COMP3026: LINEAR ALGEBRA EXERCISES

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1. Linear dependence

(1) Is the vector $\mathbf{v} \in \mathbb{R}^3$ in the span of the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$?

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}?$$

(2) The toy example in the first set of slides that mentioned recommender systems had written out the following SVD $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}}_{\mathbf{A}} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}}_{\mathbf{U}} \underbrace{\begin{pmatrix} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}^T}_{\mathbf{V}^T}.$$

(a) Let the columns of $\mathbf{U}$ be denoted $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, and those of $\mathbf{V}$ denoted $\mathbf{v}_i$, $i = 1, \dots, 5$. Show that these singular vectors $\{\mathbf{u}_i\}$ and $\{\mathbf{v}_i\}$ form orthonormal sets.

(b) Calculate $\mathbf{U}\mathbf{U}^T$, $\mathbf{U}^T\mathbf{U}$, $\mathbf{V}\mathbf{V}^T$ and $\mathbf{V}^T\mathbf{V}$.

(c) Express the original individual user $\tilde{\mathbf{u}}_2$ and movie vector $\tilde{\mathbf{v}}_3$ as linear combinations of these singular vectors as basis vectors. In other words, for

$$\tilde{\mathbf{u}}_2 \triangleq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \tilde{\mathbf{v}}_3 \triangleq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Find $\tilde{\mathbf{u}}_2 = \alpha_1^{(2)}\mathbf{u}_1 + \alpha_2^{(2)}\mathbf{u}_2 + \alpha_3^{(2)}\mathbf{u}_3$, $\tilde{\mathbf{v}}_3 = \beta_1^{(3)}\mathbf{v}_1 + \cdots + \beta_5^{(3)}\mathbf{v}_5$ and $\tilde{\mathbf{v}}_3 = \beta_1^{(3)}\mathbf{v}_1 + \cdots + \beta_5^{(3)}\mathbf{v}_5$. (Note the difference between $\tilde{\mathbf{u}}_i$ and $\mathbf{u}_i$, etc.)

2. Matrix polynomials

(1) For $\mathbf{A} = \begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix}$, and $f(x) = x^2 + 3x - 10$, calculate $f(\mathbf{A})$.
- The answer $f(\mathbf{A})$ is a matrix.
- $3\mathbf{A}$ is a matrix.
- The number 10 in $f(x)$ has to be multiplied by the 2×2 identity matrix to make it a matrix.
- *Hint*: Verify $\mathbf{A}^2 = \begin{pmatrix} 22 & -6 \\ -9 & 7 \end{pmatrix}$.

(2) Solve for x: $f(x) = x^2 + 3x - 10 = 0$. Call the solutions x_1 and x_2.

(3) Define the matrices $\mathbf{B}_1 = \mathbf{A} - x_1\mathbf{I}$ and $\mathbf{B}_2 = \mathbf{A} - x_2\mathbf{I}$ where $\mathbf{I}$ is the 2×2 identity matrix. Evaluate the determinants of $\mathbf{B}_1$ and $\mathbf{B}_2$ They should both be zero.

(4) The columns of $\mathbf{B}_1$ and $\mathbf{B}_2$ must thus be linearly dependent. Find numbers v_1 and v_2 such that

$$v_1 \times (\mathbf{B}_1)_{\text{col }1} + v_2 \times (\mathbf{B}_1)_{\text{col }2} = 0.$$

Similarly, find numbers w_1 and w_2 such that

$$w_1 \times (\mathbf{B}_2)_{\text{col }1} + w_2 \times (\mathbf{B}_2)_{\text{col }2} = 0.$$

(5) *Partial answer:* $v_1 = -2$, $v_2 = 1$.

3. Computing eigenvalues and eigenvectors

The eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$ is the following: find, for a matrix $\mathbf{A}$, the eigenvectors $\mathbf{x}$ and eigenvalues λ.

(1) Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & 2 & -4 \end{pmatrix}$$

- **STEP I:** Compute the characteristic polynomial of $\mathbf{A}$ and find its roots. Verify:

$$\chi_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 - \lambda^2 + 10\lambda + 10$$

  and note that $\chi_{\mathbf{A}}(\lambda) = (\lambda^2 - 10)(\lambda + 1)$.
  What are the eigenvalues of $\mathbf{A}$?
- **STEP II:**

For each eigenvalue λ_i, $i = 1, 2, 3$, we need to compute the corresponding eigenvectors. Find x, y, z so that

$$\begin{pmatrix} 2 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_i \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

- *Hint:* The 3 eigenvectors are:

$$\frac{1}{(1+\sqrt{10})} \begin{pmatrix} \frac{3}{2}\left(\sqrt{10}-4\right) \\ 3 \\ 1+\sqrt{10} \end{pmatrix}, \frac{1}{(\sqrt{10}-1)} \begin{pmatrix} \frac{3}{2}\left(4+\sqrt{10}\right) \\ -3 \\ \sqrt{10}-1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

