Green's Functions

Ryan Coyne

March 26, 2023

Contents

1 Introduction

The first part of this text is primarily concerned with the solutions to differential equations of the form

$$\mathbf{L}u = \phi, \tag{1.1}$$

over an interval a $\leq x \leq$ b and subject to certain boundary conditions, where $\mathbf{L}$ is an nth order linear ordinary differential operator and where the function ϕ is integrable on the given interval. For $\mathbf{L}$ to be linear, it must satisfy the condition

$$\mathbf{L}(\alpha v + \beta w) = \alpha \mathbf{L} v + \beta \mathbf{L} w \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant. We claim without proof that for this condition to be met, $\mathbf{L}$ must be of the form

$$\mathbf{L} = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(0). \tag{1.3}$$

Since $\mathbf{L}$ is of order n, there will be n boundary conditions of the general form

$$\mathbf{B_j}(u) = c_j; \quad j = 1, 2, \ldots, n \tag{1.4}$$

where the $\mathbf{B_j}$'s are prescribed functionals [1] and c_j's are prescribed constants. We will only consider $\mathbf{B_j}$'s that are linear combinations of u and its derivatives through order $n-1$ and evaluated at the endpoints, a and b.

For $\mathbf{B_j}$ to be linear, it must satisfy the condition

$$\mathbf{B_j}(\alpha v + \beta w) = \alpha \mathbf{B_j}(v) + \beta \mathbf{B_j}(w). \tag{1.6}$$

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and its boundary conditions, begin by finding the adjoint operator, denoted $\mathbf{L}^*$. The adjoint operator consists of the formal adjoint, $\mathbf{L}^*$, and the boundary conditions associated with the Green's function. To determine these, first form the product, vLu, and integrate it over the interval of interest. By repeated integration by parts, we can express the integral in the form

$$\int_{\mathrm{a}}^{\mathrm{b}} v\mathbf{L}u dx = [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u\mathbf{L}^* v dx, \tag{2.1}$$

where $[\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}}$ represents the boundary terms resulting from successive integration by parts. Here, u and v must be sufficiently differentiable functions so that the left and right sides are well defined.

[1] A functional is a transformation with a set of functions as its domain and a set of numbers as its range. To illustrate, consider the functional

$$\mathscr{F}(u) = \int_0^1 u^2(x) dx. \tag{1.5}$$

The domain of this functional might be the set of functions defined over the interval $(0,1)$ and for which the integral of u^2 from 0 to 1 exists, and the range is $(0, \infty)$.

Example 2.1 Consider the linear differentiable operator

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x). \tag{2.2}$$

To find $\mathbf{L}^*$, perform integration by parts on each term of the product vLu until there is no derivative of u within the integral. That is to say, integrate by parts twice on the first term, once on the second, and not at all on the third. Doing this, we are left with

$$\begin{aligned}\int_{\mathrm{a}}^{\mathrm{b}} vLudx &= \int_{\mathrm{a}}^{\mathrm{b}} (vau'' + vbu' + vc)dx \\ &= (vau' + vbu)\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} (-(va)'u' - (vb)'u + vcu)dx \\ &= (vau' + vbu - (va)'u)\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} ((va)''u - (vb)'u + vcu)dx \\ &= (vau' + vbu - (va)'u)\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u((va)'' - (bv)' + cv)dx.\end{aligned} \tag{2.3}$$

From this, it is clear that

$$\begin{aligned}\mathbf{L}^*v &= (av)'' - (bv)' + cv \\ &= (a'v + av')' - b'v - bv' + cv \\ &= av'' + (2a' - b)v' + (a'' - b' + c)\end{aligned} \tag{2.4}$$

and so the formal adjoint of the second-order linear differential operator L must be of the form

$$\mathbf{L}^* = a\frac{d^2}{dx^2} + (2a' - b)\frac{d}{dx} + (a'' - b' + c). \tag{2.5}$$

If $\mathbf{L}^* = \mathbf{L}$, then $\mathbf{L}$ is called formally self-adjoint. By comparing equations (2.2) and (2.5), we can see that for a second-order linear differentiable operator to be formally self-adjoint, it is sufficient that $a' = b$. You may notice that $a'' - b' + c$ must also be equal to c, but this is always true given that a' equals b.

