Bardasis-Schrieffer Polaritons

Our general goal is to find a superconducting collective mode that will hybridize with cavity photons to produce a polariton. In this project we explore how this can happen with a d-wave excitation appearing above an s-wave condenced state. It will turn out that the so-called Bardasis-Schrieffer (BS) mode is the excitation that fulfills this goal. The BS mode is the component of the (complex) d-wave order parameter Δ^d that is 90° out-of-phase with the s-wave order parameter Δ^s. If we choose Δ^s to be purely real, then this means the BS mode is the imaginary component of Δ^d. The d-wave component that is in-phase with Δ^s will turn out to be overdamped and its coupling to photons will be small in the particle-hole asymmetry.

I. MINIMALLY COUPLED BDG ACTION WITH SUPERCURRENT

We start from the action of interacting electrons, with the interaction assumed to be separable and decomposed in angular momentum channels:

$$S[\bar{\psi},\psi] = \sum_{k,\sigma} \bar{\psi}_{k,\sigma} \left(-i\epsilon_n + \frac{k^2}{2m} - \mu \right) \psi_{k,\sigma} - \frac{1}{\beta} \sum_q \sum_{l=s,d} g_l \bar{\phi}^l_q \phi^l_q, \tag{1}$$

with $k = (\epsilon_n, \mathbf{k})$, $q = (\Omega_m, \mathbf{q})$, and the interaction written in terms of the fermion bilinears,

$$\phi^l_q = \sum_k f_l(\mathbf{k}) \psi_{-k+\frac{q}{2},\downarrow} \psi_{k+\frac{q}{2},\uparrow} \tag{2}$$

where the angular functions are $f_s(\mathbf{k}) = 1$ and $f_d(\mathbf{k}) = \sqrt{2}\cos(2\theta_k)$. Though our system is completely rotationally symmetric (C_∞ symmetry) this choice of d-wave form factor picks an explicit x-axis and mimics the form factor of the $d_{x^2-y^2}$ basis function of the C_4 symmetry group of a square lattice, which is a much more physically relevant symmetry. We assume that $g_s > g_d$ so the system favors purely s-wave pairing, but also that the difference is small so that there isn't a prohibitively large energy cost to create d-wave excitations. The interaction can be straightforwardly decoupled in the Cooper channel with a Hubbard-Stratonovich transformation (both s- and d-wave simultaneously) to give

$$S[\bar{\psi},\psi,\bar{\Delta},\Delta] = \sum_{k,\sigma} \bar{\psi}_{k,\sigma} \left(-i\epsilon_n + \frac{k^2}{2m} - \mu \right) \psi_{k,\sigma} + \frac{1}{\beta} \sum_{q,l} \frac{1}{g_l} \left|\Delta^l_q\right|^2 - \frac{1}{\beta} \sum_{q,l} \left(\bar{\Delta}^l_q \phi^l_q + \Delta^l_q \bar{\phi}^l_q \right) \tag{3}$$

$$\begin{aligned} &= \sum_{k,\sigma} \bar{\psi}_{k,\sigma} \left(-i\epsilon_n + \frac{k^2}{2m} - \mu \right) \psi_{k,\sigma} + \frac{1}{\beta} \sum_{q,l} \frac{1}{g_l} \left|\Delta^l_q\right|^2 \\ &\qquad - \frac{1}{\beta} \sum_{k,q,l} \left(f_l(\mathbf{k}) \bar{\Delta}^l_q \; \psi_{-k+\frac{q}{2},\downarrow} \psi_{k+\frac{q}{2},\uparrow} + f_l(\mathbf{k}) \Delta^l_q \; \bar{\psi}_{k+\frac{q}{2},\uparrow} \bar{\psi}_{-k+\frac{q}{2},\downarrow} \right) \end{aligned} \tag{4}$$

$$= \sum_k \bar{\Psi}_k \begin{pmatrix} -i\epsilon_n + \xi_k & 0 \\ 0 & -i\epsilon_n - \xi_k \end{pmatrix} \Psi_k - \frac{1}{\beta} \sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \begin{pmatrix} 0 & \sum_l f_l(\mathbf{k}) \Delta^l_q \\ \sum_l f_l(\mathbf{k}) \bar{\Delta}^l_q & 0 \end{pmatrix} \Psi_{k-\frac{q}{2}} + \frac{1}{\beta} \sum_{q,l} \frac{1}{g_l} \left|\Delta^l_q\right|^2. \tag{5}$$

In the last line we have defined the Nambu spinor $\Psi_k = (\psi_{k,\uparrow}, \bar{\psi}_{-k,\downarrow})^T$. This gives us our BdG action.

A. Minimal coupling with a supercurrent

The result of minimally coupling to the photon field is the addition of two new terms, the paramagnetic and diamagnetic terms. They are given by

$$\begin{aligned} S_{\psi-A} &= \frac{1}{\beta} \sum_{k,q,\sigma} \bar{\psi}_{k+\frac{q}{2},\sigma} \left[-\frac{e}{m} \mathbf{k} \cdot \mathbf{A}_q + \frac{e^2}{2m\beta} \sum_{q'} \mathbf{A}_{\frac{q+q'}{2}} \cdot \mathbf{A}_{\frac{q-q'}{2}} \right] \psi_{k-\frac{q}{2},\sigma} \\ &= \frac{1}{\beta} \sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \left[-\frac{e}{m} \mathbf{k} \cdot \mathbf{A}_q \; \hat{\tau}_0 + \frac{e^2}{2m\beta} \sum_{q'} \mathbf{A}_{\frac{q+q'}{2}} \cdot \mathbf{A}_{\frac{q-q'}{2}} \; \hat{\tau}_3 \right] \Psi_{k-\frac{q}{2}}, \end{aligned}$$

where in the second line we have changed to the Nambu basis written in terms of Pauli matrices $\hat{\tau}_i$, with $\hat{\tau}_0$ representing the identity matrix.

Now we consider driving a supercurrent in the system. A supercurrent corresponds to a nontrivial phase winding of the superconducting order parameter, $\Delta^s \to \Delta^s\ e^{i\theta(\mathbf{r})}$. Instead of keeping this phase dependence, which would be very complicated to do in momentum space, we can perform a gauge transformation to remove it from Δ and add a new contribution to the gauge field proportional to the gradient of this phase. The energetically favored configuration is a spacially uniform supercurrent, meaing a constant phase gradient, so this new part of $\mathbf{A}$ is just a constant vector, $\mathbf{A}(x) \to \mathbf{A}(x) + \mathbf{A}_S$. Performing a Fourier transform to momentum space we then have $\mathbf{A}_q \to \mathbf{A}_q + \mathbf{A}_S \beta\delta_{q,0}$.

Plugging this into the photon coupling terms above and keeping only terms linear in the gauge field $\mathbf{A}_q$, we arrive at the form of the coupling that we will use from this point forward,

$$S_{\psi-A} \to \frac{1}{\beta}\sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \left(-e\mathbf{v}_\mathbf{k}\ \hat{\tau}_0 - e\mathbf{v}_S\ \hat{\tau}_3\right)\cdot \mathbf{A}_q\ \Psi_{k-\frac{q}{2}} + \sum_k \bar{\Psi}_k \left(\mathbf{k}\cdot\mathbf{v}_S\ \hat{\tau}_0 + \frac{1}{2}mv_S^2\hat{\tau}_3\right)\Psi_k + \frac{1}{\beta}\sum_{k,q}\bar{\Psi}_{k+\frac{q}{2}}\hat{\tau}_3\Psi_{k-\frac{q}{2}}\frac{e^2}{2m\beta}\sum_{q'}\mathbf{A}_{\frac{q+q'}{2}}\cdot\mathbf{A}_{\frac{q-q'}{2}}. \tag{6}$$

We have written the coupling expressions in terms of the electron velocity $\mathbf{v}_\mathbf{k} = \frac{k}{m}$ and the superfluid velocity $\mathbf{v}_S = -\frac{e}{m}\mathbf{A}_s$ for convenience. Note that the terms coupling photons to the normal current (first term) and to the externally imposed supercurrent (second term) come with different Nambu structure, so it's possible that superconducting excitations may couple to photons through one of these vertices and not the other. The second set of terms, which do not couple electrons to the gauge field, add new terms to the free part of the action. The first is a Doppler shift contribution to the energy and the second gives a shift of the chemical potential due to the supercurrent. The last term is simply the diamagnetic term, which in a clean system gives an unphysical contribution to the response, so we discard it going forward.

Introducing some more compact notation, the action is

$$S = S_\Delta + S_A - \sum_k \bar{\Psi}_k\ \hat{G}_{0,k}^{-1}\ \Psi_k + \frac{1}{\beta}\sum_{k,q}\bar{\Psi}_{k+\frac{q}{2}}\left(\hat{\chi}^P_{k,q}[A] + \hat{\chi}^S_{k,q}[A] - \hat{\Delta}^s_{k,q} - \hat{\Delta}^d_{k,q}\right)\Psi_{k-\frac{q}{2}}, \tag{7}$$

where $S_\Delta = \frac{1}{\beta}\sum_{q,l}\frac{1}{g_l}\left|\Delta^l_q\right|^2$, S_A is the free action for photons in the cavity, $\hat{\chi}^P$ and $\hat{\chi}^S$ are the paramagnetic ($\hat{\tau}_0$) and supercurrent ($\hat{\tau}_3$) couplings to the photon field given above, and

$$\hat{G}_{0,k}^{-1} = (i\epsilon_n - \mathbf{k}\cdot\mathbf{v}_S)\ \hat{\tau}_0 - \underbrace{\left(\frac{k^2}{2m} - \mu + \frac{1}{2}mv_S^2\right)}_{\equiv\xi^S_k}\ \hat{\tau}_3, \quad \hat{\Delta}^l_{k,q} = f_l(\mathbf{k})\begin{pmatrix} 0 & \Delta^l_q \\ \bar{\Delta}^l_q & 0\end{pmatrix}. \tag{8}$$

At this point we make the mean field approximation which amounts to the replacement $\Delta^s_q \to \Delta\,\beta\delta_{q,0}$ with $\Delta = const \in \mathbb{R}$, and combine this into the Green's function, giving $\hat{G}_k^{-1} = \hat{G}_{0,k}^{-1} + \Delta\ \hat{\tau}_1$. The ground state is then a homogeneous s-wave superconductor with constant supercurrent, with the terms $\hat{\chi}^P$, $\hat{\chi}^S$, and $\hat{\Delta}^d$ as small perturbations.