4. Singular Value Decomposition

For a matrix $\mathbf{X}$:

$$\mathbf{X} = \begin{pmatrix} -1 & 2 & -1 & -3 \\ 2 & 1 & 3 & 1 \end{pmatrix},$$

the singular value decomposition (SVD) of $\mathbf{X}$ is written as $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^T$ where

$$\mathbf{U} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \ \Sigma = \begin{pmatrix} \sqrt{21} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{V} = \begin{pmatrix} \sqrt{\frac{3}{14}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{5}{3\sqrt{7}} \\ -\frac{1}{\sqrt{42}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{7}} \\ 2\sqrt{\frac{2}{21}} & \frac{\sqrt{2}}{3} & 0 & \frac{5}{3\sqrt{7}} \\ 2\sqrt{\frac{2}{21}} & -\frac{\sqrt{2}}{3} & \frac{1}{\sqrt{3}} & -\frac{2}{3\sqrt{7}} \end{pmatrix}$$

(1) Calculate $\mathbf{C} = \mathbf{X}\mathbf{X}^T$. You will find that $\mathbf{C} = \begin{pmatrix} 15 & -6 \\ -6 & 15 \end{pmatrix}$. The negative off-diagonal elements of $\mathbf{C}$ capture the observation that for most cases, the elements of each column of $\mathbf{X}$ are of opposite sign.

(2) Compute the eigenvalues of $\mathbf{C}$. Solve for the equation that sets the characteristic polynomial of $\mathbf{C}$ to zero. In other words,
   - calculate $\chi_{\mathbf{C}}(x) := \det(\mathbf{C} - x\mathbf{I})$ and find the values $x = x_1, x_2$ such that $\chi_{\mathbf{C}}(x_1) = \chi_{\mathbf{C}}(x_2) = 0$.
   - *Hint*: You will find that $\chi_{\mathbf{C}}(x) = x^2 - 30x + 189$, and you should use the observation that $189 = 21 \times 9$.

(3) How do the eigenvalues $x_{1,2}$ relate to the diagonal entries of Σ?

(4) Verify that the matrices $\mathbf{D}_i = \mathbf{C} - x_i\mathbf{I}$ are proportional to

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and find the nullspace for each, *i.e.*, find $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{D}_i\mathbf{v}_i = 0$. These $\mathbf{v}_i$s are the eigenvectors of $\mathbf{C}$. Normalise them and compare with $\mathbf{U}$.

(5) You might want the help of some software for this, *e.g.*, `numpy.linalg.eig`. You can check that

$$\mathbf{X}^T\mathbf{X} = \begin{pmatrix} 5 & 0 & 7 & 5 \\ 0 & 5 & 1 & -5 \\ 7 & 1 & 10 & 6 \\ 5 & -5 & 6 & 10 \end{pmatrix},$$

which should have 4 eigenvalues. Two of them should be the same as those of $\mathbf{C}$. What about the other two? Verify that the un-normalised eigenvectors of $\mathbf{X}^T\mathbf{X}$ are

$$(3,-1,4,4)^T, (-1,-3,-2,2)^T, (-1,1,0,1)^T, (-7,-1,5,0)^T,$$

and that normalising them will yield the columns of $\mathbf{V}$.

5. Low-rank approximation

We can construct the rank-1 approximation $\tilde{\mathbf{X}}_1$ of $\mathbf{X}$ by setting $\tilde{\mathbf{X}}_1 = \mathbf{u}_1\sigma_1\mathbf{v}_1^T$.

(1) Using (from the previous exercise)

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(-1,1)^T, \quad \sigma_1 = \sqrt{21}, \quad \mathbf{v}_1 = \frac{1}{\sqrt{42}}(3,-1,4,4)^T$$

confirm that the rank-1 approximation is

$$\tilde{\mathbf{X}}_1 = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} & -2 & -2 \\ \frac{3}{2} & -\frac{1}{2} & 2 & 2 \end{pmatrix}.$$

In particular, note that the rows are not independent.

(2) Compute the rank one approximation to $\mathbf{C}$ as $\tilde{\mathbf{C}}_1 = \tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T$. This should be proportional to

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Given $\tilde{\mathbf{X}}_1$, why is this not a surprise? What are its eigenvalues and eigenvectors?

(3) Verify that $\tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1$ is

$$\begin{pmatrix} \frac{9}{2} & -\frac{3}{2} & 6 & 6 \\ -\frac{3}{2} & \frac{1}{2} & -2 & -2 \\ 6 & -2 & 8 & 8 \\ 6 & -2 & 8 & 8 \end{pmatrix}.$$

What would you expect its eigenvalues to be? Check that all the rows and columns are are multiples of $(-\frac{3}{2}, \frac{1}{2}, -2, -2)$. Relate this observation to the eigenvalue spectrum and the definition of rank.