Green's Functions

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Contents

1 Introduction

1.1 Nonhomogeneous Linear Differential Equations

The first part of this text is primarily concerned with the solutions to non-homogeneous linear differential equations, which have the form

$$\mathbf{L}u = \phi, \tag{1.1}$$

over an interval a $\leq x \leq$ b and subject to certain boundary conditions, where $\mathbf{L}$ is an nth order linear ordinary differential operator and where the function ϕ is integrable on the given interval. For $\mathbf{L}$ to be linear, it must satisfy the condition

$$\mathbf{L}(\alpha v + \beta w) = \alpha \mathbf{L} v + \beta \mathbf{L} w \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant.

Theorem 1.1. $\mathbf{L}$ *is linear iff it is of the form*

$$\mathbf{L} = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x). \tag{1.3}$$

Proof $\implies$
We prove, by induction, that $\mathbf{D}^{(i)}$ is a linear differential operator for all $i \in \mathbb{N}$.
Base Case Let $i = 1$.

$$\begin{aligned}
\mathbf{D}(\alpha u(x) + \beta v(x)) &= \lim_{h \to 0} \frac{\alpha u(x+h) + \beta v(x+h) - (\alpha u(x) + \beta v(x))}{h} \\
&= \alpha \frac{u(x+h) - u(x)}{h} + \beta \frac{v(x+h) - v(x)}{h} \\
&= \alpha \mathbf{D} u + \beta \mathbf{D} v
\end{aligned} \tag{1.4}$$

Induction Step Let $k \in \mathbb{N}$ and suppose that the statement holds for all $k > 1$. Then,

$$\begin{aligned}
\mathbf{D}^{(k+1)}(\alpha u + \beta v) &= \mathbf{D}(\mathbf{D}^{(k)}(\alpha u + \beta v)) \\
&= \mathbf{D}(\alpha \mathbf{D}^{(k)} u + \beta \mathbf{D}^{(k)} v) \\
&= \alpha \mathbf{D}(\mathbf{D}^{(k)} u) + \beta \mathbf{D}(\mathbf{D}^{(k)} v) \\
&= \alpha \mathbf{D}^{(k+1)} u + \beta \mathbf{D}^{(k+1)} v
\end{aligned} \tag{1.5}$$

It follows that

$$\sum_{i=0}^{n} a_i(x) D^{(i)} \tag{1.6}$$

is also linear.
$\impliedby$
Let

$$\begin{aligned}
\mathbf{L}u &= f(x, \underbrace{u, u', u'', \ldots, u^(n)}_{\mathbf{u}}) \\
&= f(x, \mathbf{u}).
\end{aligned} \tag{1.7}$$

Then,

$$\mathbf{L}(\alpha v + \beta w) = f(x, \alpha \mathbf{v} + \beta \mathbf{w}) \tag{1.8}$$

and

$$\alpha \mathbf{L} v + \beta \mathbf{L} w = \alpha f(\mathbf{v}) + \beta f(\mathbf{w}). \tag{1.9}$$

If $\mathbf{L}$ is linear then

$$\begin{aligned} \mathbf{L}(\alpha v + \beta w) &= \alpha \mathbf{L} v + \beta \mathbf{L} w \\ f(\alpha v + \beta w) &= \alpha f(x, \mathbf{v}) + \beta f(x, \mathbf{u}). \end{aligned} \tag{1.10}$$

It follows that

$$f(x, \mathbf{u} + \epsilon \mathbf{v}) = f(x, \mathbf{u}) + \epsilon f(x, \mathbf{v}) \tag{1.11}$$

$$\frac{f(x, \mathbf{u} + \epsilon \mathbf{v}) - f(x, \mathbf{u})}{\epsilon} = f(x, \mathbf{v}) \tag{1.12}$$

Next, we take the limit as $\epsilon \to 0$.

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{f(x, \mathbf{u} + \epsilon \mathbf{v}) - f(x, \mathbf{u})}{\epsilon} &= \mathbf{D_v} f(x, \mathbf{u}) \\ &= \mathbf{v} \cdot \nabla f(x, \mathbf{u}) \end{aligned} \tag{1.13}$$

Let

$$\mathbf{u} = \mathbf{v} = \langle 0, \ldots, u^{(i)}, 0, \ldots, 0 \rangle \tag{1.14}$$

and

$$f(x, u^{(i)}) = f(x, \langle 0, \ldots, u^{(i)}, 0, \ldots, 0 \rangle). \tag{1.15}$$

Then it follows from equation (1.13) that,

$$u^{(i)} \frac{\partial f(x, u^{(i)})}{\partial u^{(i)}} = f(x, u^{(i)}). \tag{1.16}$$

$$\begin{aligned} \sum_{i=0}^{n} u^{(i)} \frac{\partial f(x, u^{(i)})}{\partial u)^{i}} &= f(x, u) + f(x, u') + \cdots + f(x, u^{(n)}) \\ &= a_n(x) \frac{d^n}{dx^n} u + a_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} u + \cdots + a_0(x) u \end{aligned} \tag{1.17}$$

$\blacksquare$

Since $\mathbf{L}$ is of order n, there will be n boundary conditions of the general form

$$\mathbf{B}_j(u) = c_j; \quad j = 1, 2, \ldots, n, \tag{1.18}$$

where the $\mathbf{B}_j$'s are prescribed functionals [1] and c_j's are prescribed constants. We will only consider $\mathbf{B}_j$'s that are linear combinations of u and its derivatives through order $n-1$ and evaluated at the endpoints, a and b.

