Contents

Chapter 1

Parametric Integral

1.1 Parametric Definite Integral

1.1.1 The definition of parametric definite integral

Definition 1.1. The function $f(x)$ is defined on $[a,b] \times [c,d]$. If $\forall x \in [a,b]$, the define integral $\displaystyle\int_c^d f(x,y)\,\mathrm{d}y$ exists, then it defines a function:

$$I(x) = \int_c^d f(x,y)\,\mathrm{d}y$$

We call $I(x)$ the **parametric define integral**, and call x the **parameter**.

Define integral is a process of limit. Hence research on it is **similar** to the function series. The question we are concerned with is, under what conditions does $I(x)$ to be **continuous, integrable, and differentiable?**

1.1.2 The sufficient condition for I(x) to be continuous

Theorem 1.2. *thm:The sufficient condition for I(x) to be continuous Suppose that $f(x,y)$ is continuous on $D = [a,b] \times [c,d]$. Then $I(x)$ is continuous on $[a,b]$.*[1]

Proof. We use the uniform continuity of $f(x,y)$ on D to prove the theorem.

As $f(x,y)$ in continuous on the compact set D, it is uniformly continuous on D. Hence $\forall \varepsilon > 0, \exists\, \delta > 0$ such that

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \delta \implies |f(x_1,y_1) - f(x_2,y_2)| < \frac{\varepsilon}{d-c}$$

Thus $\forall x \in [a,b]$, we have

$$\begin{aligned} |I(x) - I(x_0)| &= \left| \int_c^d f(x,y) - f(x_0,y)\,\mathrm{d}y \right| \le \int_c^d |f(x,y) - f(x_0,y)|\,\mathrm{d}y \\ &< \int_c^d \frac{\varepsilon}{d-c}\,\mathrm{d}y = \varepsilon. \quad \forall x_0 \in U(x,\delta) \end{aligned}$$

This completes the proof. □

[1]The theorem can be generalized as follows: Under the assumption of the theorem, $\forall (x,u) \in D$, $I(x,u) = \displaystyle\int_c^u f(x,y)\,\mathrm{d}y$ is continuous on $[a,b]$.

Under the condition of the theorem above, we have

$$\lim_{x\to x_0}\int_c^d f(x,y)\,\mathrm{d}y = \lim_{x\to x_0} I(x) = I(x_0) = \int_c^d \lim_{x\to x_0} f(x,y)\,\mathrm{d}y$$

Hence **the continuity of $I(x)$ can be interpreted as the changeability between limit operation and integral operation.**

By the theorem of transforming double integrals into iterated integrals, we have:

Theorem 1.3. *Suppose $f(x,y)$ in continuous on $D=[a,b]\times[c,d]$. Then $I_1(x)=\int_c^d f(x,y)\,\mathrm{d}y$ and $I_2(y)=\int_a^b f(x,y)\,\mathrm{d}x$ are integrable on $[a,b],[c,d]$ respectively, and*

$$\int_a^b I_1(x)\,\mathrm{d}x = \int_c^d I_2(y)\,\mathrm{d}y$$

The theorem states that under the condition of the theorem, integral operations can be changed.

$$\int_a^b \mathrm{d}x\int_c^d f(x,y)\,\mathrm{d}y = \int_c^d \mathrm{d}y\int_a^b f(x,y)\,\mathrm{d}x$$

1.1.3 The Integrability of Parametric Define Integral

Theorem 1.4. *Suppose $f(x,y)$ and its partial derivative $f'_x(x,y)$ is continuous on $D=[a,b]\times[c,d]$. Then $I(x)=\int_c^d f(x,y)\,\mathrm{d}y$ is differentiable on $[a,b]$ and*

$$I'(x)=\int_c^d f'_x(x,y)\,\mathrm{d}y.$$

Proof. By the theorem **??**, $g(x)=\int_c^d f'_x(x,y)\,\mathrm{d}y$ is continuous on $[a,b]$. Thus we have

$$\begin{aligned}\int_a^x g(u)\,\mathrm{d}u &= \int_a^x \mathrm{d}u\int_c^d f'_u(u,y)\,\mathrm{d}y = \int_c^d \mathrm{d}y\int_a^x f'_u(u,y)\,\mathrm{d}x\\ &= \int_c^d [f(x,y)-f(a,y)]\,\mathrm{d}y = I(x)-I(a)\end{aligned}$$

Therefore $g(x)=I'(x)$. This completes the proof. $\square$

The differentiability means the changeability between differential and integral.

The theorem can be generalized as follows

Theorem 1.5. *Suppose $f(x,y), f'_x(x,y)$ are continuous on D and $\varphi(x)$ is a function defined on $[a,b]$ that $c\le\varphi(x)\le d$. Then $I(x)=\int_c^{\varphi(x)} f(x,y)\,\mathrm{d}y$ is differentiable and*

$$I'(x)=\int_c^{\varphi(x)} f'_x(x,y)\,\mathrm{d}y + f(x,\varphi(x))\varphi'(x)$$

Proof. Let $u = \varphi(x)$. Then $I(x) = F(x, u) = \displaystyle\int_c^u f(x, y)\,\mathrm{d}y$. Take the partial derivative of F with respect to x

$$\frac{\partial F(x, u)}{\partial x} = \int_c^u f'_x(x, y)\,\mathrm{d}y$$

As $f'_x(x, y)$ is continuous on D, $\dfrac{\partial F(x, u)}{\partial x}$ is continuous on $[a, b]$. Similarly

$$\frac{\partial F(x, u)}{\partial u} = f(x, u) \text{ is continuous on } [c, d]$$

That is the two partial derivatives of F are both continuous. Hence

$$I'(x) = \frac{\partial F(x, u)}{\partial x} + \frac{\partial F(x, u)}{\partial u} \cdot \frac{\partial u}{\partial x}$$

This completes the proof. □