II. EMPTY CAVITY PHOTON ACTION S_A

Before integrating out the electrons and expanding the resulting trace-log, we explicity consider the free part of the photon action. Inside of a cavity this is

$$S_A = \frac{1}{2\beta}\sum_{q,n} A_{\alpha,n,-q}\left[(i\Omega_m)^2 - \omega^2_{n,\mathbf{q}}\right]A_{\alpha,n,q}, \tag{9}$$

with L the size of the cavity, α indexing the two cavity modes, n labeling the discrete modes resulting from the confinement in z, and $\omega^2_{n,\mathbf{q}} = |\mathbf{q}|^2 + \omega^2_{n,0}$. Note that this is written using Lorentz-Heaviside units, which is missing a factor of $1/4\pi$ compared to gaussian units, as well as with $c = 1$. As is the case throughout these notes, $\mathbf{q}$ represents 2D momentum in the unconfined directions. From this point onwards we consider a fixed single mode n and then drop the index. In terms of the cavity polarizations the vector potential is

$$\mathbf{A}_q(z) = \sum_\alpha A_{\alpha,q}\,\hat{\boldsymbol{\epsilon}}_{\alpha,\mathbf{q}}(z), \tag{10}$$

where the polarization vectors are

$$\hat{\boldsymbol{\epsilon}}_{1,\mathbf{q}}(z) = i\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi z}{L}\right)\frac{\hat{\mathbf{z}}\times\mathbf{q}}{|\mathbf{q}|} \qquad \hat{\boldsymbol{\epsilon}}_{2,\mathbf{q}}(z) = \sqrt{\frac{2}{L}}\frac{1}{\omega_q}\left[q\cos\left(\frac{n\pi z}{L}\right)\hat{\mathbf{z}} - i\omega_0\sin\left(\frac{n\pi z}{L}\right)\frac{\mathbf{q}}{|\mathbf{q}|}\right]. \tag{11}$$

Due to the cavity confinement, the free photon spectrum can be approximated as a parabolic dispersion for small momentum,

$$\omega_{\mathbf{q}} \approx \omega_0 + \frac{q^2}{2m_{\text{phot}}}, \qquad \omega_0 = m_{\text{phot}} = \frac{n\pi}{L}. \tag{12}$$

To relate the polarization amplitudes and the Cartesian components of the vector potential we restrict our attention to $z = L/2$, where the z-component of the polarization vectors vanishes so we can relate two independent components in each basis. (There are only 2 independent components in either basis, but at most other points exactly how the 3 Cartesian components reduce to just 2, so that we can write a 2×2 matrix relating the two, is not immediately apparent in general.) This is also conveniently where the superconducting layer is situated, so the Cartesian components at this point are those that actual couple to fermions. Simplify the notation by defining $\mathbf{A}_q(z = L/2) \equiv \mathbf{A}_q$ and

$$\hat{\boldsymbol{\epsilon}}_{1,q}(z = L/2) \equiv \hat{\boldsymbol{\epsilon}}_{1,q} = i\sqrt{\frac{2}{L}}\frac{\hat{\mathbf{z}}\times\mathbf{q}}{|\mathbf{q}|} \qquad \hat{\boldsymbol{\epsilon}}_{2,q}(z = L/2) \equiv \hat{\boldsymbol{\epsilon}}_{2,q} = -i\sqrt{\frac{2}{L}}\frac{\omega_0}{\omega_q}\frac{\mathbf{q}}{|\mathbf{q}|}. \tag{13}$$

III. EFFECTIVE ACTION FOR Δ (INTEGRATING OUT Ψ)

The partition function can be written in terms of field integrals, letting us integrate out the electron fields.

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}(\bar{\psi},\psi,\bar{\Delta},\Delta,A)e^{-S[\bar{\psi},\psi,\bar{\Delta},\Delta,A]} \\
&= \int \mathcal{D}(\bar{\Delta},\Delta,A)e^{-(S_\Delta+S_A)}\int \mathcal{D}(\bar{\psi},\psi)\exp\left[\sum\bar{\Psi}\left(\hat{G}^{-1} - \frac{1}{\beta}\left(\hat{\chi}^P[A] + \hat{\chi}^S[A] - \hat{\Delta}^d\right)\right)\Psi\right] \\
&= \int \mathcal{D}(\bar{\Delta},\Delta)\exp\left[-S_\Delta - S_A + \operatorname{Tr}\ln\left(\beta\hat{G}^{-1} - \hat{\chi}^P[A] - \hat{\chi}^S[A] + \hat{\Delta}^d\right)\right].
\end{aligned}$$

This gives the action for photons and d-wave excitations above the s-wave superconducting ground state, and the coupling between them. We write the three perturbations as a single matrix $\hat{V}$ and produce an effective action by expanding the trace-log, keeping terms up to second order in this small perturbation.

$$S = S_\Delta + S_A - \operatorname{Tr}\ln\left(\beta\hat{G}^{-1} - \hat{V}\right) \to S_{\text{eff}} = S_\Delta + S_A - \operatorname{Tr}\ln\left(\beta\hat{G}^{-1}\right) + \operatorname{Tr}\left(\frac{1}{\beta}\hat{G}\hat{V}\right) + \frac{1}{2}\operatorname{Tr}\left(\frac{1}{\beta}\hat{G}\hat{V}\right)^2. \tag{14}$$

The saddle point solution of this action ignoring all small perturbations gives the s-wave gap equation,

$$\frac{1}{g_s} - \sum_{\mathbf{k}}\frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{2\lambda_k} = 0 \tag{15}$$

Notice that the traces over powers of the Green's function time the perturbations will result in delta functions enforcing conservation of momentum and frequency. The number of these delta functions will be one less than the power. Since the frequency delta functions each come with a factor of β, all but one of the $1/\beta$'s will be cancelled. We absorb this single factor of $1/\beta = T$ into the definition of Tr in these expanded terms.

The terms of most interest to us in our search for hybridized modes are those at second order in $\hat{V}$, which contain the coupling of photons to the d-wave superconducting mode as well as the photon and d-wave bubbles that are vital in determining their excitation spectra. Expanding this term,

$$\frac{1}{2}\operatorname{Tr}(\hat{G}\hat{V}\hat{G}\hat{V}) = \frac{1}{2}\operatorname{Tr}\left(\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\right) + \frac{1}{2}\operatorname{Tr}\left(\hat{G}\hat{\Delta}^d\hat{G}\hat{\Delta}^d\right) - \operatorname{Tr}\left(\hat{G}(\hat{\chi}^P + \hat{\chi}^S)\hat{G}\hat{\Delta}^d\right). \tag{16}$$

The first term describes the coupling of photons the s-wave superconductor. The second term contributes to the action of the d-wave excitations. The last term is the coupling of photons to the d-wave excitations. Finding if this term has a finite component is our primary concern.

A. Bardasis-Schrieffer mode (d-wave excitations)

The bubble diagram of d-wave excitations is written

$$\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}(\hat{G}\hat{\Delta}^d\hat{G}\hat{\Delta}^d) &= \frac{T^2}{2}\sum_{k,q}\mathrm{tr}\left(\hat{G}_{k+\frac{q}{2}}\hat{\Delta}^d_{k,q}\hat{G}_{k-\frac{q}{2}}\hat{\Delta}^d_{k,-q}\right)\\
&\approx \frac{T^2}{2}\sum_{k,q}\mathrm{tr}\left(\hat{g}^+_{\mathbf{k}}\,\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}}\,\hat{g}^-_{\mathbf{k}}\,\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,-q}\hat{U}_{\mathbf{k}}\right)\\
&= \frac{T^2}{2}\sum_{k,q}\sum_{\alpha,\alpha'=\pm}\frac{1}{i\epsilon^+_n - E^\alpha_{\mathbf{k}}}\frac{1}{i\epsilon^-_n - E^{\alpha'}_{\mathbf{k}}}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}}\right)_{\alpha\alpha'}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,-q}\hat{U}_{\mathbf{k}}\right)_{\alpha'\alpha}\\
&= \frac{T}{2}\sum_{\mathbf{k},q}\sum_{\alpha,\alpha'}\frac{n_F(E^{\alpha'}_{\mathbf{k}}) - n_F(E^\alpha_{\mathbf{k}})}{i\Omega_m - (E^\alpha_{\mathbf{k}} - E^{\alpha'}_{\mathbf{k}})}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}}\right)_{\alpha\alpha'}\left(\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,-q}\hat{U}_{\mathbf{k}}\right)_{\alpha'\alpha},
\end{aligned} \tag{17}$$

Going to the second line we have made the approximation of discarding the Green's functions' dependence on the small momentum transfer $\mathbf{q}$, but we keep the dependence on the frequency Ω_m. (Notation: $\epsilon^\pm_n = \epsilon_n \pm \frac{\Omega_m}{2}$) We've then diagonalized the Green's functions with the Bogoliubov transformation $\hat{U}_{\mathbf{k}}$,

$$\hat{U}_{\mathbf{k}} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}} \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix}, \quad u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1+\frac{\xi^S_k}{\lambda_k}\right)}, \quad v_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1-\frac{\xi^S_k}{\lambda_k}\right)}, \quad \lambda_k = \sqrt{\xi^{S\,2}_k + \Delta^2}. \tag{18}$$

The quasiparticle excitation spectrum is $E^\pm_k = \pm\lambda_k + \mathbf{k}\cdot\mathbf{v}_S$. This differs from the usual spectrum of a superconductor by the Doppler shift term resulting from the supercurrent. The last line is the result of performing the Matsubara sum over fermionic frequency, producing a difference of Fermi functions, $n_F(E)$. These approximations will only let us produce the $\mathbf{q}=0$ value of the d-wave mode dispersion. Since the kinetic mass of the d-wave mode is much greater than the cavity photons' effective inertial mass then this is a good approximation.

The matrix element involving the d-wave pairing function is

$$\hat{U}^\dagger_{\mathbf{k}}\hat{\Delta}^d_{k,q}\hat{U}_{\mathbf{k}} = \frac{1}{2}f_d(\mathbf{k})\left[\frac{1}{\lambda_k}\left(\Delta^d_q + \bar{\Delta}^d_{-q}\right)\left(\xi^S_k\;\hat{\tau}_1 - \Delta\;\hat{\tau}_3\right) + \left(\Delta^d_q - \bar{\Delta}^d_{-q}\right)(i\hat{\tau}_2)\right]. \tag{19}$$

Only the terms in this matrix element that are off-diagonal in α,α' will give a nonzero result due to the difference of Fermi functions cancelling otherwise. The above expression becomes

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}(\hat{G}\hat{\Delta}^d\hat{G}\hat{\Delta}^d)\\
&\approx \sum_{\mathbf{k},q} f_d(\mathbf{k})^2\left(n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})\right)\frac{2\lambda_k}{(i\Omega_m)^2 - (2\lambda_k)^2}\left[\frac{\xi^{S\,2}_k}{\lambda^2_k}\frac{\Delta^d_q + \bar{\Delta}^d_{-q}}{2}\frac{\Delta^d_{-q} + \bar{\Delta}^d_q}{2} - \frac{\Delta^d_q - \bar{\Delta}^d_{-q}}{2}\frac{\Delta^d_{-q} - \bar{\Delta}^d_q}{2}\right.\\
&\qquad\left. + \frac{\xi^S_k}{\lambda_k}\left(\frac{\Delta^d_q + \bar{\Delta}^d_{-q}}{2}\frac{\Delta^d_{-q} - \bar{\Delta}^d_q}{2} - \frac{\Delta^d_{-q} + \bar{\Delta}^d_q}{2}\frac{\Delta^d_q - \bar{\Delta}^d_{-q}}{2}\right)\right]\\
&\approx 2\sum_{\mathbf{k},q} f_d(\mathbf{k})^2\frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2}\left[\frac{\xi^{S\,2}_k}{\lambda_k}\;\bar{d}^R_q d^R_q + \lambda_k\;\bar{d}^I_q d^I_q\right].
\end{aligned}$$

We have dropped the term linear in ξ^S_k, noting that in the ξ-approximation this term goes as the particle-hole asymmetry, which we assume to be small. In the last line we have addtionally rewritten the d-wave gap function in terms of its real and imaginary components, $\Delta^d_q = d^R_q + id^I_q$, with d^R, d^I themselves real, so that $\bar{d}_q = d_{-q}$ for each of them. Because we have chosen a purely real s-wave Δ, the imaginary component of the d-wave gap is the Bardasis-Schrieffer mode. (In general, the BS mode is the component $\frac{\pi}{2}$-out-of-phase with the s-wave gap. If the s-wave gap isn't taken to be real, then parametrize the d-wave gap in terms of components in-phase and $\frac{\pi}{2}$-out-of-phase with it to arrive at the same result.)

