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Conformal Mapping

Abstract

Conformal mapping deals with mathematical functions which preserve angles locally. The topic of conformal mapping often appears in a pure math context, however, there are real-world uses and applications as well. In order to understand conformal mapping, there are certain topics which are introduced: differences between the real and complex plane, the geometry of complex numbers, angle and orientation preservation, and behavior at critical points. Afterwards some examples of physical problems with problematic geometries that can be solved more easily using conformal mappings are provided.

In plain words, a conformal mapping is a mathematical transformation that preserves local angles. From this context the preservation of local angles refers to the comparison of the argument of a complex number before and after a complex transformation is applied.

The argument of a complex number is its angle of inclination with respect to the horizontal axis. In order for this to make sense, we must first cover the geometry of complex numbers. Afterwards we introduce the mathematical definition of an argument of a complex number.

In the real plane $\mathbb{R}^2$, the horizontal axis is known as the x-axis and the vertical axis is known as the y-axis. Then the vector $(a,\, b) \in \mathbb{R}^2$, where both a and b are fixed real-valued numbers, represents a point in the plane that is a units along the x-axis and b units along the y-axis.

The complex plane $\mathbb{C}$ is very similar. In the complex plane, the horizontal axis is known as the real axis and the vertical axis is known as the imaginary axis. Then the vector $(c,\, d) \in \mathbb{C}$, where both c and d are fixed real-valued numbers, represents a point in the plane that is c units along the real axis and d units along the imaginary axis.

Consider a complex number of the form $z = (x,\, y)$, where x and y are any real-valued numbers. Algebraically, this can be represented in the form

$$z = x + iy,$$

where x is the real part of a complex number and y is the imaginary part of a complex number, denoted by $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Now consider a complex number of the form $z = re^{i\theta}$. The magnitude of the complex number z is denoted as $r = |z|$. This follows the same form as the magnitude of a two-dimensional vector

$$r = |z| = |(x,\, y)| = \sqrt{x^2 + y^2}.$$

Here we denote the argument of the complex number z as $\theta = \arg z$, and below we introduce the mathematical definition.

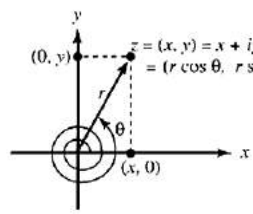

Figure 1

Definition (Argument)**.** The angle of inclination θ of a point z with respect to the real axis is called the *argument* of z. We write

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right), \quad (x \neq 0).$$

Previously it was shown that a complex number can be split apart into its real and imaginary parts. In the same vein, the image of a complex function can be split apart into its real and imaginary parts. Consider a complex number $z = x + iy$ and a complex mapping f. Then it holds that

$$f(z) = f(x + iy) = u(x,\, y) + iv(x,\, y).$$

In other words, the components $u = u(x,\, y)$ and $v = v(x,\, y)$ of f are real-valued functions of the two variables x and y.

As of now, we've covered enough of the algebra and the geometry of complex numbers to better understand the topic of conformal mappings. Below are two of the mathematical definitions to which we will adhere.

Definition (Conformal mapping I)**.** A mapping $w = f(z)$ is said to be *conformal* at a point z_0 if it preserves angles between oriented curves in magnitude as well as in orientation.

Definition (Conformal mapping II)**.** A mapping defined on an open set is *conformal* if it is holomorphic (infinitely differentiable) and one-to-one.

There are many different types of conformal mappings. For now we will focus on a type of conformal mapping known as a linear transformation. Afterwards we demonstrate how the geometric properties of linear transformations apply to the general case.

A linear transformation is a complex function that can be considered as the composition of a (i) rotation, (ii) magnification, and (iii) translation.

(i) Visualized as a rigid rotation about the origin by an angle α.

(ii) Stretches or shrinks the distance between points by a factor of $K \neq 0$.

(iii) Visualized as a rigid translation through a vector $B = a + ib$.

The rotation, magnification, and translation are each mappings from the z-plane onto the w-plane. It can be shown that each mapping is one-to-one. The mathematical definitions of each mapping are provided below.

