Bardasis-Schrieffer Polaritons

Our general goal is to find a superconducting collective mode that will hybridize with cavity photons to produce a polariton. In this project we explore how this can happen with a d-wave excitation appearing above an s-wave condenced state. It will turn out that the so-called Bardasis-Schrieffer (BS) mode is the excitation that fulfills this goal. The BS mode is the component of the (complex) d-wave order parameter Δ^d that is 90° out-of-phase with the s-wave order parameter Δ^s. If we choose Δ^s to be purely real, then this means the BS mode is the imaginary component of Δ^d. The d-wave component that is in-phase with Δ^s will turn out to be overdamped and its coupling to photons will be small in the particle-hole asymmetry.

I. MINIMALLY COUPLED BDG ACTION WITH SUPERCURRENT

We start from the action of interacting electrons, with the interaction assumed to be separable and decomposed in angular momentum channels:

$$S[\bar{\psi},\psi] = \sum_{k,\sigma} \bar{\psi}_{k,\sigma} \left(-i\epsilon_n + \frac{k^2}{2m} - \mu \right) \psi_{k,\sigma} - \frac{1}{\beta} \sum_q \sum_{l=s,d} g_l \bar{\phi}^l_q \phi^l_q, \tag{1}$$

with $k = (\epsilon_n, \mathbf{k})$, $q = (\Omega_m, \mathbf{q})$, and the interaction written in terms of the fermion bilinears,

$$\phi^l_q = \sum_k f_l(\mathbf{k}) \psi_{-k+\frac{q}{2},\downarrow} \psi_{k+\frac{q}{2},\uparrow} \tag{2}$$

where the angular functions are $f_s(\mathbf{k}) = 1$ and $f_d(\mathbf{k}) = \sqrt{2}\cos(2\theta_k)$. Though our system is completely rotationally symmetric (C_∞ symmetry) this choice of d-wave form factor picks an explicit x-axis and mimics the form factor of the $d_{x^2-y^2}$ basis function of the C_4 symmetry group of a square lattice, which is a much more physically relevant symmetry. We assume that $g_s > g_d$ so the system favors purely s-wave pairing, but also that the difference is small so that there isn't a prohibitively large energy cost to create d-wave excitations. The interaction can be straightforwardly decoupled in the Cooper channel with a Hubbard-Stratonovich transformation (both s- and d-wave simultaneously) to give

$$S[\bar{\psi},\psi,\bar{\Delta},\Delta] = \sum_{k,\sigma} \bar{\psi}_{k,\sigma} \left(-i\epsilon_n + \frac{k^2}{2m} - \mu \right) \psi_{k,\sigma} + \frac{1}{\beta} \sum_{q,l} \frac{1}{g_l} \left| \Delta^l_q \right|^2 - \frac{1}{\beta} \sum_{q,l} \left(\bar{\Delta}^l_q \phi^l_q + \Delta^l_q \bar{\phi}^l_q \right) \tag{3}$$

$$\begin{aligned} &= \sum_{k,\sigma} \bar{\psi}_{k,\sigma} \left(-i\epsilon_n + \frac{k^2}{2m} - \mu \right) \psi_{k,\sigma} + \frac{1}{\beta} \sum_{q,l} \frac{1}{g_l} \left| \Delta^l_q \right|^2 \\ &\qquad - \frac{1}{\beta} \sum_{k,q,l} \left(f_l(\mathbf{k}) \bar{\Delta}^l_q \; \psi_{-k+\frac{q}{2},\downarrow} \psi_{k+\frac{q}{2},\uparrow} + f_l(\mathbf{k}) \Delta^l_q \; \bar{\psi}_{k+\frac{q}{2},\uparrow} \bar{\psi}_{-k+\frac{q}{2},\downarrow} \right) \end{aligned} \tag{4}$$

$$= \sum_k \bar{\Psi}_k \begin{pmatrix} -i\epsilon_n + \xi_k & 0 \\ 0 & -i\epsilon_n - \xi_k \end{pmatrix} \Psi_k - \frac{1}{\beta} \sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \begin{pmatrix} 0 & \sum_l f_l(\mathbf{k}) \Delta^l_q \\ \sum_l f_l(\mathbf{k}) \bar{\Delta}^l_q & 0 \end{pmatrix} \Psi_{k-\frac{q}{2}} + \frac{1}{\beta} \sum_{q,l} \frac{1}{g_l} \left| \Delta^l_q \right|^2. \tag{5}$$

In the last line we have defined the Nambu spinor $\Psi_k = (\psi_{k,\uparrow}, \bar{\psi}_{-k,\downarrow})^T$. This gives us our BdG action.

A. Minimal coupling with a supercurrent

The result of minimally coupling to the photon field is the addition of two new terms, the paramagnetic and diamagnetic terms. They are given by

$$\begin{aligned} S_{\psi-A} &= \frac{1}{\beta} \sum_{k,q,\sigma} \bar{\psi}_{k+\frac{q}{2},\sigma} \left[-\frac{e}{m} \mathbf{k} \cdot \mathbf{A}_q + \frac{e^2}{2m\beta} \sum_{q'} \mathbf{A}_{\frac{q+q'}{2}} \cdot \mathbf{A}_{\frac{q-q'}{2}} \right] \psi_{k-\frac{q}{2},\sigma} \\ &= \frac{1}{\beta} \sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \left[-\frac{e}{m} \mathbf{k} \cdot \mathbf{A}_q \; \hat{\tau}_0 + \frac{e^2}{2m\beta} \sum_{q'} \mathbf{A}_{\frac{q+q'}{2}} \cdot \mathbf{A}_{\frac{q-q'}{2}} \; \hat{\tau}_3 \right] \Psi_{k-\frac{q}{2}}, \end{aligned}$$

where in the second line we have changed to the Nambu basis written in terms of Pauli matrices $\hat{\tau}_i$, with $\hat{\tau}_0$ representing the identity matrix.

Now we consider driving a supercurrent in the system. A supercurrent corresponds to a nontrivial phase winding of the superconducting order parameter, $\Delta^s \to \Delta^s\, e^{i\theta(\mathbf{r})}$. Instead of keeping this phase dependence, which would be very complicated to do in momentum space, we can perform a gauge transformation to remove it from Δ and add a new contribution to the gauge field proportional to the gradient of this phase. The energetically favored configuration is a spacially uniform supercurrent, meaing a constant phase gradient, so this new part of $\mathbf{A}$ is just a constant vector, $\mathbf{A}(x) \to \mathbf{A}(x) + \mathbf{A}_S$. Performing a Fourier transform to momentum space we then have $\mathbf{A}_q \to \mathbf{A}_q + \mathbf{A}_S \beta \delta_{q,0}$.

Plugging this into the photon coupling terms above and keeping only terms linear in the gauge field $\mathbf{A}_q$, we arrive at the form of the coupling that we will use from this point forward,

$$\begin{aligned} S_{\psi-A} \to \frac{1}{\beta} \sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \left(-e\mathbf{v}_{\mathbf{k}}\, \hat{\tau}_0 - e\mathbf{v}_S\, \hat{\tau}_3\right) \cdot \mathbf{A}_q\, \Psi_{k-\frac{q}{2}} \\ + \sum_k \bar{\Psi}_k \left(\mathbf{k}\cdot\mathbf{v}_S\, \hat{\tau}_0 + \frac{1}{2} m v_S^2 \hat{\tau}_3\right) \Psi_k + \frac{1}{\beta}\sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \hat{\tau}_3 \Psi_{k-\frac{q}{2}} \frac{e^2}{2m\beta} \sum_{q'} \mathbf{A}_{\frac{q+q'}{2}} \cdot \mathbf{A}_{\frac{q-q'}{2}}. \end{aligned} \tag{6}$$

We have written the coupling expressions in terms of the electron velocity $\mathbf{v}_{\mathbf{k}} = \frac{k}{m}$ and the superfluid velocity $\mathbf{v}_S = -\frac{e}{m}\mathbf{A}_s$ for convenience. Note that the terms coupling photons to the normal current (first term) and to the externally imposed supercurrent (second term) come with different Nambu structure, so it's possible that superconducting excitations may couple to photons through one of these vertices and not the other. The second set of terms, which do not couple electrons to the gauge field, add new terms to the free part of the action. The first is a Doppler shift contribution to the energy and the second gives a shift of the chemical potential due to the supercurrent. The last term is simply the diamagnetic term, which in a clean system gives an unphysical contribution to the response, so we discard it going forward.

Introducing some more compact notation, the action is

$$S = S_\Delta + S_{\text{cav}} - \sum_k \bar{\Psi}_k\, \hat{G}_{0,k}^{-1}\, \Psi_k + \frac{1}{\beta}\sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \left(\hat{\chi}^P_{k,q}[A] + \hat{\chi}^S_{k,q}[A] - \hat{\Delta}^s_{k,q} - \hat{\Delta}^d_{k,q}\right) \Psi_{k-\frac{q}{2}}, \tag{7}$$

where $S_\Delta = \frac{1}{\beta}\sum_{q,l} \frac{1}{g_l} \left|\Delta^l_q\right|^2$, S_{cav} is the free action for photons in the cavity, $\hat{\chi}^P$ and $\hat{\chi}^S$ are the paramagnetic ($\hat{\tau}_0$) and supercurrent ($\hat{\tau}_3$) couplings to the photon field given above and

$$\hat{G}_{0,k}^{-1} = \left(i\epsilon_n - \mathbf{k}\cdot\mathbf{v}_S\right)\, \hat{\tau}_0 - \underbrace{\left(\frac{k^2}{2m} - \mu + \frac{1}{2} m v_S^2\right)}_{\equiv \xi^S_k}\, \hat{\tau}_3, \quad \hat{\Delta}^l_{k,q} = f_l(\mathbf{k}) \begin{pmatrix} 0 & \Delta^l_q \\ \bar{\Delta}^l_q & 0 \end{pmatrix}. \tag{8}$$

At this point we make the mean field approximation which amounts to the replacement $\Delta^s_q \to \Delta\, \beta\delta_{q,0}$ with $\Delta = const \in \mathbb{R}$, and combine this into the Green's function, giving $\hat{G}_k^{-1} = \hat{G}_{0,k}^{-1} + \Delta\, \hat{\tau}_1$. The ground state is then a homogeneous s-wave superconductor with constant supercurrent, with the terms $\hat{\chi}^P$, $\hat{\chi}^S$, and $\hat{\Delta}^d$ as small perturbations.

B. Empty cavity photon action S_A

Before integrating out the electrons and expanding the resulting trace-log, we explicity consider the free part of the photon action. Inside of a cavity this is

$$S_{\text{cav}} = -\frac{1}{2\beta}\sum_{q,n} A_{\alpha,n,-q} \left[(i\Omega_m)^2 - \omega_{n,\mathbf{q}}^2\right] A_{\alpha,n,q}, \tag{9}$$

with α indexing the two cavity polarizations, n labeling the discrete modes resulting from the confinement in z, and $\omega_{n,\mathbf{q}}^2 = |\mathbf{q}|^2 + \omega_{n,0}^2$. (Note that this is written using Lorentz-Heaviside units, which is missing a factor of $1/4\pi$ compared to gaussian units, as well as with $c = 1$.) As is the case throughout these notes, $\mathbf{q}$ represents 2D momentum in the unconfined directions. From this point onwards we consider a fixed single mode n and then drop the index. In terms of the cavity polarizations the vector potential that appears in minimal coupling is

$$\mathbf{A}_q(z) = \sum_\alpha A_{\alpha,q}\, \hat{\boldsymbol{\epsilon}}_{\alpha,\mathbf{q}}(z), \tag{10}$$

where the polarization vectors are

$$\hat{\boldsymbol{\epsilon}}_{1,\mathbf{q}}(z) = i\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi z}{L}\right)\frac{\hat{\mathbf{z}}\times\mathbf{q}}{|\mathbf{q}|} \qquad \hat{\boldsymbol{\epsilon}}_{2,\mathbf{q}}(z) = \sqrt{\frac{2}{L}}\frac{1}{\omega_q}\left[q\,\cos\left(\frac{n\pi z}{L}\right)\hat{\mathbf{z}} - i\omega_0\sin\left(\frac{n\pi z}{L}\right)\frac{\mathbf{q}}{|\mathbf{q}|}\right]. \tag{11}$$

Due to the cavity confinement, the free photon spectrum can be approximated as a parabolic dispersion for small momentum,

$$\omega_{\mathbf{q}} = \sqrt{\omega_0^2 + |\mathbf{q}|^2} \approx \omega_0 + \frac{|\mathbf{q}|^2}{2\omega_0}, \qquad \omega_0 = \frac{n\pi}{L}. \tag{12}$$

To relate the polarization amplitudes and the Cartesian components of the vector potential we restrict our attention to $z = L/2$, where the z-component of the polarization vectors vanish so we can relate two independent components in each basis. (There are only 2 independent components in either basis, but at most other points exactly how the 3 Cartesian components reduce to just 2, so that we can write a 2×2 matrix relating the two, is not immediately apparent in general.) This is also conveniently where the superconducting layer is situated, so the Cartesian components at this point are those that actual couple to fermions. Simplify the notation by defining $\mathbf{A}_q(z = L/2) \equiv \mathbf{A}_q$ and

$$\hat{\boldsymbol{\epsilon}}_{1,\mathbf{q}}(z = L/2) \equiv \hat{\boldsymbol{\epsilon}}_{1,q} = i\sqrt{\frac{2}{L}}\frac{\hat{\mathbf{z}}\times\mathbf{q}}{|\mathbf{q}|} \qquad \hat{\boldsymbol{\epsilon}}_{2,q}(z = L/2) \equiv \hat{\boldsymbol{\epsilon}}_{2,\mathbf{q}} = -i\sqrt{\frac{2}{L}}\frac{\omega_0}{\omega_q}\frac{\mathbf{q}}{|\mathbf{q}|}, \tag{13}$$

so that

$$\mathbf{A}_q(z = L/2) \equiv \mathbf{A}_q = \sum_\alpha A_{\alpha,q}\hat{\boldsymbol{\epsilon}}_{\alpha,\mathbf{q}}. \tag{14}$$

II. EFFECTIVE ACTION FOR Δ AND A (INTEGRATING OUT Ψ)

The partition function can be written in terms of field integrals, letting us integrate out the electron fields.