We can likewise rewrite the d-wave part of the Hubbard-Stratonovich action S_Δ in terms of these two components. Combining this with the bubble diagram calculated above and separating the distinct modes, we have the overall action for d-wave excitations at $\mathbf{q}=0$,

$$S_{d^I} = \sum_q \left[\frac{1}{g_d} + \sum_{\mathbf{k}} f_d(\mathbf{k})^2 \, \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} \, 2\lambda_k \right] \bar{d}^I_q d^I_q, \tag{20}$$

$$S_{d^R} = \sum_q \left[\frac{1}{g_d} + \sum_{\mathbf{k}} f_d(\mathbf{k})^2 \, \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} \, 2\frac{\xi^{S\,2}_k}{\lambda_k} \right] \bar{d}^R_q d^R_q. \tag{21}$$

Since the real mode is generally overdamped we will ignore it from this point onward and rename $d^I_q \to d_q$. We can use the s-wave gap equation to simplify things a bit more,

$$\begin{aligned}
S_d &= \frac{1}{\beta} \sum_q \left[\frac{1}{g_d} + \sum_{\mathbf{k}} f_d(\mathbf{k})^2 \, \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} \, 2\lambda_k \underbrace{- \frac{1}{g_s} + \sum_{\mathbf{k}} \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{2\lambda_k}}_{=0} \right] d_{-q} d_q \\
&= \frac{1}{\beta} \sum_q \left[\frac{1}{g_d} - \frac{1}{g_s} + \sum_{\mathbf{k}} \left(n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}}) \right) \frac{1}{2\lambda_k} \left(\frac{(i\Omega_m)^2 + (2\lambda_k)^2 (f_d(\mathbf{k})^2 - 1)}{(i\Omega_m)^2 - (2\lambda_k)^2} \right) \right] d_{-q} d_q \\
&= \frac{1}{\beta} \sum_q \left[\frac{1}{g_d} - \frac{1}{g_s} + \sum_{\mathbf{k}} \frac{(i\Omega_m)^2}{2\lambda_k} \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} + \sum_{\mathbf{k}} 2\lambda_k \cos(4\theta_k) \frac{n_F(E^-_{\mathbf{k}}) - n_F(E^+_{\mathbf{k}})}{(i\Omega_m)^2 - (2\lambda_k)^2} \right] d_{-q} d_q.
\end{aligned}$$

In the last line we put $f_d(\mathbf{k})^2 - 1 = 2\cos(2\theta_k) - 1 = \cos(4\theta_k)$. Since $g_s > g_d$, the first two terms give the finite positive contribution to the energy threshold for excitation. Note that in the absence of supercurrent the angular integral in the last term gives exactly 0, and the second term reduces to the known form for the BS mode.

Extending the imaginary frequency to be a general complex number, the value where this action vanishes ($i\Omega_m = \Omega_{\text{BS}}$, where Ω_{BS} is real) defines the BS mode frequency. Within our approximations, the BS mode has a flat dispersion because it has a high kinetic mass relative to the cavity photons.

B. Photon Self-Energy

The next term we examine is the self-energy of the photons due to the superconductor. Unlike for the previous term, here we keep the dependence on $\mathbf{q}$ in the Green's functions because the momentum dependence of the photon dispersion will be an important detail for the hybridization.

$$\begin{aligned}
\frac{1}{2} \operatorname{Tr} \left[\hat{G}(\hat{\chi}^P + \hat{\chi}^S) \hat{G}(\hat{\chi}^P + \hat{\chi}^S) \right] &= \frac{T}{2} \sum_{k,q} \operatorname{tr} \left[\hat{G}_{k+\frac{q}{2}} \left(\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q} \right) \hat{G}_{k-\frac{q}{2}} \left(\hat{\chi}^P_{k,-q} + \hat{\chi}^S_{k,-q} \right) \right] \\
&= \frac{1}{2} \sum_{\mathbf{k},q} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \left(\hat{U}^\dagger_{\mathbf{k}+\mathbf{q}/2} (\hat{\chi}^P_{k,q} + \hat{\chi}^S_{k,q}) \hat{U}_{\mathbf{k}-\mathbf{q}/2} \right)_{\alpha,\alpha'} \left(\hat{U}^\dagger_{\mathbf{k}-\mathbf{q}/2} (\hat{\chi}^P_{k,-q} + \hat{\chi}^S_{k,-q}) \hat{U}_{\mathbf{k}+\mathbf{q}/2} \right)_{\alpha',\alpha}.
\end{aligned}$$

Using the forms of the coupling to photons we arrive at the expression

$$\begin{aligned}
&\frac{e^2}{2} \sum_{\mathbf{k},q} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \\
&\quad \times \Big\{ (\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_q)(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_{-q}) \left(\ell^2_{\mathbf{k},\mathbf{q}} \delta_{\alpha,\alpha'} + p^2_{\mathbf{k},\mathbf{q}} \delta_{\alpha,-\alpha'} \right) + (\mathbf{v}_S \cdot \mathbf{A}_q)(\mathbf{v}_S \cdot \mathbf{A}_{-q}) \left(n^2_{\mathbf{k},\mathbf{q}} \delta_{\alpha,\alpha'} + m^2_{\mathbf{k},\mathbf{q}} \, \delta_{\alpha,-\alpha'} \right) \\
&\qquad + \left[(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_q)(\mathbf{v}_S \cdot \mathbf{A}_{-q}) + (\mathbf{v}_S \cdot \mathbf{A}_q)(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_{-q}) \right] \alpha \left(\ell_{\mathbf{k},\mathbf{q}} n_{\mathbf{k},\mathbf{q}} \delta_{\alpha,\alpha'} + p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}} \delta_{\alpha,-\alpha'} \right) \Big\},
\end{aligned}$$

where we have introduced the superconducting coherence factors with their usual definitions,

$$\begin{aligned}
\ell_{\mathbf{k},\mathbf{q}} &= u_+ u_- + v_+ v_- \qquad & p_{\mathbf{k},\mathbf{q}} &= u_+ v_- - v_+ u_- \\
n_{\mathbf{k},\mathbf{q}} &= u_+ u_- - v_+ v_- \qquad & m_{\mathbf{k},\mathbf{q}} &= u_+ v_- + v_+ u_-.
\end{aligned}$$

We use the shorthand $u_\pm = u_{\mathbf{k}\pm\mathbf{q}/2}$ and similarly for $v_\pm$, with $u_\mathbf{k}, v_\mathbf{k}$ as given in Eq. 18. We rewrite this part of the trace-log in terms of the photon self-energy $\hat{\Pi}_q$ as

$$\frac{1}{2}\,\mathrm{Tr}\left[\hat{G}(\hat{\chi}^P+\hat{\chi}^S)\hat{G}(\hat{\chi}^P+\hat{\chi}^S)\right] = \frac{1}{2\beta}\sum_q\sum_{\mu,\nu} A^\mu_{-q}\Pi^{\mu\nu}_q A^\nu_q = \frac{1}{2\beta}\sum_q\sum_{\alpha,\alpha'} A_{\alpha,-q}\Pi_{\alpha\alpha',q}A_{\alpha',q}, \tag{22}$$

where μ,ν index the Cartesian components of the photon field and α,α' index the cavity modes. The second equality uses the polarization vectors to translate between real space coordinates and cavity modes,

$$\Pi_{\alpha\alpha',q} = \sum_{\mu,\nu}\hat{\epsilon}^\mu_{\alpha,-\mathbf{q}}\Pi^{\mu\nu}_q\hat{\epsilon}^\nu_{\alpha',\mathbf{q}}. \tag{23}$$

We will leave the photon self-energy in this compact form until we need to explicitly expand it later on.

C. Coupling of photons to d-wave excitations

Finally we arrive at the term that couples photons and d-wave excitations has two components, one with the paramagnetic vertex and the other with the supercurrent vertex. For now we can consider both at the same time.

$$\begin{aligned}
\mathrm{Tr}(\hat{G}(\hat{\chi}^P+\hat{\chi}^S)\hat{G}\hat{\Delta}^d) &= T\sum_{k,q}\mathrm{tr}\left(\hat{G}_{k+\frac{q}{2}}(\hat{\chi}^P_{k,q}+\hat{\chi}^S_{k,q})\hat{G}_{k-\frac{q}{2}}\hat{\Delta}^d_{k,-q}\right)\\
&\approx T\sum_{k,q}\mathrm{tr}\left(\hat{g}^+_\mathbf{k}\;\hat{U}^\dagger_\mathbf{k}(\hat{\chi}^P_{k,q}+\hat{\chi}^S_{k,q})\hat{U}_\mathbf{k}\;\hat{g}^-_\mathbf{k}\;\hat{U}^\dagger_\mathbf{k}\hat{\Delta}^d_{k,-q}\hat{U}_\mathbf{k}\right)\\
&= T\sum_{k,q}\sum_{\alpha,\alpha'=\pm}\frac{1}{i\epsilon^+_n - E^\alpha_\mathbf{k}}\;\frac{1}{i\epsilon^-_n - E^{\alpha'}_\mathbf{k}}\left(\hat{U}^\dagger_\mathbf{k}(\hat{\chi}^P_{k,q}+\hat{\chi}^S_{k,q})\hat{U}_\mathbf{k}\right)_{\alpha,\alpha'}\left(\hat{U}^\dagger_\mathbf{k}\hat{\Delta}^d_{k,-q}\hat{U}_\mathbf{k}\right)_{\alpha',\alpha}\\
&= \sum_{\mathbf{k},q}\sum_{\alpha,\alpha'}\frac{n_F(E^{\alpha'}_\mathbf{k})-n_F(E^\alpha_\mathbf{k})}{i\Omega_m-(E^\alpha_\mathbf{k}-E^{\alpha'}_\mathbf{k})}\left(\hat{U}^\dagger_\mathbf{k}(\hat{\chi}^P_{k,q}+\hat{\chi}^S_{k,q})\hat{U}_\mathbf{k}\right)_{\alpha,\alpha'}\left(\hat{U}^\dagger_\mathbf{k}\hat{\Delta}^d_{k,-q}\hat{U}_\mathbf{k}\right)_{\alpha',\alpha},
\end{aligned}$$

using the same approximations as discussed above for the d-wave bubble. The new matrix elements that appear in this expression are

$$\begin{aligned}
\hat{U}^\dagger_\mathbf{k}\hat{\chi}^P_{k,q}\hat{U}_\mathbf{k} &= \hat{\chi}^P_{k,q} = -e\mathbf{v}_\mathbf{k}\cdot\mathbf{A}_q\;\hat{\tau}_0\\
\hat{U}^\dagger_\mathbf{k}\hat{\chi}^S_{k,q}\hat{U}_\mathbf{k} &= -e\mathbf{v}_S\cdot\mathbf{A}_q\frac{1}{\lambda_k}\left(\xi^S_k\;\hat{\tau}_3+\Delta\;\hat{\tau}_1\right).
\end{aligned} \tag{24}$$

