Problem Set 11

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1. Suppose that A is a set with exactly 4 elements. What is the maximum number of elements that a relation, $\mathcal{R}$, on A can contain so that $\mathcal{R} \cap \mathcal{R}^{-1} = \emptyset$? The maximum number of elements of $\mathcal{R}$ is 6.
   PROOF: Let $(a, b) \in \mathcal{R}$. Now, by definition, $(b, a) \in \mathcal{R}^{-1}$. Since $\mathcal{R} \cap \mathcal{R}^{-1} = \emptyset$, it follows that $(b, a) \notin \mathcal{R}$. Thus, $(a, b) \neq (b, a)$ and $a \neq b$. With 4 elements of A, there are four elements of $\mathcal{R}$ for which $a = b$. Since $|A \times A| = 2^{|A|} = 2^4 = 16$, there are 12 elements, (a, b), of $A \times A$ such that $a \neq b$. Now, for each element $(a, b) \in A \times A$, there exists $(b, a) \in A \times A$. Thus there are six pairs of elements $\{(a, b), (b, a)\} \subset A \times A$. If both elements in any of those pairs are in $\mathcal{R}$, then they are also both in $\mathcal{R}^{-1}$. So, at most one element from each pair may be in $\mathcal{R}$. Therefore there can be at most 6 elements in $\mathcal{R}$.

2. (a)

3. Determine the maximum number of elements in a relation $\mathcal{R}$ on $A = \{a, b, c\}$ such that $\mathcal{R}$ has none of the properties reflexive, symmetric and transitive. Note that $A \times A = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, a), (b, c), (c, a), (c, b)\}$ Suppose that $R = \{(a, a), (b, b), (a, b), (a, c)\}$. Now, none of the other elements of $A \times A$ can be included in $\mathcal{R}$. The inclusion element (c, c) would result in the relation being reflexive. Including (b, a), or (c, a), would result in the

4. $\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (4, 5), (5, 4), (1, 5), (5, 1), (2, 6), (6, 2)\}$

5. Let $H = \{2^m | m \in \mathbb{Z}\}$. Define $\mathcal{R}$ to be the relation defined on $\mathbb{Q}^+$ by $a\mathcal{R}b$ if $a/b \in H$.

   (a) Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{Q}^+$.
   To show that the relation is reflexive, consider $a \in \mathbb{Q}^+$. Now, $a/a = 1 = 2^0 \in H$. To show that the relation is symmetric consider $a, b \in \mathbb{Q}^+$. Suppose, $a/b \in H$. Now, $a/b = 2^m$ for some $m \in \mathbb{Z}$. Since, $b/a = (a/b)^{-1} = 2^{-m} \in H$. Thus, if $(a, b) \in \mathcal{R}$ then $(b, a) \in \mathcal{R}$ for all $a, b \in \mathbb{Q}^+$. To show that the relation is transitive, suppose that $a/b \in H$ and $b/c \in H$ for some $a, b, c \in \mathbb{Q}^+$. Now, $a/b = 2^k$ and $b/c = 2^l$ for some $k, l \in \mathbb{Q}^+$. Since, $(a/b)(b/c) = a/c$, it follows that $a/c = 2^k \cdot 2^l = 2^{k+l} \in H$. Therefore $\mathcal{R}$ is an equivalence relation.

   (b) Describe the elements in the equivalence class [3].
   $[3] = \{3 \cdot 2^n | n \in \mathbb{Z}\}$.

6. Recall that relations on a set A are, by definition, subsets of $A \times A$.

   (a) PROVE: The intersection of two equivalence relations on a non-empty set A is also an equivalence relation on A. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be equivalence relatiosn on a set, A. Now, by the reflexivity property of equivalence relations, for all $a \in A$, $(a, a) \in \mathcal{R}_1$ and $(a, a) \in \mathcal{R}_2$. Thus, $(a, a) \in \mathcal{R}_1 \cap \mathcal{R}_2$, so the intersection of two equivalence relations defined on A is reflexive. Now, for all $a, b \in A$, if

7. Define a relation $\mathcal{R}$ on $\mathbb{Z}$ by $x\mathcal{R}y$ exactly when $x^3 \equiv y^3 \pmod 4$, and assume $\mathcal{R}$ is an equivalence relation. Determine the equivalence classes of $\mathcal{R}$.
   PROOF: