Problem Set 6

Ryan Coyne

October 16, 2023

1. Let $x, y, z \in \mathbb{Z}$. Prove: If exactly two of x, y, z are even, then $3x+5y+7z$ is odd.
   Case 1: Let x, y be even and let z be odd. Then, $x = 2k$, $y = 2l$, and $z = 2m + 1$, for some $k, l, m \in \mathbb{Z}$. Now,

$$\begin{aligned} 3x + 5y + 7z &= 6k + 10l + 14m + 7 \\ &= 6k + 10l + 14m + 6 + 1 \\ &= 2(3k + 5l + 7m + 3) + 1 \end{aligned}$$

   which is odd, by definition.
   Case 2: Let x, z be even and let y be odd. Then, $x = 2k$, $y = 2l + 1$, and $z = 2m$, for some $k, l, m \in \mathbb{Z}$. Now,

$$\begin{aligned} 3x + 5y + 7z &= 6k + 10l + 5 + 14m \\ &= 6k + 10l + 14m + 4 + 1 \\ &= 2(3k + 5l + 7m + 2) + 1 \end{aligned}$$

   which is odd, by definition.
   Case 3: Let y, z be odd and let x be odd. Then, $x = 2k + 1$, $y = 2l$, and $z = 2m$, for some $k, l, m \in \mathbb{Z}$. Now,

$$\begin{aligned} 3x + 5y + 7z &= 6k + 3 + 10l + 14m \\ &= 6k + 10l + 14m + 2 + 1 \\ &= 2(3k + 5l + 7m + 1) + 1 \end{aligned}$$

   which is odd, by definition.

2. Let $a, b \in \mathbb{Z}$. Prove: If $ab = 4$, then $(a-b)^3 - 9(a-b) = 0$.
   Case 1: Let $a = 1$ and $b = 4$. Then,
$$\begin{aligned}(a-b)^3 - 9(a-b) &= (1-4)^3 - 9(1-4)\\ &= -3^3 - 9 \cdot -3\\ &= -27 + 27\\ &= 0.\end{aligned}$$
   Case 2: Let $a = 1$ and $b = 4$. Then,
$$\begin{aligned}(a-b)^3 - 9(a-b) &= (-1+4)^3 - 9(-1+4)\\ &= 3^3 - 9 \cdot 3\\ &= 27 - 27\\ &= 0.\end{aligned}$$
   Case 3: Let $a = 2$ and $b = 2$. Then,
$$\begin{aligned}(a-b)^3 - 9(a-b) &= (2-2)^3 - 9(2-2)\\ &= 0^3 - 9 \cdot 0\\ &= 0.\end{aligned}$$
   Case 4: Let $a = -2$ and $b = -2$. Then,
$$\begin{aligned}(a-b)^3 - 9(a-b) &= (-2+2)^3 - 9(-2+2)\\ &= 0^3 - 9 \cdot 0\\ &= 0.\end{aligned}$$
   Therefore, by exhaustion, $(a-b)^3 - 9(a-b) = 0$.

3. Let $a \in \mathbb{Z}$. Prove: If $3 \mid 2a$, then $3 \mid a$.
   By Result 4.8 from the textbook, if $3 \mid cd$, then $3 \mid c$ or $3 \mid d$, for some $c, d \in \mathbb{Z}$. Since $3 \mid 2a$ and $3 \nmid 2$, then it must be the case that $3 \mid a$.

4. Let $x, y \in \mathbb{Z}$. Prove: If 3 divides neither x or y, then $3 \mid (x^2 - y^2)$.
   Since $(x^2 - y^2)$ can be factored into $(x+y)(x-y)$, $3 \mid (x^2 - y^2)$ exactly when $3 \mid (x+y)$ or $3 \mid (x-y)$. Proceeding by cases according to the remainder of 3 divided by x and the remainder of 3 divided by y.
   (i) Let $x = 3k + 1$, and $y = 3l + 1$ for some $k, l \in \mathbb{Z}$. Then,
$$\begin{aligned}x - y &= 3k + 1 - 3l - 1\\ &= 3k - 3l\\ &= 3(k - l),\end{aligned}$$

which is divisible by 3.
(ii) Let $x = 3k + 2$, and $y = 3l + 2$ for some $k, l \in \mathbb{Z}$. Then,