Since it is purely diagonal, the coupling through the paramagnetic vertex picks out only the diagonal matrix elements $\alpha=\alpha'$. Within our approximations this leads to an exact cancellation of the Fermi functions, making this term identically zero. The supercurrent vertex has off-diagonal components so it gives the only nonzero contribution. The result is

$$\begin{aligned}
\mathrm{Tr}(\hat{G}(\hat{\chi}^P+\hat{\chi}^S)\hat{G}\hat{\Delta}^d) &= -e\Delta\sum_{\mathbf{k},q}\sum_{\alpha=\pm}\alpha\frac{n_F(E^{-\alpha}_\mathbf{k})-n_F(E^\alpha_\mathbf{k})}{i\Omega_m-(E^\alpha_\mathbf{k}-E^{-\alpha}_\mathbf{k})}\;f_d(\mathbf{k})\;\frac{1}{\lambda_k}\mathbf{v}_S\cdot\mathbf{A}_q\frac{\Delta^d_{-q}-\bar{\Delta}^d_q}{2}\\
&\quad - e\Delta\sum_{\mathbf{k},q}\sum_{\alpha=\pm}\frac{n_F(E^{-\alpha}_\mathbf{k})-n_F(E^\alpha_\mathbf{k})}{i\Omega_m-(E^\alpha_\mathbf{k}-E^{-\alpha}_\mathbf{k})}\;f_d(\mathbf{k})\;\frac{\xi^S_k}{\lambda^2_k}\mathbf{v}_S\cdot\mathbf{A}_q\frac{\Delta^d_{-q}+\bar{\Delta}^d_q}{2}\\
&\approx -2e\Delta\sum_{\mathbf{k},q}\left(n_F(E^-_\mathbf{k})-n_F(E^+_\mathbf{k})\right)\frac{i\Omega_m}{(i\Omega_m)^2-(E^+_\mathbf{k}-E^-_\mathbf{k})^2}\;\frac{f_d(\mathbf{k})}{\lambda_k}\mathbf{v}_S\cdot\mathbf{A}_q\frac{\Delta^d_{-q}-\bar{\Delta}^d_q}{2}\\
&= -2ie\Delta\sum_{\mathbf{k},q}\left(n_F(E^-_\mathbf{k})-n_F(E^+_\mathbf{k})\right)\frac{i\Omega_m}{(i\Omega_m)^2-(2\lambda_k)^2}\;\frac{f_d(\mathbf{k})}{\lambda_k}\;\mathbf{v}_S\cdot\mathbf{A}_q\;d_{-q}\\
&= -ie\Delta\sum_{\mathbf{k},q}\left(n_F(E^-_\mathbf{k})-n_F(E^+_\mathbf{k})\right)\frac{i\Omega_m}{(i\Omega_m)^2-(2\lambda_k)^2}\;\frac{f_d(\mathbf{k})}{\lambda_k}\;\mathbf{v}_S\cdot\left(\mathbf{A}_q\;d_{-q}-\mathbf{A}_{-q}d_q\right).
\end{aligned} \tag{25}$$

In the second equality we have dropped the term linear in ξ_k^S, which is small in particle-hole asymmetry.

Note that this term is nonzero only because of the supercurrent. First, it explicitly contains $\mathbf{v}_S \cdot \mathbf{A}_q$, so will go to zero if $\mathbf{v}_S \to 0$. Additionally, the angular integration in $\mathbf{k}$ does not give identically zero only because the Fermi functions depend on the angle of $\mathbf{k}$ through the supercurrent Doppler shift term in the energy. The integral over the single power of the d-wave form factor would exactly cancel were it not for this. In the last line we use the properties of the BS mode and the vector potential to write the fields in a more symmetric way.

IV. EXPANDING THE PHOTON SELF-ENERGY

Recombine the terms in the expression for the photon self-energy based on the structure of their dependence on α, α' to give 4 distinct types of terms,

$$
\begin{aligned}
& e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \, \sigma^0_{\alpha,\alpha'} \left[\ell^2_{\mathbf{k},\mathbf{q}} (\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_q)(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_{-q}) + n^2_{\mathbf{k},\mathbf{q}} (\mathbf{v}_S \cdot \mathbf{A}_q)(\mathbf{v}_S \cdot \mathbf{A}_{-q})\right] \\
& \quad + e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \, \sigma^1_{\alpha,\alpha'} \left[p^2_{\mathbf{k},\mathbf{q}} (\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_q)(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_{-q}) + m^2_{\mathbf{k},\mathbf{q}} (\mathbf{v}_S \cdot \mathbf{A}_q)(\mathbf{v}_S \cdot \mathbf{A}_{-q})\right] \\
& \quad + e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \, \sigma^3_{\alpha,\alpha'} \ell_{\mathbf{k},\mathbf{q}} n_{\mathbf{k},\mathbf{q}} \left[(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_q)(\mathbf{v}_S \cdot \mathbf{A}_{-q}) + (\mathbf{v}_S \cdot \mathbf{A}_q)(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_{-q})\right] \\
& \quad + e^2 \sum_{\mathbf{k}} \sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \, i\sigma^2_{\alpha,\alpha'} \, p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}} \left[(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_q)(\mathbf{v}_S \cdot \mathbf{A}_{-q}) + (\mathbf{v}_S \cdot \mathbf{A}_q)(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{A}_{-q})\right]
\end{aligned} \tag{26}
$$

We will now expand these expressions to second order in momentum and first order in frequency, starting with the coherence factors. When expanded for small q, the functions ℓ, m, and n all have a part independent of q, no linear term, and a q^2 term. The function p, on the other hand, up to order q^2 has only a linear in q term, so the expansion of p^2 is order q^2. Explicitly the expansions we will need are

$$
\begin{aligned}
\ell^2_{\mathbf{k},\mathbf{q}} &\approx 1 - \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q})^2 & p^2_{\mathbf{k},\mathbf{q}} &= 1 - \ell^2_{\mathbf{k},\mathbf{q}} \approx \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q})^2 \\
n^2_{\mathbf{k},\mathbf{q}} &\approx \frac{\xi_k^{S\,2}}{\lambda_k^2} + \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}\left[\frac{\xi_k^S}{m} q^2 - 2\frac{\xi_k^{S\,2}}{\lambda_k^2}(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q})^2\right] & m^2_{\mathbf{k},\mathbf{q}} &\approx \frac{\Delta^2}{\lambda_k^2} - \frac{1}{4}\frac{\Delta^2}{\lambda_k^4}\left[\frac{\xi_k^S}{m} q^2 - 2\frac{\xi_k^{S\,2}}{\lambda_k^2}(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q})^2\right] \\
\ell_{\mathbf{k},\mathbf{q}} n_{\mathbf{k},\mathbf{q}} &\approx \frac{\xi_k^S}{\lambda_k} + \frac{1}{8}\frac{\Delta^2}{\lambda_k^4}\left[\frac{\lambda_k}{m} q^2 - 3\frac{\xi_k^S}{\lambda_k}(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q})^2\right] & p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}} &\approx \frac{1}{2}\frac{\Delta^2}{\lambda_k^3}\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q}
\end{aligned} \tag{27}
$$

We see here that when expanded the last of these functions, $p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}}$, is linear in q and contains no term independent of q, so the rest of the expression containing it only needs to be expanded to first order in small parameters.

Start by expanding the energies and the occupation functions in $|\mathbf{q}|$,

$$
E^{\alpha}_{\mathbf{k}\pm\mathbf{q}/2} \approx E^{\alpha}_{\mathbf{k}} \pm \frac{\mathbf{q}}{2} \cdot \nabla E^{\alpha}_{\mathbf{k}} + \frac{1}{2}\frac{q_i}{2}\frac{q_j}{2}\partial_i\partial_j E^{\alpha}_{\mathbf{k}} \equiv E^{\alpha}_{\mathbf{k}} \pm \frac{\mathbf{q}}{2} \cdot \mathbf{v}^{\alpha}_{\mathbf{k}} + \alpha \frac{q_i q_j}{8 m_{\mathbf{k},ij}} \tag{28}
$$

$$
n_F\left(E^{\alpha}_{\mathbf{k}\pm\mathbf{q}/2}\right) \approx n_F(E^{\alpha}_{\mathbf{k}}) + \left(\pm\frac{\mathbf{q}}{2} \cdot \mathbf{v}^{\alpha}_{\mathbf{k}} + \alpha \frac{q_i q_j}{8 m_{\mathbf{k},ij}}\right) n'_F(E^{\alpha}_{\mathbf{k}}) + \frac{1}{2}\left(\frac{\mathbf{q}}{2} \cdot \mathbf{v}^{\alpha}_{\mathbf{k}}\right)^2 n''_F(E^{\alpha}_{\mathbf{k}}), \tag{29}
$$

where we have defined

$$
\mathbf{v}^{\alpha}_{\mathbf{k}} = \nabla E^{\alpha}_{\mathbf{k}} = \alpha \frac{\xi_k^S}{\lambda_k}\mathbf{v}_{\mathbf{k}} + \mathbf{v}_S, \quad \frac{1}{m_{\mathbf{k},ij}} = \partial_i\partial_j\lambda_k = \frac{1}{\lambda_k}\left(\frac{\xi_k^S}{m}\delta_{ij} + \frac{v^i_{\mathbf{k}} v^j_{\mathbf{k}} \Delta^2}{\lambda_k^2}\right) \tag{30}
$$

With these approximations and notation the differences of energies are

$$E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha}_{\mathbf{k}-\mathbf{q}/2} \approx \mathbf{q}\cdot\mathbf{v}^{\alpha}_{\mathbf{k}} \tag{31}$$

$$\begin{aligned} E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2} &\approx E^{\alpha}_{\mathbf{k}} - E^{-\alpha}_{\mathbf{k}} + \mathbf{q}\cdot\frac{\mathbf{v}^{\alpha}_{\mathbf{k}} + \mathbf{v}^{-\alpha}_{\mathbf{k}}}{2} + \alpha\,\frac{q_i q_j}{4m_{\mathbf{k},ij}} \\ &= 2\alpha\,\lambda_k + \mathbf{q}\cdot\mathbf{v}_S + \alpha\,\frac{q_i q_j}{4m_{\mathbf{k},ij}}, \end{aligned} \tag{32}$$

and the differences of the Fermi functions are

$$n_F(E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}) - n_F(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}) \approx -\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}} + \mathbf{v}_S\right) n'_F(E^{\alpha}_{\mathbf{k}}) \tag{33}$$

$$\begin{aligned} n_F(E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}) - n_F(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}) &\approx n_F(E^{-\alpha}_{\mathbf{k}}) - n_F(E^{\alpha}_{\mathbf{k}}) - \frac{\mathbf{q}}{2}\cdot\left(\mathbf{v}^{-\alpha}_{\mathbf{k}}\; n'_F(E^{-\alpha}_{\mathbf{k}}) + \mathbf{v}^{\alpha}_{\mathbf{k}}\; n'_F(E^{\alpha}_{\mathbf{k}})\right) \\ &\quad - \alpha\frac{q_i q_j}{8m_{\mathbf{k},ij}}\left(n'_F(E^{-\alpha}_{\mathbf{k}}) + n'_F(E^{\alpha}_{\mathbf{k}})\right) + \frac{q_i q_j}{8}\left(v^{-\alpha}_{\mathbf{k},i}v^{-\alpha}_{\mathbf{k},j}n''_F(E^{-\alpha}_{\mathbf{k}}) - v^{\alpha}_{\mathbf{k},i}v^{\alpha}_{\mathbf{k},j}n''_F(E^{\alpha}_{\mathbf{k}})\right) \\ &= \alpha\left(n_F(E^{-}_{\mathbf{k}}) - n_F(E^{+}_{\mathbf{k}})\right) + \frac{\mathbf{q}}{2}\cdot\left[\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}}\left(n'_F(E^{-}_{\mathbf{k}}) - n'_F(E^{+}_{\mathbf{k}})\right) - \mathbf{v}_S\left(n'_F(E^{+}_{\mathbf{k}}) + n'_F(E^{-}_{\mathbf{k}})\right)\right] \\ &\quad -\alpha\,\frac{q_i q_j}{8m_{\mathbf{k},ij}}\left(n'_F(E^{+}_{\mathbf{k}}) + n'_F(E^{-}_{\mathbf{k}})\right) + \alpha\,\frac{q_i q_j}{8}\left[\frac{\xi^{S\,2}_k}{\lambda^2_k}v^i_{\mathbf{k}}v^j_{\mathbf{k}} + v^i_S v^j_S\right]\left(n''_F(E^{-}_{\mathbf{k}}) - n''_F(E^{+}_{\mathbf{k}})\right) \\ &\quad -\alpha\,\frac{q_i q_j}{8}\,\frac{\xi^S_k}{\lambda_k}\left(v^i_{\mathbf{k}}v^j_S + v^i_S v^j_{\mathbf{k}}\right)\left(n''_F(E^{+}_{\mathbf{k}}) + n''_F(E^{-}_{\mathbf{k}})\right) \\ &\equiv \alpha\,\delta n_{\mathbf{k}} + \frac{\mathbf{q}}{2}\cdot\left[\frac{\xi^S_k}{\lambda_k}\mathbf{v}_{\mathbf{k}}\delta n'_{\mathbf{k}} - \mathbf{v}_S N'_{\mathbf{k}}\right] \\ &\quad -\alpha\frac{q_i q_j}{8}\left[\frac{N'_{\mathbf{k}}}{m_{\mathbf{k},ij}} - \left(\frac{\xi^{S\,2}_k}{\lambda^2_k}v^i_{\mathbf{k}}v^j_{\mathbf{k}} + v^i_S v^j_S\right)\delta n''_{\mathbf{k}} + \frac{\xi^S_k}{\lambda_k}\left(v^i_{\mathbf{k}}v^j_S + v^i_S v^j_{\mathbf{k}}\right)N''_{\mathbf{k}}\right], \end{aligned} \tag{34}$$

where we have introduced the notation $\delta n_{\mathbf{k}} = n_F(E^{-}_{\mathbf{k}}) - n_F(E^{+}_{\mathbf{k}})$ and $N_{\mathbf{k}} = n_F(E^{+}_{\mathbf{k}}) + n_F(E^{-}_{\mathbf{k}})$.

