p)\| = \|x(n_0+p)\| \leqq |x| \leqq$ $\leqq \lambda' K'_Y |f|_{\boldsymbol{b}} \leqq e^{-1} \|y(n_0)\|$. The proof of (Ei_0) is completed as in Case I.

Let z' be a non-trivial solution of (I) with $q(z'(0)) = z'(0)$, and let $n_0 \geqq 0$ be given. Define $x \in \boldsymbol{s}(X)$ by $x(n) = z'(n)$, $0 \leqq n \leqq n_0$; $x(n) = \sigma^{-1} z'(n) \sum_{j=n+1}^{n_0+p} \varphi(j-n_0)$, $n_0 \leqq n \leqq n_0 + p$; and $x(n) = 0$, $n \geqq n_0 + p$. Thus x is a solution of (II) with $f = -\sigma^{-1} \sum_{n=n_0+1}^{n_0+p} \chi_{[1]}^n \varphi(n-n_0)\, z'(n)$. By (Dii′), $\|f\| \leqq \sigma^{-1} N''_0 \|z'(n_0+p)\| (T^{n_0}\varphi)_{[1]}$, so that $|f|_b \leqq \sigma^{-1} N_0'' \|z'(n_0+p)\|$. Now x lies eventually in Y, and $q(x(0)) = x(0)$. Thus Theorem 8.7 yields $\|z'(n_0+p)\| \geqq \sigma N_0''^{-1} |f|_b \geqq \sigma(\lambda' K_Y' N_0'')^{-1} |x| \geqq e\|x(n_0)\| = e\|z'(n_0)\|$. The proof of (Eii′) is completed as in Case I.

10.3. Theorem. *If the regular covariant sequence Y induces an exponential dichotomy for A, then every ℓ-pair or $\ell^{\rightarrow}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ with $\boldsymbol{b}$ stronger than $\boldsymbol{d}$ is (regularly) admissible for A; and $Y = X_0 = X_{0\boldsymbol{d}}$ is a closed $(\boldsymbol{b}, \boldsymbol{d})$-sequence for A for every ℓ-pair or $\ell^{\rightarrow}$-pair $(\boldsymbol{b}, \boldsymbol{d})$ with $\mathrm{k}\boldsymbol{b}$ stronger than $\boldsymbol{d}$.*

Proof. By Lemma 7.6, $Y = X_{0\boldsymbol{d}}$ under either condition on $(\boldsymbol{b}, \boldsymbol{d})$. The last part of the statement follows from the first: indeed, the admissibility of $(\mathrm{k}\boldsymbol{b}, \boldsymbol{d})$ implies that $Y = X_{0\boldsymbol{d}}$ is a $(\mathrm{k}\boldsymbol{b}, \boldsymbol{d})$-sequence (Theorem 8.8), hence a $(\boldsymbol{b}, \boldsymbol{d})$-sequence, since $\boldsymbol{b}$ is stronger than $\mathrm{lck}\boldsymbol{b} = \mathrm{lc}\boldsymbol{b}$ (Lemma 8.4, (a) and Lemma 3.4). The theorem will be proved if we show that $(\boldsymbol{d}, \boldsymbol{d})$ is admissible for every $\boldsymbol{d} \in \mathrm{b}\ell$.

We proceed to construct, precisely as in the proof of Theorem 9.2, the solution $x_{m,u}$ of (II) with $f = \chi_{[1]}^m u$ for each $m \in \omega_{[1]}$, $u \in X$; but now the use of (Ei$_0$), (Eii′), (Diii′) allows us to replace (9.2) by

$$(10.2)\qquad \begin{aligned} \|x_{m,u}(n)\| &\leqq \lambda_0' N'' e^{-\nu'(m-n)} \|u\|, & 0 \leqq n < m, \\ \|x_{m,u}(n)\| &\leqq (1+\lambda_0') N e^{-\nu(n-m)} \|u\|, & n \geqq m. \end{aligned}$$