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}(\bar{\psi},\psi,\bar{\Delta},\Delta,A)e^{-S[\bar{\psi},\psi,\bar{\Delta},\Delta,A]} \\
&= \int \mathcal{D}(\bar{\Delta},\Delta,A)e^{-(S_\Delta + S_{\text{cav}})}\int \mathcal{D}(\bar{\psi},\psi)\exp\left[\sum \bar{\Psi}\left(\hat{G}^{-1} - \hat{\chi}^P[A] - \hat{\chi}^S[A] + \hat{\Delta}^d\right)\Psi\right] \\
&= \int \mathcal{D}(\bar{\Delta},\Delta)\exp\left[-S_\Delta - S_{\text{cav}} + \operatorname{Tr}\ln\left(\hat{G}^{-1} - \hat{\chi}^P[A] - \hat{\chi}^S[A] + \hat{\Delta}^d\right)\right].
\end{aligned}$$

This gives the action for photons and d-wave excitations above the s-wave superconducting ground state, and the coupling between them. We write the three perturbations as a single matrix $\hat{V}$ and produce an effective action by expanding the trace-log, keeping terms up to second order in this small perturbation.

$$S = S_\Delta + S_{\text{cav}} - \operatorname{Tr}\ln\left(\hat{G}^{-1} - \hat{V}\right) \to S_{\text{eff}} = S_\Delta + S_{\text{cav}} - \operatorname{Tr}\ln\left(\hat{G}^{-1}\right) + \operatorname{Tr}\left(\hat{G}\hat{V}\right) + \frac{1}{2}\operatorname{Tr}\left(\hat{G}\hat{V}\right)^2. \tag{15}$$

The saddle point solution of this action ignoring all small perturbations gives the s-wave gap equation,

$$\frac{1}{g_s} - \sum_{\mathbf{k}}\frac{n_F(E_{\mathbf{k}}^-) - n_F(E_{\mathbf{k}}^+)}{2\lambda_k} = 0. \tag{16}$$

The terms of most interest to us in our search for hybridized modes are those at second order in $\hat{V}$, which contain the coupling of photons to the d-wave superconducting mode as well as the photon and d-wave bubbles that are vital in determining their spectra. Expanding this term,

$$\frac{1}{2}\operatorname{Tr}(\hat{G}\hat{V}\hat{G}\hat{V}) = \frac{1}{2}\operatorname{Tr}\left(\hat{G}\hat{\Delta}^d\hat{G}\hat{\Delta}^d\right) + \frac{1}{2}\operatorname{Tr}\left(\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\right) - \operatorname{Tr}\left(\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\hat{G}\hat{\Delta}^d\right). \tag{17}$$

The first term combines with the d-wave part of S_Δ to give the action of the d-wave excitations. The second term describes the coupling of photons the s-wave superconductor, giving a photon self-energy. The last term is the coupling of photons to the d-wave excitations. Thanks only to the supercurrent this last term will prove to be finite.

A. Bardasis-Schrieffer mode (d-wave excitations)

The bubble diagram of d-wave excitations is written

$$
\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}(\hat{G}\hat{\Delta}^d\hat{G}\hat{\Delta}^d) &= \frac{T^2}{2}\sum_{k,q}\mathrm{tr}\left(\hat{G}_{k+\frac{q}{2}}\hat{\Delta}^d_{k,q}\hat{G}_{k-\frac{q}{2}}\hat{\Delta}^d_{k,-q}\right)\\
&\approx \frac{T^2}{2}\sum_{k,q}\mathrm{tr}\left(\hat{g}^+_{\mathbf{k}}\,\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}}\,\hat{g}^-_{\mathbf{k}}\,\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,-q}\hat{U}_{\mathbf{k}}\right)\\
&= \frac{T^2}{2}\sum_{k,q}\sum_{\alpha,\alpha'=\pm}\frac{1}{i\epsilon^+_n - E^\alpha_{\mathbf{k}}}\frac{1}{i\epsilon^-_n - E^{\alpha'}_{\mathbf{k}}}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}}\right)_{\alpha\alpha'}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,-q}\hat{U}_{\mathbf{k}}\right)_{\alpha'\alpha}\\
&= \frac{T}{2}\sum_{\mathbf{k},q}\sum_{\alpha,\alpha'}\frac{n_F(E^{\alpha'}_{\mathbf{k}}) - n_F(E^\alpha_{\mathbf{k}})}{i\Omega_m - (E^\alpha_{\mathbf{k}} - E^{\alpha'}_{\mathbf{k}})}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}}\right)_{\alpha\alpha'}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,-q}\hat{U}_{\mathbf{k}}\right)_{\alpha'\alpha},
\end{aligned}
\tag{18}
$$

Going to the second line we have made the approximation of discarding the Green's functions' dependence on the small momentum transfer $\mathbf{q}$, but we keep the dependence on the frequency Ω_m. We make this approximation because we assume that the kinetic mass of the eventual Bardasis-Schrieffer mode is much larger than the effective mass of the cavity photons, so that we can safely approximate its dispersion for the purposes of hybridization by its value at $|\mathbf{q}| = 0$. (Notation: $\epsilon^\pm_n = \epsilon_n \pm \frac{\Omega_m}{2}$) We've then diagonalized the Green's functions with the Bogoliubov transformation $\hat{U}_{\mathbf{k}}$,

$$
\hat{U}_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}} \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix}, \quad u_{\mathbf{k}}, v_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 \pm \frac{\xi^S_k}{\lambda_k}\right)}, \quad \lambda_k = \sqrt{\xi^{S\,2}_k + \Delta^2}.
\tag{19}
$$

The quasiparticle excitation spectrum is $E^\pm_k = \pm\lambda_k + \mathbf{k}\cdot\mathbf{v}_S$. This differs from the usual spectrum of a superconductor by the Doppler shift term resulting from the supercurrent.

The matrix element involving the d-wave pairing function is

$$
\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}} = \frac{1}{2}f_d(\mathbf{k})\left[\frac{1}{\lambda_k}\left(\Delta^d_q + \bar{\Delta}^d_{-q}\right)\left(\xi^S_k\;\hat{\tau}_1 - \Delta\;\hat{\tau}_3\right) + \left(\Delta^d_q - \bar{\Delta}^d_{-q}\right)(i\hat{\tau}_2)\right].
\tag{20}
$$

Only the terms in this matrix element that are off-diagonal in α, α' will give a nonzero result due to the difference of Fermi functions cancelling otherwise. The trace becomes

$$
\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}(\hat{G}\hat{\Delta}^d\hat{G}\hat{\Delta}^d)\\
&\quad\approx \frac{1}{\beta}\sum_{\mathbf{k},q} f_d(\mathbf{k})^2\left(n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})\right)\frac{2\lambda_k}{(i\Omega_m)^2 - (2\lambda_k)^2}\left[\frac{\xi^{S\,2}_k}{\lambda^2_k}\frac{\Delta^d_q + \bar{\Delta}^d_{-q}}{2}\frac{\Delta^d_{-q} + \bar{\Delta}^d_q}{2} - \frac{\Delta^d_q - \bar{\Delta}^d_{-q}}{2}\frac{\Delta^d_{-q} - \bar{\Delta}^d_q}{2}\right.\\
&\qquad\left. + \frac{\xi^S_k}{\lambda_k}\left(\frac{\Delta^d_q + \bar{\Delta}^d_{-q}}{2}\frac{\Delta^d_{-q} - \bar{\Delta}^d_q}{2} - \frac{\Delta^d_{-q} + \bar{\Delta}^d_q}{2}\frac{\Delta^d_q - \bar{\Delta}^d_{-q}}{2}\right)\right]\\
&\quad\approx \frac{1}{\beta}\sum_{\mathbf{k},q} f_d(\mathbf{k})^2\frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2}\,2\lambda_k\left[\frac{\xi^{S\,2}_k}{\lambda^2_k}\;\bar{d}^R_q d^R_q + \bar{d}^I_q d^I_q\right].
\end{aligned}
$$

We have dropped the term linear in ξ^S_k, noting that in the ξ-approximation this term goes as the particle-hole asymmetry which we assume to be small. In the last line we have addtionally rewritten the d-wave gap function in terms of its real and imaginary components, $\Delta^d_q = d^R_q + i d^I_q$, with d^R, d^I themselves real, so that $\bar{d}_q = d_{-q}$ for each of them. Because we have chosen a purely real s-wave Δ, the imaginary component of the d-wave gap is the Bardasis-Schrieffer mode. (In general, the BS mode is the component $\frac{\pi}{2}$-out-of-phase with the s-wave gap. If the s-wave gap isn't taken to be real, then parametrize the d-wave gap in terms of components in-phase and $\frac{\pi}{2}$-out-of-phase with it to arrive at the same result.)

We can likewise rewrite the d-wave part of the Hubbard-Stratonovich action S_Δ in terms of these two components. Combining this with the bubble diagram calculated above and separating the distinct modes, we have the overall action for d-wave excitations at $|\mathbf{q}|=0$,

$$S_{d^R} = \frac{1}{\beta}\sum_q \left[\frac{1}{g_d} + \sum_{\mathbf{k}} f_d(\mathbf{k})^2 \; \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} \; 2\frac{\xi_k^{S\,2}}{\lambda_k}\right] \bar{d}^R_q d^R_q, \tag{21}$$

$$S_{d^I} = \frac{1}{\beta}\sum_q \left[\frac{1}{g_d} + \sum_{\mathbf{k}} f_d(\mathbf{k})^2 \; \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} \; 2\lambda_k\right] \bar{d}^I_q d^I_q. \tag{22}$$

Since the real mode is generally overdamped we will ignore it from this point onward and rename $d^I_q \to d_q$. We can use the s-wave gap equation to simplify this action a bit more,

$$\begin{aligned} S_d &= \frac{1}{\beta}\sum_q \left[\frac{1}{g_d} + \sum_{\mathbf{k}} f_d(\mathbf{k})^2 \; \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} \; 2\lambda_k \underbrace{- \frac{1}{g_s} + \sum_{\mathbf{k}} \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{2\lambda_k}}_{=0}\right] d_{-q} d_q \\ &= \frac{1}{\beta}\sum_q \left[\frac{1}{g_d} - \frac{1}{g_s} + \sum_{\mathbf{k}} \left(n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})\right) \frac{1}{2\lambda_k}\left(\frac{(i\Omega_m)^2 + (2\lambda_k)^2(f_d(\mathbf{k})^2 - 1)}{(i\Omega_m)^2 - (2\lambda_k)^2}\right)\right] d_{-q} d_q. \end{aligned}$$

Using the explicit form of the d-wave form factor we have $f_d(\mathbf{k})^2 - 1 = 2\cos(2\theta_k) - 1 = \cos(4\theta_k)$. This lets us write the Bardasis-Schrieffer action as,

$$\boxed{S_d = \frac{1}{\beta}\sum_q \left[\frac{1}{g_d} - \frac{1}{g_s} + (i\Omega_m)^2 \sum_{\mathbf{k}} \frac{1}{2\lambda_k}\frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} + \sum_{\mathbf{k}} 2\lambda_k \cos(4\theta_k) \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2}\right] d_{-q} d_q}. \tag{23}$$

Since $g_s > g_d$, the first two terms give a positive contribution to the energy threshold for excitation, though this alone is not the energy of the mode. Extending the imaginary frequency to be a general complex number, the value where this action vanishes ($i\Omega_m \to z = \Omega_{\mathrm{BS}} \in \mathbb{R}$) defines the BS mode frequency.

B. Photon Self-Energy

The next term we examine is the self-energy of the photons due to the superconductor. Unlike for the previous term, here we keep the dependence on $\mathbf{q}$ in the Green's functions because the momentum dependence of the photon dispersion will be important for the hybridization.

$$\begin{aligned} \frac{1}{2}\operatorname{Tr}\left[\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\right] &= \frac{T^2}{2}\sum_{k,q} \operatorname{tr}\left[\hat{G}_{k+\frac{q}{2}}\left(\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q}\right)\hat{G}_{k-\frac{q}{2}}\left(\hat{\chi}^P_{k,-q} + \hat{\chi}^S_{k,-q}\right)\right] \\ = \frac{T}{2}\sum_{\mathbf{k},q}\sum_{\alpha,\alpha'} &\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \left(\hat{U}^\dagger_{\mathbf{k}+\mathbf{q}/2}(\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q})\hat{U}_{\mathbf{k}-\mathbf{q}/2}\right)_{\alpha,\alpha'} \left(\hat{U}^\dagger_{\mathbf{k}-\mathbf{q}/2}(\hat{\chi}^P_{k,-q} + \hat{\chi}^S_{k,-q})\hat{U}_{\mathbf{k}+\mathbf{q}/2}\right)_{\alpha',\alpha}. \end{aligned}$$

Using the forms of the coupling to photons we arrive at the expression

$$\begin{aligned} &\frac{e^2}{2\beta}\sum_{\mathbf{k},q}\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \\ &\quad\times\Big\{(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{A}_q)(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{A}_{-q})\left(\ell^2_{\mathbf{k},\mathbf{q}}\delta_{\alpha,\alpha'} + p^2_{\mathbf{k},\mathbf{q}}\delta_{\alpha,-\alpha'}\right) + (\mathbf{v}_S\cdot\mathbf{A}_q)(\mathbf{v}_S\cdot\mathbf{A}_{-q})\left(n^2_{\mathbf{k},\mathbf{q}}\delta_{\alpha,\alpha'} + m^2_{\mathbf{k},\mathbf{q}}\;\delta_{\alpha,-\alpha'}\right) \\ &\qquad + \left[(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{A}_q)(\mathbf{v}_S\cdot\mathbf{A}_{-q}) + (\mathbf{v}_S\cdot\mathbf{A}_q)(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{A}_{-q})\right]\alpha\left(\ell_{\mathbf{k},\mathbf{q}}n_{\mathbf{k},\mathbf{q}}\delta_{\alpha,\alpha'} + p_{\mathbf{k},\mathbf{q}}m_{\mathbf{k},\mathbf{q}}\delta_{\alpha,-\alpha'}\right)\Big\}, \end{aligned} \tag{24}$$

where we have introduced the superconducting coherence factors with their usual definitions,

$$\begin{aligned} \ell_{\mathbf{k},\mathbf{q}} &= u_+u_- + v_+v_- \qquad & p_{\mathbf{k},\mathbf{q}} &= u_+v_- - v_+u_- \\ n_{\mathbf{k},\mathbf{q}} &= u_+u_- - v_+v_- \qquad & m_{\mathbf{k},\mathbf{q}} &= u_+v_- + v_+u_- . \end{aligned}$$

We use the shorthand $u_\pm = u_{\mathbf{k}\pm\mathbf{q}/2}$ and similarly for $v_\pm$, with $u_\mathbf{k}, v_\mathbf{k}$ as given in Eq. 19. We rewrite this trace in terms of the photon response function $\hat{\Pi}_q$ as

$$S_\Pi = \frac{1}{2}\,\mathrm{Tr}\left[\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\right] = \frac{1}{2\beta}\sum_q\sum_{\mu,\nu} A^\mu_{-q}\Pi^{\mu\nu}_q A^\nu_q = \frac{1}{2\beta}\sum_q\sum_{\alpha,\alpha'} A_{\alpha,-q}\Pi_{\alpha\alpha',q}A_{\alpha',q}, \tag{25}$$

where μ,ν index the Cartesian components of the photon field and α,α' index the cavity modes. The last equality uses the polarization vectors to translate between the real space orthogonal Cartesian basis and cavity modes,

$$\Pi_{\alpha\alpha',q} = \sum_{\mu,\nu}\hat{\epsilon}^\mu_{\alpha,-\mathbf{q}}\Pi^{\mu\nu}_q\hat{\epsilon}^\nu_{\alpha',\mathbf{q}}. \tag{26}$$

We can combine this with the empty cavity action in Sec. I B to give the total action for the photons,

$$\boxed{S_A = -\frac{1}{2\beta}\sum_q\sum_{\alpha,\alpha'} A_{\alpha,-q}\left[\left((i\Omega_m)^2 - \omega_q^2\right)\delta_{\alpha\alpha'} - \Pi_{\alpha\alpha',q}\right]A_{\alpha',q}.} \tag{27}$$

We will leave the response function in this compact form until we need to explicitly expand it later on.

