Ordinary Differential Equations

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Chapter 1

Introduction To Differential Equations

1.1 Definitions and Terminology

Ordinary differential equations, ODEs, have only one independent variable but may have one or more dependant variables. The equations

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \text{ and } \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

are all examples of ordinary differential equations.

Partial differential equations have more than one independent variable and therefore involve partial derivatives. The equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ and } \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

are examples of partial differential equations. This text will only discuss ordinary DEs.

The order of a differential equation is the order of the highest order derivative in the DE. For example, the equation

$$5y'' + y' - 10xy = 0$$

is a second order differential equation.

Linear ODEs are equations if the dependant variable(s) are linear. In other words the coefficients must only contain the dependant variable and/or constants and no functions such as sine or logarithms are applied to the dependant variable(s). Linear ODEs have the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y - g(x) = 0$$

The equations

$$y'' - 3y = 0, \quad x^3\tfrac{d^3y}{dx^3} + x\tfrac{dy}{dx} - 5y = e^x, \text{ and } (y-x)dx + 4xdy = 0$$

are examples of linear ODEs and the equations

$$(1-y)y' + 2y = e^x,\ \tfrac{d^2y}{dx^2} + \sin y = 0, \text{ and } \tfrac{d^4y}{dx^4} + y^2 = 0$$

are nonlinear ODEs.

The solution of a differential equation is a function ϕ such that it reduces the ODE to an identity when substituted into the equation. In other words ϕ has at least n derivatives for which

$$F(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)) = 0.$$

The solution will often be denoted by y or $y(x)$.

A relation $G(x, y) = 0$ is said to be an implicit solution of an ODE provided there is at least one function ϕ that satisfies the relation as well as the differential equation.

If a differential equation has n solutions, y_1, $y_2, ..., y_n$, then we can say a general solution is $y = c_1y_1 + c_2y_2 + ... + c_ny_n$ where c_1 to c_n are constant coefficients. A solution to a differential equation that is free of unknown coefficients is called a particular solution.

1.2 Initial-Value Problems

Sometimes we want to find a solution $y(x)$ that satisfies certain conditions, these are called initial-value problems. The conditions for an initial value problem are called initial conditions. For an initial value problem involving an ODE of order n there are n initial conditions

$$y(x_0) = y_0,\ y'(x_0) = y_0, ...,\ y^{(n-1)}(x_0) = y_{n-1}.$$

To find the particular solution(s) to an IVP we plug the given conditions into the general solution and it's applicable derivatives and then solve the system of equations to find the constant coefficients.

Chapter 2

First Order Differential Equations

2.1 Separable Equations

The simplest of all differential equations are first order differential equations with separable variables. These are equations of the form

$$\tfrac{dy}{dx} = g(x)h(y).$$

For example the equation

$$(1+x)\tfrac{dy}{dx} - y = 0$$

is separable. We can rewrite this equation as

$$(1+x)dy - ydx = 0$$

This looks like we have multiplied by dx but that is not what happened since $\frac{dy}{dx}$ is not actually a faction. If you want to know how we got there you can ask me but it doesn't really matter because you can just think of it as multiplying both sides by dx and just know that's not actually what's happening. My professor didn't explain it to us and I didn't know what was actually going on until I read about it in the textbook.

By dividing by $(1+x)y$ we can rewrite this equation as

$$\tfrac{dy}{y} = \tfrac{dx}{1+x}.$$

Now that we have the equation in that form we can integrate both sides

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

and we find the implicit solution is.

$$\ln|y| = \ln|1+x| + c$$

The explicit solution is

$$|y| = |1+x|e^c$$
$$y = \pm e^c(1+x)$$

Relabeling $\pm e^c$ as c gives us

$$y = c(1+x)$$

2.2 Integrating Factors

A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be linear in the variable y.

By dividing both sides by $a_1(x)$ we obtain a more useful form of the equation, referred to as standard form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

In some instances these equations can be solved by separation of variables such as these:

$$\frac{dy}{dx} + 2xy = 0 \quad \text{and} \quad \frac{dy}{dx} = y + 5$$

The linear equation

$$\frac{dy}{dx} + y = x$$

however is not separable.

The method for solving these hinges on the fact that the left hand side can be converted into the form of the derivative of the product of two functions of x. This is done by multiplying both sides by a special function $\mu(x)$. The form we want the left sie to be in looks like

$$\tfrac{d}{dx}[\mu(x)y] = \mu\tfrac{dy}{dx} + \tfrac{d\mu}{dx}y = \mu\tfrac{dy}{dx} + \mu P(x)y$$

Based on this equation we know that $\frac{d\mu}{dx}y = \mu P(x)y$. Which is true provided that

$$\tfrac{d\mu}{dx} = \mu P(x)$$

which can be solved for μ by separation of variables.

$$\begin{aligned}\int \frac{d\mu}{\mu} &= \int P(x)dx \\ \ln|\mu(x)| &= \int P(x)dx \\ \mu(x) &= e^{\int P(x)dx}\end{aligned}$$

Here the function $\mu(x)$ is what we call an integrating factor.

Multiplying the standard form of a first order linear DE by $\mu(x)$ gives us

$$\begin{aligned}e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y &= e^{\int P(x)dx}f(x) \\ \frac{d}{dx}\left[e^{\int P(x)dx}y\right] &= e^{\int P(x)dx}f(x)\end{aligned}$$

As an example we will solve the equation $\frac{dy}{dx} - 3y = 0$

$$\begin{aligned}P(x) &= -3 \\ \mu &= e^{\int -3dx} \\ \mu &= e^{-3x} \\ e^{-3x}\frac{dy}{dx} - 3e^{-3x}y &= 0 \\ \frac{d}{dx}\left[e^{-3x}y\right] &= 0 \\ e^{-3x}y &= c \\ y &= ce^{3x}\end{aligned}$$

Exercises:

(1) $x\frac{dy}{dx} - 4y = x^6e^x$

(2) $(x^2 - 9)\frac{dy}{dx} + xy = 0$

2.3 Exact Equations