Let now $f \in \boldsymbol{d}_{[1]}(X)$ be given. We have $\sum_{j=0}^{\infty} e^{-\nu j} |T^j f_{[0]}|_{\boldsymbol{d}} = (1-e^{-\nu})^{-1} |f|_{\boldsymbol{d}} < \infty$, $\sum_{j=1}^{\infty} e^{-\nu' j} |T^{-j} f_{[0]}|_{\boldsymbol{d}} \leqq (e^{\nu'}-1)^{-1} |f|_{\boldsymbol{d}} < \infty$. Since $\boldsymbol{d}$ is complete and stronger than $\boldsymbol{s}$, it follows that the sum

$$\varphi = (1+\lambda_0') N \sum_{j=0}^{\infty} e^{-\nu j} \|T^j f_{[0]}\| + \lambda_0' N'' \sum_{j=1}^{\infty} e^{-\nu' j} \|T^{-j} f_{[0]}\|,$$

taken pointwise, exists and is finite-valued, and $\varphi \in \boldsymbol{d}$. By (10.2),

$$\begin{aligned} \sum_{m=1}^{\infty} \|x_{m,f(m)}(n)\| &\leqq (1+\lambda_0') N \sum_{m=1}^{n} e^{-\nu(n-m)} \|f(m)\| + \lambda_0' N'' \sum_{m=n+1}^{\infty} e^{-\nu'(m-n)} \|f(m)\| \\ &= (1+\lambda_0') N \sum_{j=0}^{n-1} e^{-\nu j} \|T^j f_{[0]}(n)\| + \lambda_0' N'' \sum_{j=1}^{\infty} e^{-\nu' j} \|T^{-j} f_{[0]}(n)\| \leqq \\ &\leqq \varphi(n), \qquad n \in \omega. \end{aligned}$$

Therefore the series $\sum_{m=1}^{\infty} x_{m,f(m)}$ converges pointwise to a solution x of (II) with the given f, and $\|x\| \leqq \varphi$, whence $x \in \boldsymbol{d}(X)$. Since $f \in \boldsymbol{d}_{[1]}(X)$ was arbitrary, $(\boldsymbol{d}, \boldsymbol{d})$ is admissible.

We need not formulate a theorem, analogous to Theorem 9.3, that would combine the preceding results into necessary and sufficient conditions. We do show, however, that the choice of admissible pairs in Theorem 10.3 was not capricious, but is actually best possible under rather mild assumptions. A corresponding remark about the choice of pairs $(\boldsymbol{b}, \boldsymbol{d})$ for which Y is a $(\boldsymbol{b}, \boldsymbol{d})$-sequence would then follow, since for every such pair $(\mathrm{k}\boldsymbol{b}, \boldsymbol{d})$ is admissible (Theorem 8.8).

10.4. Theorem. *Assume that the regular covariant sequence Y induces an exponential dichotomy for A, and that either $Y(n)=\{0\}$ for at most finitely many $n\in\omega$, or $A\in \boldsymbol{l}^{\infty}_{[1]}(\tilde{X})$. If the ℓ-pair or $\ell^{\rightarrow}$-pair $(\boldsymbol{b},\boldsymbol{d})$ is admissible, then $\boldsymbol{b}$ is stronger than $\boldsymbol{d}$.*

Proof. Case I: $Y(n)=\{0\}$ for only finitely many $n\in\omega$. There exists $m\in\omega$ such that $Y(n)\neq\{0\}$ for all $n\geqq m$.

There exists a function $u\in \boldsymbol{s}_{[m]}(X)$ such that $u(n)\in Y(n)\backslash\{0\}$, $n\geqq m$, and $A(n)\,u(n-1)=-u(n)$ or 0, $n>m$ (the indices n for which the latter alternative holds, together with the index m, are termed *renewal indices*): indeed, u is constructed inductively by choosing $u(m)\in Y(m)\backslash\{0\}$ arbitrarily, and setting $u(n)=-A(n)\,u(n-1)$ if the second member is $\neq 0$ (whence $u(n)\in Y(n)\backslash\{0\}$ by (5.1)), and choosing $u(n)\in Y(n)\backslash\{0\}$ arbitrarily otherwise. If $n\geqq n_0\geqq m$ and none of $n_0+1,\ldots,n$ is a renewal index, then $u(n)=U(n,n_0)\,u(n_0)$, and (Ei$_0$) yields