C. Coupling of photons to d-wave excitations

Finally we arrive at the term that couples photons and d-wave excitations. It has two components, one with the paramagnetic vertex and the other with the supercurrent vertex. For now we can consider both at the same time.

$$\begin{aligned} \mathrm{Tr}(\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\hat{G}\hat{\Delta}^d) &= T^2\sum_{k,q}\mathrm{tr}\left(\hat{G}_{k+\frac{q}{2}}(\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q})\hat{G}_{k-\frac{q}{2}}\hat{\Delta}^d_{k,-q}\right) \\ &\approx T^2\sum_{k,q}\mathrm{tr}\left(\hat{g}^+_\mathbf{k}\ \hat{U}^\dagger_\mathbf{k}(\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q})\hat{U}_\mathbf{k}\ \hat{g}^-_\mathbf{k}\ \hat{U}^\dagger_\mathbf{k}\hat{\Delta}^d_{k,-q}\hat{U}_\mathbf{k}\right) \\ &= T^2\sum_{k,q}\sum_{\alpha,\alpha'=\pm}\frac{1}{i\epsilon^+_n - E^\alpha_\mathbf{k}}\ \frac{1}{i\epsilon^-_n - E^{\alpha'}_\mathbf{k}}\left(\hat{U}^\dagger_\mathbf{k}(\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q})\hat{U}_\mathbf{k}\right)_{\alpha,\alpha'}\left(\hat{U}^\dagger_\mathbf{k}\hat{\Delta}^d_{k,-q}\hat{U}_\mathbf{k}\right)_{\alpha',\alpha} \\ &= T\sum_{\mathbf{k},q}\sum_{\alpha,\alpha'}\frac{n_F(E^{\alpha'}_\mathbf{k}) - n_F(E^\alpha_\mathbf{k})}{i\Omega_m - (E^\alpha_\mathbf{k} - E^{\alpha'}_\mathbf{k})}\left(\hat{U}^\dagger_\mathbf{k}(\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q})\hat{U}_\mathbf{k}\right)_{\alpha,\alpha'}\left(\hat{U}^\dagger_\mathbf{k}\hat{\Delta}^d_{k,-q}\hat{U}_\mathbf{k}\right)_{\alpha',\alpha}, \end{aligned}$$

using the same approximations as discussed above for the d-wave bubble. The new matrix elements that appear in this expression are

$$\begin{aligned} \hat{U}^\dagger_\mathbf{k}\hat{\chi}^P_{k,q}\hat{U}_\mathbf{k} &= \hat{\chi}^P_{k,q} = -e\mathbf{v}_\mathbf{k}\cdot\mathbf{A}_q\ \hat{\tau}_0 \\ \hat{U}^\dagger_\mathbf{k}\hat{\chi}^S_{k,q}\hat{U}_\mathbf{k} &= -e\mathbf{v}_S\cdot\mathbf{A}_q\frac{1}{\lambda_k}\left(\xi^S_k\ \hat{\tau}_3 + \Delta\ \hat{\tau}_1\right). \end{aligned} \tag{28}$$

The coupling through the paramagnetic vertex has only $\alpha = \alpha'$, and within our approximations this leads to an exact cancellation of the Fermi functions, making this term identically zero. The supercurrent vertex has off-diagonal components so it gives the only nonzero contribution to the coupling. The result is

$$\begin{aligned}
\mathrm{Tr}(\hat{G}(\hat{\chi}^P+\hat{\chi}^S)\hat{G}\hat{\Delta}^d) &= -e\Delta\sum_{\mathbf{k},q}\sum_{\alpha=\pm}\alpha\frac{n_F(E^{-\alpha}_{\mathbf{k}})-n_F(E^{\alpha}_{\mathbf{k}})}{i\Omega_m-(E^{\alpha}_{\mathbf{k}}-E^{-\alpha}_{\mathbf{k}})}\; f_d(\mathbf{k})\;\frac{1}{\lambda_k}\mathbf{v}_S\cdot\mathbf{A}_q\;\underbrace{\frac{\Delta^d_{-q}-\bar{\Delta}^d_q}{2}}_{=id^I_{-q}} \\
&\quad -e\Delta\sum_{\mathbf{k},q}\sum_{\alpha=\pm}\frac{n_F(E^{-\alpha}_{\mathbf{k}})-n_F(E^{\alpha}_{\mathbf{k}})}{i\Omega_m-(E^{\alpha}_{\mathbf{k}}-E^{-\alpha}_{\mathbf{k}})}\; f_d(\mathbf{k})\;\frac{\xi^S_k}{\lambda^2_k}\mathbf{v}_S\cdot\mathbf{A}_q\;\underbrace{\frac{\Delta^d_{-q}+\bar{\Delta}^d_q}{2}}_{=d^R_{-q}} \\
&\approx -2ie\Delta\sum_{\mathbf{k},q}\Big(n_F(E^-_{\mathbf{k}})-n_F(E^+_{\mathbf{k}})\Big)\,\frac{i\Omega_m}{(i\Omega_m)^2-(2\lambda_k)^2}\;\frac{f_d(\mathbf{k})}{\lambda_k}\;\mathbf{v}_S\cdot\mathbf{A}_q\;d_{-q}
\end{aligned}$$

$$\Rightarrow \boxed{S_{A-d}=-ie\Delta\sum_{\mathbf{k},q}\Big(n_F(E^-_{\mathbf{k}})-n_F(E^+_{\mathbf{k}})\Big)\,\frac{i\Omega_m}{(i\Omega_m)^2-(2\lambda_k)^2}\;\frac{f_d(\mathbf{k})}{\lambda_k}\;\mathbf{v}_S\cdot\big(\mathbf{A}_q\;d_{-q}-\mathbf{A}_{-q}d_q\big)\,.} \tag{29}$$

In the second equality we have dropped the coupling to the real component d^R because the term linear is in ξ^S_k, which is small in particle-hole asymmetry. This is further justification to ignore it in Sec. II A besides just being overdamped. In the final expression we use that the BS mode and the vector potential are real fields to write the coupling in a more symmetric way.

Note that this term is nonzero only because of the supercurrent. First, it explicitly contains $\mathbf{v}_S\cdot\mathbf{A}_q$, so will go to zero if $\mathbf{v}_S\to 0$. Additionally, the angular integration does not give identically zero because the Fermi functions depend on the angle of $\mathbf{k}$ through the supercurrent Doppler shift term in the energy. The integral over the single power of the d-wave form factor would exactly cancel were it not for this.

III. FIELDS TO MODE OPERATORS

We now have the effective action for the system in terms of the Bardasis-Schrieffer collective mode d_q, the photon fields $A_{\alpha,q}$, and the coupling between them. It will be more useful, however, to rewrite this not in terms of the fields but in terms of the mode operators, for which we may be able to separate out a Hamiltonian describing the hybridized system. Here we briefly outline the procedure to translate between the two pictures for a general bosonic field in the Matsubara formalism, before then applying it to our specific problem.

A. General bosonic mode operators

Start from a bosonic action for the real field φ,

$$S_\varphi=\int d\tau\sum_{\mathbf{q}}\frac{m}{2}\Big(\partial_\tau\varphi_{-\mathbf{q}}(\tau)\,\partial_\tau\varphi_{\mathbf{q}}(\tau)+\omega_q^2\;\varphi_{-\mathbf{q}}(\tau)\varphi_{\mathbf{q}}(\tau)\Big)=-\frac{m}{2\beta}\sum_q\varphi_{-q}\Big((i\Omega_m)^2-\omega_q^2\Big)\,\varphi_q. \tag{30}$$

The partition function is then just $\mathcal{Z}=\int\mathcal{D}\varphi\exp[-S_\varphi]$. Now introduce the auxiliary real field $\pi_{\mathbf{q}}(\tau)$ with the action

$$S_\pi=\int d\tau\sum_{\mathbf{q}}\frac{1}{2m}\pi_{-\mathbf{q}}(\tau)\pi_{\mathbf{q}}(\tau), \tag{31}$$

which is normalized so that it's corresponding partition function is equal to 1, as in a Hubbard-Stratonovich decomposition. Shift this field by the τ-derivative of φ,

$$\begin{aligned}
&\pi_{\mathbf{q}}(\tau)\to\pi_{\mathbf{q}}(\tau)-im\partial_\tau\varphi_{\mathbf{q}}(\tau) \\
\Rightarrow S_\pi=\int d\tau\sum_{\mathbf{q}}\frac{1}{2m}\pi_{-\mathbf{q}}(\tau)\pi_{\mathbf{q}}(\tau)\to\int d\tau\sum_{\mathbf{q}}&\left[\frac{1}{2m}\pi_{-\mathbf{q}}(\tau)\pi_{\mathbf{q}}(\tau)-\frac{m}{2}\partial_\tau\varphi_{-\mathbf{q}}(\tau)\,\partial_\tau\varphi_{\mathbf{q}}(\tau)-i\partial_\tau\varphi_{\mathbf{q}}(\tau)\,\pi_{-\mathbf{q}}(\tau)\right],
\end{aligned} \tag{32}$$

which we can do because we are free to deform the integration contour however we like as long as we do not cross a pole. Simply shifting the funtional integration countour for π back down returns π to a real field. After cancellations between this shifted S_π and S_φ the partition function of the whole system now becomes

$$\mathcal{Z} = \int \mathcal{D}\varphi \mathcal{D}\pi \exp\left[-\int d\tau \sum_{\mathbf{q}} \left(-i\partial_\tau \varphi_{\mathbf{q}}(\tau)\pi_{-\mathbf{q}}(\tau) + \frac{1}{2m}\pi_{-\mathbf{q}}(\tau)\pi_{\mathbf{q}}(\tau) + \frac{m}{2}\omega_q^2\ \varphi_{-\mathbf{q}}(\tau)\varphi_{\mathbf{q}}(\tau)\right)\right]. \tag{33}$$

We now change variables, going from the two fields φ and π to the complex fields b and $\bar{b}$ through the transformation

$$\begin{aligned} b_{\mathbf{q}}(\tau) &= \frac{1}{\sqrt{2m\omega_q}}\left(m\omega_q\varphi_{\mathbf{q}}(\tau) + i\pi_{\mathbf{q}}(\tau)\right), \quad \bar{b}_{\mathbf{q}}(\tau) = \frac{1}{\sqrt{2m\omega_q}}\left(m\omega_q\varphi_{-\mathbf{q}}(\tau) - i\pi_{-\mathbf{q}}(\tau)\right) \\ \Rightarrow \varphi_{\mathbf{q}}(\tau) &= \frac{1}{\sqrt{2m\omega_q}}\left(b_{\mathbf{q}}(\tau) + \bar{b}_{-\mathbf{q}}(\tau)\right), \quad \pi_{\mathbf{q}}(\tau) = -i\sqrt{\frac{m\omega_q}{2}}\left(b_{\mathbf{q}}(\tau) - \bar{b}_{-\mathbf{q}}(\tau)\right). \end{aligned} \tag{34}$$

Substituting this into the action above, simplifying, and Fourier transforming τ to Matsubara frequency yields

$$\mathcal{Z} = \int \mathcal{D}(\bar{b}, b) \exp\left[-\frac{1}{\beta}\sum_q \bar{b}_q\left(-i\Omega_m + \omega_q\right) b_q\right]. \tag{35}$$

This gives a straightforward way to translate an action in terms of a real bosonic field to its corresponding complex mode basis:

$$\boxed{S_\varphi = -\frac{m}{2\beta}\sum_q \varphi_{-q}\left((i\Omega_m)^2 - \omega_q^2\right)\varphi_q \Leftrightarrow S_b = \frac{1}{\beta}\sum_q \bar{b}_q(-i\Omega_m + \omega_q)b_q \quad \text{with} \quad \varphi_q = \frac{b_q + \bar{b}_{-q}}{\sqrt{2m\omega_q}}} \tag{36}$$

B. Bardasis-Schrieffer mode operators

We start from the action Eq. 23, reproduced here,

$$S_d = \frac{1}{\beta}\sum_q \left[\frac{1}{g_d} - \frac{1}{g_s} + (i\Omega_m)^2 \sum_{\mathbf{k}} \frac{1}{2\lambda_k}\frac{n_F(E_{\mathbf{k}}^-) - n_F(E_{\mathbf{k}}^+)}{(i\Omega_m)^2 - (2\lambda_k)^2} + \sum_{\mathbf{k}} 2\lambda_k \cos(4\theta_k)\frac{n_F(E_{\mathbf{k}}^-) - n_F(E_{\mathbf{k}}^+)}{(i\Omega_m)^2 - (2\lambda_k)^2}\right] d_{-q}d_q. \tag{37}$$

