Homework #4

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2.5. a)

N_1	N_2	N_3	N_1	N_2	N_3	N_1	N_2	N_3
4	0	0	2	0	2	0	4	0
3	1	0	1	3	0	0	3	1
3	0	1	1	2	1	0	2	2
2	2	0	1	1	2	0	1	3
2	1	1	1	0	3	0	0	4

total = 15

$$\Omega = \binom{6}{4} = \frac{6!}{4!\,2!} = \frac{720}{24 \cdot 2} = \frac{720}{48} = 15$$

b)

N_1	N_2	N_3	N_1	N_2	N_3	N_1	N_2	N_3
5	0	0	2	2	1	1	0	4
4	1	0	2	1	2	0	5	0
4	0	1	2	0	3	0	4	1
3	2	0	1	4	0	0	3	2
3	1	1	1	3	1	0	2	3
3	0	2	1	2	2	0	1	4
2	3	0	1	1	3	0	0	5

total = 21

$$\Omega = \binom{7}{5} = \frac{7!}{5! \cdot 2!} = \frac{5040}{120 \cdot 2} = 21$$

c)

N_1	N_2	N_3	N_1	N_2	N_3	N_1	N_2	N_3	N_1	N_2	N_3
6	0	0	3	2	1	2	0	4	0	6	0
5	1	0	3	1	2	1	5	0	0	5	1
5	0	1	3	0	3	1	4	1	0	4	2
4	2	0	2	4	0	1	3	2	0	3	3
4	1	1	2	3	1	1	2	3	0	2	4
4	0	2	2	2	2	1	1	4	0	1	5
3	3	0	2	1	3	1	0	5	0	0	6

total = 28

$$\Omega = \binom{8}{6} = \frac{8!}{6! \cdot 2!} = 28$$

d)

N_1	N_2	N_3	N_4	N_1	N_2	N_3	N_4
2	0	0	0	0	1	1	0
1	1	0	0	0	1	0	1
1	0	1	0	0	0	2	0
1	0	0	1	0	0	1	1
0	2	0	0	0	0	0	2

total = 10 $\quad \Omega = \binom{5}{2} = 10$

e)

N_1	N_2	N_3	N_4
3	0	0	0
2	1	0	0
2	0	1	0
2	0	0	1
1	2	0	0
1	1	1	0
1	1	0	1

N_1	N_2	N_3	N_4
1	0	2	0
1	0	1	1
1	0	0	2
0	3	0	0
0	2	1	0
0	2	0	1
0	1	2	0

N_1	N_2	N_3	N_4
0	1	1	1
0	1	0	2
0	0	3	0
0	0	2	1
0	0	1	2
0	0	0	3

total = 20

$$\Omega = \binom{6}{3} = 20$$

f) N_1

9

total = 1

$$\Omega = \binom{9}{9} = \frac{9!}{9! \cdot 0!} = \frac{9!}{9!} = 1$$

g)

$$\begin{array}{ccccc} N_1 & N_2 & N_3 & \cdots & N_n \end{array}$$

$$\left.\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{array}\right\} N$$

total = N

$$\Omega = \binom{N}{1} = \frac{N!}{1!\,(N-1)!} = \frac{N\cancel{(N-1)!}}{\cancel{(N-1)!}} = N$$

2.8. a) $N_A = 10 \quad N_B = 10 \quad q = 20$

$$q_A = \underbrace{0, 1, 2, \ldots, 20}_{21} \qquad q_B = q - q_A$$

21 macrostates

b) $\Omega(20, 20) = \binom{39}{20} = 6.89 \times 10^{10}$

c) $\Omega_A(10, 20) = \binom{29}{20} = 1.00 \times 10^{7}$

$\Omega_B = 1$

$\Omega_{total} = \Omega_A \Omega_B = 1.00 \times 10^{7}$

$$P = \frac{1.00 \times 10^{7}}{6.89 \times 10^{10}}$$
$$= 1.45 \times 10^{-4}$$

d) $\Omega_A(10, 10) = \binom{19}{10} = 92378$

$\Omega_B = \Omega_A$

$$\Omega_{total} = \Omega_A^2$$
$$= 8.53 \times 10^{9}$$

$$P = \frac{8.53 \times 10^{9}}{6.89 \times 10^{10}}$$
$$= 0.124$$

e) If the system is in a state with small multiplicity it will change so that it has a large mult. and will not revert to the improbable state.