$$\|u(n)\|\leqq Ne^{-\nu(n-n_0)}\|u(n_0)\|\,. \tag{10.3}$$

Let $\varphi\in\boldsymbol{b}$, $\varphi\geqq 0$, be given; φ is of course bounded, by Lemma 3.7. Define $x\in\boldsymbol{s}(X)$ by $x(n)=0$, $0\leqq n<m$; and $x(n)=u(n)\sum_{k=m_n}^{n}\varphi(k)\|u(k)\|^{-1}$, $n\geqq m$, where m_n is the largest renewal index $\leqq n$. It is then easily verified that x is a solution of (II) with $f=(\varphi_{[m]}\operatorname{sgn}u)_{[1]}$, so that $\|f\|\leqq\varphi_{[1]}$, whence $f\in\boldsymbol{b}_{[1]}(X)$. Since $(\boldsymbol{b},\boldsymbol{d})$ is admissible, there exists a $\boldsymbol{d}$-solution x' of (II); and x' is a fortiori bounded. Now by (10.3) $\|x(n)\|\leqq N\sum_{k=m_n}^{n}\varphi(k)\,e^{-\nu(n-k)}\leqq N(1-e^{-\nu})^{-1}|\varphi|$, $n\geqq m$, so that x is bounded and differs from x' by a solution of (I) lying in $X_0=Y=X_{0d}$ (Lemma 7.6); therefore x is itself a $\boldsymbol{d}$-solution. But $\|x(n)\|\geqq\varphi(n)$, $n\geqq m$, so that $\varphi_{[m]}\in\boldsymbol{d}_{[m]}$; hence $\varphi\in\boldsymbol{d}$ (Lemma 3.9). Since φ was an arbitrary positive element of $\boldsymbol{b}$, $\boldsymbol{b}$ is stronger than $\boldsymbol{d}$ (Lemma 3.2 and condition (f)).

Case II: $A\in\boldsymbol{l}^{\infty}_{[1]}(\tilde{X})$. If $Y(0)=X$, then $Y=X$ since Y is regular (Lemma 5.4), and we are in Case I. We therefore assume without loss that $Y(0)\neq X$; the splitting q in condition (c) of Theorem 7.1 is therefore non-trivial, and there exists a non-trivial solution z' of (I) with $q(z'(0))=z'(0)$.

Let $\varphi\in\boldsymbol{b}$, $\varphi\geqq 0$ be given. Define $x\in\boldsymbol{s}(X)$ by $x(n)=z'(n)\sum_{k=n+1}^{\infty}\varphi(k)\|z'(k)\|^{-1}$, $n\in\omega$; the series converges on account of (Eii′), and in fact $\|x(n)\|\leqq$ $\leqq N''\sum_{k=n+1}^{\infty}\varphi(k)\,e^{-\nu'(k-n)}\leqq N''(e^{\nu'}-1)^{-1}|\varphi|$, so that x is bounded. Now x is a solution of (II) with $f=-(\varphi\operatorname{sgn}z')_{[1]}$, so that $\|f\|=\varphi_{[1]}$, $f\in\boldsymbol{b}_{[1]}(X)$. It follows as in Case I that $x\in\boldsymbol{d}(X)$. Now $\|x(n)\|\geqq\varphi(n+1)\|z'(n)\|\,\|z'(n+1)\|^{-1}\geqq$ $\geqq\varphi(n+1)\|A(n+1)\|^{-1}$, $n\in\omega$, so that $\Theta_1\varphi\leqq|A|\,\|T^{+}x\|$, whence $\Theta_1\varphi\in\boldsymbol{d}$, whence $\varphi\in\boldsymbol{d}$. The conclusion follows as in Case I.

10.5. Corollary. *Assume that X is finite-dimensional and that the regular covariant sequence Y induces an exponential dichotomy for A; if $Y=\{0\}$ (constant sequence), assume in addition that $A\in\boldsymbol{l}^{\infty}_{[1]}(\tilde{X})$. If the ℓ-pair or $\ell^{\rightarrow}$-pair $(\boldsymbol{b},\boldsymbol{d})$ is admissible, then $\boldsymbol{b}$ is stronger than $\boldsymbol{d}$.*

Proof. Theorem 10.4 and Lemma 5.4 (last part).