In order to apply Eq. 36 we need to get this into the appropriate form, which means we need to expand the expression in brackets to second order in Matsubara frequency. The quantity inside the brackets, up to a sign, is the inverse propagator of the BS mode. Write it in terms of a general complex argument as

$$D_{\text{BS}}^{-1}(z^2) = -\left[\frac{1}{g_d} - \frac{1}{g_s} + z^2\sum_{\mathbf{k}} \frac{1}{2\lambda_k}\frac{n_F(E_{\mathbf{k}}^-) - n_F(E_{\mathbf{k}}^+)}{z^2 - (2\lambda_k)^2} + \sum_{\mathbf{k}} 2\lambda_k\cos(4\theta_k)\frac{n_F(E_{\mathbf{k}}^-) - n_F(E_{\mathbf{k}}^+)}{z^2 - (2\lambda_k)^2}\right]. \tag{38}$$

The Bardasis-Schrieffer frequency is defined as the values of z such that $D_{\text{BS}}^{-1}(\Omega_{\text{BS}}^2) = 0$. Expand this in a Taylor series around this point:

$$D_{\text{BS}}^{-1}(z^2) \approx D_{\text{BS}}^{-1}(\Omega_{\text{BS}}^2) + (z^2 - \Omega_{\text{BS}}^2)\left.\frac{\partial D_{\text{BS}}^{-1}(z^2)}{\partial(z^2)}\right|_{z^2=\Omega_{\text{BS}}^2} = (z^2 - \Omega_{\text{BS}}^2)\left.\frac{\partial D_{\text{BS}}^{-1}(z^2)}{\partial(z^2)}\right|_{z=\Omega_{\text{BS}}} \equiv \frac{M_{\text{BS}}}{2}(z^2 - \Omega_{\text{BS}}^2), \tag{39}$$

where we have defined the derivative of the Green's function as half the "mass" M_{BS}. (It doesn't actually have units of mass, it just appears in the same place as the parameter m in the general derivation.) Putting $z \to i\Omega_m$ again, we have now rewritten the BS action as

$$S_d \approx -\frac{M_{\text{BS}}}{2\beta}\sum_q d_{-q}\left((i\Omega_m)^2 - \Omega_{\text{BS}}^2\right)d_q, \tag{40}$$

which is just the form we need to apply Eq. 36, with $d_q = \frac{1}{\sqrt{2M_{\text{BS}}\Omega_{\text{BS}}}}(b_q + \bar{b}_{-q})$.

C. Photon mode operators

The empty cavity action Eq. 9 is already in the form needed to apply the translation to mode operators,

$$S_{\mathrm{cav},A} = -\frac{1}{2\beta}\sum_{q,\alpha} A_{\alpha,-q}\left((i\Omega_m)^2 + \omega_q^2\right)A_{\alpha,q} \Rightarrow S_{\mathrm{cav},a} = \frac{1}{\beta}\sum_{q,\alpha}\bar{a}_{\alpha,q}(-i\Omega_m + \omega_q)a_{\alpha,q}, \tag{41}$$

with $A_{\alpha,q} = \frac{1}{\sqrt{2\omega_q}}(a_{\alpha,q} + \bar{a}_{\alpha,-q})$. The difficulty in the photon sector is in now handling the self-energy Π. Since the free part of the action has already taken care of the nontrivial aspect of translating from the fields A to the modes a, we can simply replace $A_{\alpha,q}$ with it's definition in terms of the mode operators in the self-energy term. The result is

$$\begin{aligned}
S_\Pi &= \frac{1}{2\beta}\sum_{q,\alpha,\alpha'} A_{\alpha,-q}\Pi_{\alpha\alpha',q}A_{\alpha',q} = \frac{1}{2\beta}\sum_{q,\alpha,\alpha'}\frac{1}{2\omega_q}(a_{\alpha,-q} + \bar{a}_{\alpha,q})\Pi_{\alpha\alpha',q}(a_{\alpha',q} + \bar{a}_{\alpha',-q}) \\
&= \frac{1}{2\beta}\sum_{q,\alpha,\alpha'}\frac{1}{2\omega_q}\left[a_{\alpha,-q}\Pi_{\alpha\alpha',q}a_{\alpha',q} + \bar{a}_{\alpha,-q}\Pi_{\alpha\alpha',q}\bar{a}_{\alpha',q} + \bar{a}_{\alpha,-q}\Pi_{\alpha\alpha',q}a_{\alpha',q} + a_{\alpha,-q}\Pi_{\alpha\alpha',q}\bar{a}_{\alpha',q}\right] \\
&\approx \frac{1}{2\beta}\sum_{q,\alpha,\alpha'}\frac{1}{2\omega_q}\left[\bar{a}_{\alpha,-q}\Pi_{\alpha\alpha',q}a_{\alpha',q} + a_{\alpha,-q}\Pi_{\alpha\alpha',q}\bar{a}_{\alpha',q}\right] \\
&= \frac{1}{\beta}\sum_{q,\alpha,\alpha'}\frac{1}{2\omega_q}\bar{a}_{\alpha,-q}\Pi_{\alpha\alpha',q}a_{\alpha',q},
\end{aligned} \tag{42}$$

where in the last line we have used that $\Pi_{\alpha\alpha',q} = \Pi_{\alpha'\alpha,-q}$. Furthermore, we have made the approximation of throwing out the counterrotating terms, which contain either two a's or two $\bar{a}$'s. Defining the self-energy itself as $\tilde{\Pi}_{\alpha\alpha',q} = \frac{1}{2\omega_q}\Pi_{\alpha\alpha',q}$ we can then write the full photon mode action as

$$S_a = \frac{1}{\beta}\sum_{q,\alpha,\alpha'}\bar{a}_{\alpha,q}\left[\left(-i\Omega_m + \omega_q\right)\delta_{\alpha\alpha'} + \tilde{\Pi}_{\alpha\alpha',q}\right]a_{\alpha',q}. \tag{43}$$

We would like to write the action in terms of some effective photon Hamiltonian so that the part inside the brackets (the inverse Green's function, up to a sign) has the form $-i\Omega_m + \hat{H}_{\mathbf{q}}$, where $\hat{H}_{\mathbf{q}}$ is a function of momentum only. To do this we need to remove the frequency dependence of the self-energy, which we can do by simply expanding it to first order in the deviation of the frequency from some reference frequency,

$$\hat{\tilde{\Pi}}(i\Omega_m, \mathbf{q}) \approx \hat{\tilde{\Pi}}(\omega_r, \mathbf{q}) + (i\Omega_m - \omega_r)\left.\frac{\partial\hat{\tilde{\Pi}}(z,\mathbf{q})}{\partial z}\right|_{z=\omega_r}. \tag{44}$$

The term proportional to $i\Omega_m$ combines with the linear in frequency part to give a wave function renormalization, and the rest contributes to the Hamiltonian. The question then is what reference frequency do we expand around? The natural candidate is the frequency at $\mathbf{q} = 0$, which is just ω_0 plus the shift due to the self-energy, but this shift will generically be different for the two polarizations. However, it turns out that this shift, in certain approximations, is identically zero for one mode and incredibly small for the other, as will be shown in Sec. IV, so we use simply $\omega_r = \omega_0$. Define

$$\left.\frac{\partial\hat{\tilde{\Pi}}(z,\mathbf{q})}{\partial z}\right|_{z=\omega_0} \equiv \hat{1} - \hat{Z}_{\mathbf{q}} \quad \Rightarrow \quad \hat{\tilde{\Pi}}(i\Omega_m, \mathbf{q}) \approx \hat{\tilde{\Pi}}(\omega_0, \mathbf{q}) + (\hat{1} - \hat{Z}_{\mathbf{q}})(i\Omega_m - \omega_0) \tag{45}$$

so that

$$(-i\Omega_m + \omega_q) + \hat{\tilde{\Pi}}(i\Omega_m, \mathbf{q}) \approx (-i\Omega_m + \omega_0)\hat{Z}_{\mathbf{q}} + \omega_q - \omega_0 + \hat{\tilde{\Pi}}(\omega_0, \mathbf{q}). \tag{46}$$

If $\hat{Z}_{\mathbf{q}}$ is positive-definite, then a Cholesky decomposition lets us write it in terms of a lower-triangular matrix as $\hat{Z}_{\mathbf{q}} = \hat{L}_{\mathbf{q}}\hat{L}_{\mathbf{q}}^\dagger$. With this we can absorb the prefactor in front of the frequency into a redefinition of the mode operators,

$\tilde{a}_{\alpha,q} = \sum_\beta \left(\hat{L}^\dagger_{\mathbf{q}}\right)_{\alpha\beta} a_{\beta,q}$, giving the form we want:

$$\sum_{\alpha,\alpha'} \bar{a}_{\alpha,q} \left[(-i\Omega_m + \omega_0) \left(\hat{Z}_{\mathbf{q}}\right)_{\alpha\alpha'} + (\omega_q - \omega_0)\delta_{\alpha\alpha'} + \tilde{\Pi}(\omega_0, \mathbf{q})_{\alpha\alpha'} \right] a_{\alpha',q}$$
$$= \sum_{\alpha,\alpha'} \bar{\tilde{a}}_{\alpha,q} \left[(-i\Omega_m + \omega_0)\, \delta_{\alpha\alpha'} + (\omega_q - \omega_0) \left(\hat{L}^\dagger_{\mathbf{q}} \hat{L}_{\mathbf{q}}\right)^{-1}_{\alpha\alpha'} + \left(\hat{L}^{-1}_{\mathbf{q}} \hat{\tilde{\Pi}}(\omega_0, \mathbf{q}) (\hat{L}^\dagger_{\mathbf{q}})^{-1}\right)_{\alpha\alpha'} \right] \tilde{a}_{\alpha',q}. \quad (47)$$

Note also that this redefinition of the photon mode operators will need to be done in the coupling between photons and the BS mode as well. The photon Hamiltonian we seek can then be simply picked out,

$$\hat{H}_{\mathbf{q}} = \omega_0 \hat{1} + \underbrace{(\omega_q - \omega_0)}_{=\frac{|\mathbf{q}|^2}{2\omega_0}} \left(\hat{L}^\dagger_{\mathbf{q}} \hat{L}_{\mathbf{q}}\right)^{-1} + \hat{L}^{-1}_{\mathbf{q}} \hat{\tilde{\Pi}}(\omega_0, \mathbf{q}) (\hat{L}^\dagger_{\mathbf{q}})^{-1}. \quad (48)$$

D. Photon-BS coupling in terms of mode operators

Finally we rewrite the coupling between the photons and the Bardasis-Schrieffer mode in terms of the mode operators we have introduced. Simply substituting the expressions of the fields in terms of the operators into the coupling term in the action, Eq. 29, we have

$$S_{a-b} = -\frac{1}{\beta} \sum_{q,\alpha} \frac{ie\Delta}{\sqrt{4M_{\mathrm{BS}}\Omega_{\mathrm{BS}}\omega_q}} \sum_{\mathbf{k}} \left(n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})\right) \frac{i\Omega_m}{(i\Omega_m)^2 - (2\lambda_k)^2} \frac{f_d(\mathbf{k})}{\lambda_k} \mathbf{v}_S \cdot \hat{\boldsymbol{\epsilon}}_{\alpha,\mathbf{q}} \left(\bar{b}_q a_{\alpha,q} + \bar{a}_{\alpha,q} b_q\right). \quad (49)$$

To obtain this form, as in the case of the photon self-energy term, we must discard counterrotating terms proportional to $\bar{b}\bar{a}$ and ba. Despite having an explicit factor of i the coulpling is real because the polarization vectors $\hat{\boldsymbol{\epsilon}}$ also contain a factor of i to cancel it. Now we must also take into account the rotation of the photon mode operators from the wave function renormalization. This gives

$$S_{a-b} = \frac{1}{\beta} \sum_q \sum_\alpha \left(\bar{b}_q,\ \bar{\tilde{a}}_{\alpha,q}\right) \begin{pmatrix} 0 & g_{\alpha,q} \\ g^*_{\alpha,q} & 0 \end{pmatrix} \begin{pmatrix} b_q \\ \tilde{a}_{\alpha,q} \end{pmatrix} \quad (50)$$

$$g_{\alpha,q} = -\frac{ie\Delta}{\sqrt{4M_{\mathrm{BS}}\Omega_{\mathrm{BS}}\omega_q}} \sum_{\alpha'} \mathbf{v}_S \cdot \hat{\boldsymbol{\epsilon}}_{\alpha',\mathbf{q}} \left[\left(\hat{L}^\dagger_{\mathbf{q}}\right)^{-1}\right]_{\alpha'\alpha} \sum_{\mathbf{k}} \left(n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})\right) \frac{i\Omega_m}{(i\Omega_m)^2 - (2\lambda_k)^2} \frac{f_d(\mathbf{k})}{\lambda_k} \quad (51)$$

$$g^*_{\alpha,q} = -\frac{ie\Delta}{\sqrt{4M_{\mathrm{BS}}\Omega_{\mathrm{BS}}\omega_q}} \sum_{\alpha'} \mathbf{v}_S \cdot \hat{\boldsymbol{\epsilon}}_{\alpha',\mathbf{q}} \left[\hat{L}^{-1}_{\mathbf{q}}\right]_{\alpha\alpha'} \sum_{\mathbf{k}} \left(n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})\right) \frac{i\Omega_m}{(i\Omega_m)^2 - (2\lambda_k)^2} \frac{f_d(\mathbf{k})}{\lambda_k}. \quad (52)$$

Note that the coupling is still a function of frequency. We remove this dependence by simply replacing it with the Bardasis-Schrieffer frequency, since this is the frequency at which the photon dispersion and the BS mode dispersion cross. Therefore we finally define

$$g^{\mathrm{eff}}_{\alpha,\mathbf{q}} = -ie\Delta \sqrt{\frac{\Omega_{\mathrm{BS}}}{M_{\mathrm{BS}}\, \omega_q}} \sum_{\alpha'} \mathbf{v}_S \cdot \hat{\boldsymbol{\epsilon}}_{\alpha',\mathbf{q}} \left[\left(\hat{L}^\dagger_{\mathbf{q}}\right)^{-1}\right]_{\alpha'\alpha} \sum_{\mathbf{k}} \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{\Omega^2_{\mathrm{BS}} - (2\lambda_k)^2} \frac{f_d(\mathbf{k})}{2\lambda_k}. \quad (53)$$

E. Mode action

Putting it all together, we arrive at the action in terms of the mode operators,

$$S = \frac{1}{\beta} \sum_q \left(\bar{b}_q, \bar{\tilde{a}}_{\alpha,q}\right) \begin{pmatrix} -i\Omega_m + \Omega_{\mathrm{BS}} & g^{\mathrm{eff}}_{\alpha,\mathbf{q}}\ \delta_{\alpha\alpha'} \\ g^{\mathrm{eff}*}_{\alpha,\mathbf{q}}\ \delta_{\alpha\alpha'} & -i\Omega_m \delta_{\alpha\alpha'} + H_{\alpha\alpha',\mathbf{q}} \end{pmatrix} \begin{pmatrix} b_q \\ \tilde{a}_{\alpha',q} \end{pmatrix}, \quad (54)$$

with an implicit sum over the repeated indices α, α'.

