Problem Set 7

Ryan Coyne

October 27, 2023

1. Let $a, b, c, d \in \mathbb{R}$. Prove: $ac + bd \leq \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}$.
   Proof: Consider the square of each side of the relation,

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$$
$$a^2c^2 + 2abcd + b^2d^2 \leq a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2$$
$$2abcd \leq b^2c^2 + a^2d^2.$$

   Now, we rearrange the relation to relate it to a known quantity. In this case, 0,

$$b^2c^2 - 2abcd + a^2d^2 \geq 0$$
$$.$$

   The left side can be easily factored, leading to

$$(bc - ad)^2 \geq 0,$$

   which is trivially true in the real numbers. ■

2. Let $x, y, z \in \mathbb{R}$. Prove: $|x - z| \leq |x - y| + |y - z|$.
   Proof: Note that the sum of arguments on the right is equal to the argument on the left. That is to say,

$$(x - y) + (y - z) = x - z.$$

   Therefore, by the triangle inequality, the statement must be true. ■

3. Prove: For every two positive real numbers, a and b.

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

   First, make a common denominator and add the fractions,

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2}{ab} + \frac{b^2}{ab}$$
$$= \frac{a^2 + b^2}{ab}.$$

Next, multiply both sides by ab and rearrange,

$$\begin{aligned}\frac{a^2+b^2}{ab} &\geq 2 \\ a^2+b^2 &\geq 2ab \\ a^2-2ab+b^2 &\geq 0.\end{aligned}$$

Now, factor the right side,

$$(a-b)^2 \geq 0.$$

This relation is true because the square of any real number is at least 0. Thus, the initial statement is true. ■
To find the solution set, we begin from $(a-b)^2 = 0$, we take the square root, finding $x - y = 0$, and therefore, $y = x$. However, we must not divide by zero, so the complete solution set is $y = x$ where $x \neq 0$.

4. Let A and B be sets. Prove: $A \cup B = A \cap B$ if and only if $A = B$.
   ($\Longrightarrow$) Let $a \in A$. Then, $a \in A \cup B$, and by hypothesis, $a \in A \cap B$. Now, $a \in B$, for all $a \in A$. Thus, $A \subseteq B$.
   Let $b \in B$. Then, $b \in A \cup B$, and by hypothesis, $b \in A \cap B$. Now, $b \in A$, for all $b \in B$. Thus, $B \subseteq A$.
   Now, since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$ by definition.
   ($\Longleftarrow$) Since, $A = B$, it follows that

$$\begin{aligned}A \cup B &= A \cup A \\ &= A\end{aligned}$$

   and that

$$\begin{aligned}A \cap B &= A \cap A \\ &= A.\end{aligned}$$

   Therefore, $A \cup B = A \cap B$. ■

5. Let A, B, C be sets. Prove: $A \cap \overline{(B \cap C)} = \overline{(\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})}$.
   Proof: We begin by showing that $A \cap \overline{(B \cap C)} \subseteq \overline{(\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})}$. For all $x \in A \cap \overline{(B \cap C)}$, $x \in A$. Thus,$x \notin \overline{A}$, and so $x \notin \overline{A} \cap \overline{C}$. Now, $x \notin (\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})$, and so $x \in \overline{(\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})}$. Therfore, $A \cap \overline{(B \cap C)} \subseteq \overline{(\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})}$.
   Next we show that $\overline{(\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})} \subseteq A \cap \overline{(B \cap C)}$. Let $y \in \overline{(\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})}$, so $y \notin (\overline{A} \cup B) \cap (\overline{A} \cap \overline{C})$.

6. For sets A and B, find a necessary and sufficient condition for

$$(A \times B) \cap (B \times A) = \emptyset.$$

   $A \times B = \{(a,b) | a \in A \text{ and } b \in B\}$
   $B \times A = \{(b,a) | b \in B \text{ and } a \in A\}$
   If $(A \times B) \cap (B \times A) = \emptyset$, then $(a,b) = (b,a)$ for some $a \in A$ and $b \in B$.
   Now, $(A \times B) \cap (B \times A) = \emptyset$ if and only if $a \neq b$ for all $a \in A$ and $b \in B$.

7. Let A, B, C, and D be sets. Prove: $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
Proof: First we showt that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$. Let $(x, y) \in (A \times B) \cap (C \times D)$. Then,$(x, y) \in A \times B$, and $(x, y) \in C \times D$. Thus, by definition, $x \in A,\ y \in B,\ x \in C$, and $y \in D$. Since x is in both A and C, $x \in A \cap C$, and since y is in both B and D, $y \in B \cap D$. Therefore, $(x, y) \in (A \cap C) \times (B \cap D)$. By following the steps in the reverse order, we can see that $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$. Therefore, the statement holds for all A, B, C, and D. ■