Definition If the boundary conditions on $\mathbf{L}$ are homogeneous[2], then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(Lu, v) = (u, \mathbf{L}^*v) \tag{2.6}$$

where (f, g) is the inner product of f and g,

$$(f, g) = \int_a^b f(x)g(x)dx. \tag{2.7}$$

This means that the adjoint operator $\mathscr{L}^*$ consists of $\mathbf{L}^*$ and boundary conditions for which the boundary terms of the integral are zero.

[2]By homogeneous, we mean that each boundary condition is of the form $\mathbf{B_j}(u) = 0$.

Example 2.2 Consider $\mathscr{L}$ to consist of $L = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \le x \le 1$. Then

$$\begin{aligned}
(Lu, v) &= \int_0^1 u'v dx \\
&= (uv)\Big|_0^1 - \int_0^1 uv' dx \\
&= u(1)v(1) - u(0)v(0) + \int_0^1 u\mathbf{L}^* v dx \\
&= u(1)(v(1) - 3v(0)) + \int_0^1 u\mathbf{L}^* v dx
\end{aligned} \tag{2.8}$$

Since the particular value of $u(1)$ is not given, we must impose the condition $v(1) - 3v(0) = 0$, because choosing $u(1) = 0$ would undully restrict our solution. Therefore $\mathscr{L}^*$ consists of $\mathbf{L}^*$ which is $-\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$.

As a final note, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called self-adjoint.

3 the Dirac delta function

3.1 Delta Sequences

In physics, we often consider the idea of a point mass. Suppose we have a unit point mass at $x = 0$ with mass density given by $w(x)$. We are interested in the mass but do not know the details of its density. We do, however, know that the $w(x)$ will be highly localized in space and that

$$\int_{-\infty}^{\infty} w(x) dx = 1 \tag{3.1}$$

so that the net mass is unity.

We expect two highly concentrated unit mass densities to produce masses with nearly identical physical effects. As such, we might simplify the problem by deciding, a priori, on a definite form for w, such as

$$w_k(x) = \begin{cases} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases} \tag{3.2}$$

or

$$w_k(x) = \frac{k}{\pi(1 + k^2x^2)}, \tag{3.3}$$

where k is some larger natural number. In Fig 3.2, we see that w becomes highly concentrated at $x = 0$ when k is large.

If we let $k \to \infty$, then the mass distribution approaches our idea of a point mass at $x = 0$. Calling this $\delta(x)$, then