IV. EVALUATING THE PHOTON RESPONSE FUNCTION

In order to evaluate the effective photon Hamiltonian and the effective coupling we need to calculate the photon self-energy, the matrix $\hat{Z}$, and the matrix $\hat{L}$ with which it can be rewritten. Recombine the terms Eq. 24 based on the structure of their dependence on α, α' to give 4 distinct types of terms,

$$\begin{aligned}
&e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\, \sigma^0_{\alpha,\alpha'} \left[\ell^2_{\mathbf{k},\mathbf{q}}\, (\mathbf{v_k}\cdot\mathbf{A}_q)(\mathbf{v_k}\cdot\mathbf{A}_{-q}) + n^2_{\mathbf{k},\mathbf{q}}\, (\mathbf{v}_S\cdot\mathbf{A}_q)(\mathbf{v}_S\cdot\mathbf{A}_{-q})\right] \\
&+ e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\, \sigma^1_{\alpha,\alpha'} \left[p^2_{\mathbf{k},\mathbf{q}}\, (\mathbf{v_k}\cdot\mathbf{A}_q)(\mathbf{v_k}\cdot\mathbf{A}_{-q}) + m^2_{\mathbf{k},\mathbf{q}}\, (\mathbf{v}_S\cdot\mathbf{A}_q)(\mathbf{v}_S\cdot\mathbf{A}_{-q})\right] \\
&+ e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\, \sigma^3_{\alpha,\alpha'} \ell_{\mathbf{k},\mathbf{q}} n_{\mathbf{k},\mathbf{q}} \left[(\mathbf{v_k}\cdot\mathbf{A}_q)(\mathbf{v}_S\cdot\mathbf{A}_{-q}) + (\mathbf{v}_S\cdot\mathbf{A}_q)(\mathbf{v_k}\cdot\mathbf{A}_{-q})\right] \\
&+ e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\, i\sigma^2_{\alpha,\alpha'}\, p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}} \left[(\mathbf{v_k}\cdot\mathbf{A}_q)(\mathbf{v}_S\cdot\mathbf{A}_{-q}) + (\mathbf{v}_S\cdot\mathbf{A}_q)(\mathbf{v_k}\cdot\mathbf{A}_{-q})\right]
\end{aligned} \tag{55}$$

Based on the expressions above, we only need the self-energy to first order in the deviation of the frequency from ω_0, and furthermore we will expand it to second order in the momentum, approximating the photon dispersion as parabolic. We start the expansion with the coherence factors. When expanded for small q, the functions ℓ, m, and n all have a part independent of q, no linear term, and a q^2 term. The function p, on the other hand, up to order q^2 has only a linear in q term, so the expansion of p^2 is order q^2. Explicitly the expansions we will need are

$$\begin{aligned}
\ell^2_{\mathbf{k},\mathbf{q}} &\approx 1 - \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}(\mathbf{v_k}\cdot\mathbf{q})^2 & p^2_{\mathbf{k},\mathbf{q}} &= 1 - \ell^2_{\mathbf{k},\mathbf{q}} \approx \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}(\mathbf{v_k}\cdot\mathbf{q})^2 \\
n^2_{\mathbf{k},\mathbf{q}} &\approx \frac{\xi_k^{S\,2}}{\lambda_k^2} + \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}\left[\frac{\xi_k^S}{m}\,q^2 - 2\frac{\xi_k^{S\,2}}{\lambda_k^2}(\mathbf{v_k}\cdot\mathbf{q})^2\right] & m^2_{\mathbf{k},\mathbf{q}} &\approx \frac{\Delta^2}{\lambda_k^2} - \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}\left[\frac{\xi_k^S}{m}\,q^2 - 2\frac{\xi_k^{S\,2}}{\lambda_k^2}(\mathbf{v_k}\cdot\mathbf{q})^2\right] \\
\ell_{\mathbf{k},\mathbf{q}} n_{\mathbf{k},\mathbf{q}} &\approx \frac{\xi_k^S}{\lambda_k} + \frac{1}{8}\frac{\Delta^2}{\lambda_k^4}\left[\frac{\lambda_k}{m}\,q^2 - 3\frac{\xi_k^S}{\lambda_k}(\mathbf{v_k}\cdot\mathbf{q})^2\right] & p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}} &\approx \frac{1}{2}\frac{\Delta^2}{\lambda_k^3}\mathbf{v_k}\cdot\mathbf{q}
\end{aligned} \tag{56}$$

We see here that when expanded the last of these functions, $p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}}$, is linear in q and contains no term independent of q, so the rest of the expression containing it only needs to be expanded to first order in small parameters.

Now we expand the energies and the occupation functions in q,

$$E^{\alpha}_{\mathbf{k}\pm\mathbf{q}/2} \approx E^{\alpha}_{\mathbf{k}} \pm \frac{\mathbf{q}}{2}\cdot\nabla E^{\alpha}_{\mathbf{k}} + \frac{1}{2}\frac{q_i}{2}\frac{q_j}{2}\partial_i\partial_j E^{\alpha}_{\mathbf{k}} \equiv E^{\alpha}_{\mathbf{k}} \pm \frac{\mathbf{q}}{2}\cdot\mathbf{v}^{\alpha}_{\mathbf{k}} + \alpha\,\frac{q_i q_j}{8 m_{\mathbf{k},ij}} \tag{57}$$

$$n_F\left(E^{\alpha}_{\mathbf{k}\pm\mathbf{q}/2}\right) \approx n_F(E^{\alpha}_{\mathbf{k}}) + \left(\pm\frac{\mathbf{q}}{2}\cdot\mathbf{v}^{\alpha}_{\mathbf{k}} + \alpha\,\frac{q_i q_j}{8 m_{\mathbf{k},ij}}\right) n'_F(E^{\alpha}_{\mathbf{k}}) + \frac{1}{2}\left(\frac{\mathbf{q}}{2}\cdot\mathbf{v}^{\alpha}_{\mathbf{k}}\right)^2 n''_F(E^{\alpha}_{\mathbf{k}}), \tag{58}$$

where we have defined

$$\mathbf{v}^{\alpha}_{\mathbf{k}} = \nabla E^{\alpha}_{\mathbf{k}} = \alpha\frac{\xi_k^S}{\lambda_k}\mathbf{v_k} + \mathbf{v}_S, \qquad \frac{1}{m_{\mathbf{k},ij}} = \partial_i\partial_j\lambda_k = \frac{1}{\lambda_k}\left(\frac{\xi_k^S}{m}\delta_{ij} + \frac{v^i_{\mathbf{k}} v^j_{\mathbf{k}}\Delta^2}{\lambda_k^2}\right) \tag{59}$$

With these approximations and notation the differences of energies are

$$E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha}_{\mathbf{k}-\mathbf{q}/2} \approx \mathbf{q}\cdot\mathbf{v}^{\alpha}_{\mathbf{k}} \tag{60}$$

$$\begin{aligned}
E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2} &\approx E^{\alpha}_{\mathbf{k}} - E^{-\alpha}_{\mathbf{k}} + \mathbf{q}\cdot\frac{\mathbf{v}^{\alpha}_{\mathbf{k}} + \mathbf{v}^{-\alpha}_{\mathbf{k}}}{2} + \alpha\,\frac{q_i q_j}{4 m_{\mathbf{k},ij}} \\
&= 2\alpha\,\lambda_k + \mathbf{q}\cdot\mathbf{v}_S + \alpha\,\frac{q_i q_j}{4 m_{\mathbf{k},ij}},
\end{aligned} \tag{61}$$

and the differences of the Fermi functions are

$$n_F(E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}) - n_F(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}) \approx -\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right) n'_F(E^{\alpha}_\mathbf{k}) \tag{62}$$

$$\begin{aligned}
n_F(E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}) - n_F(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}) &\approx n_F(E^{-\alpha}_\mathbf{k}) - n_F(E^{\alpha}_\mathbf{k}) - \frac{\mathbf{q}}{2}\cdot\left(\mathbf{v}^{-\alpha}_\mathbf{k}\; n'_F(E^{-\alpha}_\mathbf{k}) + \mathbf{v}^{\alpha}_\mathbf{k}\; n'_F(E^{\alpha}_\mathbf{k})\right) \\
&\quad - \alpha\frac{q_i q_j}{8 m_{\mathbf{k},ij}}\left(n'_F(E^{-\alpha}_\mathbf{k}) + n'_F(E^{\alpha}_\mathbf{k})\right) + \frac{q_i q_j}{8}\left(v^{-\alpha}_{\mathbf{k},i} v^{-\alpha}_{\mathbf{k},j} n''_F(E^{-\alpha}_\mathbf{k}) - v^{\alpha}_{\mathbf{k},i} v^{\alpha}_{\mathbf{k},j} n''_F(E^{\alpha}_\mathbf{k})\right) \\
&= \alpha\left(n_F(E^-_\mathbf{k}) - n_F(E^+_\mathbf{k})\right) + \frac{\mathbf{q}}{2}\cdot\left[\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}\left(n'_F(E^-_\mathbf{k}) - n'_F(E^+_\mathbf{k})\right) - \mathbf{v}_S\left(n'_F(E^+_\mathbf{k}) + n'_F(E^-_\mathbf{k})\right)\right] \\
&\quad -\alpha\,\frac{q_i q_j}{8 m_{\mathbf{k},ij}}\left(n'_F(E^+_\mathbf{k}) + n'_F(E^-_\mathbf{k})\right) + \alpha\,\frac{q_i q_j}{8}\left[\frac{\xi^{S\,2}_k}{\lambda_k^2} v^i_\mathbf{k} v^j_\mathbf{k} + v^i_S v^j_S\right]\left(n''_F(E^-_\mathbf{k}) - n''_F(E^+_\mathbf{k})\right) \\
&\quad -\alpha\,\frac{q_i q_j}{8}\frac{\xi^S_k}{\lambda_k}\left(v^i_\mathbf{k} v^j_S + v^i_S v^j_\mathbf{k}\right)\left(n''_F(E^+_\mathbf{k}) + n''_F(E^-_\mathbf{k})\right) \\
&\equiv \alpha\,\delta n_\mathbf{k} + \frac{\mathbf{q}}{2}\cdot\left[\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}\delta n'_\mathbf{k} - \mathbf{v}_S N'_\mathbf{k}\right] \\
&\quad -\alpha\frac{q_i q_j}{8}\left[\frac{N'_\mathbf{k}}{m_{\mathbf{k},ij}} - \left(\frac{\xi^{S\,2}_k}{\lambda_k^2} v^i_\mathbf{k} v^j_\mathbf{k} + v^i_S v^j_S\right)\delta n''_\mathbf{k} + \frac{\xi^S_k}{\lambda_k}\left(v^i_\mathbf{k} v^j_S + v^i_S v^j_\mathbf{k}\right) N''_\mathbf{k}\right],
\end{aligned} \tag{63}$$

where we have introduced the notation $\delta n_\mathbf{k} = n_F(E^-_\mathbf{k}) - n_F(E^+_\mathbf{k})$ and $N_\mathbf{k} = n_F(E^+_\mathbf{k}) + n_F(E^-_\mathbf{k})$.

We can now expand the factors containing the difference of Fermi functions to second order in both $|\mathbf{q}|$ and in $\delta\Omega$, defined as the fluctuation in imaginary frequency away from the $\mathbf{q} = 0$ on-shell condition ($\delta\Omega = i\Omega_m - \omega_0$) and then do the sums over α, α'.

We start with the factor in the first line of Eq. 55 involving $\sigma^0_{\alpha,\alpha'}$ which gives

$$\begin{aligned}
&\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\sigma^0_{\alpha,\alpha'} \\
&\approx \sum_\alpha \frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right) n'_F(E^\alpha_\mathbf{k})}{\omega_0 + \delta\Omega - \mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right)} \approx \sum_\alpha \frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right) n'_F(E^\alpha_\mathbf{k})}{\omega_0}\left(1 - \frac{\delta\Omega - \mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right)}{\omega_0}\right) \\
&= -q_i\sum_\alpha\frac{\alpha\frac{\xi^S_k}{\lambda_k}v^i_\mathbf{k} + v^i_S}{\omega_0}n'_F(E^\alpha_\mathbf{k})\left(1 - \frac{\delta\Omega}{\omega_0}\right) - q_i q_j\sum_\alpha\frac{\left(\alpha\frac{\xi^S_k}{\lambda_k}v^i_\mathbf{k} + v^i_S\right)\left(\alpha\frac{\xi^S_k}{\lambda_k}v^j_\mathbf{k} + v^j_S\right)}{\omega_0^2}n'_F(E^\alpha_\mathbf{k}) \\
&= q_i\frac{\frac{\xi^S_k}{\lambda_k}v^i_\mathbf{k}\delta n'_\mathbf{k} - v^i_S N'_\mathbf{k}}{\omega_0}\left(1 - \frac{\delta\Omega}{\omega_0}\right) - q_i q_j\frac{\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2 v^i_\mathbf{k} v^j_\mathbf{k} + v^i_S v^j_S\right]N'_\mathbf{k} - \frac{\xi^S_k}{\lambda_k}\left(v^i_\mathbf{k} v^j_S + v^i_S v^j_\mathbf{k}\right)\delta n'_\mathbf{k}}{\omega_0^2}
\end{aligned}$$

Because these terms are all at least first order in small quantities, the order q^2 parts of the expansions of $\ell_{\mathbf{k},\mathbf{q}}$ and $n_{\mathbf{k},\mathbf{q}}$ are never needed. As a result, these terms only ever multiply functions that are even in ξ^S_k, so all of the terms here that are odd in ξ can be dropped because they are small in particle-hole asymmetry. This simplification gives

$$\begin{aligned}
\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\sigma^0_{\alpha,\alpha'} &\to -\frac{1}{\omega_0}N'_\mathbf{k}\left(1 - \frac{\delta\Omega}{\omega_0}\right)\mathbf{v}_S\cdot\mathbf{q} - \frac{\left(\frac{\xi^S_k}{\lambda_k}\right)^2 v^i_\mathbf{k} v^j_\mathbf{k} + v^i_S v^j_S}{\omega_0^2}N'_\mathbf{k}\,q_i q_j \\
&\equiv X^{(1,0)}_{0,\mathbf{k},i}\left(1 - \frac{\delta\Omega}{\omega_0}\right)q_i + X^{(2,0)}_{0,\mathbf{k},ij}\;q_i q_j.
\end{aligned} \tag{64}$$

