

Problem Set 9

Ryan Coyne

November 5, 2023

Result 1: There exist two distinct irrational numbers a and b such that a^b is rational.

PROOF: Consider the number $\sqrt{6}^{\sqrt{2}}$. Now, there are two cases.

Case 1: The number $\sqrt{6}^{\sqrt{2}}$ is rational. Then $a = \sqrt{6}$, $b = \sqrt{2}$, and a^b is rational.

Case 2: The number $\sqrt{6}^{\sqrt{2}}$ is irrational. Now, raise this number to the power of $\sqrt{2}$, $(\sqrt{6}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{6}^2 = 6$, which is rational. Then, $a = \sqrt{6}^{\sqrt{2}}$, $b = \sqrt{2}$, and a^b is rational. ■

Result 2: There exists four distinct positive integers such that each of the integers divides (evenly) the sum of the remaining three integers.

PROOF: Consider the numbers 2, 4, 6, 12. Now sum each combination of three numbers, $4 + 6 + 12 = 22$, $2 + 6 + 12 = 20$, $2 + 4 + 12 = 18$, and $2 + 4 + 6 = 12$. Then, $2|22$, $4|20$, $6|18$, and $12|12$. Therefore there are four such integers. ■

Result 3: There are no integers $a \geq 2$ and $n \geq 1$ such that $a^2 + 1 = 2^n$

PROOF: Suppose to the contrary, that $a^2 + 1 = 2^n$. Now rearrange into, $a^2 = 2^n - 1$. Then, consider two cases: $n = 1$ and $n \geq 2$.

Case 1: If $n = 1$, then $2^1 - 1 = 1$. However, this is incompatible because $a \geq 2$.

Case 2: Let $n \geq 2$. Now, $a^2 = 2^n - 1$ suggests that $2^n - 1$ is a perfect square because a is assumed to be an integer. However, $2^n - 1 = (2^{n/2} + 1)(2^{n/2} - 1)$ and thus cannot be a perfect square.

Therefore, by contradiction, there cannot be integers $a \geq 2$ and $n \geq 1$ such that $a^2 + 1 = 2^n$. ■

Result 4: There do not exist real numbers a and b in the open interval $(0, 1)$ such that $4a(1 - b) > 1$ and $4b(1 - a) > 1$.

PROOF: Start by manipulating the first equation,

$$\begin{aligned} 4a(1 - b) &> 1 \\ a &> \frac{1}{4(1 - b)} \end{aligned}$$

Now, consider the second equation,

$$\begin{aligned}
4b(1-a) &> 1 \\
4b - 4ab &> 1 \\
-4ab &> 1 - 4b \\
a &< \frac{4b-1}{4b}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{4b-1}{4b} &> \frac{1}{4(1-b)} \\
4(4b-1)(1-b) &> 4b \\
(4b-1)(1-b) &> b \\
4b - 4b^2 - 1 + b &> b \\
-4b^2 + 4b - 1 &> 0.
\end{aligned}$$

In this manipulation, we can be sure that we never divide by zero or implicitly multiply by a negative number because we have already restricted ourselves to the interval $(0, 1)$. Now, the sole value for b at which $-4b^2 + 4b - 1 = 0$ is $b = \frac{1}{2}$. This value is not valid because we require $-4b^2 + 4b - 1 > 0$. Now, we choose values within the interval $(0, 1)$ and on either side of $b = \frac{1}{2}$ to check the truth of the hypothesis.

Case 1: Let $b = \frac{1}{4}$. Now, $-4b^2 + 4b - 1 = -1 < 0$. Therefore there are no values on the interval $(0, \frac{1}{2}]$ which satisfy both inequalities.

Case 2: Let $b = \frac{3}{4}$. Now, $-4b^2 + 4b - 1 = -1 < 0$. Therefore there are no values on the interval $(\frac{1}{2}, 1]$ which satisfy both inequalities.

Therefore there are no values of b on the interval $(0, 1)$ that can satisfy both $4a(1-b) > 1$ and $4b(1-a) > 1$, for any value of a . ■