Green's Functions

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1 Introduction

The first part of this text is primarily concerned with solutions to differential equations of the form

$$Lu = \phi \tag{1.1}$$

over an interval $a \leq x \leq b$ and subject to boundary conditions $\{B_1, \ldots, B_n\}$, where L is an nth order linear ordinary differential operator. For L to be linear it must satisfy the conditition

$$L(\alpha v + \beta w) = \alpha L v + \beta L w \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant. For this condition to be met L must be of the form

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x) \tag{1.3}$$

The boundary conditions are linear functionals of the form

$$B_j(u) = c_j; \quad j = 1, 2, \ldots, n \tag{1.4}$$

where c_j is an arbitrary constant.

Here, functional, refers to a transformation which has a set of functions as its domain and a set of numbers. As an example

$$B(u) = u(0) = 0 \tag{1.5}$$

is a simple boundary condition for a 1st order differential operator. Specifically, our B_j's will be limited to linear combinations of u and it's derivatives up to order $n-1$. These boundary conditions have the same linearity constraints as the differential operator L.

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and it's boundary conditions we will need the formal adjoint operator. This operator, which we will call L^*, can be found via repeated integration by parts. In general

$$\int_a^b vLu\,dx = [\cdots]\Big|_a^b + \int_a^b uL^*v\,dx \tag{2.1}$$

Here, u and v are completely arbitrary while being sufficiently differentiable for L and $L*$ to exist.

As an example, consider

$$L = A(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + C(x) \tag{2.2}$$

To find L^* perform integration by parts on the on each term of the product vLu a number of times equal to the order of the derivative that is a part of the term. That is to say, twice on the first

term, once, on the second, and not at all on the third. Doing this, we are left with

$$\int_a^b vLudx = \int_a^b (vAu'' + vBu' + vC)dx$$

$$= (vAu' + vBu)\Big|_a^b + \int_a^b (-(vA)'u' - (vB)'u + vCu)dx$$

$$= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b ((vA)''u - (vB)'u + vCu)dx \qquad (2.3)$$

$$= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b u((vA)'' - (Bv)' + Cv)dx$$

From this it is clear that

$$L^*v = (Av)'' - (Bv)' + Cv$$
$$= (A'v + Av')' - B'v - Bv' + Cv \qquad (2.4)$$
$$= Av'' + (2A' - B)v' + (A'' - B' + C)$$

and so the formal adjoint of a second order linear differential operator L must be of the form

$$L^* = A\frac{d^2}{dx^2} + (2A' - B)\frac{d}{dx} + (A'' - B' + C) \qquad (2.5)$$

If L^* is found to be equal to L then L is called formally self adjoint. By comparing equations (2.2) and (2.5) we can see that for a second order linear differentiable operator to be formally self adjoint, A' must be equal to B. You may notice that $A'' - B' + C$ must also be equal to C but this is always true given that A' equals B.

If the boundary conditions on L are homogenous then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(Lu, v) = (L^*v, u) \qquad (2.6)$$

where (f, g) is the inner produce of f and g

$$(f, g) = \int_a^b f(x)g(x)dx \qquad (2.7)$$

We now can understand that the adjoint operator $\mathscr{L}^*$ must consist of L^* and boundary conditions to force the terms that come about from integrating by parts to be zero.

As an example of $\mathscr{L}$ and $\mathscr{L}^*$, consider $\mathscr{L}$ to consist of $L = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \le x \le 1$. Then

$$(Lu, v) = \int_0^1 u'vdx$$

$$= (uv)\Big|_0^1 - \int_0^1 uv'dx$$

$$= u(1)v(1) - u(0)v(0) + \int_0^1 uL^*vdx \qquad (2.8)$$

$$= u(1)(v(1) - 3v(0)) + \int_0^1 uL^*vdx$$

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Since the particular value of $u(1)$ is not given, we must make $v(1) - 3v(0)$ equal zero, because choosing $u(1) = 0$ would undully restrict our solution. Therefore $\mathscr{L}^*$ consists of L^* which is $-\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$. As a final note, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called self-adjoint.

3 The Delta Function

The Dirac delta function is integral to the method of green's functions. It is typically represented by the symbol δ and has the properties

$$\delta(x) = 0 \quad \text{for } t \neq 0 \tag{3.1}$$

and

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{3.2}$$

but its most important property is

$$\int_a^b f(x)\delta(x - t)dt = f(t), \quad a \leq t \leq b \tag{3.3}$$

If, in the previous integral, t is not between a and b, then the integral is equal to 0.

To attempt to find an intuition for the delta function, we can consider the idea of

While commonly referred to as a function, δ is more appropriately defined as a generalized function or a distribution.