For $\mathbf{B}_j$ to be **linear**, it must satisfy the condition

$$\mathbf{B}_j(\alpha v + \beta w) = \alpha \mathbf{B}_j(v) + \beta \mathbf{B}_j(w). \tag{1.20}$$

[1]A **functional** is a transformation with a set of functions as its domain and a set of numbers as its range. To illustrate, consider the functional

$$\mathscr{F}(u) = \int_0^1 u^2(x)\,dx. \tag{1.19}$$

The domain of this functional might be the set of functions defined over the interval $[0, 1]$ and for which the integral of u^2 from 0 to 1 exists. The range is $(0, \infty)$.

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and its boundary conditions, begin by finding the adjoint operator, denoted $\mathbf{L}^*$. The **adjoint operator** consists of the formal adjoint, $\mathbf{L}^*$, and the boundary conditions associated with the Green's function. To determine these, first form the product, vLu, and integrate it over the interval of interest. By repeated integration by parts, we can express the integral in the form

$$\int_{\mathrm{a}}^{\mathrm{b}} v\mathbf{L}u dx = [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u\mathbf{L}^* v dx, \tag{2.1}$$

where $[\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}}$ represents the boundary terms resulting from successive integration by parts. Here, u and v must be sufficiently differentiable functions so that the left and right sides are well defined.

Example 2.1 Consider the linear differentiable operator

$$\mathbf{L} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x). \tag{2.2}$$

To find $\mathbf{L}^*$, perform integration by parts on each term of the product $v\mathbf{L}u$ until the integrand has the form $u\mathbf{L}^*v$. That is to say, integrate by parts twice on the first term, once on the second, and not at all on the third. Doing this, we are left with

$$\begin{aligned}\int_{\mathrm{a}}^{\mathrm{b}} v\mathbf{L}u dx &= \int_{\mathrm{a}}^{\mathrm{b}} (vau'' + vbu' + vc)dx \\ &= (vau' + vbu)\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} (-(va)'u' - (vb)'u + vcu)dx \\ &= (vau' + vbu - (va)'u)\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} ((va)''u - (vb)'u + vcu)dx \\ &= (vau' + vbu - (va)'u)\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u((va)'' - (bv)' + cv)dx.\end{aligned} \tag{2.3}$$

From this, it is clear that

$$\begin{aligned}\mathbf{L}^*v &= (av)'' - (bv)' + cv \\ &= (a'v + av')' - b'v - bv' + cv \\ &= av'' + (2a' - b)v' + (a'' - b' + c)\end{aligned} \tag{2.4}$$

and so the formal adjoint of the second-order linear differential operator L must be of the form

$$\mathbf{L}^* = a\frac{d^2}{dx^2} + (2a' - b)\frac{d}{dx} + (a'' - b' + c). \tag{2.5}$$

If $\mathbf{L}^* = \mathbf{L}$, then $\mathbf{L}$ is called **formally self-adjoint**. By comparing equations (2.2) and (2.5), we can see that for a second-order linear differentiable operator to be formally self-adjoint, it is sufficient that $a' = b$ since this implies $2a' - b = a'$ and $a'' - b' + c = a'' - a'' + c = c$.

Definition If the boundary conditions on $\mathbf{L}$ are homogeneous[2], then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(\mathbf{L}u, v) = (u, \mathbf{L}^* v) \tag{2.6}$$

where (f, g) is the **inner product** of f and g,

$$(f, g) = \int_a^b f(x)g(x)dx. \tag{2.7}$$

This means that the adjoint operator $\mathscr{L}^*$ consists of $\mathbf{L}^*$ and boundary conditions for which the boundary terms of the integral are zero.

Example 2.2 Consider $\mathscr{L}$ to consist of $\mathbf{L} = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \leq x \leq 1$. Then

$$\begin{aligned}
(\mathbf{L}u, v) &= \int_0^1 u'v dx \\
&= (uv)\Big|_0^1 - \int_0^1 uv' dx \\
&= u(1)v(1) - u(0)v(0) + \int_0^1 u\mathbf{L}^* v dx \\
&= u(1)(v(1) - 3v(0)) + \int_0^1 u\mathbf{L}^* v dx
\end{aligned} \tag{2.8}$$

Since the particular value of $u(1)$ is not given, we must impose the condition $v(1) - 3v(0) = 0$, because choosing $u(1) = 0$ would unduly restrict our solution. Therefore $\mathscr{L}^*$ consists of $\mathbf{L}^* = -\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$.

