Problem Set 9

Ryan Coyne

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Result 1: There exist two distinct irrational numbers a and b such that a^b is rational.

PROOF: Consider the number $\sqrt{6}^{\sqrt{2}}$. Now, there are two cases.
Case 1: The number $\sqrt{6}^{\sqrt{2}}$ is rational. Then $a = \sqrt{6}$, $b = \sqrt{2}$, and a^b is rational.
Case 2: The number $\sqrt{6}^{\sqrt{2}}$ is irrational. Now, raise this number to the power of $\sqrt{2}$, $(\sqrt{6}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{6}^2 = 6$, which is rational. Then, $a = \sqrt{6}^{\sqrt{2}}$, $b = \sqrt{2}$, and a^b is rational. ■

Result 2: There exists four distinct positive integers such that each of the integers divides (evenly) the sum of the remaining three integers.

PROOF: Consider the numbers $2, 4, 6, 12$. Now sum each combination of three numbers, $4+6+12 = 22$, $2+6+12 = 20$, $2+4+12 = 18$, and $2+4+6 = 12$. Then, $2|22$, $4|20$, $6|18$, and $12|12$. Therefore there are four such integers. ■

Result 3: There are no integers $a \geq 2$ and $n \geq 1$ such that $a^2 + 1 = 2^n$

PROOF: Suppose to the contrary, that $a^2 + 1 = 2^n$. Now rearrange into, $a^2 = 2^n - 1$. Then, consider two cases: $n = 1$ and $n \geq 2$.
Case 1: If $n = 1$, then $2^1 - 1 = 1$. However, this is incompatible because $a \geq 2$.
Case 2: Let $n \geq 2$. Now, $a^2 = 2^n - 1$ suggests that $2^n - 1$ is a perfect square because a is assumed to be an integer. However, $2^n - 1 = (2^{n/2} + 1)(2^{n/2} - 1)$ and thus cannot be a perfect square.
Therefore, by contradiction, there cannot be integers $a \geq 2$ and $n \geq 1$ such that $a^2 + 1 = 2^n$. ■

Result 4: There do not exists real numbers a and b in the open interval $(0, 1)$ such that $4a(1-b) > 1$ and $4b(1-a) > 1$.

PROOF: Start by manipulating the first equation,

$$4a(1-b) > 1$$
$$a > \frac{1}{4(1-b)}$$

Now, consider the second equation,

$$\begin{aligned}4b(1-a) &> 1\\ 4b-4ab &> 1\\ -4ab &> 1-4b\\ a &< \frac{4b-1}{4b}.\end{aligned}$$

Then,

$$\begin{aligned}\frac{4b-1}{4b} &> \frac{1}{4(1-b)}\\ 4(4b-1)(1-b) &> 4b\\ (4b-1)(1-b) &> b\\ 4b-4b^2-1+b &> b\\ -4b^2+4b-1 &> 0.\end{aligned}$$

In this manipulation, we can be sure that we never divide by zero or implicitly multiply by a negative number because we have already restricted ourselves to the interval $(0,1)$. Now, the sole value for b at which $-4b^2+4b-1=0$ is $b=\frac{1}{2}$. This value is not valid because we require $-4b^2+4b-1>0$. Now, we choose values within the interval $(0,1)$ and on either side of $b=\frac{1}{2}$ to check the truth of the hypothesis.
Case 1: Let $b=\frac{1}{4}$. Now, $-4b^2+4b-1=-1<0$. Therefore there are no values on the interval $(0,\frac{1}{2}]$ which satisfy both inequalities.
Case 2: Let $b=\frac{3}{4}$. Now, $-4b^2+4b-1=-1<0$. Therefore there are no values on the interval $(\frac{1}{2},1]$ which satisfy both inequalities.
Therefore there are no values of b on the interval $(0,1)$ that can satisfy both $4a(1-b)>1$ and $4b(1-a)>1$, for any value of a. $\blacksquare$