We can now expand the factors containing the difference of Fermi functions to second order in both $|\mathbf{q}|$ and in $\delta\Omega$, defined as the fluctuation in frequency away from the $\mathbf{q}=0$ on-shell condition ($\Omega = \operatorname{sgn}\Omega\,\omega_0 + \delta\Omega$) and then do the sums over α, α'. Note that the imaginary parts of these functions, proportional to delta functions, describe the decay of photons into Bogoliubov quasiparticles within the superconductor. Because the Bardasis-Schrieffer mode exists within the superconducting gap, for small q and $\delta\Omega$ photons tuned to be near the BS mode are not permitted to decay into Bogoliubov quasiparticles, so we do not need to worry about the imaginary parts and will drop them everywhere.

We start with the factor in the first line of Eq. 26 involving $\sigma^0_{\alpha,\alpha'}$ which gives

$$
\begin{aligned}
&\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\sigma^0_{\alpha,\alpha'}\\
&\quad\to \mathrm{PV}\sum_{\alpha}\frac{n_F\left(E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{\Omega-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)}\approx \mathrm{PV}\sum_{\alpha}\frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}+\mathbf{v}_S\right)n'_F(E^\alpha_\mathbf{k})}{\omega_0\,\mathrm{sgn}\,\Omega+\delta\Omega-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}+\mathbf{v}_S\right)}\\
&\quad\approx\sum_\alpha\frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}+\mathbf{v}_S\right)n'_F(E^\alpha_\mathbf{k})}{\omega_0\,\mathrm{sgn}\,\Omega}\left(1-\frac{\delta\Omega-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}+\mathbf{v}_S\right)}{\omega_0\,\mathrm{sgn}\,\Omega}\right)\\
&\quad=-q_i\sum_\alpha\frac{\alpha\frac{\xi^S_k}{\lambda_k}v^i_\mathbf{k}+v^i_S}{\omega_0}n'_F(E^\alpha_\mathbf{k})\left(\mathrm{sgn}\,\Omega-\frac{\delta\Omega}{\omega_0}\right)-q_iq_j\sum_\alpha\frac{\left(\alpha\frac{\xi^S_k}{\lambda_k}v^i_\mathbf{k}+v^i_S\right)\left(\alpha\frac{\xi^S_k}{\lambda_k}v^j_\mathbf{k}+v^j_S\right)}{\omega_0^2}n'_F(E^\alpha_\mathbf{k})\\
&\quad=q_i\frac{\frac{\xi^S_k}{\lambda_k}v^i_\mathbf{k}\delta n'_\mathbf{k}-v^i_SN'_\mathbf{k}}{\omega_0}\left(\mathrm{sgn}\,\Omega-\frac{\delta\Omega}{\omega_0}\right)-q_iq_j\frac{\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2v^i_\mathbf{k}v^j_\mathbf{k}+v^i_Sv^j_S\right]N'_\mathbf{k}-\frac{\xi^S_k}{\lambda_k}\left(v^i_\mathbf{k}v^j_S+v^i_Sv^j_\mathbf{k}\right)\delta n'_\mathbf{k}}{\omega_0^2}
\end{aligned}
$$

Because these terms are all at least first order in small quantities, the order q^2 parts of the expansions of $\ell_{\mathbf{k},\mathbf{q}}$ and $n_{\mathbf{k},\mathbf{q}}$ are never needed. As a result, these terms only ever multiply functions that are even in ξ^S_k, so all of the terms here that are odd in xi can be dropped because they are small in particle-hole asymmetry. This simplification gives

$$
\begin{aligned}
\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\sigma^0_{\alpha,\alpha'}&\to-\frac{1}{\omega_0}N'_\mathbf{k}\left(\mathrm{sgn}\,\Omega-\frac{\delta\Omega}{\omega_0}\right)\mathbf{v}_S\cdot\mathbf{q}-\frac{\left(\frac{\xi^S_k}{\lambda_k}\right)^2v^i_\mathbf{k}v^j_\mathbf{k}+v^i_Sv^j_S}{\omega_0^2}N'_\mathbf{k}\,q_iq_j\\
&\equiv X^{(1,0)}_{0,\mathbf{k},i}\left(\mathrm{sgn}\,\Omega-\frac{\delta\Omega}{\omega_0}\right)q_i+X^{(2,0)}_{0,\mathbf{k},ij}\;q_iq_j. \qquad (35)
\end{aligned}
$$

The corresponding factor in the second line of Eq. 26, involving $\sigma^1_{\alpha,\alpha'}$, is significantly more complicated and contains a term for all powers and combinations of our small parameters.

$$
\begin{aligned}
&\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\sigma^1_{\alpha,\alpha'}\to\mathrm{PV}\sum_\alpha\frac{n_F\left(E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{\omega_0\,\mathrm{sgn}\,\Omega+\delta\Omega-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)}\\
&\quad\approx\sum_\alpha\left\{\frac{\alpha\,\delta n_\mathbf{k}}{\omega_0\,\mathrm{sgn}\,\Omega-2\alpha\,\lambda_k}\left[1-\frac{\delta\Omega-\mathbf{q}\cdot\mathbf{v}_S-\alpha\frac{q_iq_j}{4m_{\mathbf{k},ij}}}{\omega_0\,\mathrm{sgn}\,\Omega-2\alpha\,\lambda_k}+\left(\frac{\delta\Omega-\mathbf{q}\cdot\mathbf{v}_S}{\omega_0\,\mathrm{sgn}\,\Omega-2\alpha\,\lambda_k}\right)^2\right]\right.\\
&\qquad+\frac{\mathbf{q}}{2}\cdot\frac{\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}\delta n'_\mathbf{k}-\mathbf{v}_SN'_\mathbf{k}}{\omega_0\,\mathrm{sgn}\,\Omega-2\alpha\,\lambda_k}\left(1-\frac{\delta\Omega-\mathbf{q}\cdot\mathbf{v}_S}{\omega_0\,\mathrm{sgn}\,\Omega-2\alpha\,\lambda_k}\right)\\
&\qquad\left.-\alpha\frac{q_iq_j}{8}\frac{\frac{N'_\mathbf{k}}{m_{\mathbf{k},ij}}-\left(\frac{\xi^{s\,2}_k}{\lambda_k^2}v^i_\mathbf{k}v^j_\mathbf{k}+v^i_Sv^j_S\right)\delta n''_\mathbf{k}+\frac{\xi^S_k}{\lambda_k}\left(v^i_\mathbf{k}v^j_S+v^i_Sv^j_\mathbf{k}\right)N''_\mathbf{k}}{\omega_0\,\mathrm{sgn}\,\Omega-2\alpha\,\lambda_k}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}} - \operatorname{sgn}\Omega\frac{2\omega_0\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\,\delta\Omega + 4\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}\,\delta\Omega^2\\
&\quad+\operatorname{sgn}\Omega\,\omega_0\left[\frac{8\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\,v_S^i\,\delta n_{\mathbf{k}}+\frac{1}{\omega_0^2-(2\lambda_k)^2}\left(\frac{\xi_k^S}{\lambda_k}v_{\mathbf{k}}^i\delta n'_{\mathbf{k}}-v_S^iN'_{\mathbf{k}}\right)\right]q_i\\
&\quad-\left[8\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\,v_S^i\,\delta n_{\mathbf{k}}+\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\left(\frac{\xi_k^S}{\lambda_k}v_{\mathbf{k}}^i\delta n'_{\mathbf{k}}-v_S^iN'_{\mathbf{k}}\right)\right]q_i\,\delta\Omega\\
&\quad+\left\{\frac{1}{2m_{\mathbf{k},ij}}\left[\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}-\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]+\frac{1}{2}\left(\frac{\xi_k^S}{\lambda_k}\right)^2v_{\mathbf{k}}^iv_{\mathbf{k}}^j\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right.\\
&\qquad+v_S^iv_S^j\left[4\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}-\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}+\frac{1}{2}\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right]\\
&\qquad\left.+\frac{1}{2}\frac{\xi_k^S}{\lambda_k}\left(v_{\mathbf{k}}^iv_S^j+v_S^iv_{\mathbf{k}}^j\right)\left[\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n'_{\mathbf{k}}-\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N''_{\mathbf{k}}\right]\right\}q_iq_j
\end{aligned}
$$

These terms are mostly even in ξ, and the only ones that are odd in ξ are at least first order in small quantities. The only function that multiplies these terms that is itself odd in ξ is also second order in q (the q^2 part of the expansion of $m^2_{\mathbf{k},\mathbf{q}}$), so we can freely drop these terms. Additionally we can drop the odd in ξ parts of the expansion of m^2.

$$
\begin{aligned}
&\sum_{\alpha,\alpha'}\frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)-n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m-\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}-E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\sigma^1_{\alpha,\alpha'}\\
&\to \frac{4\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}} - \operatorname{sgn}\Omega\frac{2\omega_0\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\,\delta\Omega + 4\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}\,\delta\Omega^2\\
&\quad+\operatorname{sgn}\Omega\,\omega_0\left[\frac{8\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\,\delta n_{\mathbf{k}}-\frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}\\
&\quad-\left[8\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\,\delta n_{\mathbf{k}}-\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}\,\delta\Omega\\
&\quad+\left\{\frac{1}{2}v_{\mathbf{k}}^iv_{\mathbf{k}}^j\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}-\left(\frac{\Delta}{\lambda_k}\right)^2\frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}+\left(\frac{\xi_k^S}{\lambda_k}\right)^2\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right]\right.\\
&\qquad\left.+v_S^iv_S^j\left[4\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}-\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}+\frac{1}{2}\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right]\right\}q_iq_j\\
&= X^{(0,0)}_{1,\mathbf{k}}+\operatorname{sgn}\Omega\,X^{(0,1)}_{1,\mathbf{k}}\;\delta\Omega+X^{(0,2)}_{1,\mathbf{k}}\;\delta\Omega^2+\operatorname{sgn}\Omega\,X^{(1,0)}_{1,\mathbf{k},i}\;q_i+X^{(1,1)}_{1,\mathbf{k},i}\;q_i\;\delta\Omega+X^{(2,0)}_{1,\mathbf{k},ij}\;q_iq_j.
\end{aligned}
\tag{36}
$$

