Problem Set 6

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October 18, 2023

1. Let $x, y, z \in \mathbb{Z}$. Prove: If exactly two of x, y, z are even, then $3x + 5y + 7z$ is odd.
   Case 1: Let x, y be even and let z be odd. Then, $x = 2k$, $y = 2l$, and $z = 2m + 1$, for some $k, l, m \in \mathbb{Z}$. Now,

$$\begin{aligned} 3x + 5y + 7z &= 6k + 10l + 14m + 7 \\ &= 6k + 10l + 14m + 6 + 1 \\ &= 2(3k + 5l + 7m + 3) + 1 \end{aligned}$$

   which is odd, by definition.
   Case 2: Let x, z be even and let y be odd. Then, $x = 2k$, $y = 2l + 1$, and $z = 2m$, for some $k, l, m \in \mathbb{Z}$. Now,

$$\begin{aligned} 3x + 5y + 7z &= 6k + 10l + 5 + 14m \\ &= 6k + 10l + 14m + 4 + 1 \\ &= 2(3k + 5l + 7m + 2) + 1 \end{aligned}$$

   which is odd, by definition.
   Case 3: Let y, z be odd and let x be odd. Then, $x = 2k + 1$, $y = 2l$, and $z = 2m$, for some $k, l, m \in \mathbb{Z}$. Now,

$$\begin{aligned} 3x + 5y + 7z &= 6k + 3 + 10l + 14m \\ &= 6k + 10l + 14m + 2 + 1 \\ &= 2(3k + 5l + 7m + 1) + 1 \end{aligned}$$

   which is odd, by definition. ■

2. Let $a, b \in \mathbb{Z}$. Prove: If $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$.
   Case 1: Let $a = 1$ and $b = 4$. Then,

$$\begin{aligned} (a - b)^3 - 9(a - b) &= (1 - 4)^3 - 9(1 - 4) \\ &= -3^3 - 9 \cdot -3 \\ &= -27 + 27 \\ &= 0. \end{aligned}$$

Case 2: Let $a = 1$ and $b = 4$. Then,

$$\begin{aligned}(a-b)^3 - 9(a-b) &= (-1+4)^3 - 9(-1+4) \\ &= 3^3 - 9 \cdot 3 \\ &= 27 - 27 \\ &= 0.\end{aligned}$$

Case 3: Let $a = 2$ and $b = 2$. Then,

$$\begin{aligned}(a-b)^3 - 9(a-b) &= (2-2)^3 - 9(2-2) \\ &= 0^3 - 9 \cdot 0 \\ &= 0.\end{aligned}$$

Case 4: Let $a = -2$ and $b = -2$. Then,

$$\begin{aligned}(a-b)^3 - 9(a-b) &= (-2+2)^3 - 9(-2+2) \\ &= 0^3 - 9 \cdot 0 \\ &= 0.\end{aligned}$$

Therefore, $(a-b)^3 - 9(a-b) = 0$. ■

3. Let $a \in \mathbb{Z}$. Prove: If $3 \mid 2a$, then $3 \mid a$.
   By Result 4.8 from the textbook, if $3 \mid cd$, then $3 \mid c$ or $3 \mid d$, for some $c, d \in \mathbb{Z}$. Since $3 \mid 2a$ and $3 \nmid 2$, then it must be the case that $3 \mid a$.■

4. Let $x, y \in \mathbb{Z}$. Prove: If 3 divides neither x or y, then $3 \mid (x^2 - y^2)$.
   Since $(x^2 - y^2)$ can be factored into $(x+y)(x-y)$, $3 \mid (x^2 - y^2)$ exactly when $3 \mid (x+y)$ or $3 \mid (x-y)$. Proceeding by cases according to the remainder of 3 divided by x and the remainder of 3 divided by y.
   (i) Let $x = 3k+1$, and $y = 3l+1$ for some $k, l \in \mathbb{Z}$. Then,

$$\begin{aligned}x - y &= 3k + 1 - 3l - 1 \\ &= 3k - 3l \\ &= 3(k - l),\end{aligned}$$

   which is divisible by 3.
   (ii) Let $x = 3k+2$, and $y = 3l+2$ for some $k, l \in \mathbb{Z}$. Then,

$$\begin{aligned}x - y &= 3k + 2 - 3l - 2 \\ &= 3k - 3l \\ &= 3(k - l),\end{aligned}$$

