Homework 3

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Question 1

Let $(G, \circ)$ be a group, $g \in G$ and $n \in \mathbb{Z}$. We define the notation

$$\begin{aligned} g^0 &:= e \\ g^n &:= \underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}} \\ g^{n-1} &:= \underbrace{g^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}}_{n \text{ times}}. \end{aligned}$$

a) Show that

$$\langle g \rangle := \{g^n | n \in \mathbb{Z}\}$$

is a subgroup of G.

Proof The set $\langle g \rangle$ is non-empty because $e = g^0 \in \langle g \rangle$. Next, let $g^n, g^m \in \langle g \rangle$ for some $m, n \in \mathbb{Z}$. Thus,

$$g^n \circ g^{-m} = \underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}} \circ \underbrace{g^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}}_{m \text{ times}}.$$

If $n > m$, then each g^{-1} is canceled by a g. Thus,

$$\begin{aligned} g^n \circ g^{-m} &= \underbrace{g \circ g \circ \cdots \circ g}_{n-m \text{ times}} \\ &= g^{n-m}. \end{aligned}$$

Now, $g^n \circ g^{-m} \in \langle g \rangle$ because $n - m \in \mathbb{Z}$.
If $n < m$, then each g is cancelled by a g^{-1} and we have

$$\begin{aligned} g^n \circ g^{-m} &= \underbrace{g^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}}_{m-n \text{ times}} \\ &= (g^{-1})^{m-n} \\ &= g^{n-m} \end{aligned}$$

Now, $g^n \circ g^{-m} \in \langle g \rangle$ because $n - m \in \mathbb{Z}$.
If $n = m$, then each g is cancelled by a g^{-1} and vice-versa. Therefore,

$$g^n \circ g^{-m} = e.$$

Now, $g^n \circ g^{-m} \in \langle g \rangle$ because $e \in \langle g \rangle$. Therefore, $g^n \circ g^{-m} \in \langle g \rangle$ for all $m, n \in \mathbb{Z}$. ■

b) Consider the group $(S_5, 0)$. Determine the elements of $\langle (12)(345) \rangle$.
The set $\langle (12)(345) \rangle = \{(), (12)(345), (354), (12), (345), (12)(543)\}$

Question 2

Consider the group $(S_4, \circ)$. Find one subgroup of S_4 with:

a) 2 elements: $\{(), (12)\}$

b) 3 elements: $\{()$, (123), $(132)\}$

c) 4 elements: $\{()$, (1234), $(13)(24)$, $(1432)\}$

d) 6 elements: $\{()$, (12), (23), (123), (132), $(13)\}$

e) 8 elements: $\{()$, $\}$

f) 12 elements: $\{()$, $\}$

You do not need to justify that your answers are subgroups.