It is not possible in general to remove the additional conditions in Theorem 10.4 or Corollary 10.5:

10.6. Example. Assume that $X=R$, $A(n)\neq 0$, $n\in\omega_{[1]}$, and $\lim_{n\to\infty}|A(n)|=\infty$. Then the constant sequence $\{0\}$ induces an exponential dichotomy: indeed, set $a=\min_{n\geqq 1}|A(n)|>0$, and assume that $|A(n)|\geqq e$ for all $n\geqq n_1$. Then $|x(n)| = |x(n_0)| \prod_{k=n_0+1}^{n} |A(k)| \geqq (ae^{-1})^{n_1} e^{n-n_0}|x(n_0)|$ for every solution x of (I) and all $n\geqq n_0\geqq 0$. Therefore (l^∞, l^∞) is admissible (Theorem 10.3). If $f\in l^\infty_{[1]}$ and x is a (or rather, the unique) l^∞-solution of (II), we have

$$|x(n)|=|A(n+1)|^{-1}|f(n+1)-x(n+1)|\leqq|A(n+1)|^{-1}(|f|+|x|)\to 0$$

as $n\to\infty$, so that $x\in l^\infty_0$, and (l^∞, l^∞_0) is admissible.

11. Another kind of inhomogeneous equation

In the Introduction we pointed out the inherent asymmetry of the theory of our difference equations, arising from the possibility that the operators $A(n)$ might not be invertible; an additional, though related, asymmetry stems from the fact that the equation "proceeds" from the "bounded" end, 0, of the domain ω to the "unbounded" end, ∞. Reversal of either of these asymmetries gives rise to a parallel theory, which we do not describe.

Still another asymmetry affects only the nonhomogeneous equation: namely, the fact that the independent term $f(n)$ is added *after* $A(n)$ has acted on $x(n-1)$. It is clear, incidentally, that if all the $A(n)$ are invertible this asymmetry is closely related to the second type mentioned above. We shall therefore consider very briefly, in addition to (I) and (II), the equation

$$\text{(III)}\qquad x(n)+A(n)\,(x(n-1)-f(n-1))=0,\qquad n=1,2,\ldots,$$

where $f\in s(X)$, and its restrictions $(\mathrm{III}_{[m]})$ to $\omega_{[m]}$, $m\in\omega$. The algebraic structure is very simple: $x\in s_{[m]}(X)$ is a solution of $(\mathrm{III}_{[m]})$ if and only if x is a solution of $(\mathrm{II}_{[m]})$ with $f(n)$ replaced by $A(n)f(n-1)$, i.e., f replaced by $A(T^+f)_{[1]}$; in other words, if and only if (cf. (4.4), (4.1))

$$(11.1)\qquad x(n)=U(n,m)\,x(m)-\sum_{i=m}^{n-1}U(n,i)\,f(i)\,,\qquad n\in\omega_{[m]}\,;$$

and such a solution satisfies (cf. (4.6))

$$(11.2)\qquad x(n)=U(n,n_0)\,x(n_0)-\sum_{i=n_0}^{n-1}U(n,i)\,f(i)\,,\qquad n\geqq n_0\geqq m\,.$$

Most of the results for (II) discussed in the preceding sections are equally valid for (III), with the obvious modifications in statement and proof; indeed, the modifications in the proofs hinge on the shift of summation indices (and change of sign) in (11.1) and (11.2) with respect to (4.4) and (4.6). We summarize this analogy and establish one fundamental result for whose *proof* the analogy is not perfect.

An ℓ-pair $(\boldsymbol{b}, \boldsymbol{d})$ is *left-admissible (for A)* if (III) has a $\boldsymbol{d}$-solution for every $f \in \boldsymbol{b}(X)$, *regularly left-admissible* if, in addition, $X_{0\boldsymbol{d}}$ is closed.

11.1. Theorem. *If the* ℓ*-pair* $(\boldsymbol{b}, \boldsymbol{d})$ *is left-admissible for* A, *there exists a number* $K>0$ *such that for every* $f \in \boldsymbol{b}(X)$ *and every* $\varrho>1$ *there exists a* $\boldsymbol{d}$–*solution* x *of* (II) *with* $|x|_{\boldsymbol{d}} \leqq \varrho K |f|_{\boldsymbol{b}}$.