$$\delta(x)\text{''} = \text{''} \lim_{k\to\infty} w_k(x). \tag{3.4}$$

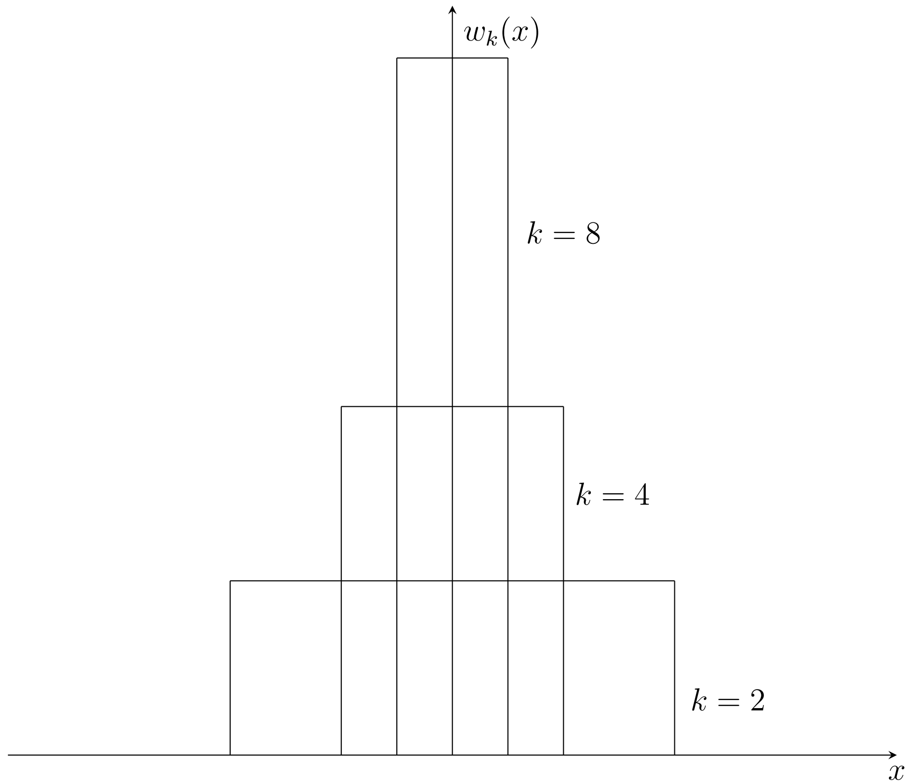

Figure 3.1: Mass Density; eq 3.2

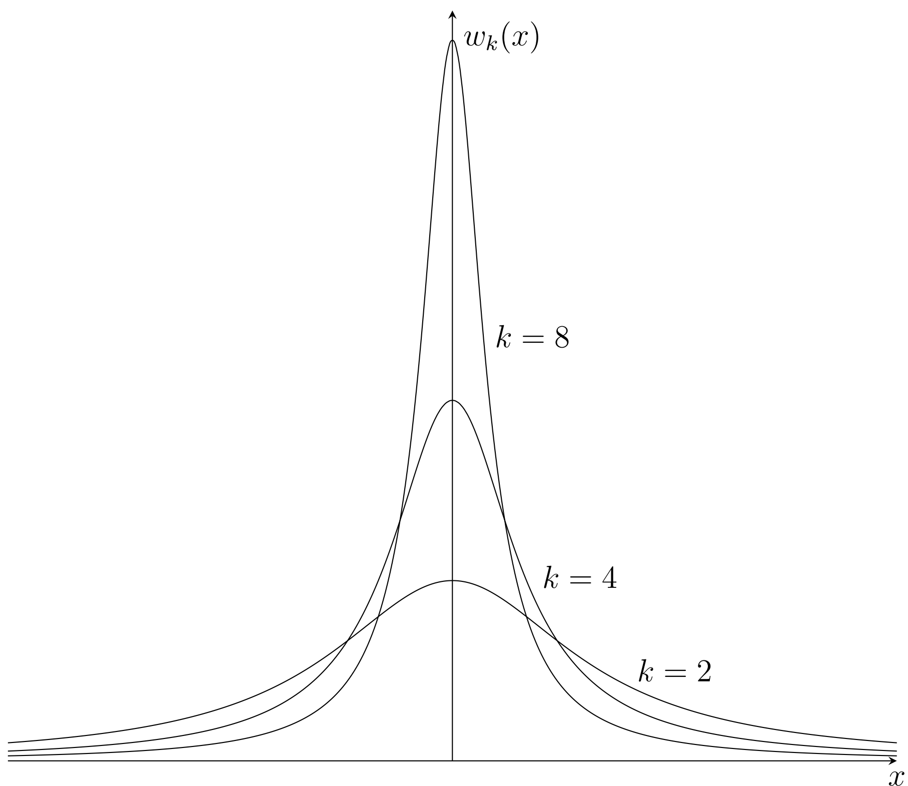

Figure 3.2: Mass Density; eq. 3.3

This definition feels intuitive, but it is not a rigorous definition of the Dirac delta function because the limit is infinite for $x = 0$. That is, it is not a function. We can instead define the Dirac delta function, $\delta(x)$, in the following way.

$$\int_{-\infty}^{\infty} h(x)\delta(x)dx = \lim_{k\to\infty} \int_{-\infty}^{\infty} h(x)w_k(x)dx \tag{3.5}$$

This way of defining the Dirac delta function is more rigorous while still being intuitive because it is related to our understanding of delta sequences. However, keep in mind that the delta function is not a function.