2.16. a) $N! \approx N^N e^{-N} \sqrt{2\pi N} \qquad \ln(N!) \approx N\ln(N) - N$

$$\binom{1000}{500} = \frac{1000!}{(500!)^2}$$
$$\approx \frac{1000^{1000} e^{-1000} \cdot \sqrt{2000\pi}}{\left(500^{500} \cdot e^{-500} \cdot \sqrt{1000\pi}\right)^2}$$
$$= \frac{\cancel{500^{1000}} \cdot 2^{1000} \cancel{e^{-1000}}}{\cancel{500^{1000}} \cancel{e^{-1000}}} \cdot \frac{\sqrt{2000\pi}}{\sqrt{1000\pi}} \cdot \frac{1}{\sqrt{1000\pi}}$$
$$= 2^{1000} \cdot \frac{1}{\sqrt{500\pi}}$$

$$P = \frac{1}{\sqrt{500\pi}}$$
$$= 0.025$$

b) $\binom{1000}{600} = \dfrac{1000!}{600!\,400!}$

$$\approx \frac{1000^{1000}\cancel{e^{-1000}}\sqrt{2000\pi}}{600^{600}\cancel{e^{-600}}\cdot 400^{400}\cancel{e^{-400}}\sqrt{4\cdot 600\cdot 400\cdot\pi}}$$

$$= \frac{1000^{1000}}{600^{600}\cdot 400^{400}}\sqrt{\frac{2000}{960000\pi}}$$

$$= \frac{\cancel{200^{1000}}\,5^{1000}}{\cancel{200^{600}}\cdot 3^{600}\cdot\cancel{200^{400}}\cdot 2^{400}}\;\frac{1}{\sqrt{480\pi}}$$

$$= \frac{5^{1000}}{3^{600}\,2^{400}\sqrt{480\pi}}$$

$$P = \frac{5^{1000}}{2^{1000}\cdot 3^{600}\cdot 2^{400}\sqrt{480\pi}}$$

$$= 4.6\times 10^{-11}$$

2.17. $\Omega = \binom{q+N-1}{q}$

$$= \frac{(q+N-1)!}{q!\,(N-1)!} \approx \frac{(q+N)!}{q!\,N!}$$

$$\ln(\Omega) = \ln\left(\frac{(q+N)!}{q!\,N!}\right)$$

$$= \ln((q+N)!) - \ln(q!) - \ln(N!)$$

$$\approx \ln\left((q+N)^{q+N}e^{-(q+N)}\right) - \ln(q^q e^{-q}) - \ln(N^N e^{-N})$$

$$= (q+N)\ln(q+N) \cancel{-q} \cancel{-N} - q\ln q \cancel{+q} - N\ln N \cancel{-N}$$

$$= (q+N)\ln(q+N) - q\ln q - N\ln N$$

$$= (q+N)\ln\left(N\left(1+\frac{q}{N}\right)\right) - q\ln q - N\ln N$$

$$= (q+N)\left(\ln N + \ln\left(1+\frac{q}{N}\right)\right) - q\ln q - N\ln N$$

assume $q \ll N \therefore \frac{q}{N} \ll 1$

$$\ln(\Omega) \approx (q+N)\left(\ln N + \frac{q}{N}\right) - q\ln q - N\ln N$$

$$= q\ln N + \cancel{N\ln N} + q + \frac{q^2}{N} \cancel{- N\ln N} - q\ln q$$

$$= q\ln\frac{N}{q} + q + \frac{q^2}{N}$$

$\frac{q^2}{N}$ is negligible

$$\ln\Omega \approx q\ln\frac{N}{q} + q$$

$$\Omega \approx \left(\frac{eN}{q}\right)^q$$

2.19. $\Omega = \binom{N}{N_\downarrow}$; $N_\uparrow = N - N_\downarrow$

$= \frac{N!}{N_\uparrow ! N_\downarrow !}$

$$\ln\Omega = \ln N! - \ln(N_\uparrow !) - \ln(N_\downarrow !)$$

$$\approx \ln(N^N e^{-N}) - \ln(N_\uparrow^{N_\uparrow} e^{-N_\uparrow}) - \ln(N_\downarrow^{N_\downarrow} e^{-N_\downarrow})$$

$$= N\ln N \cancel{-N} - (N-N_\downarrow)\ln(N-N_\downarrow) \cancel{+N} - \cancel{N_\downarrow} - N_\downarrow \ln(N_\downarrow) + \cancel{N_\downarrow}$$

$$= N\ln N - (N-N_\downarrow)\ln(N-N_\downarrow) - N_\downarrow \ln(N_\downarrow)$$

$$= N\ln N - (N-N_\downarrow)\ln\left(N\left(1-\frac{N_\downarrow}{N}\right)\right) - N_\downarrow \ln(N_\downarrow)$$

assume $N_\downarrow \ll N$.

$$\ln\Omega \approx N\ln N - (N-N_\downarrow)\left(\ln N + \frac{N_\downarrow}{N}\right) - N_\downarrow \ln(N_\downarrow)$$

$$= \cancel{N\ln N} - \cancel{N\ln N} + N_\downarrow \ln N + N_\downarrow - \frac{N_\downarrow^2}{N} - N_\downarrow \ln N_\downarrow$$

$$= N_\downarrow \ln\frac{N}{N_\downarrow} + N_\downarrow - \frac{N_\downarrow^2}{N}$$

$\frac{N_\downarrow^2}{N}$ is negligible

$$\Omega \approx \left(\frac{eN}{N_\downarrow}\right)^{N_\downarrow}$$

These systems are similar because when $q \ll N$ there is not enough energy for every oscillator to be energized so there will be some with one energy unit but most will have none. In much the same way when $N_\downarrow \ll N$ some particles in the paramagnet are "down" but most are not.