Definition (Rotation)**.** Let α denote some fixed real number. Then a *rotation* is defined by the transformation

$$w = R(z) = ze^{i\alpha} = re^{i(\theta+\alpha)}.$$

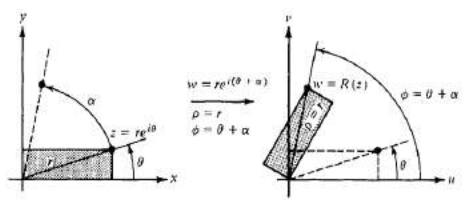

Figure 2

Definition (Magnification)**.** Let $K \neq 0$ be a fixed real number. Then a *magnification* is defined by the transformation

$$w = M(z) = Kz = Kx + iKy.$$

A magnification has the effect of stretching the distance between points by the factor $K > 0$, or it has the effect of shrinking the distance between points by a factor of $K < 0$.

Definition (Translation)**.** Let $B = a + ib$ denote some fixed complex number. Then a *translation* is defined by the transformation

$$w = T(z) = z + B = (x + a) + i(y + b).$$

A translation can be visualized as a rigid translation a units along the real axis and b units along the imaginary axis. Putting it all together, a linear transformation is a composition of a rotation, a magnification, and a translation with the following definition.

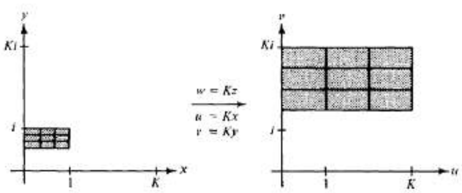

Figure 3

Definition (Linear transformation)**.** Let $A = Ke^{i\alpha}$ and let $B = a + ib$, where α and $K \neq 0$ are fixed real numbers. Then a *linear transformation* is defined by the transformation

$$w = L(z) = Az + B.$$

A linear transformation has the effect of rotating about the origin by an angle $\alpha = \operatorname{Arg} A$, then magnifying by a factor $K = |A|$, and then translating by the vector B.

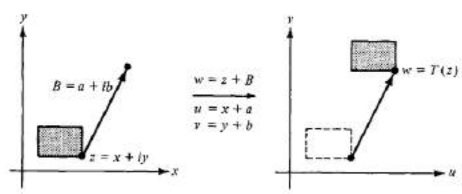

Figure 4

Consider the fact that the composition of one-to-one functions is a one-to-one function. A linear translation is a composition of a rotation, a magnification, and a translation, each of which are one-to-one mappings. As a result, the linear transformation is a one-to-one mapping.

Additionally it can be shown that the linear transformation is holomorphic, and since it is one-to-one, the linear transformation satisfies our second definition of a conformal mapping.

Getting back on to the topic of conformal mapping, which we defined informally as a mapping which preserves local angles, what exactly does it mean for angles to be preserved locally? There are two aspects to this question: the notion of angle preservation, and the notion of local behavior.

A function $w = f(z)$ preserves angles if and only if it increments the argument of any complex number z by some fixed positive number (except for points where the derivative of f vanishes). In other words, if the argument of a point z is α, then the argument of a point $w = f(z)$ is $\alpha + \beta$ for some fixed real number β.

Taking the finite Taylor series of a complex function centered at z_0 gives us an approximation of that function for points near z_0. We use this notion of local behavior to describe the properties of conformal mapping in relation to linear transformations.

Suppose $w = f(z)$ is a complex analytic function in the domain of definition D, and let z_0 be a point in D. Then the function f can be approximated by its finite Taylor series

$$T_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

To find the linear approximation of f one needs only to plug in $n = 1$ into the previous equation to obtain the following function

$$L(z) = f(z_0) + f'(z_0)(z - z_0) = A + B(z - z_0),$$

where $A = f(z_0)$ and $B = f'(z_0) \neq 0$. For points z near z_0, the mapping f behaves much like the linear mapping $w = L(z)$.

From a previous section, the effect of the linear transformation is a rotation about the origin by an angle $\alpha = \operatorname{Arg} A$, then a magnification by a factor $K = |B|$, and then a translation by the vector $A - Bz_0$.