Theorem 3.1. *If $w(x)$ is non-negative $\int_{-\infty}^{\infty} w(x)dx = 1$, and $w(x) = O(1/x^{1+\alpha})$ as $|x| \to \infty$ with $\alpha > 0$, then $kw(kx) \equiv w_k(x)$ is a δ-sequence.*

Proof

First $\lim w_k(x = 0)$ as $k \to \infty$, for each fixed $x \neq 0$, because $w_k(x) = kw(kx) = O(k \cdot k^{-1-\alpha}x^{-1-\alpha}) = O(k^-\alpha) \to 0$ as $k \to \infty$.

Then

$$\begin{aligned}\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(x)dx &= \lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)[h(x) - h(0)]dx + \lim_{k\to\infty} w_k(x)h(0)dx \\ &= I + J, \text{ say.}\end{aligned} \tag{3.6}$$

Consider J.

$$\begin{aligned}J &= h(0) \lim_{h\to\infty} \int_{-\infty}^{\infty} kw(kx)dx \\ &= h(0) \lim_{k\to\infty} \int_{-\infty}^{\infty} w(\xi)d\xi \\ &= h(0)\end{aligned} \tag{3.7}$$

Next, to show that $I = 0$ so that the right hand side of equation (3.6) is $h(0)$. Select a number $\beta > 0$. Since h is assumed to be a continuous at $x = 0$, there must exist a number $\gamma > 0$ such that $|h(x) - H(0)| < \beta$ whenever $|x - 0| = x < \gamma$. Breaking up the integral in I,

$$\begin{aligned}I = &\lim_{k\to\infty} \int_{-\infty}^{-\gamma} w_k(x)(h(x) - h(0))dx + \lim_{k\to\infty} \int_{-\gamma}^{\gamma} w_k(x)(h(x) - h(0))dx \\ &+ \lim_{k\to\infty} \int_{\gamma}^{\infty} w_k(x)(h(x) - h(0))dx \\ &\equiv I_1 + I_2 + I_3.\end{aligned} \tag{3.8}$$

Since $w_k(x) \to 0$ uniformly, over $-\infty < x < -\gamma$ and $\gamma < x < \infty$, then $I_1 = I_3 = 0$, as $k \to \infty$. I_2 must also be zero because $|h(x) - H(0)|$ must be less than an arbitrarily small choice of β. Thus, showing that $I = 0$ and that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(x)dx = J = h(0). \tag{3.9}$$

∎

3.2 Delta as a Generalized Function

The Dirac delta function is more appropriately defined as a generalized function. To understand this way of defining δ, we will begin by defining some terms.

Definition A **closed interval** is one that includes its endpoints.

Definition A function, f, is **uniformly continuous** if

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall a \in X\ \forall b \in X : |a-b| < \delta \implies |f(a) - f(b)| < \epsilon. \tag{3.10}$$

Definition A function has **compact support** if the subset of its domain for which its range is non-zero is closed and bounded.

We will call the space of infinitely differentiable functions with compact support $\mathscr{D}$.

Definition Generalized functions are linear functionals that are uniformly continuous on $\mathscr{D}$, such that all generalized functions have derivatives which are also generalized functions.

