Green's Functions

Ryan Coyne

February 27, 2023

Contents

1 Introduction

The first part of this text is primarily concerned with solutions to differential equations of the form

$$Lu = \phi \tag{1.1}$$

over an interval $a \leq x \leq b$ and subject to boundary conditions $\{B_1, \ldots, B_n\}$, where L is an nth order linear ordinary differential operator. For L to be linear it must satisfy the condition

$$L(\alpha v + \beta w) = \alpha Lv + \beta Lw \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant. For this condition to be met L must be of the form

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x) \tag{1.3}$$

The boundary conditions are linear functionals of the form

$$B_j(u) = c_j; \quad j = 1, 2, \ldots, n \tag{1.4}$$

where c_j is an arbitrary constant.

Here, functional refers to a transformation that has a set of functions as its domain and a set of numbers. As an example

$$B(u) = u(0) = 0 \tag{1.5}$$

is a simple boundary condition for a 1st order differential operator. Specifically, our B_j's will be limited to linear combinations of u and its derivatives up to order $n-1$. These boundary conditions have the same linearity constraints as the differential operator L.

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and its boundary conditions we will need the formal adjoint operator. This operator, which we will call L^*, can be found via repeated integration by parts. In general

$$\int_a^b vLudx = [\cdots]\Big|_a^b + \int_a^b uL^*vdx \tag{2.1}$$

Here, u and v are completely arbitrary while being sufficiently differentiable for L and $L*$ to exist.

As an example, consider

$$L = A(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + C(x) \tag{2.2}$$

To find L^* perform integration by parts on each term of the product vLu a number of times equal to the order of the derivative that is a part of the term. That is to say, twice on the first term, once, on the second, and not at all on the third. Doing this, we are left with

$$
\begin{aligned}
\int_a^b vLudx &= \int_a^b (vAu'' + vBu' + vC)dx \\
&= (vAu' + vBu)\Big|_a^b + \int_a^b (-(vA)'u' - (vB)'u + vCu)dx \\
&= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b ((vA)''u - (vB)'u + vCu)dx \\
&= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b u((vA)'' - (Bv)' + Cv)dx
\end{aligned}
\tag{2.3}
$$

From this it is clear that

$$
\begin{aligned}
L^*v &= (Av)'' - (Bv)' + Cv \\
&= (A'v + Av')' - B'v - Bv' + Cv \\
&= Av'' + (2A' - B)v' + (A'' - B' + C)
\end{aligned}
\tag{2.4}
$$

and so the formal adjoint of a second order linear differential operator L must be of the form

$$L^* = A\frac{d^2}{dx^2} + (2A' - B)\frac{d}{dx} + (A'' - B' + C) \tag{2.5}$$

If L^* is found to be equal to L then L is called formally self-adjoint. By comparing equations (2.2) and (2.5) we can see that for a second order linear differentiable operator to be formally self-adjoint, A' must be equal to B. You may notice that $A'' - B' + C$ must also be equal to C, but this is always true given that A' equals B.

If the boundary conditions on L are homogenous then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(Lu, v) = (L^*v, u) \tag{2.6}$$

where (f, g) is the inner produce of f and g

$$(f, g) = \int_a^b f(x)g(x)dx \tag{2.7}$$

We now can understand that the adjoint operator $\mathscr{L}^*$ must consist of L^* and boundary conditions to force the terms that come about from integrating by parts to be zero.

As an example of $\mathscr{L}$ and $\mathscr{L}^*$, consider $\mathscr{L}$ to consist of $L = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \le x \le 1$. Then

$$
\begin{aligned}
(Lu, v) &= \int_0^1 u'vdx \\
&= (uv)\Big|_0^1 - \int_0^1 uv'dx \\
&= u(1)v(1) - u(0)v(0) + \int_0^1 uL^*vdx \\
&= u(1)(v(1) - 3v(0)) + \int_0^1 uL^*vdx
\end{aligned}
\tag{2.8}
$$

Since the particular value of $u(1)$ is not given, we must make $v(1) - 3v(0)$ equal zero, because choosing $u(1) = 0$ would unduly restrict our solution. Therefore $\mathscr{L}^*$ consists of L^* which is $-\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$. As a final note, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called self-adjoint.