As a final note, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called **self-adjoint**.

3 The Dirac delta function

3.1 Delta Sequences

In physics, we often consider the idea of a point mass. Suppose we have a unit point mass at $x = 0$ with mass density given by $w(x)$. We are interested in the mass but do not know the details of its density. We do, however, know that the $w(x)$ will be highly localized in space and that

$$\int_{-\infty}^{\infty} w(x)dx = 1, \tag{3.1}$$

so that the net mass is unity.

[2]By **homogeneous**, we mean that each boundary condition is of the form $\mathbf{B}_j(u) = 0$.

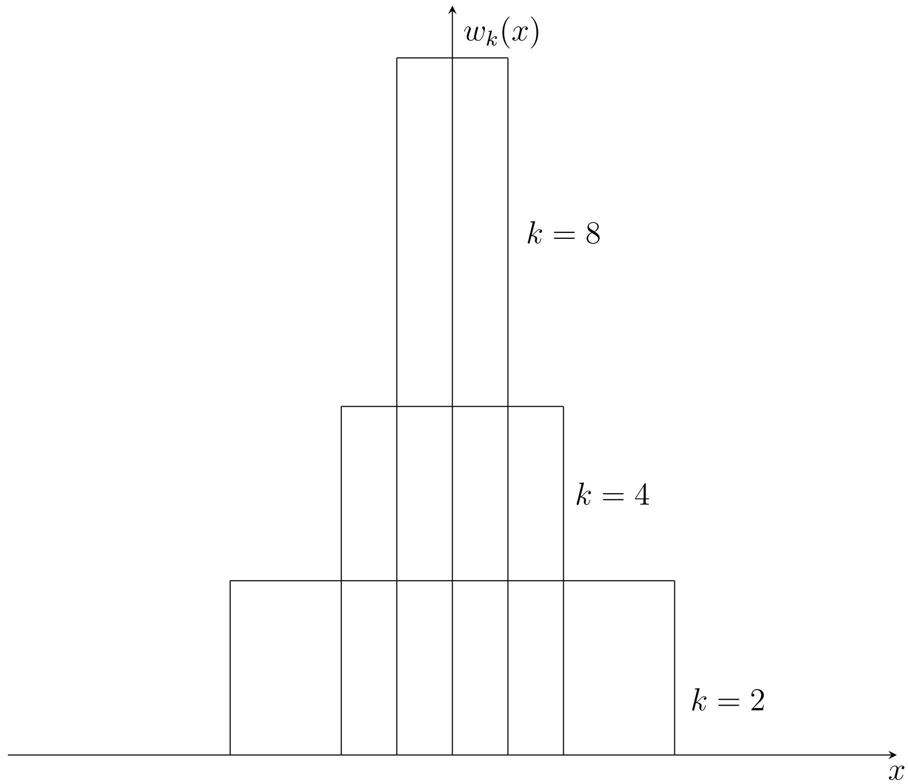


Figure 3.1: Mass Density; eq 3.2

We expect two highly concentrated unit mass densities to produce masses with nearly identical physical effects. As such, we might simplify the problem by deciding, a priori, on a definite form for w, such as

$$w_k(x) = \begin{cases} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases} \tag{3.2}$$

or

$$w_k(x) = \frac{k}{\pi(1 + k^2 x^2)}, \tag{3.3}$$

where k is some larger natural number. In Fig 3.2, we see that w becomes highly concentrated at $x = 0$ when k is large.

If we let $k \to \infty$, then the mass distribution approaches our idea of a point mass at $x = 0$. Calling this $\delta(x)$, we would like to write,

$$\delta(x) \text{ " = " } \lim_{k \to \infty} w_k(x). \tag{3.4}$$

This definition feels intuitive, but it is not a rigorous definition of the Dirac delta function because the limit is infinite for $x = 0$. That is, the right-hand side is not a function. We instead define the Dirac delta function, $\delta(x)$, in the following way

$$\begin{aligned} \lim_{k \to \infty} \int_{-\infty}^{\infty} h(x) w_k(x) dx &= \int_{-\infty}^{\infty} h(x) \delta(x) dx \\ &= h(0) \end{aligned} \tag{3.5}$$