Consider the following functional,

$$\mathscr{F}(h) = \int_{-\infty}^{\infty} g(x)h(x)dx. \tag{3.11}$$

This functional assigns a numerical value, $\mathscr{F}(h)$, for each function h within the domain, $\mathscr{D}$, of $\mathscr{F}$. We will take $\mathscr{D}$ to be the set of all functions that are defined over $-\infty < x < \infty$, are infinitely differentiable, and approach zero outside of some finite interval,

Example 3.1 Suppose $\mathscr{F}(h)$ is the integral of h from ξ to ∞.

$$\int_{\infty}^{\infty} g(x)h(x)dx = \int_{\xi}^{\infty} h(x)dx \tag{3.12}$$

Then, $g(x)$ must be the Heaviside step function,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ \frac{1}{2}, & x = 0 \end{cases} \tag{3.13}$$

which is a function in the classical sense. [3]

If $\mathscr{F}(h)$ is $h(0)$ so that

$$\int_{-\infty}^{\infty} g(x)h(x)dx = h(0) \tag{3.14}$$

[3] We have defined $H(0)$ to be $\frac{1}{2}$, which is the most commonly accepted value. However, for our purposes, the value at any particular point is not important since we are only ever interested in integrating the function.

then it can be shown that there is no function, $g(x)$, which exists such that equation (3.14) is true for all functions, $h(x)$, in the domain, $\mathscr{D}$. We call g defined by equation (3.14) a generalized function, and in particular, it is the Dirac delta function. As such, δ is defined in the following way.

$$\int_{-\infty}^{\infty} \delta(x)h(x)dx = h(0) \tag{3.15}$$

Although $\delta(x)$ acts at $x = 0$, it can be adjusted to act at any point by shifting the argument. Thus, $\delta(x - \xi)$ acts at $x = \xi$,

$$\int_{-infty}^{\infty} \delta(x - \xi)h(x)dx = h(\xi). \tag{3.16}$$

As a generalized function, δ is also differentiable. By referring to (3.5), one can see that defining the derivative of a generalized function involves determining the functional, $\mathscr{F}(h)$ for

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = \mathscr{F}(h). \tag{3.17}$$

Integrating by parts

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = g(x)h(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(x)h'(x)dx. \tag{3.18}$$

The integral term is fairly simple to interpret since it is of the same form as (3.5), but the boundary term is not as nice because it involves knowing the values of g. However, our restriction that h has compact support means that it must vanish at infinity, and since we are integrating from $-\infty$ to inf, the boundary term must be zero.

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = -\int_{-\infty}^{\infty} g(x)h'(x)dx \tag{3.19}$$

For the Dirac delta function, this means

$$\begin{aligned} \int_{-\infty}^{\infty} \delta'(x - \xi)h(x)dx &= -\int_{-\infty}^{\infty} \delta(x - \xi)h'(x)dx \\ &= -h'(\xi). \end{aligned} \tag{3.20}$$

Theorem 3.2. *The jth derivative of the Dirac delta function is defined by*

$$\int_{-\infty}^{\infty} \delta^{(j)}(\xi - x)h(\xi)d\xi = (-1)^j h^{(j)}(x). \tag{3.21}$$

Proof ____

Base case: Proven to be true for $j = 1$ in equation (3.20).
Induction step: Let $k \in \mathbb{N}$ and suppose the statements holds for $k > 1$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \delta^{(k+1)}(\xi - x)h(\xi)d\xi &= \delta^{(k)}(\xi - x)h(\xi)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta^{(k)}(\xi - x)h'(\xi)d\xi \\ &= -\int_{-\infty}^{\infty} \delta^{(k)}(\xi - x)h'(\xi)d\xi \\ &= -(-1)^k (h')^{(k)}(x) \\ &= (-1)^{k+1} h^{(k+1)}(x) \end{aligned}$$

∎

Note that because of the discontinuity in $H(x-\xi)$ at the point $x=\xi$, the derivative of H does not exist as an ordinary function. However, the previous method does allow us to find $H'(x-\xi)$ as a generalized function,

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} H(x-\xi)h'(x)dx \\ &= -\int_{\xi}^{\infty} h'(x)dx = h(\xi).\end{aligned} \tag{3.22}$$

