Problem Set 9

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1: Find a formula for

$$1 + 4 + 7 + \cdots (3n - 2)$$

for positive integers then verify your formula by mathematical induction.
The formula is

$$1 + 4 + 7 + \cdots (3n - 2) = \frac{n(3n-1)}{2}$$

for all positive integers, n.

Proof: We will prove by induction that

$$1 + 4 + 7 + \cdots (3n - 2) = \frac{n(3n-1)}{2}$$

for all positive integers n. Base Step: Let $n = 1$. Now, $3n - 2 = 3 - 2 = 1$, and so the sum is equal to one. Now, consider the right side

$$\begin{aligned} \frac{1 \cdot (3 \cdot 1 - 1)}{2} &= \frac{3-1}{2} \\ &= \frac{2}{2} \\ &= 1. \end{aligned}$$

Thus, the formula is valid for $n = 1$.
We show that,

$$1 + 4 + 7 + \cdots (3(k+1) - 2) = \frac{(k+1)(3(k+1)-1)}{2}$$

Now, assume that $1 + 4 + 7 + \cdots (3k - 2) = \frac{k(3k-1)}{2}$ is valid for all $k \in \mathbb{N}$. Add $3(k+1) - 2$ to both sides,

$$1 + 4 + 7 + \cdots (3k - 2) + (3k + 1) = \frac{k(3k-1)}{2} + 3k + 1.$$

Next, write the right side with a common denominator and rearrange the numerator into our desired form,

$$\begin{aligned}
1+4+7+\cdots(3k-2)+(3k+1) &= \frac{k(3k-1)+6k+2}{2} \\
&= \frac{3k^2-k+6k+2}{2} \\
&= \frac{3k^2+5k+2}{2} \\
&= \frac{(k+1)(3k+2)}{2} \\
&= \frac{(k+1)(3(k+1)-1)}{2}.
\end{aligned}$$

Thus, by the principle of mathematical induction, $1+4+7+\cdots(3n-2)=\frac{n(3n-1)}{2}$. ■

2: Prove the following inequality for every positive integer n:

$$2!\cdot 4!\cdot 6!\cdots(2n)! \geq ((n+1)!)^n.$$

Proof: We proceed by induction. Since $(2\cdot 1)! = (2!)^1$, the statement is true when $n=1$. Assume that

$$2!\cdot 4!\cdot 6!\cdots(2k)! \geq ((k+1)!)^k \tag{1}$$

for some integer, k. We show,

$$2!\cdot 4!\cdot 6!\cdots(2k)!\cdot(2(k+1)) \geq ((k+2)!)^{k+1}.$$

Now, multiply either side of equation (1) by $(2(k+1))!$,

$$2!\cdot 4!\cdot 6!\cdots(2k)!\cdot(2(k+1))! \geq ((k+1)!)^k\cdot(2(k+1))!.$$

Now, we show that $((k+1)!)^k\cdot(2(k+1))! \geq ((k+2)!)^{k+1}$ directly. Divide both sides by $((k+1)!)^k$,

$$\begin{aligned}
\frac{((k+1)!)^k\cdot(2(k+1))!}{((k+1)!)^k} &\geq \frac{((k+2)!)^{k+1}}{((k+1)!)^k} \\
(2k+2)! &\geq \frac{((k+1)!)^{k+1}(k+2)^{k+1}}{((k+1)!)^k} \\
&\geq (k+1)!(k+2)^{k+1}.
\end{aligned}$$

Now, divide by $(k+1)!$,

$$\begin{aligned}
\frac{(2k+2)!}{(k+1)!} &\geq \frac{(k+1)!(k+2)^{k+1}}{(k+1)!} \\
\underbrace{(k+2)\cdot(k+3)\cdots(2k+1)\cdot(2k+2)}_{k+1\text{ factors}} &\geq (k+2)^{k+1}.
\end{aligned}$$

Note that both sides involve the multiplication of integers $k+1$ times. However, all of the factors on the right are $k+2$, and exactly one of the factors on the left is $k+2$. Additionally, k of the factors on the left are greater than $k+2$. Thus, it is true that $((k+1)!)^k \cdot (2(k+1))! \geq ((k+2)!)^{k+1}$, and by induction. ■

3: Prove that for every real number $x > -1$ and every positive integer n,

$$(1+x)^n \geq 1+nx$$

Proof: ■

4: Prove that $81|(10^{n+1} - 9n - 10)$ for every positive integer n.

Proof: Base Case: Let $n = 1$. Now, $10^2 - 9 - 10 = 81$. Thus, $81|(10^{n+1} - 9n - 10)$ when $n = 1$.
Induction Step: Now, assume that $81|(10^{k+1} - 9k - 10)$. Next, consider $10^{k+2} - 9(k+1) - 10 = 10^{k+2} - 9k - 19$. Now, subtract $10(10^{k+1} - 9k - 10)$ from both sides

$$\begin{aligned} 10^{k+2} - 9(k+1) - 10 - 10(10^{k+1} - 9k - 10) &= 10^{k+2} - 9k - 19 - 10(10^{k+1} - 9k - 10) \\ &= 10^{k+2} - 9k - 19 - 10^{k+2} + 90k + 100 \\ &= 81(k+1). \end{aligned}$$

Then, add $10(10^{k+1} - 9k - 10)$ to both sides,

$$10^{k+2} - 9(k+1) - 10 = 81(k+1) + 10(10^{k+1} - 9k - 10).$$

Since, $81|(10^{k+1} - 9k - 10)$, then $10^{k+1} - 9k - 10 = 81j$, for some $j \in \mathbb{N}$. Thus,

$$\begin{aligned} 10^{k+2} - 9(k+1) - 10 &= 81(k+1) + 10(81j) \\ &= 81(k+1) + 81(10j) \\ &= 81(k+1+10j). \end{aligned}$$

Thus, $81|(10^{k+2} - 9(k+1) - 10)$. Therefore, $81|(10^{n+1} - 9n - 10)$ for all $n \in \mathbb{N}$. ■

5: A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 1,\ a_2 = 2;\ a_n = a_{n-1} + 2a_{n-2},$$

for $n \geq 3$. Conjecture a formula for a_n and verify that your conjecture is correct.

Proof: We prove by induction that for the sequence defined above, $a_n = 2^{n-1}$.
Since, $a_1 = 2^0 = 1$, the formula holds for $n = 1$. Assume for an arbitrary k that $a_i = 2^{i-1}$ for every integer i with $1 \leq i \leq k$. We show that $a_{k+1} = 2^k$. If $k = 1$, then $a_{k+1} = a_2 = 2^1 = 2$. Since, $a_2 = 2$, it follows that $a_{k+1} = 2^k$ for $k = 1$. Now, we may assume that $k \geq 2$. Since $k+1 \geq 3$, it follows that

$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} \\ &= 2^{k-1} + 2 \cdot 2^{k-2} \\ &= 2^{k-1} + 2^{k-1} \\ &= 2 \cdot 2^{k-1} \\ &= 2^k, \end{aligned}$$

which is the desired result. By the strong principle of mathematical induction, $a_n = 2^{n-1}$, for all $n \in \mathbb{N}$. ■

6: Consider the sequence of Fibonacci numbers $\{F_n\}$, where

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2},$$

for $n \geq 3$.
(a) Prove $2|F_n$ if and only if $3|n$.

PROOF: Since, $3|n$, it is clear that $n = 3m$ for some $m \in N$. In the case where $m = 1$, we have

$$F_n = F_3 = F_2 + F_1 = 1 + 1 = 2$$

■