The last two lines of Eq. 26 containing σ^3 and $i\sigma^2$, which arise from the cross terms from the trace-log with one paramagnetic vertex and one supercurrent vertex, have similar factors as the first two but with an extra overall factor of α. The factor in the first, containing $\sigma^3_{\alpha,\alpha'}$, is

$$
\begin{aligned}
&\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \sigma^3_{\alpha,\alpha'} \\
&\quad \to \mathrm{PV}\sum_{\alpha} \alpha \frac{n_F\left(E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{\Omega - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)} \approx \mathrm{PV}\sum_{\alpha} \alpha \frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right) n'_F(E^\alpha_\mathbf{k})}{\omega_0 \,\mathrm{sgn}\,\Omega + \delta\Omega - \mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right)} \\
&\quad \approx \sum_{\alpha} \alpha \frac{-\mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right) n'_F(E^\alpha_\mathbf{k})}{\omega_0 \,\mathrm{sgn}\,\Omega} \left(1 - \frac{\delta\Omega - \mathbf{q}\cdot\left(\alpha\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k} + \mathbf{v}_S\right)}{\omega_0 \,\mathrm{sgn}\,\Omega}\right) \\
&\quad = -q_i \sum_{\alpha} \frac{\frac{\xi^S_k}{\lambda_k} v^i_\mathbf{k} + \alpha v^i_S}{\omega_0} n'_F(E^\alpha_\mathbf{k}) \left(\mathrm{sgn}\,\Omega - \frac{\delta\Omega}{\omega_0}\right) - q_i q_j \sum_{\alpha} \alpha \frac{\left(\alpha\frac{\xi^S_k}{\lambda_k} v^i_\mathbf{k} + v^i_S\right)\left(\alpha\frac{\xi^S_k}{\lambda_k} v^j_\mathbf{k} + v^j_S\right)}{\omega_0^2} n'_F(E^\alpha_\mathbf{k}) \\
&\quad = q_i \frac{-\frac{\xi^S_k}{\lambda_k} v^i_\mathbf{k} N'_\mathbf{k} + v^i_S \delta n'_\mathbf{k}}{\omega_0} \left(\mathrm{sgn}\,\Omega - \frac{\delta\Omega}{\omega_0}\right) + q_i q_j \frac{\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2 v^i_\mathbf{k} v^j_\mathbf{k} + v^i_S v^j_S\right]\delta n'_\mathbf{k} - \frac{\xi^S_k}{\lambda_k}\left(v^i_\mathbf{k} v^j_S + v^i_S v^j_\mathbf{k}\right) N'_\mathbf{k}}{\omega_0^2}.
\end{aligned}
$$

This is very similar to the σ^0 case, but with the δn's and N's swapped around along with some sign changes. Also like that case, these terms are all already at least first order in small quantities, so we do not need the part of $\ell_{\mathbf{k},\mathbf{q}} n_{\mathbf{k},\mathbf{q}}$ that is second order in q. The part of that function that is zeroth order in q is proportional to ξ, so because we are dropping terms that are odd in ξ overall due to the smallness of particle-hole asymmetry, we only need to keep the terms here that are themselves odd in ξ. The result is

$$
\begin{aligned}
\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \sigma^3_{\alpha,\alpha'} &\to -\frac{\xi^S_k}{\lambda_k}\frac{1}{\omega_0} N'_\mathbf{k} \left(\mathrm{sgn}\,\Omega - \frac{\delta\Omega}{\omega_0}\right) \mathbf{v}_\mathbf{k}\cdot\mathbf{q} - \frac{\xi^S_k}{\lambda_k}\frac{\left(v^i_\mathbf{k} v^j_S + v^i_S v^j_\mathbf{k}\right)}{\omega_0^2} N'_\mathbf{k}\, q_i q_j \\
&\equiv \frac{\xi^S_k}{\lambda_k}\left[X^{(1,0)}_{3,\mathbf{k},i}\left(\mathrm{sgn}\,\Omega - \frac{\delta\Omega}{\omega_0}\right) q_i + X^{(2,0)}_{3,\mathbf{k},ij}\; q_i q_j\right]. \qquad (37)
\end{aligned}
$$

Finally, the last line involving $i\sigma^2_{\alpha,\alpha'}$ only needs to be expanded to 1st order. Additionally, the function it multiplies (i.e. $p_{\mathbf{k},\mathbf{q}} m_{\mathbf{k},\mathbf{q}}$) is even in ξ so we can drop all terms in this expansion that are odd in ξ. The result is

$$
\begin{aligned}
&\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\, i\sigma^2_{\alpha,\alpha'} \to \mathrm{PV}\sum_{\alpha}\alpha\, \frac{n_F\left(E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{\omega_0\,\mathrm{sgn}\,\Omega + \delta\Omega - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{-\alpha}_{\mathbf{k}-\mathbf{q}/2}\right)} \\
&\quad \approx \sum_{\alpha}\left[\frac{\delta n_\mathbf{k}}{\omega_0\,\mathrm{sgn}\,\Omega - 2\alpha\,\lambda_k}\left(1 - \frac{\delta\Omega - \mathbf{q}\cdot\mathbf{v}_S}{\omega_0\,\mathrm{sgn}\,\Omega - 2\alpha\,\lambda_k}\right) + \alpha\,\frac{\mathbf{q}}{2}\cdot\frac{\frac{\xi^S_k}{\lambda_k}\mathbf{v}_\mathbf{k}\delta n'_\mathbf{k} - \mathbf{v}_S N'_\mathbf{k}}{\omega_0\,\mathrm{sgn}\,\Omega - 2\alpha\,\lambda_k}\right] \\
&\quad \approx \mathrm{sgn}\,\Omega\,\frac{2\omega_0}{\omega_0^2 - (2\lambda_k)^2}\delta n_\mathbf{k} - 2\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\delta n_\mathbf{k}\delta\Omega + \left[2\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\delta n_\mathbf{k} - \frac{2\lambda_k}{\omega_0^2 - (2\lambda_k)^2}N'_\mathbf{k}\right]\mathbf{v}_S\cdot\mathbf{q} \\
&\quad = \mathrm{sgn}\,\Omega\, X^{(0,0)}_{2,\mathbf{k}} + X^{(0,1)}_{2,\mathbf{k}}\;\delta\Omega + X^{(1,0)}_{2,\mathbf{k},i}\; q_i. \qquad (38)
\end{aligned}
$$

Now combining these expansions with the expansions of the coherence factors and dropping all terms higher than second order in small quantities q and $\delta\Omega$ we have

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\ \sigma^0_{\alpha,\alpha'}\ell^2_{\mathbf{k},\mathbf{q}} \approx X^{(1,0)}_{0,\mathbf{k},i}\ q_i\left(\operatorname{sgn}\Omega - \frac{\delta\Omega}{\omega_0}\right) + X^{(2,0)}_{0,\mathbf{k},ij}\ q_i q_j \tag{39}$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\ \sigma^0_{\alpha,\alpha'}n^2_{\mathbf{k},\mathbf{q}} \approx \frac{\xi^{S\,2}_k}{\lambda^2_k}\left(X^{(1,0)}_{0,\mathbf{k},i}\ q_i\left(\operatorname{sgn}\Omega - \frac{\delta\Omega}{\omega_0}\right) + X^{(2,0)}_{0,\mathbf{k},ij}\ q_i q_j\right) \tag{40}$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\ \sigma^1_{\alpha,\alpha'}p^2_{\mathbf{k},\mathbf{q}} \approx \frac{1}{4}\frac{\Delta^2}{\lambda^4_k}X^{(0,0)}_{1,\mathbf{k}}(\mathbf{v}_\mathbf{k}\cdot\mathbf{q})^2 \tag{41}$$

$$\begin{aligned}\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\ \sigma^1_{\alpha,\alpha'}m^2_{\mathbf{k},\mathbf{q}} &\approx \frac{1}{2}\frac{\Delta^2\xi^{S\,2}_k}{\lambda^6_k}X^{(0,0)}_{1,\mathbf{k}}(\mathbf{v}_\mathbf{k}\cdot\mathbf{q})^2 \\ &+ \frac{\Delta^2}{\lambda^2_k}\left(X^{(0,0)}_{1,\mathbf{k}} + \operatorname{sgn}\Omega\, X^{(0,1)}_{1,\mathbf{k}}\ \delta\Omega + X^{(0,2)}_{1,\mathbf{k}}\ \delta\Omega^2 + \operatorname{sgn}\Omega\, X^{(1,0)}_{1,\mathbf{k},i}\ q_i + X^{(1,1)}_{1,\mathbf{k},i}\ q_i\ \delta\Omega + X^{(2,0)}_{1,\mathbf{k},ij}\ q_i q_j\right)\end{aligned} \tag{42}$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\ \sigma^3_{\alpha,\alpha'}\ \ell_{\mathbf{k},\mathbf{q}}n_{\mathbf{k},\mathbf{q}} \approx \frac{\xi^{S\,2}_k}{\lambda^2_k}\left(X^{(1,0)}_{3,\mathbf{k},i}\ q_i\left(\operatorname{sgn}\Omega - \frac{\delta\Omega}{\omega_0}\right) + X^{(2,0)}_{3,\mathbf{k},ij}\ q_i q_j\right) \tag{43}$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\ i\sigma^2_{\alpha,\alpha'}\ p_{\mathbf{k},\mathbf{q}}m_{\mathbf{k},\mathbf{q}} \approx \frac{1}{2}\frac{\Delta^2}{\lambda^3_k}\left(\operatorname{sgn}\Omega\, X^{(0,0)}_{2,\mathbf{k}} + X^{(0,1)}_{2,\mathbf{k}}\ \delta\Omega + X^{(1,0)}_{2,\mathbf{k},i}\ q_i\right)\mathbf{v}_\mathbf{k}\cdot\mathbf{q} \tag{44}$$

Since the maximum possible value of the superfluid velocity is of order Δ/k_F, i.e. that corresponding to the critical current, we know that $v_S \ll v_F \sim v_\mathbf{k}$ since $\Delta \ll E_F$. This gives us another small parameter that we can use to simplify our expansions above, namely v_S/v_F. In particular, we can drop many of the terms arising from the bubbles involving 2 supercurrent vertices, since they appear alongside a corresponding bubble with 2 paramagnetic vertices, and are therefore small by comparison. All of the terms multiplying $n^2_{\mathbf{k},\mathbf{q}}$ can be dropped (since $(\xi/\lambda)^2 \sim 1$ and they match up 1-to-1 with terms multiplying $\ell^2_{\mathbf{k},\mathbf{q}}$). We therefore discard Eq. 40 entirely, as well as the terms in Eq. 42 that have a corresponding partner in Eq. 41. This turns out to be just the first term; the remaining terms have no partner in Eq. 41.