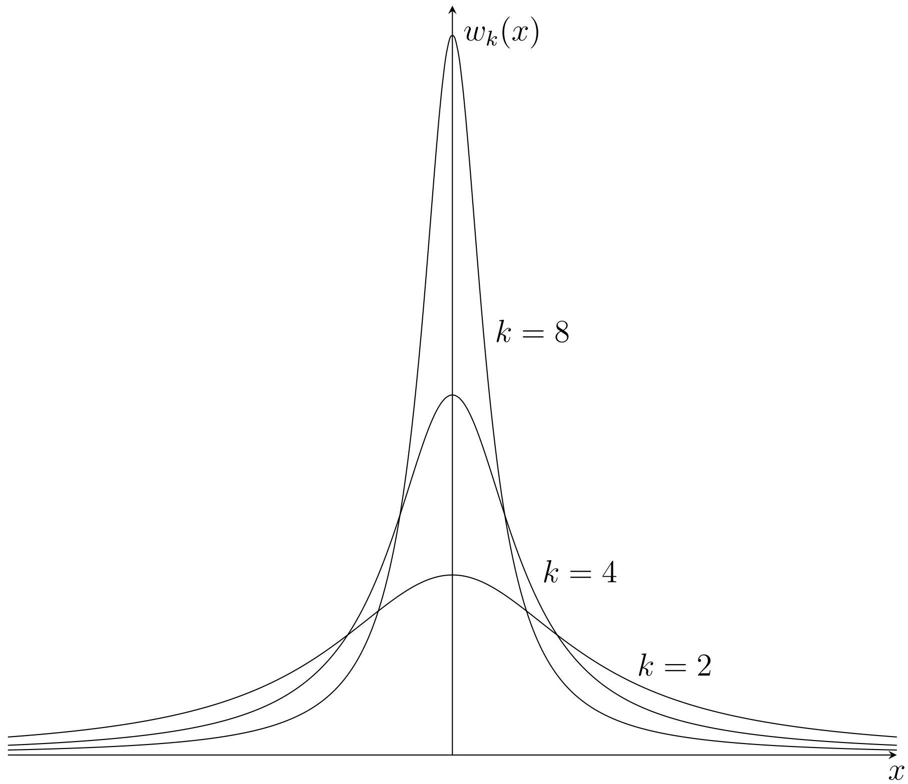


Figure 3.2: Mass Density; eq. 3.3

and call $w_k(x)$ a **δ-sequence**. This way of defining the Dirac delta function is more rigorous while still being intuitive because it is related to our understanding of delta sequences. However, keep in mind that the delta function is not a function.

Theorem 3.1. *If $w(x)$ is non-negative $\int_{-\infty}^{\infty} w(x)dx = 1$, and $w(x) = O(1/x^{1+\alpha})$ as $|x| \to \infty$ with $\alpha > 0$, then $kw(kx) \equiv w_k(x)$ is a δ-sequence.*

Proof First, $\lim_{k\to\infty} w_k(x) = 0$, for each fixed $x \neq 0$, because $w_k(x) = kw(kx) = O(k \cdot k^{-1-\alpha} x^{-1-\alpha}) = O(k^{-\alpha}) \to 0$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(x)dx = \underbrace{\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)[h(x) - h(0)]dx}_{I} + \underbrace{\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(0)dx}_{J} . \tag{3.6}$$

Consider J,

$$\begin{aligned} J &= h(0) \lim_{h\to\infty} \int_{-\infty}^{\infty} kw(kx)dx \\ \text{Let } \xi &= kx \\ J &= h(0) \lim_{k\to\infty} \int_{-\infty}^{\infty} w(\xi)d\xi \\ &= h(0) \end{aligned} \tag{3.7}$$

Next, we wish to show that $I = 0$, so that the right hand side of equation (3.6) is $h(0)$. Select a number $\epsilon > 0$. Since h is assumed to be continuous at $x = 0$, there must exist a number $\delta > 0$ such that $|h(x) - H(0)| < \epsilon$ whenever $|x - 0| = x < \delta$. Breaking up the integral I,

$$\begin{aligned} I = & \underbrace{\lim_{k\to\infty} \int_{-\infty}^{-\delta} w_k(x)(h(x) - h(0))dx}_{I_1} + \underbrace{\lim_{k\to\infty} \int_{-\delta}^{\delta} w_k(x)(h(x) - h(0))dx}_{I_2} \\ & + \underbrace{\lim_{k\to\infty} \int_{\delta}^{\infty} w_k(x)(h(x) - h(0))dx}_{I_3}. \end{aligned} \tag{3.8}$$

Since $w_k(x) \to 0$ uniformly, over $-\infty < x < -\delta$ and $\delta < x < \infty$, then $I_1 = I_3 = 0$. I_2 must also be zero because $|h(x) - h(0)|$ must be less than any positive real β. Thus, $I = 0$ and so

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(x)dx = h(0). \tag{3.9}$$

∎

3.2 The Dirac Delta Function as a Generalized Function

The Dirac delta function is more appropriately defined as a generalized function. To understand this way of defining δ, we will begin by defining some terms.

Definition A **closed interval** is one that includes its endpoints.

Definition A function, f, is **uniformly continuous** if

$$\forall\epsilon > 0\ \exists\delta > 0\ \forall a \in X\ \forall b \in X : |a - b| < \delta \implies |f(a) - f(b)| < \epsilon. \tag{3.10}$$

Definition A function has **compact support** if the subset of its domain for which its range is non-zero is closed and bounded.