Since

$$\int_{-\infty}^{\infty} \delta(x-\xi)h(x)dx = h(\xi) \tag{3.23}$$

it must be the case that, in the sense of generalized functions,

$$H'(x-\xi) = \delta(x-\xi). \tag{3.24}$$

Such equalities between generalized functions, as seen in (3.18), are understood in the sense that if some h in $\mathscr{D}$ is multiplied through, and we integrate over $(-\infty,\infty)$ then the result will hold. To wit, we consider generalized functions, g_1 and g_2, to be equal if, for all $h \in \mathscr{D}$,

$$\int_{-\infty}^{\infty} g_1(x)h(x)dx = \int_{-\infty}^{\infty} g_2(x)h(x)dx. \tag{3.25}$$

Example 3.2 Consider the sequence

$$w_k(x) = \begin{cases} -k^2, 0 \le x < 1/k \\ k^2, -1/k < x < 0 \\ 0, |x| \ge 1/k \end{cases} \tag{3.26}$$

We want to verify that w_k is a δ' sequence. If this is true, then

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} h(x)w_k(x)dx = \int_{-\infty}^{\infty} h(x)\delta'(x)dx = -h'(0). \tag{3.27}$$

Separating the integral for each case of w_k,

$$\lim_{k\to\infty}\left(k^2\int_{-1/k}^{0} h(x)dx\right) + \lim_{k\to\infty}\left(-k^2\int_{0}^{1/k} h(x)dx\right). \tag{3.28}$$

Selecting a and b such that, $-1/k \le \mathrm{a} \le 0$ and $0 < \mathrm{b} < 1/k$. It follows from the mean value theorem of integral calculus that

$$\lim_{k\to\infty}\left(k^2\cdot h(\mathrm{a})\cdot\left(\frac{1}{k}\right)\right) + \lim_{k\to\infty}\left(-k^2\cdot h(\mathrm{b})\cdot\left(\frac{1}{k}\right)\right) = 0. \tag{3.29}$$

since a and b $\to 0$ as $k \to \infty$.