The mixed terms, those with one supercurrent vertex and one paramagnetic vertex, cannot be straighforwardly simplified without further inspection since they may contain terms that are the same order as the coupling between the BS mode and the photon sector, which is itself proportional to v_S. Discarding all terms that contain powers of

v_S whenever there is a corresponding term using $v_{\mathbf{k}}$, the expressions above simplify to

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)} \left(\sigma^0_{\alpha,\alpha'}\ell^2_{\mathbf{k},\mathbf{q}} + \sigma^1_{\alpha,\alpha'}p^2_{\mathbf{k},\mathbf{q}}\right) \approx -\frac{1}{\omega_0}N'_{\mathbf{k}}\left(\operatorname{sgn}\Omega - \frac{\delta\Omega}{\omega_0}\right)\mathbf{v}_S\cdot\mathbf{q} \tag{45}$$

$$-\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{1}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{1}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})^2 \tag{46}$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\,\sigma^1_{\alpha,\alpha'}m^2_{\mathbf{k},\mathbf{q}}$$

$$\approx \frac{\Delta^2}{\lambda_k^2}\frac{4\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}} - \operatorname{sgn}\Omega\frac{\Delta^2}{\lambda_k^2}\frac{2\omega_0\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\,\delta\Omega + 4\lambda_k\frac{\Delta^2}{\lambda_k^2}\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}\,\delta\Omega^2$$

$$+\operatorname{sgn}\Omega\,\omega_0\frac{\Delta^2}{\lambda_k^2}\left[\frac{8\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}$$

$$-\frac{\Delta^2}{\lambda_k^2}\left[8\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}} - \frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}\right]\mathbf{v}_S\cdot\mathbf{q}\,\delta\Omega$$

$$+\frac{1}{2}\frac{\Delta^2}{\lambda_k^2}\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \left(\frac{\Delta}{\lambda_k}\right)^2\frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}} + \left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})^2$$

$$\sum_{\alpha,\alpha'} \frac{n_F\left(E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right) - n_F\left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2}\right)}{i\Omega_m - \left(E^{\alpha}_{\mathbf{k}+\mathbf{q}/2} - E^{\alpha'}_{\mathbf{k}-\mathbf{q}/2}\right)}\left(\sigma^3_{\alpha,\alpha'}\,\ell_{\mathbf{k},\mathbf{q}}n_{\mathbf{k},\mathbf{q}} + i\sigma^2_{\alpha,\alpha'}\,p_{\mathbf{k},\mathbf{q}}m_{\mathbf{k},\mathbf{q}}\right) \tag{47}$$

$$\approx -\operatorname{sgn}\Omega\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{1}{\omega_0}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q}) \tag{48}$$

$$+\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{1}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\right]\delta\Omega(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q}) \tag{49}$$

$$-\left[\left(\frac{\xi^S_k}{\lambda_k}\right)^2\frac{2}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\left(\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right)\right](\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q})(\mathbf{v}_S\cdot\mathbf{q}) \tag{50}$$

Putting these terms back into their respective expressions we can then define the integrals over $\mathbf{k}$ as simple coefficients of our expansion, the result being terms entering the quadratic action for the photon field.

It is as this point that we change the basis for the photon field as discussed in Eq. **??**, and addtionally rotate the bases for $\mathbf{k}$ and $\mathbf{q}$ to be measured from that same axis as well so that $\mathbf{v}_S\cdot\mathbf{q} = v_S q\cos\theta_q$ and $\mathbf{v}_{\mathbf{k}}\cdot\mathbf{q} = v_k q\cos(\theta_k-\theta_q)$. (In the ξ-approximation, $v_k \to v_F$, the Fermi velocity.) The terms coming with ℓ^2 and p^2 appear in expressions with two paramagnetic vertices, and so they couple to both the parallel and perpendicular components of $\mathbf{A}$. The terms coming with m^2, however, come with two supercurrent verticies, and so only couple to $\mathbf{A}^{\parallel}$. The cross terms couple the parallel component of $\mathbf{A}$ to either component. The terms in the expansion are

$$x_P^{10,\mu\nu}(\theta_q)\; q\left(\operatorname{sgn}\Omega - \frac{\delta\Omega}{\omega_0}\right) = -e^2 v_S \sum_{\mathbf{k}} \frac{1}{\omega_0} N'_{\mathbf{k}} v^\mu_{\mathbf{k}} v^\nu_{\mathbf{k}}\; q\cos\theta_q \left(\operatorname{sgn}\Omega - \frac{\delta\Omega}{\omega_0}\right) \tag{51}$$

$$x_P^{20,\mu\nu}(\theta_q)\; q^2 = -\frac{e^2}{\omega_0^2}\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right] v^\mu_{\mathbf{k}} v^\nu_{\mathbf{k}}\; v_k^2\, \cos^2(\theta_k-\theta_q)\; q^2 \tag{52}$$

$$x_S^{00} = 4e^2 v_S^2 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}} \tag{53}$$

$$x_S^{01}\operatorname{sgn}\Omega\,\delta\Omega = -2e^2 v_S^2\omega_0 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\;\operatorname{sgn}\Omega\,\delta\Omega \tag{54}$$

$$x_S^{02}\delta\Omega^2 = 4e^2\, v_S^2 \sum_{\mathbf{k}} \lambda_k \frac{\Delta^2}{\lambda_k^2}\frac{3\omega_0^3+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\delta n_{\mathbf{k}}\;\delta\Omega^2 \tag{55}$$

$$x_S^{10}(\theta_q) q\;\operatorname{sgn}\Omega = e^2\omega_0 v_S^3 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\left[\frac{8\lambda_k}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\,\delta n_{\mathbf{k}} - \frac{1}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right]\; q\cos\theta_q\;\operatorname{sgn}\Omega \tag{56}$$

$$x_S^{11}(\theta_q)\, q\,\delta\Omega = -e^2 v_S^3 \sum_{\mathbf{k}} \frac{\Delta^2}{\lambda_k^2}\left[8\lambda_k\frac{3\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^3}\,\delta n_{\mathbf{k}} - \frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}N'_{\mathbf{k}}\right]\; q\cos\theta_q\;\delta\Omega \tag{57}$$

$$x_S^{20}(\theta_q) q^2 = \frac{e^2 v_S^2}{2}\sum_{\mathbf{k}}\frac{\Delta^2}{\lambda_k^2}\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \left(\frac{\Delta}{\lambda_k}\right)^2\frac{N'_{\mathbf{k}}}{\omega_0^2-(2\lambda_k)^2} + \left(\frac{\xi_k^S}{\lambda_k}\right)^2\frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}\delta n''_{\mathbf{k}}\right] v_k^2\,\cos^2(\theta_k-\theta_q)\, q^2 \tag{58}$$

$$x_{SP}^{10,\mu}(\theta_q)\; q\;\operatorname{sgn}\Omega = -\frac{e^2 v_S}{\omega_0}\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2-(2\lambda_k)^2}\delta n_{\mathbf{k}}\right] v^\mu_{\mathbf{k}}\; v_k\,\cos(\theta_k-\theta_q)\; q\operatorname{sgn}\Omega \tag{59}$$

$$x_{SP}^{11,\mu}(\theta_q)\; q\,\delta\Omega = e^2 v_S\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{1}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}}\right] v^\mu_{\mathbf{k}}\; v_k\,\cos(\theta_k-\theta_q)\; q\,\delta\Omega \tag{60}$$

$$x_{SP}^{20,\mu}(\theta_q)\, q^2 = -e^2 v_S\sum_{\mathbf{k}}\left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{2}{\omega_0^2}N'_{\mathbf{k}} - \frac{\Delta^2}{\lambda_k^3}\left(\frac{\omega_0^2+(2\lambda_k)^2}{\left[\omega_0^2-(2\lambda_k)^2\right]^2}\delta n_{\mathbf{k}} - \frac{\lambda_k}{\omega_0^2-(2\lambda_k)^2}N'_{\mathbf{k}}\right)\right] v^\mu_{\mathbf{k}}\, v_k\,\cos(\theta_k-\theta_q)\; q^2\cos\theta_q, \tag{61}$$

where the indices $\mu,\nu = \parallel,\perp$. The terms with a subscript P (for paramagnetic) multiply $\mathbf{A}^\mu_q\mathbf{A}^\nu_{-q}$, those with subscript S (for supercurrent) multiply just the parallel components $\mathbf{A}^\parallel_q\mathbf{A}^\parallel_{-q}$, and those with subscript SP (the cross terms) multiply $\mathbf{A}^\mu_q\mathbf{A}^\parallel_{-q}+\mathbf{A}^\parallel_q\mathbf{A}^\mu_{-q}$.

Before writing the full photon action, we should examine the angular integral over θ_k, which may lead to some simplifications. To do so, we need to understand the angular properties of $\delta n_{\mathbf{k}}$ and $N'_{\mathbf{k}}$. Rewriting these in a more transparent form we have

$$\delta n_{\mathbf{k}} = \frac{\sinh(\lambda_k/T)}{\cosh(\lambda_k/T)+\cosh(\mathbf{v}_S\cdot\mathbf{k}/T)} \qquad N'_{\mathbf{k}} = -\frac{1}{T}\frac{\cosh(\lambda_k/T)\cosh(\mathbf{v}_S\cdot\mathbf{k}/T)}{\left(\cosh(\lambda_k/T)+\cosh(\mathbf{v}_S\cdot\mathbf{k}/T)\right)^2}. \tag{62}$$

Because cosh is even, these expressions do not change sign under $\mathbf{k}\to-\mathbf{k}$.

We now go through the distinct angular integrals one by one. Using the identity $\cos(\theta_k-\theta_q) = \cos\theta_k\cos\theta_q +$

$\sin\theta_k \sin\theta_q$ there are 21 different integrals. They are

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \tag{63}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \sin^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos\theta_k \sin\theta_k \tag{64}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^4\theta_k, \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \sin^4\theta_k, \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^2\theta_k \sin^2\theta_k \tag{65}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^3\theta_k \sin\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos\theta_k \sin^3\theta_k \tag{66}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \tag{67}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \sin^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos\theta_k \sin\theta_k \tag{68}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^4\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \sin^4\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^2\theta_k \sin^2\theta_k \tag{69}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^3\theta_k \sin\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos\theta_k \sin^3\theta_k \tag{70}$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \cos^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \sin^2\theta_k \qquad \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \cos\theta_k \sin\theta_k \tag{71}$$

$$\tag{72}$$

The 7 integrals here with an odd power of $\sin\theta_k$ are identically zero for all k which reduces the total number of integrals to 14. Furthermore, we can put $\sin^2\theta_k = 1 - \cos^2\theta_k$ and $\sin^4\theta_k = 1 - 2\cos^2\theta_k + \cos^4\theta_k$ and rewrite all integrals involving sin as integrals just involving cos, reducing the number further to 8. These 8 nonzero integrals are

$$\chi_0^0(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \qquad \chi_0^1(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^2\theta_k \qquad \chi_0^2(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n_{\mathbf{k}} \cos^4\theta_k \tag{73}$$

$$\chi_1^0(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \qquad \chi_1^1(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^2\theta_k \qquad \chi_1^2(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ N'_{\mathbf{k}} \cos^4\theta_k \tag{74}$$

$$\chi_2^0(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \qquad \chi_2^1(k) = \int_0^{2\pi} \frac{d\theta_k}{2\pi}\ \delta n''_{\mathbf{k}} \cos^2\theta_k. \tag{75}$$

The subscript gives the order of the derivative of the appropriate combination of Fermi functions inside the integrand, and the twice superscript gives the power of the $\cos\theta_k$ factor.