The corresponding factor in the second line of Eq. 55, involving $\sigma^1_{\alpha,\alpha'}$, is significantly more complicated and contains a term for all powers and combinations of our small parameters.

$$
\begin{aligned}
&\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\sigma^1_{\alpha,\alpha'}\approx\sum_{\alpha}\frac{n_F\left(E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{\omega_0+\delta\Omega-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)}\\
&\approx\sum_\alpha\left\{\frac{\alpha\,\delta n_{\mathbf{k}}}{\omega_0-2\alpha\,\lambda_k}\left[1-\frac{\delta\Omega-\mathbf{q}\cdot\mathbf{v}_S-\alpha\frac{q_iq_j}{4m_{\mathbf{k},ij}}}{\omega_0-2\alpha\,\lambda_k}+\left(\frac{\delta\Omega-\mathbf{q}\cdot\mathbf{v}_S}{\omega_0-2\alpha\,\lambda_k}\right)^2\right]\right.\\
&\qquad+\frac{\mathbf{q}}{2}\cdot\frac{\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}}\delta n'_{\mathbf{k}}-\mathbf{v}_SN'_{\mathbf{k}}}{\omega_0-2\alpha\,\lambda_k}\left(1-\frac{\delta\Omega-\mathbf{q}\cdot\mathbf{v}_S}{\omega_0-2\alpha\,\lambda_k}\right)\\
&\qquad\left.-\alpha\frac{q_iq_j}{8}\frac{\frac{N'_{\mathbf{k}}}{m_{\mathbf{k},ij}}-\left(\frac{\xi^{s\,2}_k}{\lambda_k^2}v^i_{\mathbf{k}}v^j_{\mathbf{k}}+v^i_Sv^j_S\right)\delta n''_{\mathbf{k}}+\frac{\xi^S_k}{\lambda_k}\left(v^i_{\mathbf{k}}v^j_S+v^i_Sv^j_{\mathbf{k}}\right)N''_{\mathbf{k}}}{\omega_0-2\alpha\,\lambda_k}\right\}\\
&=\frac{4\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}-\frac{2\omega_0\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\,\delta\Omega\\
&\quad+\omega_0\left[\frac{8\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}v^i_S\,\delta n_{\mathbf{k}}+\frac{1}{\omega_0^2-(2\lambda_k)^2}\left(\frac{\xi^S_k}{\lambda_k}v^i_{\mathbf{k}}\delta n'_{\mathbf{k}}-v^i_SN'_{\mathbf{k}}\right)\right]q_i\\
&\quad-\left[8\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}v^i_S\,\delta n_{\mathbf{k}}+\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\left(\frac{\xi^S_k}{\lambda_k}v^i_{\mathbf{k}}\delta n'_{\mathbf{k}}-v^i_SN'_{\mathbf{k}}\right)\right]q_i\,\delta\Omega\\
&\quad+\left\{\frac{1}{2m_{\mathbf{k},ij}}\left[\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}-\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]+\frac{1}{2}\left(\frac{\xi^S_k}{\lambda_k}\right)^2v^i_{\mathbf{k}}v^j_{\mathbf{k}}\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right.\\
&\qquad+v^i_Sv^j_S\left[4\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}-\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}+\frac{1}{2}\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right]\\
&\qquad\left.+\frac{1}{2}\frac{\xi^S_k}{\lambda_k}\left(v^i_{\mathbf{k}}v^j_S+v^i_Sv^j_{\mathbf{k}}\right)\left[\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n'_{\mathbf{k}}-\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N''_{\mathbf{k}}\right]\right\}q_iq_j
\end{aligned}
$$

These terms are mostly even in ξ, and the only ones that are odd in ξ are at least first order in small quantities. The only function that multiplies these terms that is itself odd in ξ is also second order in q (the q^2 part of the expansion of $m^2_{\mathbf{k},\mathbf{q}}$), so we can freely drop these terms. Additionally we can drop the odd in ξ parts of the expansion of m^2.

$$\begin{aligned}
&\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \sigma^1_{\alpha,\alpha'} \\
&\quad\approx \frac{4\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\delta n_{\mathbf{k}} - \frac{2\omega_0\lambda_k}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\,\delta\Omega + \omega_0\left[\frac{8\lambda_k}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\,\delta n_{\mathbf{k}} - \frac{1}{\omega_0^2 - (2\lambda_k)^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q} \\
&\qquad - \left[8\lambda_k\frac{3\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^3}\,\delta n_{\mathbf{k}} - \frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}\;\delta\Omega \\
&\qquad + \left\{\frac{1}{2}v^i_{\mathbf{k}}v^j_{\mathbf{k}}\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \left(\frac{\Delta}{\lambda_k}\right)^2\frac{1}{\omega_0^2 - (2\lambda_k)^2}N'_{\mathbf{k}} + \left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\delta n''_{\mathbf{k}}\right]\right. \\
&\qquad\qquad \left. + v^i_S v^j_S\left[4\lambda_k\frac{3\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}} - \frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}N'_{\mathbf{k}} + \frac{1}{2}\frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\delta n''_{\mathbf{k}}\right]\right\} q_i q_j \\
&\quad= X^{(0,0)}_{1,\mathbf{k}} + X^{(0,1)}_{1,\mathbf{k}}\;\delta\Omega + X^{(1,0)}_{1,\mathbf{k},i}\;q_i + X^{(1,1)}_{1,\mathbf{k},i}\;q_i\;\delta\Omega + X^{(2,0)}_{1,\mathbf{k},ij}\;q_i q_j.
\end{aligned} \tag{65}$$

The last two lines of Eq. 55 containing σ^3 and $i\sigma^2$, which arise from the cross terms from the trace-log with one paramagnetic vertex and one supercurrent vertex, have similar factors as the first two but with an extra overall factor of α. The factor in the first, containing $\sigma^3_{\alpha,\alpha'}$, is

$$\begin{aligned}
&\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \sigma^3_{\alpha,\alpha'} \\
&\quad\approx \sum_\alpha \alpha\frac{n_F\left(E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{\Omega - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)} \approx \sum_\alpha \alpha\frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}} + \mathbf{v}_S\right)n'_F(E^\alpha_{\mathbf{k}})}{\omega_0 + \delta\Omega - \mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}} + \mathbf{v}_S\right)} \\
&\quad\approx \sum_\alpha \alpha\frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}} + \mathbf{v}_S\right)n'_F(E^\alpha_{\mathbf{k}})}{\omega_0}\left(1 - \frac{\delta\Omega - \mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}} + \mathbf{v}_S\right)}{\omega_0}\right) \\
&\quad= -q_i\sum_\alpha\frac{\frac{\xi^S_k}{\lambda_k}v^i_{\mathbf{k}} + \alpha v^i_S}{\omega_0}n'_F(E^\alpha_{\mathbf{k}})\left(1 - \frac{\delta\Omega}{\omega_0}\right) - q_i q_j\sum_\alpha \alpha\frac{\left(\alpha\frac{\xi^S_k}{\lambda_k}v^i_{\mathbf{k}} + v^i_S\right)\left(\alpha\frac{\xi^S_k}{\lambda_k}v^j_{\mathbf{k}} + v^j_S\right)}{\omega_0^2}n'_F(E^\alpha_{\mathbf{k}}) \\
&\quad= q_i\frac{-\frac{\xi^S_k}{\lambda_k}v^i_{\mathbf{k}}N'_{\mathbf{k}} + v^i_S\delta n'_{\mathbf{k}}}{\omega_0}\left(1 - \frac{\delta\Omega}{\omega_0}\right) + q_i q_j\frac{\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2 v^i_{\mathbf{k}}v^j_{\mathbf{k}} + v^i_S v^j_S\right]\delta n'_{\mathbf{k}} - \frac{\xi^S_k}{\lambda_k}\left(v^i_{\mathbf{k}}v^j_S + v^i_S v^j_{\mathbf{k}}\right)N'_{\mathbf{k}}}{\omega_0^2}.
\end{aligned}$$

This is very similar to the σ^0 case, but with the δn's and N's swapped around along with some sign changes. Also like that case, these terms are all already at least first order in small quantities, so we do not need the part of $\ell_{\mathbf{k},\mathbf{q}}n_{\mathbf{k},\mathbf{q}}$ that is second order in q. The part of that function that is zeroth order in q is proportional to ξ, so because we are dropping terms that are odd in ξ overall due to the smallness of particle-hole asymmetry, we only need to keep the terms here that are themselves odd in ξ. The result is

$$\begin{aligned}
\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \sigma^3_{\alpha,\alpha'} &\approx -\frac{\xi^S_k}{\lambda_k}\frac{1}{\omega_0}N'_{\mathbf{k}}\left(1 - \frac{\delta\Omega}{\omega_0}\right)\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q} - \frac{\xi^S_k}{\lambda_k}\frac{\left(v^i_{\mathbf{k}}v^j_S + v^i_S v^j_{\mathbf{k}}\right)}{\omega_0^2}N'_{\mathbf{k}}\,q_i q_j \\
&\equiv \frac{\xi^S_k}{\lambda_k}\left[X^{(1,0)}_{3,\mathbf{k},i}\left(1 - \frac{\delta\Omega}{\omega_0}\right)\;q_i + X^{(2,0)}_{3,\mathbf{k},ij}\;q_i q_j\right].
\end{aligned} \tag{66}$$

Finally, the last line involving $i\sigma^2_{\alpha,\alpha'}$ only needs to be expanded to 1st order. Additionally, the function it multiplies (i.e. $p_{\mathbf{k},\mathbf{q}}m_{\mathbf{k},\mathbf{q}}$) is even in ξ so we can drop all terms in this expansion that are odd in ξ. The result is

$$\begin{aligned}
\sum_{\alpha,\alpha'}&\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,i\sigma^2_{\alpha,\alpha'}\approx\sum_\alpha\alpha\,\frac{n_F\left(E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{\omega_0+\delta\Omega-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)}\\
&\approx\sum_\alpha\left[\frac{\delta n_{\mathbf{k}}}{\omega_0-2\alpha\,\lambda_k}\left(1-\frac{\delta\Omega-\mathbf{q}\cdot\mathbf{v}_S}{\omega_0-2\alpha\,\lambda_k}\right)+\alpha\,\frac{\mathbf{q}}{2}\cdot\frac{\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}}\delta n'_{\mathbf{k}}-\mathbf{v}_S N'_{\mathbf{k}}}{\omega_0-2\alpha\,\lambda_k}\right]\\
&\approx\frac{2\omega_0}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}-2\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\delta\Omega+\left[2\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}-\frac{2\lambda_k}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}\\
&=X^{(0,0)}_{2,\mathbf{k}}+X^{(0,1)}_{2,\mathbf{k}}\;\delta\Omega+X^{(1,0)}_{2,\mathbf{k},i}\;q_i.
\end{aligned}\tag{67}$$

Now combining these expansions with the expansions of the coherence factors and dropping all terms higher than second order in small quantities q and $\delta\Omega$ we have

$$\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,\sigma^0_{\alpha,\alpha'}\ell^2_{\mathbf{k},\mathbf{q}}\approx X^{(1,0)}_{0,\mathbf{k},i}\;q_i\left(1-\frac{\delta\Omega}{\omega_0}\right)+X^{(2,0)}_{0,\mathbf{k},ij}\;q_iq_j\tag{68}$$

$$\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,\sigma^0_{\alpha,\alpha'}n^2_{\mathbf{k},\mathbf{q}}\approx\frac{\xi^{S\,2}_k}{\lambda_k^2}\left(X^{(1,0)}_{0,\mathbf{k},i}\;q_i\left(1-\frac{\delta\Omega}{\omega_0}\right)+X^{(2,0)}_{0,\mathbf{k},ij}\;q_iq_j\right)\tag{69}$$

$$\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,\sigma^1_{\alpha,\alpha'}p^2_{\mathbf{k},\mathbf{q}}\approx\frac{1}{4}\frac{\Delta^2}{\lambda_k^4}X^{(0,0)}_{1,\mathbf{k}}(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})^2\tag{70}$$

$$\begin{aligned}
\sum_{\alpha,\alpha'}&\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,\sigma^1_{\alpha,\alpha'}m^2_{\mathbf{k},\mathbf{q}}\approx\frac{1}{2}\frac{\Delta^2\xi^{S\,2}_k}{\lambda_k^6}X^{(0,0)}_{1,\mathbf{k}}(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})^2\\
&+\frac{\Delta^2}{\lambda_k^2}\left(X^{(0,0)}_{1,\mathbf{k}}+X^{(0,1)}_{1,\mathbf{k}}\;\delta\Omega+X^{(1,0)}_{1,\mathbf{k},i}\;q_i+X^{(1,1)}_{1,\mathbf{k},i}\;q_i\;\delta\Omega+X^{(2,0)}_{1,\mathbf{k},ij}\;q_iq_j\right)
\end{aligned}\tag{71}$$

$$\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,\sigma^3_{\alpha,\alpha'}\;\ell_{\mathbf{k},\mathbf{q}}n_{\mathbf{k},\mathbf{q}}\approx\frac{\xi^{S\,2}_k}{\lambda_k^2}\left(X^{(1,0)}_{3,\mathbf{k},i}\;q_i\left(1-\frac{\delta\Omega}{\omega_0}\right)+X^{(2,0)}_{3,\mathbf{k},ij}\;q_iq_j\right)\tag{72}$$

$$\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,i\sigma^2_{\alpha,\alpha'}\;p_{\mathbf{k},\mathbf{q}}m_{\mathbf{k},\mathbf{q}}\approx\frac{1}{2}\frac{\Delta^2}{\lambda_k^3}\left(X^{(0,0)}_{2,\mathbf{k}}+X^{(0,1)}_{2,\mathbf{k}}\;\delta\Omega+X^{(1,0)}_{2,\mathbf{k},i}\;q_i\right)\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q}\tag{73}$$

Since the maximum possible value of the superfluid velocity is of order Δ/k_F, i.e. that corresponding to the critical current, we know that $v_S\ll v_F\sim v_{\mathbf{k}}$ since $\Delta\ll E_F$. This gives us another small parameter that we can use to simplify our expansions above, namely v_S/v_F. In particular, we can drop many of the terms arising from the bubbles involving 2 supercurrent vertices, since they appear alongside a corresponding bubble with 2 paramagnetic vertices, and are therefore small by comparison. All of the terms multiplying $n^2_{\mathbf{k},\mathbf{q}}$ can be dropped (since $(\xi/\lambda)^2\sim1$ and they match up 1-to-1 with terms multiplying $\ell^2_{\mathbf{k},\mathbf{q}}$). We therefore discard Eq. 69 entirely, as well as the terms in Eq. 71 that have a corresponding partner in Eq. 70. This turns out to be just the first term; the remaining terms have no partner in Eq. 70.