Proof. In contradistinction to equation (II), knowledge of a solution x of (III) does not completely determine f unless all the $A(n)$ are invertible; we therefore require a slight modification of the approach in Theorem 8.1. We norm the outer direct sum $\boldsymbol{d}(X) \oplus \boldsymbol{b}(X)$ with the norm $\|\cdot \oplus \cdot\| = |\cdot|_{\boldsymbol{d}} + |\cdot|_{\boldsymbol{b}}$; it is a Banach space. Let Y be the linear manifold in this space of those elements $x \oplus f$ such that x is a solution of (III) with that f. The linear mapping $x \oplus f \to f$: $Y \to \boldsymbol{b}(X)$ is surjective by assumption and its graph is obviously closed in $(\boldsymbol{s}(X) \times \boldsymbol{s}(X)) \times \boldsymbol{s}(X)$, a fortiori in the stronger topology of $(\boldsymbol{d}(X) \oplus \boldsymbol{b}(X)) \times \times \boldsymbol{b}(X)$. By the Open-Mapping Theorem, there exists $k>0$ such that for each $f \in \boldsymbol{b}(X)$ there exists $x \oplus f \in Y$ that satisfies $|x|_{\boldsymbol{d}} + |f|_{\boldsymbol{b}} \leqq k |f|_{\boldsymbol{b}}$. The conclusion follows, with K the infimum of the possible values of $k-1$.

The other required definition is as follows: if $(\boldsymbol{b}, \boldsymbol{d})$ is an ℓ-pair, a covariant sequence Y is a *left-*$(\boldsymbol{b}, \boldsymbol{d})$*-sequence (for A)* if $Y \subset X_{0\boldsymbol{d}}$ and if there exists a number $K_Y>0$ such that for any $f \in \mathrm{k}_0 \boldsymbol{b}(X)$ and every $\varrho>1$ there exists a solution of (III) that lies eventually in Y and satisfies $|x|_{\boldsymbol{d}} \leqq \varrho K_Y |f|_{\boldsymbol{b}}$. If Y is also closed, it is a *closed left-*$(\boldsymbol{b}, \boldsymbol{d})$*-sequence.*

11.2. Theorem. *All theorems, lemmas, and corollaries in Sections 5—10 remain valid with* (II) *replaced by* (III), *admissibility by left-admissibility*, $(\cdot, \cdot)$-*sequence by left-*$(\cdot, \cdot)$*-sequence, and the appropriate subscripts* [1] *deleted; with the exception of Theorem 10.4 and Corollary 10.5.*

Proof. For Theorem 8.1, see Theorem 11.1 above. The rest (so far as they are at all concerned with the inhomogeneous equation) follow, as suggested above, by slight modifications of the original proofs, taking into account the shift of summation indices in (11.1) and (11.2). We do not give them in detail.

As for Theorem 10.4 and Corollary 10.5, they are replaced by the following result, of which we omit the (analogous) proof.

11.3. Theorem. *Assume that the regular covariant sequence* Y *induces an exponential dichotomy for* A, *and that* $Y \neq X$ *(constant sequence). If the* ℓ*-pair or* $\ell^{\rightarrow}$*-pair* $(\boldsymbol{b}, \boldsymbol{d})$ *is left-admissible, then* $\boldsymbol{b}$ *is stronger than* $\boldsymbol{d}$.

11.4. Example. To obtain a similar result for the case $Y=X$ it seems necessary to assume that $A(n)$ is always invertible, and that $A^{-1} \in \boldsymbol{l}^{\infty}_{[1]}(\tilde{X})$. Indeed, if $A=0_{[1]}$, $Y=X$ certainly induces an exponential dichotomy; but for *any* $f \in \boldsymbol{s}(X)$ and *any* solution x of (III), $x(n)=0$ for all $n \geqq 1$; in particular, even $(\boldsymbol{l}^{\infty}, \boldsymbol{l}^{1})$ is left-admissible. It is easy to construct an example analogous to Example 10.5 with an invertible-valued A such that $A^{-1} \notin \boldsymbol{l}^{\infty}_{[1]}(\tilde{X})$, for which $(\boldsymbol{l}^{\infty}, \boldsymbol{l}^{\infty}_{0})$ is left-admissible.

References

1. HARTMAN, P.: Ordinary differential equations. New York: Wiley 1964.
2. HILLE, E., and R. S. PHILLIPS: Functional analysis and semi-groups. Am. Math. Soc. Colloquium Publ. Vol. **31** (revised ed.). Am. Math. Soc., Providence 1957.
3. LI, TA: Die Stabilitätsfrage bei Differenzengleichungen. Acta Math. **63**, 99—141 (1934).
4. MASSERA, J. L., and J. J. SCHÄFFER: Linear differential equations and function spaces. New York: Academic Press 1966.
5. PERRON, O.: Die Stabilitätsfrage bei Differentialgleichungen. Math. Z. **32**, 703—728 (1930).
6. SCHÄFFER, J. J.: Function spaces with translations. Math. Ann. **137**, 209—262 (1959).

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