The expansion coefficients can then be rewritten as one-dimensional integrals over the magnitude of k,

$$x_P^{10,\mu\nu}(\theta_q) = -\frac{e^2 v_S}{2\pi\omega_0}\cos\theta_q \int_0^\infty dk\, k\; v_k^2 \begin{pmatrix} \chi_1^1 & 0 \\ 0 & \chi_1^0 - \chi_1^1 \end{pmatrix}_{\mu\nu} \tag{76}$$

$$x_P^{20,\mu\nu}(\theta_q) = -\frac{e^2}{2\pi\omega_0^2}\int_0^\infty dk\, k\; v_k^4 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \begin{pmatrix} \chi_1^1 \sin^2\theta_q + \chi_1^2 \cos 2\theta_q & \left(\chi_1^1 - \chi_1^2\right)\sin 2\theta_q \\ \left(\chi_1^1 - \chi_1^2\right)\sin 2\theta_q & \left(\chi_1^0 - \chi_1^1\right)\sin^2\theta_q + \left(\chi_1^1 - \chi_1^2\right)\cos 2\theta_q \end{pmatrix}_{\mu\nu} \right.$$
$$\left. -\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2 - (2\lambda_k)^2} \begin{pmatrix} \chi_0^1 \sin^2\theta_q + \chi_0^2 \cos 2\theta_q & \left(\chi_0^1 - \chi_0^2\right)\sin 2\theta_q \\ \left(\chi_0^1 - \chi_0^2\right)\sin 2\theta_q & \left(\chi_0^0 - \chi_0^1\right)\sin^2\theta_q + \left(\chi_0^1 - \chi_0^2\right)\cos 2\theta_q \end{pmatrix}_{\mu\nu} \right] \tag{77}$$

$$x_S^{00} = \frac{2e^2 v_S^2}{\pi}\int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\chi_0^0(k) \tag{78}$$

$$x_S^{01} = -\frac{e^2 v_S^2 \omega_0}{\pi}\int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\frac{\lambda_k}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\chi_0^0(k) \tag{79}$$

$$x_S^{02} = \frac{2e^2 v_S^2}{\pi}\int_0^\infty dk\, k\; \lambda_k \frac{\Delta^2}{\lambda_k^2}\frac{3\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^3}\chi_0^0(k) \tag{80}$$

$$x_S^{10}(\theta_q) = \frac{e^2 v_S^3 \omega_0}{2\pi}\cos\theta_q \int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\left[\frac{8\lambda_k}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\chi_0^0(k) - \frac{\omega_0^2 + (2\lambda_k)^2}{\omega_0^2 - (2\lambda_k)^2}\chi_1^0(k)\right] \tag{81}$$

$$x_S^{11}(\theta_q) = -\frac{e^2 v_S^3}{2\pi}\cos\theta_q \int_0^\infty dk\, k\; \frac{\Delta^2}{\lambda_k^2}\left[8\lambda_k \frac{3\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^3}\chi_0^0(k) - \frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\chi_1^0(k)\right] \tag{82}$$

$$x_S^{20}(\theta_q) = \frac{e^2 v_S^2}{4\pi}\int_0^\infty dk\, k\; v_k^2 \frac{\Delta^2}{\lambda_k^2}\left[\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\left(\chi_0^0(k)\sin^2\theta_q + \chi_0^1(k)\cos 2\theta_q\right)\right.$$
$$\left. -\left(\frac{\Delta}{\lambda_k}\right)^2 \frac{\chi_1^0(k)\sin^2\theta_q + \chi_1^1(k)\cos 2\theta_q}{\omega_0^2 - (2\lambda_k)^2} + \left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\left(\chi_2^0(k)\sin^2\theta_q + \chi_2^1(k)\cos 2\theta_q\right)\right] \tag{83}$$

$$x_{SP}^{10,\mu}(\theta_q) = -\frac{e^2 v_S}{2\pi\omega_0}\int_0^\infty dk\, k\; v_k^2 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ \left(\chi_1^0(k) - \chi_1^1(k)\right)\sin\theta_q \end{pmatrix}_\mu - \frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2}{\omega_0^2 - (2\lambda_k)^2}\begin{pmatrix} \chi_0^1(k)\cos\theta_q \\ \left(\chi_0^0(k) - \chi_0^1(k)\right)\sin\theta_q \end{pmatrix}_\mu \right] \tag{84}$$

$$x_{SP}^{11,\mu}(\theta_q) = \frac{e^2 v_S}{2\pi}\int_0^\infty dk\, k\; v_k^2 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{1}{\omega_0^2}\begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ \left(\chi_1^0(k) - \chi_1^1(k)\right)\sin\theta_q \end{pmatrix}_\mu \right.$$
$$\left. -\frac{\Delta^2}{\lambda_k^3}\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\begin{pmatrix} \chi_0^1(k)\cos\theta_q \\ \left(\chi_0^0(k) - \chi_0^1(k)\right)\sin\theta_q \end{pmatrix}_\mu \right] \tag{85}$$

$$x_{SP}^{20,\mu}(\theta_q) = -\frac{e^2 v_S}{2\pi}\cos\theta_q \int_0^\infty dk\, k\; v_k^2 \left[\left(\frac{\xi_k^S}{\lambda_k}\right)^2 \frac{2}{\omega_0^2}\begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ \left(\chi_1^0(k) - \chi_1^1(k)\right)\sin\theta_q \end{pmatrix}_\mu \right.$$
$$\left. -\frac{\Delta^2}{\lambda_k^3}\left(\frac{\omega_0^2 + (2\lambda_k)^2}{\left[\omega_0^2 - (2\lambda_k)^2\right]^2}\begin{pmatrix} \chi_0^1(k)\cos\theta_q \\ \left(\chi_0^0(k) - \chi_0^1(k)\right)\sin\theta_q \end{pmatrix}_\mu - \frac{\lambda_k}{\omega_0^2 - (2\lambda_k)^2}\begin{pmatrix} \chi_1^1(k)\cos\theta_q \\ \left(\chi_1^0(k) - \chi_1^1(k)\right)\sin\theta_q \end{pmatrix}_\mu \right)\right] \tag{86}$$

So now the full photon action, including the empty cavity part from Eq. **??** and the terms derived here, is then

$$S_A \approx \frac{1}{2}\sum_q \Bigg[\sum_{\mu=\parallel,\perp}\left(\delta\Omega^2 + 2\,\mathrm{sgn}\,\Omega\omega_0\,\delta\Omega - q^2\right)A^\mu_q A^\mu_{-q} + \sum_{\mu,\nu=\parallel,\perp}\left(x_P^{10,\mu\nu}\,q\left(\mathrm{sgn}\,\Omega - \frac{\delta\Omega}{\omega_0}\right) + x_P^{20,\mu\nu}\,q^2\right)A^\mu_q A^\nu_{-q}$$
$$+\sum_{\mu=\parallel,\perp}\left(x_{SP}^{10,\mu}\,\mathrm{sgn}\,\Omega\,q + x_{SP}^{11,\mu}\,q\,\delta\Omega + x_{SP}^{20,\mu}\,q^2\right)\left(A^\mu_q A^\parallel_{-q} + A^\parallel_q A^\mu_{-q}\right)$$
$$+\left(x_S^{00} + x_S^{01}\;\mathrm{sgn}\,\Omega\,\delta\Omega + x_S^{02}\,\delta\Omega^2 + x_S^{10}\;\mathrm{sgn}\,\Omega\,q + x_S^{11}\,q\,\delta\Omega + x_S^{20}\,q^2\right)A^\parallel_q A^\parallel_{-q}\Bigg] \tag{87}$$

Now that the expansion is done, rewrite $\delta\Omega = \Omega - \mathrm{sgn}\,\Omega\,\omega_0$ and write everything as polynomials in the frequency. The Ω^2 term of the $A^\parallel_{-q}A^\parallel_q$ term is not equal to 1 (it is $1 + x_S^{02}$), so rescale the $A^\parallel$ field to cancel it:

$$A^\parallel \to \frac{A^\parallel}{\sqrt{1 + x_S^{02}}}. \tag{88}$$

This will also affect the coupling between this field and the Bardasis-Schrieffer mode, given in Eq. **??**.

After combining terms we can now write the full photon action in the following form,

$$S_A = \frac{1}{2}\sum_q \left(A^\parallel_{-q}, A^\perp_{-q}\right)\begin{pmatrix}\Omega^2 + b_\parallel(q)\Omega + c_\parallel(q) & b_\times(q)\Omega + c_\times(q)\\ b_\times(q)\Omega + c_\times(q) & \Omega^2 + b_\perp(q)\Omega + c_\perp(q)\end{pmatrix}\begin{pmatrix}A^\parallel_q\\ A^\perp_q\end{pmatrix}, \tag{89}$$

where the coefficients are defined as

$$b_\parallel(q) = \frac{x_S^{11}(\theta_q) + 2x_{SP}^{11,1}(\theta_q) - \frac{x_P^{10,11}(\theta_q)}{\omega_0}}{1 + x_S^{02}}q + \frac{x_S^{01} - 2\omega_0\,x_S^{02}}{1 + x_S^{02}}\,\mathrm{sgn}\,\Omega$$
$$c_\parallel(q) = \frac{x_P^{20,11}(\theta_q) + x_S^{20}(\theta_q) + 2x_{SP}^{20,1}(\theta_q) - 1}{1 + x_S^{02}}q^2 + \frac{x_S^{10}(\theta_q) + \omega_0\,x_S^{11}(\theta_q) + 2\left(x_{SP}^{10,1}(\theta_q) + \omega_0\,x_{SP}^{11,1}(\theta_q)\right)}{1 + x_S^{02}}\,\mathrm{sgn}\,\Omega\,q$$
$$+\frac{\omega_0^2\left(x_S^{02} - 1\right) - \omega_0\,x_S^{01} + x_S^{00}}{1 + x_S^{02}}$$
$$b_\perp(q) = -\frac{x_P^{10,22}(\theta_q)}{\omega_0}q$$
$$c_\perp(q) = \left(x_P^{20,22}(\theta_q) - 1\right)q^2 + 2x_P^{10,22}(\theta_q)\,\mathrm{sgn}\,\Omega\,q - \omega_0^2$$
$$b_\times(q) = \frac{x_{SP}^{11,2}(\theta_q)}{\sqrt{1 + x_S^{02}}}q$$
$$c_\times(q) = \frac{x_P^{20,12}(\theta_q) + x_{SP}^{20,2}(\theta_q)}{\sqrt{1 + x_S^{02}}}q^2 + \frac{x_{SP}^{10,2}(\theta_q) - \omega_0\,x_{SP}^{11,2}(\theta_q)}{\sqrt{1 + x_S^{02}}}\,\mathrm{sgn}\,\Omega\,q.$$

Consider this approximate action in the $\mathbf{q} = 0$ limit to determine how the supercurrent shifts the two dispersions relative to each other. (Note that without the supercurrent the dispersion for the two photon fields are degenerate.) We find

$$b_\parallel(\Omega, \mathbf{q} = 0) = \frac{x_S^{01} - 2\omega_0\,x_S^{02}}{1 + x_S^{02}}\,\mathrm{sgn}\,\Omega \qquad c_\parallel(\Omega, \mathbf{q} = 0) = \frac{(x_S^{02} - 1)\omega_0^2 - x_S^{01}\omega_0 + x_S^{00}}{1 + x_S^{02}}$$
$$b_\perp(\Omega, \mathbf{q} = 0) = 0 \qquad c_\perp(\Omega, \mathbf{q}) = -\omega_0^2$$
$$b_\times(\Omega, \mathbf{q} = 0) = c_\times(\Omega, \mathbf{q} = 0) = 0,$$

so the photon action is actually diagonal at $\mathbf{q} = 0$,

$$\left.\begin{pmatrix}\Omega^2 + b_\parallel(q)\Omega + c_\parallel(q) & b_\times(q)\Omega + c_\times(q)\\ b_\times(q)\Omega + c_\times(q) & \Omega^2 + b_\perp(q)\Omega + c_\perp(q)\end{pmatrix}\right|_{\mathbf{q}=0} = \begin{pmatrix}\Omega^2 + b_\parallel(\Omega, 0)\Omega + c_\parallel(\Omega, 0) & 0\\ 0 & \Omega^2 - \omega_0^2.\end{pmatrix} \tag{90}$$

We see that the dispersion of the perpendicular component is not shifted at all by the superconductor, while the parallel component has it's dispersion pushed to a different energy by the supercurrent, leading to a breaking of the degeneracy between them. The energy splitting between the two can be found by finding the frequencies such that each component vanishes, i.e. the energies of the photon bands at $\mathbf{q} = 0$, then simply taking the difference between them. Taking the positive branch in both cases we find,

$$E_{\parallel}(\mathbf{q}=0) = -\frac{b_{\parallel}(\Omega,0)}{2} + \sqrt{\left(\frac{b_{\parallel}(\Omega,0)}{2}\right)^2 - c_{\parallel}(\Omega,0)} \qquad E_{\perp}(\mathbf{q}=0) = \omega_0, \tag{91}$$

so the difference between them is

$$E_{\parallel}(0) - E_{\perp}(0) = \frac{2\omega_0 + x_S^{01}}{2\left(1 + x_S^{02}\right)} \left[\sqrt{1 - \frac{4(1 + x_S^{02})x_S^{00}}{\left(2\omega_0 + x_S^{01}\right)^2}} - 1\right]. \tag{92}$$