3 The Delta Function

In physics and engineering, there exists a notion of "point actions". These are actions that are highly localized in space and/or time. As an example, suppose a circular coin is pressed with unit force against the edge of a metal plate that extends over, $y > 0$ and $-\infty < x < \infty$, as shown in Figure 3.1. We are interested in the resulting stress field but do not know the details of the force distribution, say $w(x)$. We do, however, know it will be very concentrated in space and that

$$\int_{-\infty}^{\infty} w(x)dx = 1 \tag{3.1}$$

so that the net force is unity.

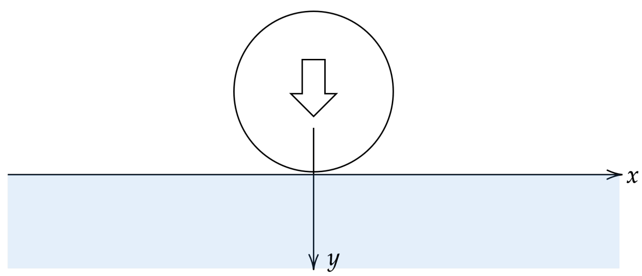

Figure 3.1: Coin pressed to the edge of a plate.

We expect that two highly concentrated force distributions would produce nearly identical stress fields except in the immediate neighborhood of the point at which the force is applied. This is provided they are statically equivalent, meaning that their resultant forces and couples are identical. As such we might simplify the problem by deciding, a priori, on a definite form for w, such as

$$w_k(x) = \begin{cases} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases} \tag{3.2}$$

or

$$w_k(x) = \frac{k}{\pi(1 + k^2x^2)} \tag{3.3}$$

where $k > 0$. In Fig 3.2 we can see that w becomes highly concentrated when k is large.

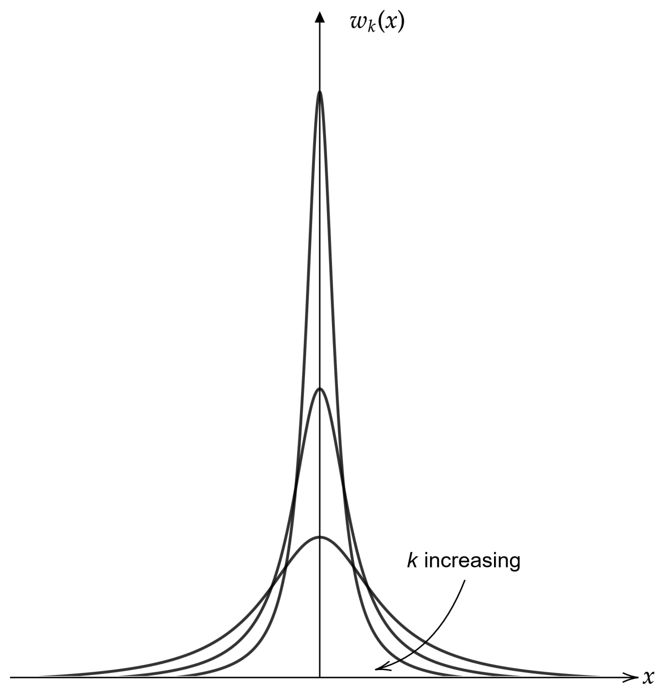

Figure 3.2: Distributed Force; eq. 3.3

If we let $k \to \infty$ then the force distribution approaches our idea of a "point-action" which in this case is a force of unit strength, acting at $x = 0$. Calling this "point-action" $\delta(x)$, then

$$\delta(x) = \lim_{k\to\infty} w_k(x) \tag{3.4}$$

This, however, cannot be considered a rigorous definition of the delta function because the limit is infinite for $x = 0$.