We will call the space of infinitely differentiable functions with compact support $\mathscr{D}$.

Definition Generalized functions are linear functionals that are uniformly continuous on $\mathscr{D}$, such that all generalized functions have derivatives which are also generalized functions.

We consider the following functional,

$$\mathscr{F}(h) = \int_{-\infty}^{\infty} g(x)h(x)dx. \tag{3.11}$$

This functional assigns a numerical value, $\mathscr{F}(h)$, for each function h within the domain, $\mathscr{D}$, of $\mathscr{F}$.

Example 3.1 Suppose $\mathscr{F}(h)$ is the integral of h from ξ to ∞.

$$\int_{\infty}^{\infty} g(x)h(x)dx = \int_{\xi}^{\infty} h(x)dx \tag{3.12}$$

Then, $g(x)$ must be the Heaviside step function,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ \frac{1}{2}, & x = 0 \end{cases} \tag{3.13}$$

which is a function in the classical sense. [3]

If $\mathscr{F}(h)$ is $h(0)$ so that

$$\int_{-\infty}^{\infty} g(x)h(x)dx = h(0) \tag{3.14}$$

then it can be shown that there is no function, $g(x)$, which exists such that equation (3.14) is true for all functions, $h(x)$, in the domain, $\mathscr{D}$. We call g defined by equation (3.14) a generalized function, and in particular, it is the Dirac delta function. As such, δ is defined in the following way.

$$\int_{-\infty}^{\infty} \delta(x)h(x)dx = h(0) \tag{3.15}$$

Although $\delta(x)$ acts at $x = 0$, it can be adjusted to act at any point by shifting the argument. Thus, $\delta(x - \xi)$ acts at $x = \xi$,

$$\int_{-\infty}^{\infty} \delta(x - \xi)h(x)dx = h(\xi). \tag{3.16}$$

As a generalized function, δ is also differentiable. By referring to (3.5), one can see that defining the derivative of a generalized function involves determining the functional, $\mathscr{F}(h)$ for

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = \mathscr{F}(h). \tag{3.17}$$

Integrating by parts

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = g(x)h(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(x)h'(x)dx. \tag{3.18}$$

The integral term is fairly simple to interpret since it is of the same form as (3.5), but the boundary term is not as nice because it involves knowing the values of g. However, our restriction that h has compact support means that it must vanish at infinity, and since we are integrating from $-\infty$ to ∞, the boundary term must be zero.

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = -\int_{-\infty}^{\infty} g(x)h'(x)dx. \tag{3.19}$$

[3] We have defined $H(0)$ to be $\frac{1}{2}$, which is a common convention. However, for our purposes, the value at any particular point is not important since we are only ever interested in integrating the function.

For the Dirac delta function, this means

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} \delta(x-\xi)h'(x)dx \\ &= -h'(\xi).\end{aligned} \tag{3.20}$$

Theorem 3.2. *The jth derivative of the Dirac delta function is*

$$\int_{-\infty}^{\infty} \delta^{(j)}(\xi - x)h(\xi)d\xi = (-1)^j h^{(j)}(x). \tag{3.21}$$

Proof We prove the statement by induction.
Base case: Proven to be true for $j = 1$ in equation (3.20).
Induction step: Let $k \in \mathbb{N}$ and suppose the statements holds for $k > 1$. Then,

$$\begin{aligned}\int_{-\infty}^{\infty} \delta^{(k+1)}(\xi - x)h(\xi)d\xi &= \delta^{(k)}(\xi - x)h(\xi)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta^{(k)}(\xi - x)h'(\xi)d\xi \\ &= -\int_{-\infty}^{\infty} \delta^{(k)}(\xi - x)h'(\xi)d\xi \\ &= -(-1)^k (h')^{(k)}(x) \\ &= (-1)^{k+1} h^{(k+1)}(x)\end{aligned}$$

∎

Note that because of the discontinuity in $H(x-\xi)$ at the point $x = \xi$, the derivative of H does not exist as an ordinary function. However, the previous method does allow us to find $H'(x-\xi)$ as a generalized function,

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} H(x-\xi)h'(x)dx \\ &= -\int_{\xi}^{\infty} h'(x)dx = h(\xi).\end{aligned} \tag{3.22}$$

Since

$$\int_{-\infty}^{\infty} \delta(x-\xi)h(x)dx = h(\xi) \tag{3.23}$$

it must be the case that, in the sense of generalized functions,

$$H'(x-\xi) = \delta(x-\xi). \tag{3.24}$$

Such equalities between generalized functions, as seen in (3.18), are understood in the sense that if some h in $\mathscr{D}$ is multiplied through, and we integrate over $(-\infty, \infty)$ then the result will hold. To wit, we consider generalized functions, g_1 and g_2, to be equal if, for all $h \in \mathscr{D}$,

$$\int_{-\infty}^{\infty} g_1(x)h(x)dx = \int_{-\infty}^{\infty} g_2(x)h(x)dx. \tag{3.25}$$