The mixed terms, those with one supercurrent vertex and one paramagnetic vertex, cannot be straighforwardly simplified without further inspection since they may contain terms that are the same order as the coupling between the BS mode and the photon sector, which is itself proportional to v_S. Discarding all terms that contain powers of

v_S whenever there is a corresponding term using $v_{\mathbf{k}}$, the expressions above simplify to

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k-q/2}}\right) - n_F\left(E^{\alpha}_{\mathbf{k+q/2}}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k+q/2}} - E^{\alpha'}_{\mathbf{k-q/2}}\right)} \left(\sigma^0_{\alpha,\alpha'}\ell^2_{\mathbf{k},\mathbf{q}} + \sigma^1_{\alpha,\alpha'}p^2_{\mathbf{k},\mathbf{q}}\right) \approx -\frac{1}{\omega_0}N'_{\mathbf{k}}\left(\operatorname{sgn}\Omega - \frac{\delta\Omega}{\omega_0}\right)\mathbf{v}_S\cdot\mathbf{q} \tag{74}$$

$$-\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2 \frac{1}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{1}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})^2 \tag{75}$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k-q/2}}\right) - n_F\left(E^{\alpha}_{\mathbf{k+q/2}}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k+q/2}} - E^{\alpha'}_{\mathbf{k-q/2}}\right)}\, \sigma^1_{\alpha,\alpha'}m^2_{\mathbf{k},\mathbf{q}}$$

$$\approx \frac{\Delta^2}{\lambda_k^2}\frac{4\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}} - \operatorname{sgn}\Omega\frac{\Delta^2}{\lambda_k^2}\frac{2\omega_0\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\,\delta\Omega + 4\lambda_k\frac{\Delta^2}{\lambda_k^2}\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}\,\delta\Omega^2$$

$$+\operatorname{sgn}\Omega\,\omega_0\frac{\Delta^2}{\lambda_k^2}\left[\frac{8\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\,\delta n_{\mathbf{k}} - \frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}$$

$$-\frac{\Delta^2}{\lambda_k^2}\left[8\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\,\delta n_{\mathbf{k}} - \frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}\,\delta\Omega$$

$$+\frac{1}{2}\frac{\Delta^2}{\lambda_k^2}\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \left(\frac{\Delta}{\lambda_k}\right)^2\frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}} + \left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})^2$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k-q/2}}\right) - n_F\left(E^{\alpha}_{\mathbf{k+q/2}}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k+q/2}} - E^{\alpha'}_{\mathbf{k-q/2}}\right)} \left(\sigma^3_{\alpha,\alpha'}\,\ell_{\mathbf{k},\mathbf{q}}n_{\mathbf{k},\mathbf{q}} + i\sigma^2_{\alpha,\alpha'}\,p_{\mathbf{k},\mathbf{q}}m_{\mathbf{k},\mathbf{q}}\right) \tag{76}$$

$$\approx -\operatorname{sgn}\Omega\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{1}{\omega_0}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q}) \tag{77}$$

$$+\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{1}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\right]\delta\Omega(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q}) \tag{78}$$

$$-\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{2}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\left(\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right)\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})(\mathbf{v}_S\cdot\mathbf{q}) \tag{79}$$

Putting these terms back into their respective expressions we can then define the integrals over $\mathbf{k}$ as simple coefficients of our expansion, the result being terms entering the quadratic action for the photon field.

It is as this point that we change the basis for the photon field and addtionally rotate the bases for $\mathbf{k}$ and $\mathbf{q}$ to be measured from that same axis as well so that $\mathbf{v}_S\cdot\mathbf{q} = v_S q\cos\theta_q$ and $\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q} = v_k q\cos(\theta_k-\theta_q)$. (In the ξ-approximation, $v_k \to v_F$, the Fermi velocity.) The terms coming with ℓ^2 and p^2 appear in expressions with two paramagnetic vertices, and so they couple to both the parallel and perpendicular components of $\mathbf{A}$. The terms coming with m^2, however, come with two supercurrent verticies, and so only couple to $\mathbf{A}^{\parallel}$. The cross terms couple the parallel component of $\mathbf{A}$

to either component. The terms in the expansion are

$$x_P^{10,\mu\nu}(\theta_q)\; q\left(1-\frac{\delta\Omega}{\omega_0}\right) = -e^2 v_S \sum_{\mathbf{k}} \frac{1}{\omega_0} N'_{\mathbf{k}} v^\mu_{\mathbf{k}} v^\nu_{\mathbf{k}}\; q\cos\theta_q \left(1-\frac{\delta\Omega}{\omega_0}\right) \tag{80}$$

$$x_P^{20,\mu\nu}(\theta_q)\; q^2 = -\frac{e^2}{\omega_0^2}\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right] v^\mu_{\mathbf{k}} v^\nu_{\mathbf{k}}\; v_k^2\, \cos^2(\theta_k-\theta_q)\; q^2 \tag{81}$$

$$x_S^{00} = 4e^2 v_S^2 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}} \tag{82}$$

$$x_S^{01}\delta\Omega = -2e^2 v_S^2\omega_0 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\;\delta\Omega \tag{83}$$

$$x_S^{10}(\theta_q)q = e^2\omega_0 v_S^3 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\left[\frac{8\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\,\delta n_{\mathbf{k}} - \frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]\; q\cos\theta_q \tag{84}$$

$$x_S^{11}(\theta_q)\, q\,\delta\Omega = -e^2 v_S^3 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\left[8\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\,\delta n_{\mathbf{k}} - \frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}\right]\; q\cos\theta_q\,\delta\Omega \tag{85}$$

$$x_S^{20}(\theta_q)q^2 = \frac{e^2 v_S^2}{2}\sum_{\mathbf{k}}\frac{\Delta^2}{\lambda_k^2}\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \left(\frac{\Delta}{\lambda_k}\right)^2\frac{N'_{\mathbf{k}}}{\omega_0^2-(2\lambda_k)^2} + \left(\frac{\xi_k^S}{\lambda_k}\right)^2\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right] v_k^2\,\cos^2(\theta_k-\theta_q)\, q^2 \tag{86}$$

$$x_{SP}^{10,\mu}(\theta_q)\; q = -\frac{e^2 v_S}{\omega_0}\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right] v^\mu_{\mathbf{k}}\; v_k\, \cos(\theta_k-\theta_q)\; q \tag{87}$$

$$x_{SP}^{11,\mu}(\theta_q)\; q\,\delta\Omega = e^2 v_S\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{1}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\right] v^\mu_{\mathbf{k}}\; v_k\, \cos(\theta_k-\theta_q)\; q\,\delta\Omega \tag{88}$$

$$x_{SP}^{20,\mu}(\theta_q)\, q^2 = -e^2 v_S\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{2}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\left(\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right)\right] v^\mu_{\mathbf{k}}\, v_k\, \cos(\theta_k-\theta_q)\; q^2\cos\theta_q, \tag{89}$$

where the indices $\mu,\nu = \parallel,\perp$. The terms with a subscript P (for paramagnetic) multiply $\mathbf{A}^\mu_q\mathbf{A}^\nu_{-q}$, those with subscript S (for supercurrent) multiply just the parallel components $\mathbf{A}^\parallel_q\mathbf{A}^\parallel_{-q}$, and those with subscript SP (the cross terms) multiply $\mathbf{A}^\mu_q\mathbf{A}^\parallel_{-q}+\mathbf{A}^\parallel_q\mathbf{A}^\mu_{-q}$.

A. Angular integration

Before writing the full photon action, we should examine the angular integral over θ_k, which may lead to some simplifications. To do so, we need to understand the angular properties of $\delta n_{\mathbf{k}}$ and $N'_{\mathbf{k}}$. Rewriting these in a more transparent form we have

$$\delta n_{\mathbf{k}} = \frac{\sinh(\lambda_k/T)}{\cosh(\lambda_k/T)+\cosh(\mathbf{v}_S\cdot\mathbf{k}/T)} \qquad N'_{\mathbf{k}} = -\frac{1}{T}\frac{\cosh(\lambda_k/T)\cosh(\mathbf{v}_S\cdot\mathbf{k}/T)}{\left(\cosh(\lambda_k/T)+\cosh(\mathbf{v}_S\cdot\mathbf{k}/T)\right)^2}. \tag{90}$$

Because cosh is even, these expressions do not change sign under $\mathbf{k}\to-\mathbf{k}$.

We now go through the distinct angular integrals one by one. Using the identity $\cos(\theta_k-\theta_q) = \cos\theta_k\cos\theta_q +$

$\sin\theta_k \sin\theta_q$ there are 21 different integrals. They are

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \tag{91}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \sin^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos\theta_k \sin\theta_k \tag{92}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^4\theta_k, \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \sin^4\theta_k, \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^2\theta_k \sin^2\theta_k \tag{93}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^3\theta_k \sin\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos\theta_k \sin^3\theta_k \tag{94}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \tag{95}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \sin^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos\theta_k \sin\theta_k \tag{96}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^4\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \sin^4\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^2\theta_k \sin^2\theta_k \tag{97}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^3\theta_k \sin\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos\theta_k \sin^3\theta_k \tag{98}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \cos^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \sin^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \cos\theta_k \sin\theta_k \tag{99}$$

$$\tag{100}$$

The 7 integrals here with an odd power of $\sin\theta_k$ are identically zero for all k which reduces the total number of integrals to 14. Furthermore, we can put $\sin^2\theta_k = 1 - \cos^2\theta_k$ and $\sin^4\theta_k = 1 - 2\cos^2\theta_k + \cos^4\theta_k$ and rewrite all integrals involving sin as integrals just involving cos, reducing the number further to 8. These 8 nonzero integrals are

$$\chi_0^0(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \qquad \chi_0^1(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^2\theta_k \qquad \chi_0^2(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^4\theta_k \tag{101}$$

$$\chi_1^0(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \qquad \chi_1^1(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^2\theta_k \qquad \chi_1^2(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^4\theta_k \tag{102}$$

$$\chi_2^0(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \qquad \chi_2^1(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \cos^2\theta_k. \tag{103}$$

The subscript gives the order of the derivative of the appropriate combination of Fermi functions inside the integrand, and the twice superscript gives the power of the $\cos\theta_k$ factor.

The expansion coefficients can then be rewritten as one-dimensional integrals over the magnitude of k,

$$x_P^{10,\mu\nu}(\theta_q) = -\frac{e^2 v_S}{2\pi\omega_0}\cos\theta_q \int_0^\infty dk\, k\; v_k^2 \begin{pmatrix} \chi_1^1 & 0 \\ 0 & \chi_1^0 - \chi_1^1 \end{pmatrix}_{\mu\nu} \tag{104}$$

$$x_P^{20,\mu\nu}(\theta_q) = -\frac{e^2}{2\pi\omega_0^2}\int_0^\infty dk\, k\; v_k^4 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \begin{pmatrix} \chi_1^1 \sin^2\theta_q + \chi_1^2 \cos 2\theta_q & \left(\chi_1^1 - \chi_1^2\right)\sin 2\theta_q \\ \left(\chi_1^1 - \chi_1^2\right)\sin 2\theta_q & \left(\chi_1^0 - \chi_1^1\right)\sin^2\theta_q + \left(\chi_1^1 - \chi_1^2\right)\cos 2\theta_q \end{pmatrix}_{\mu\nu} \right.$$
$$\left. - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2 - (2\lambda_k)^2} \begin{pmatrix} \chi_0^1 \sin^2\theta_q + \chi_0^2 \cos 2\theta_q & \left(\chi_0^1 - \chi_0^2\right)\sin 2\theta_q \\ \left(\chi_0^1 - \chi_0^2\right)\sin 2\theta_q & \left(\chi_0^0 - \chi_0^1\right)\sin^2\theta_q + \left(\chi_0^1 - \chi_0^2\right)\cos 2\theta_q \end{pmatrix}_{\mu\nu} \right] \tag{105}$$

$$x_S^{00} = \frac{2e^2 v_S^2}{\pi}\int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\chi_0^0(k) \tag{106}$$

$$x_S^{01} = -\frac{e^2 v_S^2 \omega_0}{\pi}\int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\chi_0^0(k) \tag{107}$$

$$x_S^{10}(\theta_q) = \frac{e^2 v_S^3 \omega_0}{2\pi}\cos\theta_q \int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\left[\frac{8\lambda_k}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\chi_0^0(k) - \frac{1}{\omega_0^2 - (2\lambda_k)^2}\chi_1^0(k)\right] \tag{108}$$

$$x_S^{11}(\theta_q) = -\frac{e^2 v_S^3}{2\pi}\cos\theta_q \int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\left[8\lambda_k \frac{3\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^3}\chi_0^0(k) - \frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\chi_1^0(k)\right] \tag{109}$$

$$x_S^{20}(\theta_q) = \frac{e^2 v_S^2}{4\pi}\int_0^\infty dk\, k\; v_k^2 \frac{\Delta^2}{\lambda_k^2}\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\left(\chi_0^0(k)\sin^2\theta_q + \chi_0^1(k)\cos 2\theta_q\right)\right.$$
$$\left. - \left(\frac{\Delta}{\lambda_k}\right)^2 \frac{\chi_1^0(k)\sin^2\theta_q + \chi_1^1(k)\cos 2\theta_q}{\omega_0^2 - (2\lambda_k)^2} + \left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\left(\chi_2^0(k)\sin^2\theta_q + \chi_2^1(k)\cos 2\theta_q\right)\right] \tag{110}$$

$$x_{SP}^{10,\mu}(\theta_q) = -\frac{e^2 v_S}{2\pi\omega_0}\int_0^\infty dk\, k\; v_k^2 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ (\chi_1^0(k) - \chi_1^1(k))\sin\theta_q \end{pmatrix}_\mu - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2 - (2\lambda_k)^2}\begin{pmatrix} \chi_0^1(k)\cos\theta_q \\ (\chi_0^0(k) - \chi_0^1(k))\sin\theta_q \end{pmatrix}_\mu \right] \tag{111}$$

$$x_{SP}^{11,\mu}(\theta_q) = \frac{e^2 v_S}{2\pi}\int_0^\infty dk\, k\; v_k^2 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{1}{\omega_0^2}\begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ (\chi_1^0(k) - \chi_1^1(k))\sin\theta_q \end{pmatrix}_\mu \right.$$
$$\left. - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\begin{pmatrix} \chi_0^1(k)\cos\theta_q \\ (\chi_0^0(k) - \chi_0^1(k))\sin\theta_q \end{pmatrix}_\mu \right] \tag{112}$$

$$x_{SP}^{20,\mu}(\theta_q) = -\frac{e^2 v_S}{2\pi}\cos\theta_q \int_0^\infty dk\, k\; v_k^2 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{2}{\omega_0^2}\begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ (\chi_1^0(k) - \chi_1^1(k))\sin\theta_q \end{pmatrix}_\mu \right.$$
$$\left. - \frac{\Delta^2}{\lambda_k^3}\left(\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\begin{pmatrix} \chi_0^1(k)\cos\theta_q \\ (\chi_0^0(k) - \chi_0^1(k))\sin\theta_q \end{pmatrix}_\mu - \frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ (\chi_1^0(k) - \chi_1^1(k))\sin\theta_q \end{pmatrix}_\mu \right)\right] \tag{113}$$

