Problem Set 8

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1. **Prove:** the product of an irrational number and a nonzero rational number is irrational.
   Proof: Assume, to the contrary, that $xy = z$, for some $x, z \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$. By definition, $x = \frac{a}{b}$ and $z = \frac{a'}{b'}$, for some $a, a', b, b' \in \mathbb{Z}$, with $a, a', b, b' \neq 0$. Then, $\frac{a}{b}y = \frac{a'}{b'}$, and so, $y = \frac{a'b}{b'a}$. Now, $a'b$ and $b'a$, are integers, and so y must be rational by definition. This contradicts the initial assumption that y is irrational. Thus, the product of an irrational and a nonzero rational number cannot be rational and must, therefore, be rational. ■

2. **Prove:** $\sqrt{2} + \sqrt{3}$ is an irrational number.
   Proof: Assume, to the contrary, that $\sqrt{2} + \sqrt{3}$ is rational, then $(\sqrt{2} + \sqrt{3})^2$ must also be rational. Now, $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$. If $\sqrt{6}$ is irrational, then $5 + 2\sqrt{6}$ is also irrational. Suppose that $\sqrt{6}$ is rational. Let $\sqrt{6} = \frac{a}{b}$, for some $a, b \in \mathbb{Z}$, with a and b having no common factors. Then, $6 = \frac{a^2}{b^2}$, and so $b^2 = \frac{a^2}{6}$. Since, b is an integer, b^2 is also an integer. Thus, $6|a^2$ and $6|a$. By definition, $a = 6k$ for some $k \in \mathbb{Z}$. Now, we substitute $6k$ for a and simplify,

$$
\begin{aligned}
b^2 &= \frac{(6k)^2}{6} \\
&= 6k^2.
\end{aligned}
$$

   So, b must have 6 as a factor. Since a and b both have 6 as a factor, this contradicts our assumption that they have no common factors. Now, $\sqrt{6}$ must be irrational, and therefore $\sqrt{2} + \sqrt{3}$ is also irrational. ■

3. **Prove:** there do not exist three distinct real numbers a, b, and c such that all of the numbers $a + b + c$, ab, ac, bc, and abc are equal.
   Proof: Assume, to the contrary, $a + b + c = ab = ac = bc = abc$, with a, b, c being distinct real numbers. Now, by substituting ab, bc, and ac into abc we can obtain, $abc = ac^2 = ab^2 = bc^2$. Then, $ac^2 = ab^2$ can only be true when $b = c$, $b = -c$, or $a = 0$. Since $b = c$ is disallowed, consider the cases $a = 0$ and $b = -c$.
   Case 1: Let $a = 0$. Now, $abc = 0$, and so $abc = bc = 0$. Without loss of generality, let $b = 0$. Then, $b = a = 0$, which is disallowed.
   Case 2: Let $b = -c$. Now, $bc^2 = ac^2$ is true if $a = b$ or $c = 0$. The former, $a = b$, is trivially disallowed, and if $c = 0$ and $b = -c$, then $b = 0$ and $b = c$ which is also

disallowed.
Therefore, there are no possible distinct values for a, b, and c in the real numbers. ■

4. Let a, b, c, d be real numbers. **Prove:** at most four of the numbers ab, ac, ad, bc, bd, and cd are negative.
   Proof: We will consider the possible cases for a, b, c, d being negative. All possible products of pairs of a, b, c, and d are represented so there is no qualitative difference between them, and we will examine a representative case for each number of negatives in a, b, c, d.
   Case 1: Let $a, b, c, d > 0$. In this case, none of the products are negative.
   Case 2: Let $a < 0$ and $b, c, d > 0$. Now, $ab, ac, ad < 0$ and $bc, bd, cd > 0$. Three of these are negative.
   Case 3: Let $a, b < 0$ and $c, d > 0$. Now, $ac, ad, bc, bd < 0$ and $ab, cd > 0$. Four are negative.
   Case 4: Let $a, b, c < 0$ and $d > 0$. Now $ad, cd, bd < 0$ and $ab, ac, bc > 0$. Three are negative.
   Case 5: Let $a, b, c, d < 0$. Now, $ab, ac, ad, bc, bd, cd <> 0$. Zero are negative.
   All possibilities have been exhausted and therefore, at most four of the numbers ab, ac, ad, bc, bd, and cd are negative, as in the case where $a, b < 0$ and $c, d > 0$. ■