3 The Delta Function

In physics and engineering, there exists a notion of "point actions". These are actions that are highly localized in space and/or time. As an example, suppose a circular coin is pressed with unit force against the edge of a metal plate that extends over, $y > 0$ and $-\infty < x < \infty$, as shown in Figure 3.1. We are interested in the resulting stress field but do not know the details of the force distribution, say $w(x)$. We do, however, know it will be very concentrated in space and that

$$\int_{-\infty}^{\infty} w(x)dx = 1 \tag{3.1}$$

so that the net force is unity.

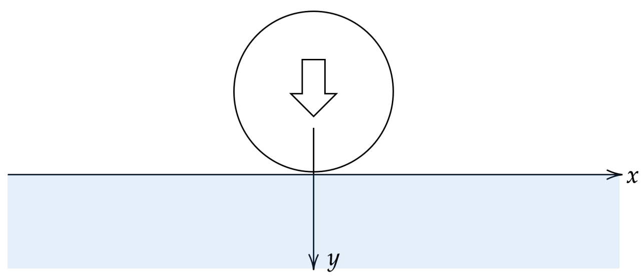

Figure 3.1: Coin pressed to the edge of a plate.

We expect that two highly concentrated force distributions would produce nearly identical stress fields except in the immediate neighborhood of the point at which the force is applied, provided they are statically equivalent, meaning that their resultant forces and couples are identical. As such we might simplify the problem by deciding, a priori, on a definite form for w, such as

$$w_k(x) = \begin{cases} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases} \tag{3.2}$$

or

$$w_k(x) = \frac{k}{\pi(1 + k^2x^2)} \tag{3.3}$$

where $k > 0$. In Fig 3.2 we can see that w becomes highly concentrated when k is large.

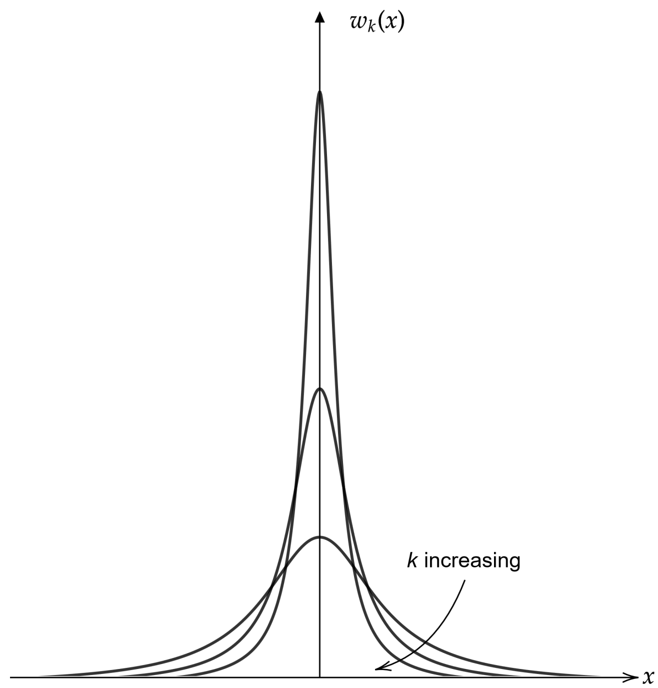

Figure 3.2: Distributed Force; eq. 3.3

If we let $k \to \infty$ then the force distribution approaches our idea of a "point-action" which in this case is a force of unit strength, acting at $x = 0$. Calling this "point-action" $\delta(x)$, then

$$\delta(x) = \lim_{k \to \infty} w_k(x) \tag{3.4}$$

This, however, cannot be considered a rigorous definition of the delta function because the limit is infinite for $x = 0$.

The delta function is more appropriately defined as a generalized function. To understand this way of defining δ consider the following functional.

$$\int_{-\infty}^{\infty} g(x)h(x) = \mathscr{F}(h) \tag{3.5}$$

This functional assigns a numerical value, $\mathscr{F}(h)$, for each function h within the domain, $\mathscr{D}$, of $\mathscr{F}$. We will take $\mathscr{D}$ to be the set of all functions that are defined over $-\infty < x < \infty$, are infinitely differentiable, and approach zero outside of some finite interval,

Suppose $\mathscr{F}(h)$ is the integral of h from ξ to ∞.

$$\int_{\infty}^{\infty} g(x)h(x)dx = \int_{\xi}^{\infty} h(x)dx \tag{3.6}$$

Then, $g(x)$ must be the Heaviside step function,

$$H(x-\xi) = \begin{cases} 1, & x > \xi \\ 0, & x < \xi \end{cases} \tag{3.7}$$

which is a function in the classical sense.

If $\mathscr{F}(h)$ is $h(0)$ so that

$$\int_{-\infty}^{\infty} g(x)h(x)dx = h(0) \tag{3.8}$$

then it can be shown that there is no function, $g(x)$, which exists such that (3.8) is true for all functions, $h(x)$, in the domain, $\mathscr{D}$. It is then the case that g must be a generalized function, which we call the delta function. This means δ can be defined as follows.

$$\int_{-\infty}^{\infty} \delta(x)h(x)dx = h(0) \tag{3.9}$$