   which is divisible by 3.
   (iii) Without loss of generality, let $x = 3k+1$, and $y = 3l+2$ for some $k, l \in \mathbb{Z}$. Then,

$$\begin{aligned}x + y &= 3k + 1 + 3l + 2 \\ &= 3(k + l + 1),\end{aligned}$$

   which is divisible by 3.
   The statement is, therefore, true. ■

5. Let $m, n \in \mathbb{N}$ such that $m \mid n$. Prove: if a and b are integers such that $a \equiv b(\text{mod } n)$, then $a \equiv b(\text{mod } m)$.
Since, $a \equiv b(\text{mod } n)$, then, $n \mid (b-a)$. Then $b-a = nc$, for some $c \in \mathbb{N}$, and given that $m \mid n$, then, $b - a = mcd$, for some $c, d \in \mathbb{N}$. Therefore, $m|(b-a)$, and by definition $a \equiv b(\text{mod } m)$. ■

6. Let $a_1, a_2, \ldots, a_n$, $n \geq 3$, be n integers such that $|a_{i+1} - a_i| \leq 1$ for $1 \leq i \leq n-1$. Prove: if k is any integer that lies strictly between a_1 and a_n, then there is an integer j with $i < j < n$ such that $a_j = k$.
Since, $|a_{i+1} - a_i| \leq 1$, and all a_j are integers, $a_{i+1} = a_i$, $a_{i+1} = a_i + 1$, or $a_{i+1} = a_i - 1$. Because each integer must be equal to or only differ from the previous by 1, then in order to progress from a_1 to a_n in the sequence, we must step through each integer between them. Each integer between a_1 and a_n must be contained in the sequence. ■

7. Let $n \in \mathbb{Z}$. Prove: $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.
($\Longrightarrow$) Since $2 \mid (n^4 - 3)$, then $n^4 - 3 = 2k$, for some $k \in \mathbb{Z}$. Then, $n^4 = 2k + 3 = 2(k+1) + 1$, and n^4 is therefore odd, and so n^2 must also be odd, and then n must be odd. Now, $n = 2l + 1$ for some $l \in \mathbb{Z}$, and therefore

$$\begin{aligned} n^2 + 3 &= 4l^2 + 4l + 4 \\ &= 4(l^2 + l + 1) \end{aligned}$$

which is divisible by 4.
($\Longleftarrow$) Since $4|(n^2 + 3)$, then, $n^2 + 3 = 4k$ for some $k \in \mathbb{Z}$. Then, $n^2 = 4b - 3$. Now,

$$\begin{aligned} n^4 - 3 &= (4k - 3)^2 - 3 \\ &= 16k^2 - 14k + 6 \\ &= 2(8k^2 - 12k + 3). \end{aligned}$$

Therefore $2|(n^4 - 3)$. ■

8. Let $a, b \in \mathbb{Z}$. Prove: $a^2 + 2b^2 \equiv 0(\text{mod } 3)$ if and only if either a and b are congruent to 0 mod 3 or neither is congruent to 0 mod 3.
We will prove the statement by the contrapositive. Assume that either a is congruent to 0 mod 3, or b is, but not both. There are 4 cases.
Case 1: Assume, $a \equiv 0(\text{mod } 3)$ and $b \equiv 1(\text{mod } 3)$. Then $a^2 + 2b^2 \equiv 0 + 2 \equiv 2$ (mod 3), and so, $a^2 + 2b^2 \not\equiv 0(\text{mod } 3)$
Case 2: Assume, $a \equiv 0(\text{mod } 3)$ and $b \equiv 2(\text{mod } 3)$. Then $a^2 + 2b^2 \equiv 0 + 2 \equiv 8 \equiv 2$ (mod 3), and so, $a^2 + 2b^2 \not\equiv 0(\text{mod } 3)$
Case 3: Assume, $a \equiv 1(\text{mod } 3)$ and $b \equiv 0(\text{mod } 3)$. Then $a^2 + 2b^2 \equiv 1 + 0 \equiv 1$ (mod 3), and so, $a^2 + 2b^2 \not\equiv 0(\text{mod } 3)$
Case 4: Assume, $a \equiv 2(\text{mod } 3)$ and $b \equiv 0(\text{mod } 3)$. Then $a^2 + 2b^2 \equiv 2 + 0 \equiv 2$ (mod 3), and so, $a^2 + 2b^2 \not\equiv 0(\text{mod } 3)$
Therefore, the contrapositive holds and we have shown the original statement to be true. ■