

Green's Functions

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1 Introduction

The first part of this text is primarily concerned with solutions to differential equations of the form

$$Lu = \phi \quad (1.1)$$

over an interval $a \leq x \leq b$ and subject to boundary conditions $\{B_1, \dots, B_n\}$, where L is an n th order linear ordinary differential operator. For L to be linear it must satisfy the condition

$$L(\alpha v + \beta w) = \alpha Lv + \beta Lw \quad (1.2)$$

for arbitrary functions v and w , with α and β being constant. For this condition to be met L must be of the form

$$L = a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_n(x) \quad (1.3)$$

The boundary conditions are linear functionals of the form

$$B_j(u) = c_j; \quad j = 1, 2, \dots, n \quad (1.4)$$

where c_j is an arbitrary constant.

Here, functional, refers to a transformation which has a set of functions as its domain and a set of numbers. As an example

$$B(u) = u(0) = 0 \quad (1.5)$$

is a simple boundary condition for a 1st order differential operator. Specifically, our B_j 's will be limited to linear combinations of u and its derivatives up to order $n-1$. These boundary conditions have the same linearity constraints as the differential operator L .

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and its boundary conditions we will need the formal adjoint operator. This operator, which we will call L^* , can be found via repeated integration by parts. In general

$$\int_a^b v L u dx = [\dots] \Big|_a^b + \int_a^b u L^* v dx \quad (2.1)$$

Here, u and v are completely arbitrary while being sufficiently differentiable for L and L^* to exist.

As an example, consider

$$L = A(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + C(x) \quad (2.2)$$

To find L^* perform integration by parts on the on each term of the product vLu a number of times equal to the order of the derivative that is a part of the term. That is to say, twice on the first

term, once, on the second, and not at all on the third. Doing this, we are left with

$$\begin{aligned}
\int_a^b vLudx &= \int_a^b (vAu'' + vBu' + vC)dx \\
&= (vAu' + vBu) \Big|_a^b + \int_a^b (-(vA)'u' - (vB)'u + vCu)dx \\
&= (vAu' + vBu - (vA)'u) \Big|_a^b + \int_a^b ((vA)''u - (vB)'u + vCu)dx \\
&= (vAu' + vBu - (vA)'u) \Big|_a^b + \int_a^b u((vA)'' - (Bv)' + Cv)dx
\end{aligned} \tag{2.3}$$

From this it is clear that

$$\begin{aligned}
L^*v &= (Av)'' - (Bv)' + Cv \\
&= (A'v + Av')' - B'v - Bv' + Cv \\
&= Av'' + (2A' - B)v' + (A'' - B' + C)v
\end{aligned} \tag{2.4}$$

and so the formal adjoint of a second order linear differential operator L must be of the form

$$L^* = A \frac{d^2}{dx^2} + (2A' - B) \frac{d}{dx} + (A'' - B' + C) \tag{2.5}$$

If L^* is found to be equal to L then L is called formally self adjoint. By comparing equations (2.2) and (2.5) we can see that for a second order linear differentiable operator to be formally self adjoint, A' must be equal to B . You may notice that $A'' - B' + C$ must also be equal to C but this is always true given that A' equals B .

If the boundary conditions on L are homogenous then we can also define an adjoint operator, \mathcal{L}^* , by the relation

$$(Lu, v) = (L^*v, u) \tag{2.6}$$

where (f, g) is the inner produce of f and g

$$(f, g) = \int_a^b f(x)g(x)dx \tag{2.7}$$

We now can understand that the adjoint operator \mathcal{L}^* must consist of L^* and boundary conditions to force the terms that come about from integrating by parts to be zero.

As an example of \mathcal{L} and \mathcal{L}^* , consider \mathcal{L} to consist of $L = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \leq x \leq 1$. Then

$$\begin{aligned}
(Lu, v) &= \int_0^1 u'vdx \\
&= (uv) \Big|_0^1 - \int_0^1 uv'dx \\
&= u(1)v(1) - u(0)v(0) + \int_0^1 uL^*vdx \\
&= u(1)(v(1) - 3v(0)) + \int_0^1 uL^*vdx
\end{aligned} \tag{2.8}$$

Since the particular value of $u(1)$ is not given, we must make $v(1) - 3v(0)$ equal zero, because choosing $u(1) = 0$ would unduly restrict our solution. Therefore \mathcal{L}^* consists of L^* which is $-\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$. As a final note, if $\mathcal{L} = \mathcal{L}^*$, then \mathcal{L} is called self-adjoint.

3 The Delta Function