

# Problem Set 9

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1: Find a formula for

$$1 + 4 + 7 + \cdots (3n - 2)$$

for positive integers then verify your formula by mathematical induction.

The formula is

$$1 + 4 + 7 + \cdots (3n - 2) = \frac{n(3n - 1)}{2}$$

for all positive integers,  $n$ .

PROOF: We will prove by induction that

$$1 + 4 + 7 + \cdots (3n - 2) = \frac{n(3n - 1)}{2}$$

for all positive integers  $n$ . Base Step: Let  $n = 1$ . Now,  $3n - 2 = 3 - 2 = 1$ , and so the sum is equal to one. Now, consider the right side

$$\begin{aligned} \frac{1 \cdot (3 \cdot 1 - 1)}{2} &= \frac{3 - 1}{2} \\ &= \frac{2}{2} \\ &= 1. \end{aligned}$$

Thus, the formula is valid for  $n = 1$ .

We show that,

$$1 + 4 + 7 + \cdots (3(k + 1) - 2) = \frac{(k + 1)(3(k + 1) - 1)}{2}$$

Now, assume that  $1 + 4 + 7 + \cdots (3k - 2) = \frac{k(3k - 1)}{2}$  is valid for all  $k \in \mathbb{N}$ . Add  $3(k + 1) - 2$  to both sides,

$$1 + 4 + 7 + \cdots (3k - 2) + (3k + 1) = \frac{k(3k - 1)}{2} + 3k + 1.$$

Next, write the right side with a common denominator and rearrange the numerator into our desired form,

$$\begin{aligned}
1 + 4 + 7 + \cdots (3k - 2) + (3k + 1) &= \frac{k(3k - 1) + 6k + 2}{2} \\
&= \frac{3k^2 - k + 6k + 2}{2} \\
&= \frac{3k^2 + 5k + 2}{2} \\
&= \frac{(k + 1)(3k + 2)}{2} \\
&= \frac{(k + 1)(3(k + 1) - 1)}{2}.
\end{aligned}$$

Thus, by the principle of mathematical induction,  $1 + 4 + 7 + \cdots (3n - 2) = \frac{n(3n-1)}{2}$ . ■

**2:** Prove the following inequality for every positive integer  $n$ :

$$2! \cdot 4! \cdot 6! \cdots (2n)! \geq ((n + 1)!)^n.$$

PROOF: We proceed by induction. Since  $(2 \cdot 1)! = (2!)^1$ , the statement is true when  $n = 1$ . Assume that

$$2! \cdot 4! \cdot 6! \cdots (2k)! \geq ((k + 1)!)^k \quad (1)$$

for some integer,  $k$ . We show,

$$2! \cdot 4! \cdot 6! \cdots (2k)! \cdot (2(k + 1)) \geq ((k + 2)!)^{k+1}.$$

Now, multiply either side of equation (1) by  $(2(k + 1))!$ ,

$$2! \cdot 4! \cdot 6! \cdots (2k)! \cdot (2(k + 1))! \geq ((k + 1)!)^k \cdot (2(k + 1))!.$$

Now, we show that  $((k + 1)!)^k \cdot (2(k + 1))! \geq ((k + 2)!)^{k+1}$  directly. Divide both sides by  $((k + 1)!)^k$ ,

$$\begin{aligned}
\frac{((k + 1)!)^k \cdot (2(k + 1))!}{((k + 1)!)^k} &\geq \frac{((k + 2)!)^{k+1}}{((k + 1)!)^k} \\
(2k + 2)! &\geq \frac{((k + 1)!)^{k+1} (k + 2)^{k+1}}{((k + 1)!)^k} \\
&\geq (k + 1)!(k + 2)^{k+1}.
\end{aligned}$$

Now, divide by  $(k + 1)!$ ,

$$\begin{aligned}
\frac{(2k + 2)!}{(k + 1)!} &\geq \frac{(k + 1)!(k + 2)^{k+1}}{(k + 1)!} \\
\underbrace{(k + 2) \cdot (k + 3) \cdots (2k + 1) \cdot (2k + 2)}_{k+1 \text{ factors}} &\geq (k + 2)^{k+1}.
\end{aligned}$$

Note that both sides involve the multiplication of integers  $k + 1$  times. However, all of the factors on the right are  $k + 2$ , and exactly one of the factors on the left is  $k + 2$ . Additionally,  $k$  of the factors on the left are greater than  $k + 2$ . Thus, it is true that  $((k + 1)!)^k \cdot (2(k + 1))! \geq ((k + 2)!)^{k+1}$ , and by induction. ■

**3:** Prove that for every real number  $x > -1$  and every positive integer  $n$ ,

$$(1 + x)^n \geq 1 + nx$$

PROOF: ■

**4:** Prove that  $81|(10^{n+1} - 9n - 10)$  for every positive integer  $n$ .

PROOF: Base Case: Let  $n = 1$ . Now,  $10^2 - 9 - 10 = 81$ . Thus,  $81|(10^{n+1} - 9n - 10)$  when  $n = 1$ .

Induction Step: Now, assume that  $81|(10^{k+1} - 9k - 10)$ . Next, consider  $10^{k+2} - 9(k+1) - 10 = 10^{k+2} - 9k - 19$ . Now, subtract  $10(10^{k+1} - 9k - 10)$  from both sides

$$\begin{aligned} 10^{k+2} - 9(k+1) - 10 - 10(10^{k+1} - 9k - 10) &= 10^{k+2} - 9k - 19 - 10(10^{k+1} - 9k - 10) \\ &= 10^{k+2} - 9k - 19 - 10^{k+2} + 90k + 100 \\ &= 81(k+1). \end{aligned}$$

Then, add  $10(10^{k+1} - 9k - 10)$  to both sides,

$$10^{k+2} - 9(k+1) - 10 = 81(k+1) + 10(10^{k+1} - 9k - 10).$$

Since,  $81|(10^{k+1} - 9k - 10)$ , then  $10^{k+1} - 9k - 10 = 81j$ , for some  $j \in \mathbb{N}$ . Thus,

$$\begin{aligned} 10^{k+2} - 9(k+1) - 10 &= 81(k+1) + 10(81j) \\ &= 81(k+1) + 81(10j) \\ &= 81(k+1 + 10j). \end{aligned}$$

Thus,  $81|(10^{k+2} - 9(k+1) - 10)$ . Therefore,  $81|(10^{n+1} - 9n - 10)$  for all  $n \in \mathbb{N}$ . ■

**5:** A sequence  $\{a_n\}$  is defined recursively by

$$a_1 = 1, \quad a_2 = 2; \quad a_n = a_{n-1} + 2a_{n-2},$$

for  $n \geq 3$ . Conjecture a formula for  $a_n$  and verify that your conjecture is correct.

One such formula is  $a_n = 2^{n-1}$ .

PROOF: ■