4.8 Show that the solution
$\ddot{x} + \omega^2 x = f(t)$ for $t \geq 0$ where
$x(0)$ and $\dot{x}$ at zero are presribed is
by the method of Green's functions, is
$$x(t) = \frac{\dot{x}(0)}{\omega}\sin\omega t + x(0)\cos\omega t + \int_0^\infty G(\tau,t)f(\tau)\,d\tau$$

where $G(\tau,t) = \begin{cases} \frac{\sin(\omega(t-\tau))}{\omega} & ; \tau < t \\ 0 & ; \tau > t \end{cases}$

$$(Lx, G) = \int_0^\infty G L x \, d\tau$$
$$= (G\dot{x} - G_\tau x)\Big|_0^\infty + \int_0^\infty x(G_{\tau\tau} + \omega^2 G)\,d\tau$$
$$= \lim_{k\to\infty} (G(k,t)\dot{x}(k) - G_\tau(k,t)x(k) - G(0,t)\dot{x}(0) + G_\tau(0,t)x(0)$$
$$+ \int_0^\infty x L^* G \, d\tau$$

$\lim_{k\to\infty} G(k,t) = 0$ , $\lim_{k\to\infty} G_\tau(k,t) = 0$ $\quad L = L^*$

$$(Lx, G) = -G(0,t)\dot{x}(0) + G_\tau(0,t)x(0) + \int_0^\infty x L G \, dt$$

$$G_{\tau\tau} + \omega^2 G = \delta(\tau - t)$$

$$e^{-i\alpha\tau} G_{\tau\tau} + \omega^2 e^{-i\alpha\tau} G = e^{-i\alpha t}\,\delta(\tau - t) \qquad e^{-i\alpha\tau} \quad G_{\tau\tau}$$

$$\int_{-\infty}^{\infty} e^{-i\alpha\tau} G_{\tau\tau}\, d\tau + \omega^2 \int_{-\infty}^{\infty} e^{-i\alpha\tau} G = e^{-i\alpha t} \qquad -i\alpha e^{-i\alpha\tau} \quad G_{\tau}$$

$$-\alpha^2 e^{-i\alpha\tau} \quad G$$

$$-\alpha^2 \hat{G} + \omega^2 \hat{G} = e^{-i\alpha t} \quad \text{where } \hat{G} = \int_{-\infty}^{\infty} G(\tau, t) e^{-i\alpha\tau}\, d\tau$$

$$\hat{G} = \frac{e^{-i\alpha t}}{\omega^2 - \alpha^2}$$

$$G(\tau, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\alpha t}}{\omega^2 - \alpha^2} \cdot e^{i\alpha\tau}\, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha(\tau - t)}}{\omega^2 - \alpha^2}\, d\alpha$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos(\alpha(\tau - t))}{\alpha^2 - \omega^2}\, d\alpha$$

$$= ?$$

4.13 show that the solution to

$$u'' = \phi(x) \quad u'(0) = u'(1) = 0$$

is given by

$$u(x) = K + \int_0^1 (\xi - x) H(\xi - x)\phi(\xi)\, d\xi = K + \int_x^1 (\xi - x)\phi(\xi)\, d\xi$$

where $K$ is an arbitrary constant and $\int_0^1 \phi(x)\,dx = 0$

$$\int_0^1 GLu\, d\xi = \cancel{G(1,x)u'(1)} - G_\xi(1,x)u(1) - \cancel{G(0,x)u'(0)} + G_\xi(0,x)u(0) + \int_0^1 uLG\, d\xi$$

$$G_\xi(0,x) = G_\xi(1,x) = 0$$

$$G_{\xi\xi} = \delta(\xi - x)$$

$$G(\xi, x) = (\xi - x)H(\xi - x) + A\xi + B$$

$H(-x) + A = 0$

$H(1-x) + A = 0$ ↑contradiction

$$H(\xi - x) = \begin{cases} 0 \,;\, \xi < x \\ 1 \,;\, \xi > x \end{cases}$$

$V_{\xi\xi} = 0 \quad V(\xi) = \xi$

$$G_{\xi\xi} = \delta(\xi - x) + F$$

$$a = -\frac{1}{\int_0^1 \xi^2 d\xi}$$

$$= -\frac{1}{\xi^3/3|_0^1}$$

$$= -3$$

$$F = -3x\xi$$

$$G_{\xi\xi} = \delta(\xi - x) - 3x\xi$$

$$G_\xi = H(\xi - x) - \frac{3}{2}x\xi^2 + A$$

$$G_\xi(0, x) = H(-x) + A = 0$$

$$A = -H(-x)$$

$$G_\xi(1, x) = H(1-x) - \frac{3}{2}x + A = 0$$

$$G(\xi, x) = (\xi - x)H(\xi - x) - \frac{1}{2}x\xi^3 - \xi H(-x) + B$$

$$\int_0^1 G(\xi, x)\,\phi(\xi) = u(x) - 3x\int_0^1 \xi u(\xi) d\xi$$

$$\int_0^1 \left((\xi - x)H(\xi - x) - \frac{1}{2}x\xi^3 - \xi H(-x) + B(x)\right)\phi(\xi)\, d\xi$$
$$+ 3kx = u(x)$$

$$\int_0^1 (\xi - x)H(\xi - x)\,\phi(\xi) d\xi - \frac{x}{8} - \frac{H(-x)}{2} + B(x) + 3kx$$
$$= u(x)$$