Notice that for all $n > 0$

$$x^n \delta(x) = 0 \tag{3.26}$$

as a result of

$$\int_{-\infty}^{\infty} x^n \delta(x) h(x) dx = [x^n h(x)]|_{x=0} = 0. \tag{3.27}$$

Example 3.2 We would like to show that the sequence

$$w_k(x) = \begin{cases} k, & 0 < x < \frac{1}{k} \\ 0, & x \leq 0 \text{ and } x \geq \frac{1}{k} \end{cases}$$

is a δ-sequence using theorem (3.1). It is clear that $w_k(x) \geq 0$ for all x and $w(x) = O(1/x^{1+\alpha})$ as $|x| \to \infty$ with $\alpha > 0$ since it is zero when ≤ 0 or $x \geq 1/k$. Lastly, we must show that the area under $w(x)$ is 1. Choosing $w(x)$ so that $kw(kx) = w_k(x)$

$$\begin{aligned} \int_{-\infty}^{\infty} w(x) dx &= \int_{-\infty}^{0} 0 dx + \int_{0}^{1} 1 dx + \int_{1}^{\infty} 0 dx \\ &= \int_{0}^{1} 1 dx \\ &= 1 \end{aligned} \tag{3.28}$$

Example 3.3 We would like to show that the sequence

$$w_k(x) = \begin{cases} -k, & |x| < \frac{1}{2k} \\ 2k, & \frac{1}{2k} \leq |x| \leq \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases} \tag{3.29}$$

is a delta sequence. Theorem (3.1) does not apply in this case becuase $w_k(x)$ is negative for some values of x so we should instead show that

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} w_k(x) h(x) dx = h(0). \tag{3.30}$$

Example 3.4 We would like to show that $e^x \delta(x) = \delta(x)$. To begin with, we integrate $e^x \delta(x) f(x)$ and would like to show that this is equal to $f(0)$ to satisfy the generalized function definition of the delta function.

$$\begin{aligned} \int_{-\infty}^{\infty} e^x \delta(x) f(x) dx &= \int_{-\infty}^{\infty} \delta(x) \underbrace{e^x f(x)}_{g(x)} dx \\ &= \int_{-\infty}^{\infty} \delta(x) g(x) dx \\ &= g(0) \\ &= e^0 f(0) \\ &= f(0) \end{aligned} \tag{3.31}$$

Therefore

$$\int_{-\infty}^{\infty} e^x \delta(x) f(x) dx = \int_{-\infty}^{\infty} \delta(x) f(x) dx \tag{3.32}$$

Example 3.5 We would like to show that $\frac{d^4}{dx^4}|x|^4 = 12\delta(x)$. Note that $|x| = 2xH(x) - x$. We begin by manipulating $|x|^3$ into a more convenient form.

$$\begin{aligned} |x|^3 &= (2xH(x) - x)^3 \\ &= x^3(2H(x) - 1)^3 \end{aligned} \tag{3.33}$$

Next, it is useful to simplify.

$$\begin{aligned} (2H(x) - 1)^3 &= (2H(x) - 1)(2H(x) - 1)^2 \\ &= (4H^2(x) - 4H(x) + 1)(2H(x) - 1) \\ &= (4H(x) - 4H(x) + 1)(2H(x) - 1) \\ &= 2H(x) - 1 \end{aligned} \tag{3.34}$$

Finally, we take successive derivatives.

$$\begin{aligned} \frac{d}{dx}(x^3(2H(x) - 1)) &= 3x^2(2H(x) - 1) + 2x^3\delta(x) \\ &= 3x^2(2H(x) - 1) \\ \frac{d}{dx}(3x^2(2H(x) - 1)) &= 6x(2H(x) - 1) + 6x^2\delta(x) \\ &= 6x(2H(x) - 1) \\ \frac{d}{dx}(6x(2H(x) - 1)) &= 12(H(x) - 1) + 12x\delta(x) \\ &= 12(H(x) - 1) \\ \frac{d}{dx}(12(H(x) - 1)) &= 12\delta(x) \end{aligned} \tag{3.35}$$

