Green's Functions

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1 Introduction

The first part of this text is primarily concerned with solutions to differential equations of the form

$$Lu = \phi \tag{1.1}$$

over an interval $a \leq x \leq b$ and subject to certain boundary conditions, where L is an nth order linear ordinary differential operator, and u and ϕ are functions of the independent variable. For L to be linear, it must satisfy the condition

$$L(\alpha v + \beta w) = \alpha L v + \beta L w \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant. We claim without proof that for this condition to be met, L must be of the form

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x). \tag{1.3}$$

Since L is of order n, there will be n boundary conditions of the general form

$$B_j(u) = c_j; \quad j = 1, 2, \ldots, n \tag{1.4}$$

where the B_j's are prescribed functionals and c_j's are prescribed constants. We will only consider B_j's that are linear combinations of u and its derivatives through order $n-1$ and evaluated at the endpoints, a and b.

Here, functional refers to a transformation with a set of functions as its domain and a set of numbers as its range. To illustrate what we mean, consider the functional

$$\mathscr{F}(u) = \int_0^1 u^2(x)dx. \tag{1.5}$$

The domain of this functional might be the set of functions defined over the interval $(0, 1)$ and for which the integral of u^2 from 0 to 1 exists, and the range is $(0, \infty)$.

For B_j to be linear, it must satisfy the condition

$$B_j(\alpha v + \beta w) = \alpha B_j(v) + \beta B_j(w) \tag{1.6}$$

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and its boundary conditions, begin by finding the adjoint operator, denoted $\mathscr{L}*$. The adjoint operator consists of the formal adjoint, L^*, and the boundary conditions associated with the Green's function. To determine these, first form the product, vLu, and integrate it over the interval of interest. By repeated integration by parts, we can express the integral in the form

$$\int_{\mathrm{a}}^{\mathrm{b}} vLudx = [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} uL^*vdx, \tag{2.1}$$

where $[\cdots]\Big|_a^b$ represents the boundary terms resulting from the successive integration by parts. Here, u and v are arbitrary and sufficiently differentiable for functions so that the left and right sides are well defined.

As an example, consider the linear differentiable operator

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x) \tag{2.2}$$

To find L^*, perform integration by parts on each term of the product vLu until there is no derivative of u within the integral. That is to say, integrate by parts twice on the first term, once on the second, and not at all on the third. Doing this, we are left with

$$\begin{aligned}\int_a^b vLudx &= \int_a^b (vau'' + vbu' + vc)dx \\ &= (vau' + vbu)\Big|_a^b + \int_a^b (-(va)'u' - (vb)'u + vcu)dx \\ &= (vau' + vbu - (va)'u)\Big|_a^b + \int_a^b ((va)''u - (vb)'u + vcu)dx \\ &= (vau' + vbu - (va)'u)\Big|_a^b + \int_a^b u((va)'' - (bv)' + cv)dx.\end{aligned} \tag{2.3}$$

From this, it is clear that

$$\begin{aligned}L^*v &= (Av)'' - (Bv)' + Cv \\ &= (A'v + Av')' - B'v - Bv' + Cv \\ &= Av'' + (2A' - B)v' + (A'' - B' + C)\end{aligned} \tag{2.4}$$

and so the formal adjoint of the second-order linear differential operator L must be of the form

$$L^* = A\frac{d^2}{dx^2} + (2A' - B)\frac{d}{dx} + (A'' - B' + C) \tag{2.5}$$

If $L^* = L$, then L is called formally self-adjoint. By comparing equations (2.2) and (2.5), we can see that for a second-order linear differentiable operator to be formally self-adjoint, A' must be equal to B. You may notice that $A'' - B' + C$ must also be equal to C, but this is always true given that A' equals B.

Definition If the boundary conditions on L are homogeneous[1], then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(Lu, v) = (u, L^*v) \tag{2.6}$$

where (f, g) is the inner product of f and g,

$$(f, g) = \int_a^b f(x)g(x)dx. \tag{2.7}$$

[1]By homogeneous, we mean that each boundary condition is of the form $B_j(u) = 0$ and only contains terms that are 0 when $u(x) = 0$.

This means adjoint operator $\mathscr{L}^*$ consists of L^* and boundary conditions for which the boundary terms of the integral are zero.