Now we actually want to show that $w = L(z)$ preserves angles at z_0. Let C denote a smooth curve that passes through the point $z(0) = z_0$. We define the curve as the following

$$C\colon z(t) = x(t) + iy(t), \quad t \in [a,\, b].$$

Let T denote the tangent vector of C at a point z_0, i.e. let $T = z'(0)$. The argument at this point is $\beta = \arg T$. Let K be the image of C under the mapping $w = L(z)$ defined by

$$K\colon w(t) = u(x(t),\, y(t)) + iv(x(t),\, y(t)).$$

Let T^* denote the tangent vector of K at w_0, i.e. let $T^* = w'(0)$. Applying the chain rule to T^* leaves us with

$$T^* = w'(0) = L'(z_0)z'(0).$$

Taking the argument at this point, and making use of one of the properties of the argument, we obtain

$$\gamma = \arg T^* = \arg L'(z_0)z'(0) = \arg L'(z_0) + \arg z'(0) = \alpha + \beta.$$

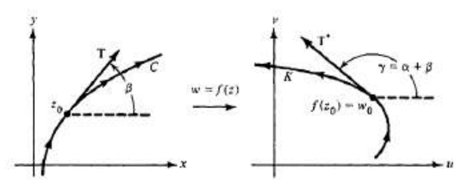

Figure 5

Since we have shown that the linear transformation increments the argument of a complex number by a fixed real number, it follows that the linear transformation preserves angles locally (wherever the derivative of the transformation is nonzero).

We will use our previous results to show that the linear transformation preserves the magnitudes and the orientations of angles as well.

Let C_1 and C_2 be smooth curves where each curve passes through the point $z(0) = z_0$. Let T_1 and T_2 be the tangent vectors of C_1 and C_2 at z_0, respectively. Then β_1 and β_2 are the respective arguments of T_1 and T_2.

The respective image curves K_1 and K_2 of C_1 and C_2 passing through w_0 have tangent vectors T_1^* and T_2^*. Let $\alpha = \arg L'(z_0)$. Then the arguments γ_1 and γ_2 of T_1^* and T_2^* are defined by

$$\gamma_1 = \arg T_1^* = \alpha + \beta_1, \quad \gamma_2 = \arg T_2^* = \alpha + \beta_2,$$

The last step is to use the difference between γ_1 and γ_2 to obtain

$$\gamma_2 - \gamma_1 = \alpha + \beta_2 - (\alpha + \beta_1) = \beta_2 - \beta_1.$$

Therefore we conclude the angle $\Delta\gamma$ from K_1 to K_2 is the same in magnitude and direction as the angle $\Delta\beta$ from C_1 to C_2.

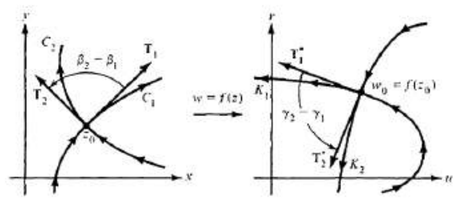

Figure 6

Not only does the linear transformation preserve angles (where the derivative is nonzero), it also preserves magnitude and direction. Hence the linear transformation satisfies our first definition of conformal mappings.

The conformal properties of preserving angles, along with their magnitudes and orientations, are not unique to the linear transformation. It can be shown that any one-to-one holomorphic function is conformal.

By now you might be wondering, what is so special about the condition that the derivative of a function is nonzero? These points are known as *critical points*. Next we examine the way a conformal mapping behaves at its critical points. Let f be a nonconstant analytic function. Suppose

$$f'(z_0) = 0,\ f''(z_0) = 0,\ \ldots,\ f^{(k-1)}(z_0) = 0,$$

and also suppose the k-th derivative of f at the point z_0 is nonzero. Since the function f is analytic then it is equivalent to its infinite Taylor series. We define each coefficient of the Taylor series a_i as