4 The Method of Green's Functions

4.1 Introduction

A **Green's function** is the solution to a differential equation of the form

$$\mathbf{L}^*_\xi G(\xi, x) = \delta(\xi - x). \tag{4.1}$$

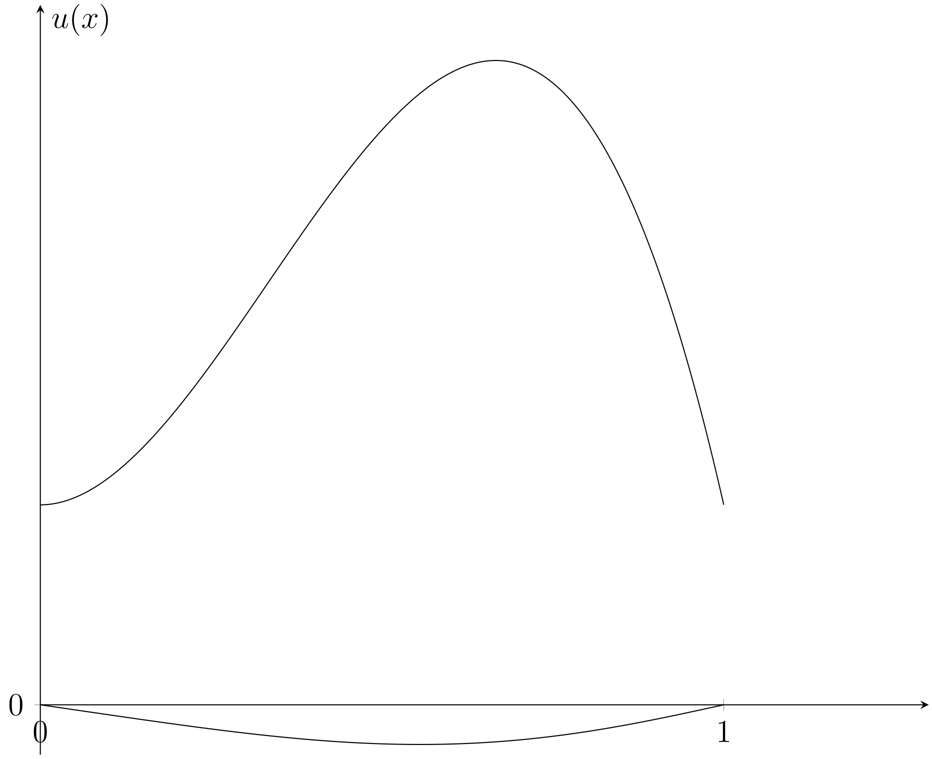
The subscript ξ on the differential operator indicates that the derivatives are being taken with respect to the variable ξ. By making use of some of the delta function's unusual properties, Green's functions can be used to solve nonhomogeneous linear differential equations.

To find the solution to the linear differential equation

$$\mathbf{L}u = \phi, \tag{4.2}$$

we start by finding the formal adjoint as in equation (2.1). If we replace v with G, and we replace x with a dummy variable ξ, we are left with an equation of the form

$$\int_{\mathrm{a}}^{\mathrm{b}} G(\xi, x)\mathbf{L}u(\xi)d\xi = [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u(\xi)\mathbf{L}^*G(\xi, x)d\xi. \tag{4.3}$$



It follows from equation (4.1) that we can replace $\mathbf{L}^*G$ with δ, and from equation (4.2) that we can replace $\mathbf{L}u$ with ϕ,

$$\begin{aligned}\int_a^b G(\xi,x)\phi(\xi)d\xi &= [\cdots]\Big|_a^b + \int_a^b u(\xi)\delta(\xi - x)d\xi \\ &= [\cdots]\Big|_a^b + u(x).\end{aligned} \tag{4.4}$$

Therefore, if we choose boundary conditions for G such that the boundary terms do not depend on u and we are able to find G, then finding u is reduced to a problem of integrating $G\phi$.

To illustrate the key ideas of the method, we will consider several examples which begin simply and become more complex. Each example will be concerned with a key concept in implementing the method of Green's functions.

4.2 Examples

Example 4.1 *Loaded String*

Consider the boundary value problem

$$u''(x) = \phi(x); \quad u(0) = u(1) = 0 \tag{4.5}$$

where $\phi(x)$ is prescribed. Equation (4.5) can be regarded as describing the static deflection of a string under unit tension between fixed endpoints and subjected to a force distribution $\phi(x)$.

To find a solution to this differential equation, we first find the formal adjoint $\mathbf{L}*$ as in equation (4.3).

$$\begin{aligned}\int_0^1 G(\xi,x)\mathbf{L}u(\xi)d\xi &= (G(\xi,x)u'(\xi) - G_\xi(\xi,x)u(\xi))\Big|_0^1 + \int_0^1 uG_{\xi\xi}d\xi \\ &= G(1,x)u'(1) - G_\xi(1,x)u(1) \\ &\quad - G(0,x)u'(0) + G_\xi(0,x)u(0) + \int_0^1 uG_{\xi\xi}d\xi.\end{aligned} \tag{4.6}$$

Therefore,

$$\mathbf{L}^* = \frac{d^2}{d\xi^2} \tag{4.7}$$

and

$$\mathbf{L}^*G = G_{\xi\xi} = \delta(\xi - x). \tag{4.8}$$

Because of the boundary conditions on u in equation (4.5), two of our boundary terms are zero. Thus,

$$\int_0^1 G(\xi,x)\mathbf{L}u(\xi)d\xi = G(1,x)u'(1) - G(0,x)u'(0) + \int_0^1 uG_{\xi\xi}d\xi. \tag{4.9}$$