Example 2.1 Consider $\mathscr{L}$ to consist of $L = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \leq x \leq 1$. Then

$$\begin{aligned}
(Lu, v) &= \int_0^1 u'v dx \\
&= (uv)\Big|_0^1 - \int_0^1 uv' dx \\
&= u(1)v(1) - u(0)v(0) + \int_0^1 uL^*v dx \\
&= u(1)(v(1) - 3v(0)) + \int_0^1 uL^*v dx
\end{aligned} \tag{2.8}$$

Since the particular value of $u(1)$ is not given, we must make $v(1) - 3v(0)$ equal zero, because choosing $u(1) = 0$ would unduly restrict our solution. Therefore $\mathscr{L}^*$ consists of L^* which is $-\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$. As a final note, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called self-adjoint.

3 The Delta Function

In physics and engineering, there exists a notion of ”point actions.” These are actions that are highly localized in space and/or time. As an example, suppose a circular coin is pressed with unit force against the edge of a metal plate that extends over, $y > 0$ and $-\infty < x < \infty$, as shown in Figure 3.1. We are interested in the resulting stress field but do not know the details of the force distribution, say $w(x)$. We do, however, know it will be very concentrated in space and that

$$\int_{-\infty}^{\infty} w(x) dx = 1 \tag{3.1}$$

so that the net force is unity.

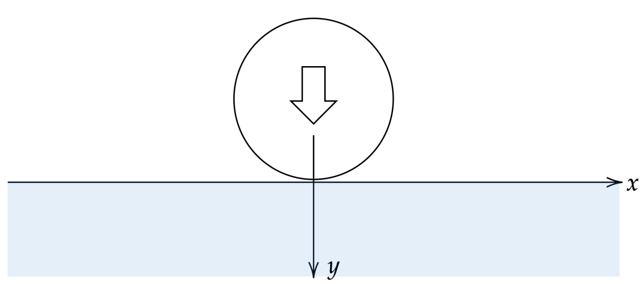

Figure 3.1: Coin pressed to the edge of a plate.

We expect that two highly concentrated force distributions would produce nearly identical stress fields except in the immediate neighborhood of the point at which the force is applied. This is provided they are statically equivalent, meaning that their resultant forces and couples are identical. As such, we might simplify the problem by deciding, a priori, on a definite form for w, such as

$$w_k(x) = \begin{cases} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases} \tag{3.2}$$

or

$$w_k(x) = \frac{k}{\pi(1 + k^2x^2)} \tag{3.3}$$

where $k > 0$. In Fig 3.2, we can see that w becomes highly concentrated when k is large.

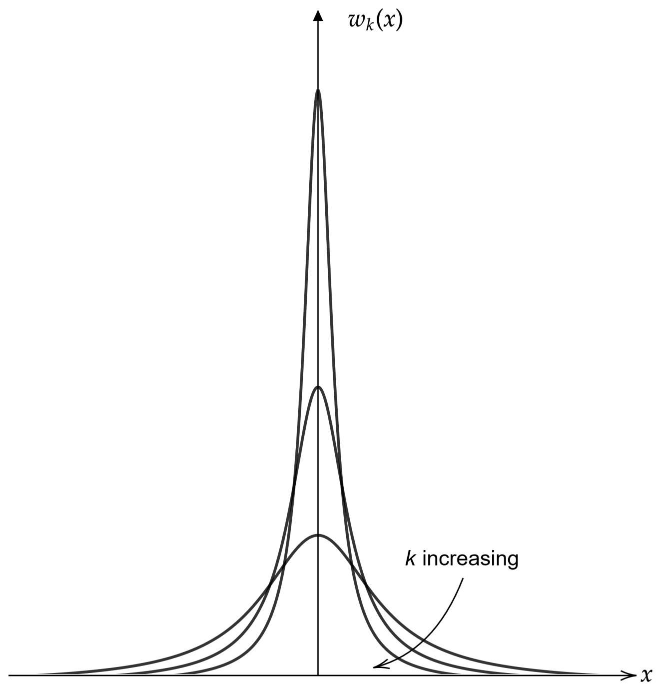

Figure 3.2: Distributed Force; eq. 3.3

If we let $k \to \infty$, then the force distribution approaches our idea of a "point-action," which in this case is a force of unit strength, acting at $x = 0$. Calling this "point-action" $\delta(x)$, then

$$\delta(x) = \lim_{k \to \infty} w_k(x) \tag{3.4}$$

However, this cannot be considered a rigorous definition of the delta function because the limit is infinite for $x = 0$.