As a final aside, notice that

$$x\delta(x) = 0 \tag{3.30}$$

as a result of

$$\int_{-\infty}^{\infty} x\delta(x)h(x)dx = [xh(x)]|_{x=0} = 0. \tag{3.31}$$

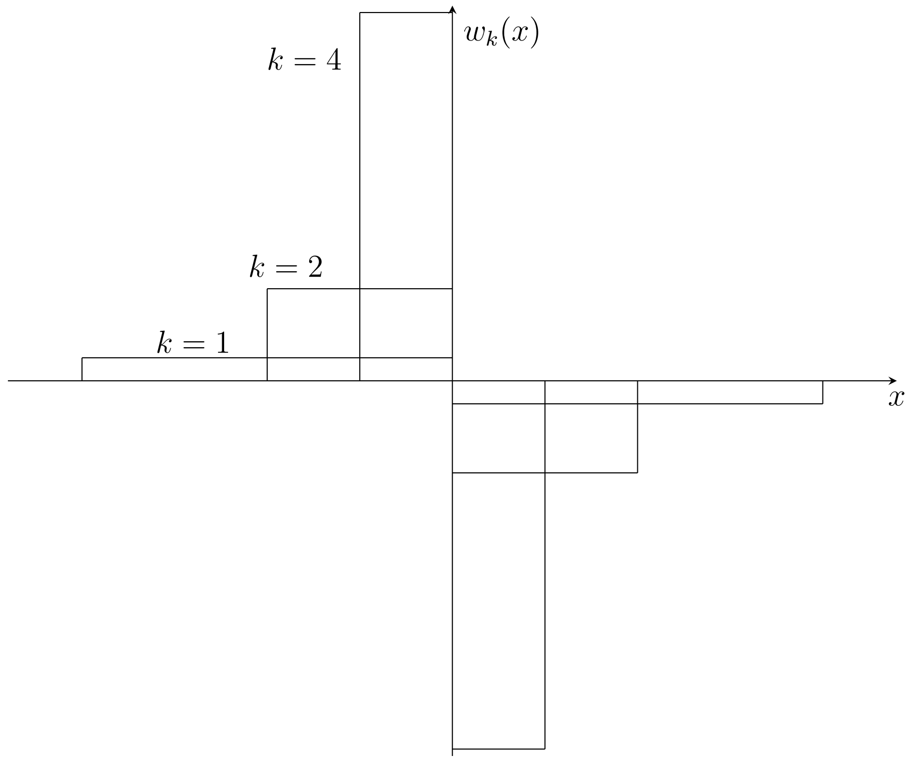

Figure 3.3: Delta sequence, eq 3.26

4 The Method of Green's Functions

A **Green's function** is the solution to a differential equation of the form

$$\mathbf{L}^*_\xi G(\xi, x) = \delta(\xi - x). \tag{4.1}$$

By making use of some of the delta function's unusual properties, Green's functions can be used to solve nonhomogeneous linear differential equations.

To find the solution to the linear differential equation

$$\mathbf{L}u = \phi, \tag{4.2}$$

we start by finding the formal adjoint as in equation (2.1). If we replace v with G, and we replace x with a dummy variable ξ, we are left with an equation of the form

$$\int_{\mathrm{a}}^{\mathrm{b}} G(\xi, x)\mathbf{L}u(\xi)d\xi = [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u(\xi)\mathbf{L}^* G(\xi, x)d\xi. \tag{4.3}$$

It follows from equation (4.1) that the we can replace $\mathbf{L}^*G$ with δ, and from equation (4.2) that we can replace $\mathbf{L}u$ with ϕ.

$$\begin{aligned} \int_{\mathrm{a}}^{\mathrm{b}} G(\xi, x)\phi(\xi)d\xi &= [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u(\xi)\delta(\xi - x)d\xi \\ &= [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + u(x). \end{aligned} \tag{4.4}$$

Therefore, if we choose boundary conditions for G such that the boundary terms do not depend on u and we are able to find G, then finding u is reduced to a problem of integrating $G\phi$.

To illustrate the key ideas of the method, we will consider several examples which begin simply and become more complex. Each example will be concerned with a key concept in implementing the method of Green's functions.

Example 4.1 *Loaded String*

Consider the boundary value problem

$$u''(x) = \phi(x); \quad u(0) = u(1) = 0 \tag{4.5}$$

where $\phi(x)$ is prescribed. **EXPLAIN LOADED STRING WITH DIAGRAM**.

To find a solution to this differential equation, we first find the formal adjoint $\mathbf{L}*$ as in equation (4.3).

$$\begin{aligned}\int_0^1 G(\xi,x)\mathbf{L}u(\xi)d\xi &= (G(\xi,x)u'(\xi) - G_\xi(\xi,x)u(\xi))\Big|_0^1 + \int_0^1 uG_{\xi\xi}d\xi \\ &= G(1,x)u'(1) - G_\xi(1,x)u(1) \\ &\quad - G(0,x)u'(0) + G_\xi(0,x)u(0) + \int_0^1 uG_{\xi\xi}d\xi.\end{aligned} \tag{4.6}$$

Therefore,

$$\mathbf{L}^* = \frac{d^2}{d\xi^2} \tag{4.7}$$

and

$$\mathbf{L}^*G = G_{\xi\xi} = \delta(\xi - x). \tag{4.8}$$

Because of the boundary conditions on u in equation (4.5), two of our boundary terms are zero, making

$$\int_0^1 G(\xi,x)\mathbf{L}u(\xi)d\xi = G(1,x)u'(1) - G(0,x)u'(0) + \int_0^1 uG_{\xi\xi}d\xi. \tag{4.9}$$

Now, we want to remove the u dependency from the boundary terms and, as such, decide that

$$G(1,x) = G(0,x) = 0. \tag{4.10}$$