The delta function is more appropriately defined as a generalized function. To understand this way of defining δ consider the following functional.

$$\int_{-\infty}^{\infty} g(x)h(x) = \mathscr{F}(h) \tag{3.5}$$

This functional assigns a numerical value, $\mathscr{F}(h)$, for each function h within the domain, $\mathscr{D}$, of $\mathscr{F}$. We will take $\mathscr{D}$ to be the set of all functions that are defined over $-\infty < x < \infty$, are infinitely differentiable, and approach zero outside of some finite interval,

Suppose $\mathscr{F}(h)$ is the integral of h from ξ to ∞.

$$\int_{\infty}^{\infty} g(x)h(x)dx = \int_{\xi}^{\infty} h(x)dx \tag{3.6}$$

Then, $g(x)$ must be the Heaviside step function,

$$H(x-\xi) = \begin{cases} 1, & x > \xi \\ 0, & x < \xi \end{cases} \tag{3.7}$$

which is a function in the classical sense.

If $\mathscr{F}(h)$ is $h(0)$ so that

$$\int_{-\infty}^{\infty} g(x)h(x)dx = h(0) \tag{3.8}$$

then it can be shown that there is no function, $g(x)$, which exists such that (3.8) is true for all functions, $h(x)$, in the domain, $\mathscr{D}$. It is then the case that g must be a generalized function, which we call the delta function. As such, δ can be defined in the following way.

$$\int_{-\infty}^{\infty} \delta(x)h(x)dx = h(0) \tag{3.9}$$

Although $\delta(x)$ acts at $x = 0$, it can be adjusted to act at any point by shifting the argument. Thus, $\delta(x - \xi)$ acts at $x = \xi$,

$$\int_{-infty}^{\infty} \delta(x-\xi)h(x)dx = h(\xi) \tag{3.10}$$

As a generalized function, δ is also differentiable. By referring to (3.5), one can see that defining the derivative of a generalized function involves determining the functional, $\mathscr{F}(h)$ for

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = \mathscr{F}(h) \tag{3.11}$$

Next, we integrate by parts

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = g(x)h(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(x)h'(x)dx \tag{3.12}$$

The integral term is fairly simple to interpret since it is of the same form as (3.5), but the boundary term is not as nice because it involves knowing the values of g. To deal with this we will simply discard the boundary term, and define g' with the formula

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = -\int_{-\infty}^{\infty} g(x)h'(x)dx \tag{3.13}$$

For the delta function, this means

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} \delta(x-\xi)h'(x)dx \\ &= -h'(\xi)\end{aligned} \tag{3.14}$$

By repeating the integration by parts for subsequent derivatives of $\delta(x - \xi)$, it can be shown that the jth derivative of the delta function is defined by

$$\int_{-\infty}^{\infty} \delta^{(j)}(x-\xi)h(x)dx = (-1)^j h^{(j)}(\xi) \tag{3.15}$$

Note that because of the discontinuity in $H(x-\xi)$ at the point $x=\xi$, the derivative of H does not exist as an ordinary function, but using the previous method does allow us to find $H'(x-\xi)$ as a generalized function.

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} H(x-\xi)h'(x)dx \\ &= -\int_{\xi}^{\infty} h'(x)dx = h(\xi)\end{aligned} \tag{3.16}$$

and because

$$\int_{-\infty}^{\infty} \delta(x-\xi)h(x)dx = h(\xi) \tag{3.17}$$

it must be the case that

$$H'(x-\xi) = \delta(x-\xi) \tag{3.18}$$

Such equalities between generalized functions, as seen in (3.18), are understood in the sense that if some h in $\mathscr{D}$ is multiplied through, and then we integrate over $(-\infty,\infty)$ then the result will be consistent. That is to say

$$\int_{-\infty}^{\infty} g_1(x)h(x)dx = \int_{-\infty}^{\infty} g_2(x)h(x)dx \tag{3.19}$$

for some equivalent generalized functions g_1 and g_2.

As a final aside, notice that

$$x\delta(x) = 0 \tag{3.20}$$

as a result of

$$\int_{-\infty}^{\infty} x\delta(x)h(x)dx = [xh(x)]|_{x=0} = 0 \tag{3.21}$$