$$a_i = \frac{f^{(i)}(z_0)}{i!}.$$

Evaluating the infinite Taylor series of f at z_0, using the fact that $a_1, a_2, \ldots, a_{k-1}$ vanish, and factoring helps us obtain

$$\begin{aligned}
f(z) &= f(z_0) + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_k(z - z_0)^k + \cdots \\
&= f(z_0) + a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots \\
&= f(z_0) + (z - z_0)^k g(z),
\end{aligned}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Rearranging $f(z)$ gives us

$$f(z) - f(z_0) = (z - z_0)^k g(z).$$

Let C be a smooth curve that passes through the point $z(0) = z_0$ with tangent vector T. Then the argument of T is given by the limit

$$\beta = \lim_{z \to z_0} \arg(z - z_0).$$

The image curve K of C passes through $w_0 = f(z_0)$ and has tangent vector T^*. The argument of T^* is given by the limit

$$\begin{aligned}
\gamma &= \lim_{w \to w_0} \arg(w - w_0) = \lim_{z \to z_0} \arg\left[f(z) - f(z_0)\right] \\
&= \lim_{z \to z_0} \arg\left[(z - z_0)^k g(z)\right] = \lim_{z \to z_0} \left[k \cdot \arg(z - z_0) + \arg g(z)\right] \\
&= k \cdot \lim_{z \to z_0} \arg(z - z_0) + \lim_{z \to z_0} \arg g(z).
\end{aligned}$$

Let $\delta = \arg g(z_0) = \arg a_k$. Combining the previous two results, we obtain

$$\gamma = k \cdot \lim_{z \to z_0} \arg(z - z_0) + \lim_{z \to z_0} \arg g(z) = k\beta + \delta.$$

We want to use the previous result to show that the angles formed by any two curves are magnified by a factor k. Let C_1 and C_2 be smooth curves that pass through z_0 and let K_1 and K_2 be their respective image curves. The next step is to use the difference between γ_1 and γ_2 to obtain

$$\gamma_2 - \gamma_2 = k\beta_2 + \delta - (k\beta_1 + \delta) = k(\beta_2 - \beta_1).$$

Therefore we conclude that the angle $\Delta\gamma$ from K_1 to K_2 is k times larger than the angle $\Delta\beta$ from C_1 and C_2.

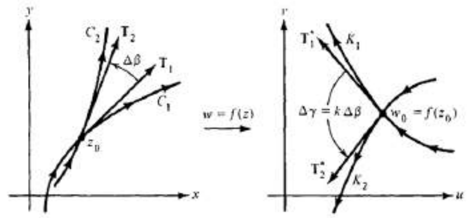

Figure 7

The trigonometric functions have a complex analog, each of which are conformal (except at the critical points of the function). Examples include the sine, cosine, and tangent functions, shown below in that order.

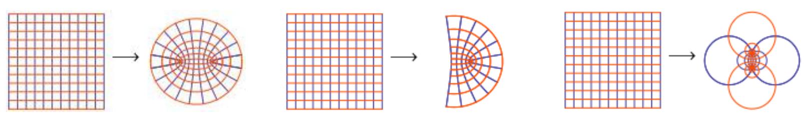

Figure 8

What is being shown is a grid of horizontal and vertical lines in the z-plane being transformed by a specified function $w = f(z)$ onto the w-plane. It is worth noting that mostly every intersection in the w-plane forms a perpendicular angle.

This occurs because the angles are orthogonal in the z-plane, and since conformal mappings preserve angles, then the w-plane has perpendicular angles as well. The points where this does not happen are critical points and we can see that angles are magnified at those critical points.

The exponential and reciprocal functions also have complex analogs which are conformal everywhere (except for critical points and undefined points) and are shown below in that order.

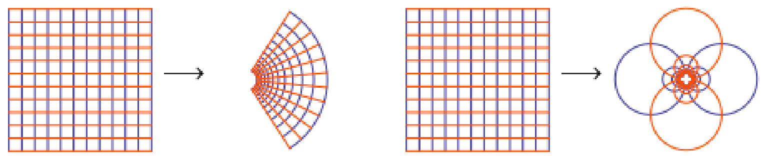

Figure 9

Many physical methods use conformal mapping to transform a problematic region into one where the solution is easier to obtain.