Definition Generalized functions are continuous linear functionals that are continuous on the set of infinitely differentiable functions with compact support such that all generalized functions have derivatives which are also generalized functions.

Definition A function has compact support if the subset of its domain for which its range is non-zero is closed and bounded.

The delta function is more appropriately defined as a generalized function. To understand this way of defining δ, consider the following functional,

$$\mathscr{F}(h) = \int_{-\infty}^{\infty} g(x)h(x). \tag{3.5}$$

This functional assigns a numerical value, $\mathscr{F}(h)$, for each function h within the domain, $\mathscr{D}$, of $\mathscr{F}$. We will take $\mathscr{D}$ to be the set of all functions that are defined over $-\infty < x < \infty$, are infinitely differentiable, and approach zero outside of some finite interval,

Suppose $\mathscr{F}(h)$ is the integral of h from ξ to ∞.

$$\int_{\infty}^{\infty} g(x)h(x)dx = \int_{\xi}^{\infty} h(x)dx \tag{3.6}$$

Then, $g(x)$ must be the Heaviside step function,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ \frac{1}{2}, & x = 0 \end{cases} \tag{3.7}$$

which is a function in the classical sense.

If $\mathscr{F}(h)$ is $h(0)$ so that

$$\int_{-\infty}^{\infty} g(x)h(x)dx = h(0) \tag{3.8}$$

then it can be shown that there is no function, $g(x)$, which exists such that (3.8) is true for all functions, $h(x)$, in the domain, $\mathscr{D}$. It is then the case that g must be a generalized function, which we call the delta function. As such, δ can be defined in the following way.

$$\int_{-\infty}^{\infty} \delta(x)h(x)dx = h(0) \tag{3.9}$$

Although $\delta(x)$ acts at $x = 0$, it can be adjusted to act at any point by shifting the argument. Thus, $\delta(x - \xi)$ acts at $x = \xi$,

$$\int_{-infty}^{\infty} \delta(x - \xi)h(x)dx = h(\xi) \tag{3.10}$$

As a generalized function, δ is also differentiable. By referring to (3.5), one can see that defining the derivative of a generalized function involves determining the functional, $\mathscr{F}(h)$ for

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = \mathscr{F}(h) \tag{3.11}$$

Next, we integrate by parts

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = g(x)h(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(x)h'(x)dx \tag{3.12}$$

The integral term is fairly simple to interpret since it is of the same form as (3.5), but the boundary term is not as nice because it involves knowing the values of g. To deal with this, we will discard the boundary term and define g' with the formula

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = -\int_{-\infty}^{\infty} g(x)h'(x)dx \tag{3.13}$$

For the delta function, this means

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} \delta(x-\xi)h'(x)dx \\ &= -h'(\xi)\end{aligned} \tag{3.14}$$

Theorem 3.1. *The jth derivative of the delta function is defined by*

$$\int_{-\infty}^{\infty} \delta^{(j)}(x-\xi)h(x)dx = (-1)^j h^{(j)}(\xi) \tag{3.15}$$

Proof By induction. I haven't figured out how to do this yet. ∎

Note that because of the discontinuity in $H(x-\xi)$ at the point $x=\xi$, the derivative of H does not exist as an ordinary function, but using the previous method does allow us to find $H'(x-\xi)$ as a generalized function.

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} H(x-\xi)h'(x)dx \\ &= -\int_{\xi}^{\infty} h'(x)dx = h(\xi)\end{aligned} \tag{3.16}$$

and because

$$\int_{-\infty}^{\infty} \delta(x-\xi)h(x)dx = h(\xi) \tag{3.17}$$

it must be the case that

$$H'(x-\xi) = \delta(x-\xi) \tag{3.18}$$

Such equalities between generalized functions, as seen in (3.18), are understood in the sense that if some h in $\mathscr{D}$ is multiplied through, and then we integrate over $(-\infty,\infty)$ then the result will be consistent. That is to say

$$\int_{-\infty}^{\infty} g_1(x)h(x)dx = \int_{-\infty}^{\infty} g_2(x)h(x)dx \tag{3.19}$$

for some equivalent generalized functions g_1 and g_2.

As a final aside, notice that

$$x\delta(x) = 0 \tag{3.20}$$

as a result of

$$\int_{-\infty}^{\infty} x\delta(x)h(x)dx = [xh(x)]|_{x=0} = 0 \tag{3.21}$$