Now, we want to remove the u dependency from the boundary terms and, as such, decide that

$$G(1,x) = G(0,x) = 0. \tag{4.10}$$

Then the solution is given by

$$u(x) = \int_0^1 G(\xi,x)\phi(\xi)d\xi. \tag{4.11}$$

To calculate the Green's function, we integrate equation (4.8), regarding x as fixed.

$$\begin{aligned}G_\xi &= H(\xi - x) + A \\ G &= (\xi - x)H(\xi - x) + A\xi + B\end{aligned} \tag{4.12}$$

Imposing the boundary conditions,

$$\begin{aligned}&G(0,x) = 0 = B \\ &G(1,x) = 0 = 1 - x + A + B,\end{aligned} \tag{4.13}$$

shows us that $B = 0$ and $A = x - 1$. Therefore

$$G(\xi,x) = (\xi - x)H(\xi - x) + (x-1)\xi. \tag{4.14}$$

Equation (4.8) can, like equation (4.5), be interepreted as the deflection of a loaded string. Specifically it is the deflection as a function of ξ due to a point load of unit strength, $\delta(\xi - x)$, instead of a load distribution, ϕ. Rewriting our Green's function as

$$G(\xi,x) = \begin{cases}(x-1)\xi, & \xi \le x \\ (\xi - 1)x, & \xi \ge x\end{cases} \tag{4.15}$$

makes it clear that $G(\xi,x)$ is symemtric, $G(\xi,x) = G(x,\xi)$. This is often referred to as "Maxwell Reciprocity." Because

A Tabular Integration by Parts

One can use a table to perform repeated integration by parts quickly. Suppose we would like to use integration by parts to change the integral of $v(x)u^{(4)}(x)$ into boundary terms plus the integral of $v^{(4)}(x)u(x)$. In the first column, we write $v(x)$ and its derivatives beneath it. In the second column, we write $u^(4)(x)$ and its anti-derivatives beneath it. At the end of this process, the table should look as follows.

$$\begin{array}{c|c} v(x) & u^{(4)}(x) \\ v'(x) & u'''(x) \\ v''(x) & u''(x) \\ v'''(x) & u'(x) \\ v^{(4)}(x) & u(x) \end{array}$$

Next, each row of the left column is given alternating signs, beginning with positive.

$$\begin{array}{c|c} +v(x) & u^{(4)}(x) \\ -v'(x) & u'''(x) \\ +v''(x) & u''(x) \\ -v'''(x) & u'(x) \\ +v^{(4)}(x) & u(x) \end{array}$$

Next we multiply diagonal terms and add each product.

$$\begin{array}{c|c} +v(x) & u^{(4)}(x) \\ -v'(x) & u'''(x) \\ +v''(x) & u''(x) \\ -v'''(x) & u'(x) \\ +v^{(4)}(x) & u(x) \end{array}$$

These are the boundary terms. Lastly, we multiply the bottom terms together.

$$\begin{array}{c|c} +v(x) & u^{(4)}(x) \\ -v'(x) & u'''(x) \\ +v''(x) & u''(x) \\ -v'''(x) & u'(x) \\ +v^{(4)}(x) \longrightarrow & u(x) \end{array}$$

This is the integrand. The result at the end of this is

$$\int_a^b v(x)u^{(4)}(x)dx = \left(v(x)u'''(x) - v'(x)u''(x) + v''(x)u'(x) - v'''(x)u(x)\right)\Big|_a^b + \int_a^b v^{(4)}(x)u(x)dx. \quad \text{(A.1)}$$

B The Mean Value Theorem

The mean value theorem states that for all $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ such that f is continuous on $[\mathrm{a}, \mathrm{b}]$, and differentiable on (a, b), then

$$\exists c \in (\mathrm{a}, \mathrm{b}) : f'(\mathrm{c}) = \frac{f(\mathrm{b}) - f(\mathrm{a})}{\mathrm{b} - \mathrm{a}} \tag{B.1}$$

and thus,

$$f'(c)(\mathrm{b} - \mathrm{a}) = f(\mathrm{b}) - f(\mathrm{a}) \tag{B.2}$$

By integrating $f'(x)$, we see that

$$\begin{aligned} \int_{\mathrm{a}}^{\mathrm{b}} f'(x)dx &= f(\mathrm{b}) - f(\mathrm{a}) \\ &= f'(\mathrm{c})(\mathrm{b} - \mathrm{a}) \end{aligned} \tag{B.3}$$

C Big O Notation

Big O notation is used to describe the limiting behavior of a function as the argument tends to some value or infinity. By $f(x) = O(g(x)$ as $x \to x_0$ we mean that $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$. For example

$$\sin 6x = O(1) \text{ as } x \to \infty. \tag{C.1}$$

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