Physical applications tend to involve solutions in three dimensions. Since complex analysis involves only two dimensions, i.e. the x-dimension and y-dimension, then we consider the case in which the solution does not vary along the axis perpendicular to the xy-plane.

Theorem (Orthogonal families of level curves)**.** Let $\phi(x,\, y)$ be harmonic in a domain D. Let $\psi(x,\, y)$ be the harmonic conjugate. Also let

$$F(z) = \phi(x,\, y) + i\psi(x,\, y)$$

be the complex potential. Then the two families of level curves

$$\{\phi(x,\, y) = K_1 \mid K_1 \in \mathbb{R}\}, \quad \{\psi(x,\, y) = K_2 \mid K_2 \in \mathbb{R}\},$$

are orthogonal to each other at points common to the curves ϕ and ψ.

Below is a table of many interpretations (all of which are physical examples) of the families of level curves.

Physical Phenomenon	**$\phi(x, y)$ = constant**	**$\psi(x, y)$ = constant**
Heat flow	Isothermals	Heat flow lines
Electrostatics	Equipotential curves	Flux lines
Fluid flow	Equipotentials	Streamlines
Gravitational field	Gravitational potential	Lines of force
Magnetism	Potential	Lines of force
Diffusion	Concentration	Lines of flow
Elasticity	Strain function	Stress lines
Current flow	Potential	Lines of flow

Figure 10

Suppose we want to find the solution to some two-dimensional physical problem, such as a heat flow problem or an electrostatics problem. The following is an example of transforming a more complicated geometry into a simpler one to work with.

Consider a function $\phi(x,\ y)$ that maps the unit disk onto the upper half plane. Let this function be harmonic with the following boundary values

$$\phi(x,\ y) = \begin{cases} 0 & \text{if } \operatorname{Arg}\theta \in (0,\ \pi) \\ 1 & \text{if } \operatorname{Arg}\theta \in (\pi,\ 2\pi) \end{cases}$$

It can be shown that the following function is one-to-one, holomorphic, and that it maps the unit disk onto the upper half plane

$$f(z) = i\frac{1-z}{1+z} = \frac{2y}{(x+1)^2+y^2} + i\frac{1-x^2-y^2}{(x+1)^2+y^2}.$$

Then this function is conformal at noncritical points by our second definition of conformal mappings. The points that lie on the upper and lower semicircle are mapped onto the positive and negative u-axis, respectively. This function becomes a new problem of finding a harmonic function $\Phi(u, v)$ with the boundary values

$$\Phi(u,\ v) = \begin{cases} 0 & \text{if } u > 0 \\ 1 & \text{if } u < 0 \end{cases}$$

We can see that the solution to the boundary value equation $\phi(x, y)$ on the left has a more complicated geometry than the solution to the boundary value equation $\Phi(x,\ y)$ on the right.

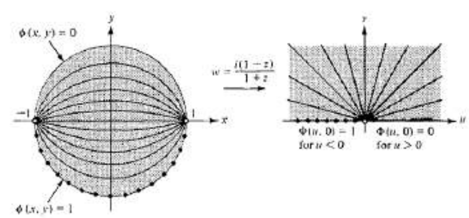

Figure 11

Keep in mind that this is just one brief example. There are infinitely many more geometries and functions to explore. Also, there are times when a solution to a physical problem can be applied to an entirely separate physical problem. Other real-world examples include the heat flow lines within a semiconductor, the flux lines of a magnet, and the lines of force in a gravitational field.

Note. Figures 1–7, 10, and 11 along with any definitions or theorems used are from the source [1]. The rest of the images are taken from source [2].

References

[1] Mathews, J. H., and Howell, R. W. (1997). *Complex Analysis for Mathematics and Engineering.* (3rd ed.). Sudbury, MA: Jones and Bartlett.

[2] Wolfram Alpha: Computational Knowledge Engine. (n.d.). Retrieved from http://www.wolframalpha.com/