B. Response function to Self-energy

So the photon self-energy term, written in the Cartesian $(\|,\perp)$ basis, is then

$$
\begin{aligned}
S_\Pi \approx \frac{1}{2\beta}\sum_q \Bigg[&\sum_{\mu,\nu=\|,\perp} \left(x_P^{10,\mu\nu}\, q\left(1-\frac{\delta\Omega}{\omega_0}\right) + x_P^{20,\mu\nu}\, q^2 \right) A^\mu_{-q} A^\nu_q \\
&+ \sum_{\mu=\|,\perp} \left(x_{SP}^{10,\mu}\, q + x_{SP}^{11,\mu}\, q\,\delta\Omega + x_{SP}^{20,\mu}\, q^2 \right)\left(A^\mu_{-q} A^\|_q + A^\|_{-q} A^\mu_q \right) \\
&+ \left(x_S^{00} + x_S^{01}\,\delta\Omega + x_S^{10}\, q + x_S^{11}\, q\,\delta\Omega + x_S^{20}\, q^2 \right) A^\|_{-q} A^\|_q \Bigg] \\
= \frac{1}{2\beta}\sum_q (A^\|_{-q}, A^\perp_{-q}) &\begin{pmatrix} b_\|(q) i\Omega_m + c_\|(q) & b_\times(q) i\Omega_m + c_\times(q) \\ b_\times(q) i\Omega_m + c_\times(q) & b_\perp(q) i\Omega_m + c_\perp(q) \end{pmatrix} \begin{pmatrix} A^\|_q \\ A^\perp_q \end{pmatrix} \\
\equiv \frac{1}{2\beta}\sum_q (A^\|_{-q}, A^\perp_{-q}) &\left(\hat{B}_{\mathbf{q}} i\Omega_m + \hat{C}_{\mathbf{q}} \right) \begin{pmatrix} A^\|_q \\ A^\perp_q \end{pmatrix}
\end{aligned} \tag{114}
$$

with

$$
b_\|(q) = x_S^{01} + \left(x_S^{11} + 2x_{SP}^{11,1} - \frac{x_P^{10,11}}{\omega_0} \right) q, \tag{115}
$$

$$
c_\|(q) = x_S^{00} - \omega_0\, x_S^{01} + \left[2x_P^{10,11} + 2(x_{SP}^{10,1} - \omega_0\, x_{SP}^{11,1}) + (x_S^{10} - \omega_0\, x_S^{11}) \right] q + \left(x_P^{20,11} + 2x_{SP}^{20,1} + x_S^{20} \right) q^2, \tag{116}
$$

$$
b_\times(q) = x_{SP}^{11,2} q, \qquad c_\times(q) = (x_{SP}^{10,2} - \omega_0\, x_{SP}^{11,2}) q + \left(x_P^{20,12} + x_{SP}^{20,2} \right) q^2, \tag{117}
$$

$$
b_\perp(q) = -\frac{x_P^{10,22}}{\omega_0} q, \qquad c_\perp(q) = 2x_P^{10,22} q + x_P^{20,22} q^2. \tag{118}
$$

We need to now change from the Cartesian $(\|,\perp)$ basis for Π to the mode basis, then define the self-energy $\tilde{\Pi}$ with which we define $\hat{Z}$ and $\hat{L}$. Putting $L=\pi/\omega_0$, these transformations on the matrix $\hat{B}_{\mathbf{q}}$ give

$$
\begin{aligned}
\tilde{B}_{\alpha\alpha',\mathbf{q}} &= \sum_{\mu,\nu=\|,\perp} \frac{\hat{\boldsymbol{\epsilon}}^\mu_{\alpha,-\mathbf{q}} B^{\mu\nu}_{\mathbf{q}} \hat{\boldsymbol{\epsilon}}^\nu_{\alpha',\mathbf{q}}}{2\omega_q} = \hat{1} - \hat{Z}_{\mathbf{q}} \\
&= \frac{1}{\pi}\frac{\omega_0}{\omega_q} \begin{pmatrix} b_\| \sin^2\theta_q + b_\perp \cos^2\theta_q - b_\times \sin(2\theta_q) & \frac{\omega_0}{\omega_q}\left(\frac{b_\|-b_\perp}{2}\sin(2\theta_q) - b_\times\cos(2\theta_q) \right) \\ \frac{\omega_0}{\omega_q}\left(\frac{b_\|-b_\perp}{2}\sin(2\theta_q) - b_\times\cos(2\theta_q) \right) & \left(\frac{\omega_0}{\omega_q}\right)^2 \left(b_\| \cos^2\theta_q + b_\perp \sin^2\theta_q + b_\times \sin(2\theta_q) \right) \end{pmatrix}_{\alpha\alpha'}.
\end{aligned} \tag{119}
$$

The same procedure on $\hat{C}_{\mathbf{q}}$ gives a matrix with the same structure but with b's replaced with c's. Call this matrix $\hat{\tilde{C}}_{\mathbf{q}}$. Comparing to the general expansion we did towards the end of Sec. III C it is

$$
\hat{\tilde{C}}_{\mathbf{q}} = \hat{\tilde{\Pi}}(\omega_0, \mathbf{q}) - \omega_0\left(\hat{1} - \hat{Z}_{\mathbf{q}} \right) \tag{120}
$$

We then have the following form for $\hat{Z}_{\mathbf{q}}$,

$$
\hat{Z}_{\mathbf{q}} = \begin{pmatrix} 1 - \frac{\omega_0}{\pi\omega_q}\left(b_\| \sin^2\theta_q + b_\perp\cos^2\theta_q - b_\times\sin(2\theta_q) \right) & -\frac{1}{\pi}\left(\frac{\omega_0}{\omega_q}\right)^2 \left(\frac{b_\|-b_\perp}{2}\sin(2\theta_q) - b_\times\cos(2\theta_q) \right) \\ -\frac{1}{\pi}\left(\frac{\omega_0}{\omega_q}\right)^2 \left(\frac{b_\|-b_\perp}{2}\sin(2\theta_q) - b_\times\cos(2\theta_q) \right) & 1 - \frac{1}{\pi}\left(\frac{\omega_0}{\omega_q}\right)^3 \left(b_\| \cos^2\theta_q + b_\perp\sin^2\theta_q + b_\times\sin(2\theta_q) \right) \end{pmatrix}. \tag{121}
$$

C. Cholesky Decomposition of Z

In general we wish to write

$$
\hat{Z} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \hat{L}\hat{L}^\dagger = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11}^* & L_{21}^* \\ 0 & L_{22}^* \end{pmatrix} = \begin{pmatrix} |L_{11}|^2 & L_{21}^* L_{11} \\ L_{21} L_{11}^* & |L_{22}|^2 + |L_{21}|^2 \end{pmatrix}. \tag{122}
$$

Since all of the elements of our $\hat{Z}_{\mathbf{q}}$ are real, we can choose all the elements of $\hat{L}_{\mathbf{q}}$ to be real as well. This gives

$$L_{11,\mathbf{q}} = \sqrt{1 - \frac{1}{\pi}\frac{\omega_0}{\omega_q}\left(b_{\|}\sin^2\theta_q + b_{\perp}\cos^2\theta_q - b_{\times}\sin(2\theta_q)\right)} \tag{123}$$

$$L_{21,\mathbf{q}} = -\frac{1}{\pi}\left(\frac{\omega_0}{\omega_q}\right)^2 \frac{\frac{b_{\|}-b_{\perp}}{2}\sin(2\theta_q) - b_{\times}\cos(2\theta_q)}{\sqrt{1 - \frac{1}{\pi}\frac{\omega_0}{\omega_q}\left(b_{\|}\sin^2\theta_q + b_{\perp}\cos^2\theta_q - b_{\times}\sin(2\theta_q)\right)}} \tag{124}$$

$$L_{22,\mathbf{q}} = \sqrt{1 - \frac{1}{\pi}\left(\frac{\omega_0}{\omega_q}\right)^3\left(b_{\|}\cos^2\theta_q + b_{\perp}\sin^2\theta_q + b_{\times}\sin(2\theta_q)\right) - L_{21}^2}. \tag{125}$$

V. THE SYSTEM HAMILTONIAN

With the Cholesky decomposition of $\hat{Z}_{\mathbf{q}}$ we now in principle have everything we need to evaluate the system Hamiltonian. Repeating our previously obtained results, the action is

$$S = \frac{1}{\beta}\sum_q \left(\bar{b}_q, \bar{\tilde{a}}_{\alpha,q}\right)\begin{pmatrix} -i\Omega_m + \Omega_{\mathrm{BS}} & g^{\mathrm{eff}}_{\alpha,\mathbf{q}}\,\delta_{\alpha\alpha'} \\ g^{\mathrm{eff}*}_{\alpha,\mathbf{q}}\,\delta_{\alpha\alpha'} & -i\Omega_m\delta_{\alpha\alpha'} + H_{\alpha\alpha',\mathbf{q}}\end{pmatrix}\begin{pmatrix} b_q \\ \tilde{a}_{\alpha',q}\end{pmatrix}, \tag{126}$$

with an implicit sum over the repeated indices α, α', with the coupling

$$g^{\mathrm{eff}}_{\alpha,\mathbf{q}} = -ie\Delta\sqrt{\frac{\Omega_{\mathrm{BS}}}{M_{\mathrm{BS}}\,\omega_q}}\sum_{\alpha'}\mathbf{v}_S\cdot\hat{\boldsymbol{\epsilon}}_{\alpha',\mathbf{q}}\left[\left(\hat{L}^{\dagger}_{\mathbf{q}}\right)^{-1}\right]_{\alpha'\alpha}\sum_{\mathbf{k}}\frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{\Omega^2_{\mathrm{BS}} - (2\lambda_k)^2}\frac{f_d(\mathbf{k})}{2\lambda_k}. \tag{127}$$

The photon Hamiltonian is

$$\hat{H}_{\mathbf{q}} = \omega_0\hat{1} + (\omega_q - \omega_0)\left(\hat{L}^{\dagger}_{\mathbf{q}}\hat{L}_{\mathbf{q}}\right)^{-1} + \hat{L}^{-1}_{\mathbf{q}}\hat{\tilde{\Pi}}(\omega_0,\mathbf{q})(\hat{L}^{\dagger}_{\mathbf{q}})^{-1} = \hat{L}^{-1}_{\mathbf{q}}\left(\omega_q\hat{1} + \hat{\tilde{C}}_{\mathbf{q}}\right)\left(\hat{L}^{\dagger}_{\mathbf{q}}\right)^{-1}. \tag{128}$$

Numerically evaluating the expansion coefficients (x_P^{10}, etc.) and the elements of $\hat{L}_{\mathbf{q}}$ using the parameters listed in Table I gives $L_{11}, L_{22} \approx 1$ and $L_{21} \approx 0$ to be a very good approximation, so that $\hat{L}_{\mathbf{q}}$ can simply be approximated as the identity matrix. The effective couplings are then

$$g^{\mathrm{eff}}_{1,\mathbf{q}} = ev_S\Delta\,\nu\,\sin\theta_q\sqrt{\frac{2\Omega_{\mathrm{BS}}}{L\,M_{\mathrm{BS}}\,\omega_q}}\int_{\Delta}^{\infty}\frac{d\lambda}{\sqrt{\lambda^2-\Delta^2}}\frac{\chi_d(\lambda)}{\Omega^2_{\mathrm{BS}} - 4\lambda^2} \tag{129}$$

$$g^{\mathrm{eff}}_{2,\mathbf{q}} = -ev_S\Delta\,\nu\,\cos\theta_q\sqrt{\frac{2\Omega_{\mathrm{BS}}}{L\,M_{\mathrm{BS}}\,\omega_q}}\frac{\omega_0}{\omega_q}\int_{\Delta}^{\infty}\frac{d\lambda}{\sqrt{\lambda^2-\Delta^2}}\frac{\chi_d(\lambda)}{\Omega^2_{\mathrm{BS}} - 4\lambda^2}, \tag{130}$$

with $\chi_d = \int_0^{2\pi}\frac{d\theta_k}{2\pi}f_d(\mathbf{k})\,\delta n_{\mathbf{k}}$, and the effective photon Hamiltonian is just $H_{\alpha\alpha',\mathbf{q}} = \omega_q\delta_{\alpha\alpha'} + \tilde{C}_{\alpha\alpha',\mathbf{q}}$.

T_c	$9.5\,\mathrm{K}$
T_F	$6.18\times10^4\,\mathrm{K}$
T	$0.4T_c$
m	$1.6m_e$
v_S	$0.6\frac{\Delta_0}{k_F}$
$\frac{1}{g_d}-\frac{1}{g_s}$	0.3ν
ω_0	$0.99\,\Omega_{\mathrm{BS}}$

TABLE I. The parameters used in our numerics. The superconducting critical temperature, Fermi temperature, effective mass correspond to the values for niobium.