Green's Functions

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Contents

1 Introduction

1.1 Nonhomogeneous Linear Differential Equations

This text is concerned with the solutions to non-homogeneous linear differential equations, which have the form

$$\mathbf{L}u = \phi, \tag{1.1}$$

over an interval a $\leq x \leq$ b and subject to certain boundary conditions, where $\mathbf{L}$ is an nth order linear ordinary differential operator and where the function ϕ is integrable on the given interval.[1]

Theorem 1.1. $\mathbf{L}$ *is linear if and only if it is of the form*

$$\mathbf{L} = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x). \tag{1.3}$$

Proof $\Longleftarrow$
We prove, by induction, that $\mathbf{D}^{(k)}$ is a linear differential operator for all $i \in \mathbb{N}$. Note: $f^{(i)} = \mathbf{D}^{(i)}f$
Base Case Let $k = 1$.

$$\begin{aligned}
\mathbf{D}(\alpha u(x) + \beta v(x)) &= \lim_{h\to 0}\frac{\alpha u(x+h) + \beta v(x+h) - (\alpha u(x) + \beta v(x))}{h} \\
&= \alpha\frac{u(x+h) - u(x)}{h} + \beta\frac{v(x+h) - v(x)}{h} \\
&= \alpha\mathbf{D}u + \beta\mathbf{D}v
\end{aligned}$$

Induction Step Suppose that the statement holds for some $k \in \mathbb{N}$, $k > 1$. Then,

$$\begin{aligned}
\mathbf{D}^{k+1}(\alpha u + \beta v) &= \mathbf{D}(\mathbf{D}^k(\alpha u + \beta v)) \\
&= \mathbf{D}(\alpha\mathbf{D}^k u + \beta\mathbf{D}^k v) \\
&= \alpha\mathbf{D}(\mathbf{D}^k u) + \beta\mathbf{D}(\mathbf{D}^k v) \\
&= \alpha\mathbf{D}^{k+1}u + \beta\mathbf{D}^{k+1}v
\end{aligned}$$

It follows that

$$\sum_{k=0}^{n} a_k(x)D^k$$

is also linear because $a_k(x)D^k$.
$\Longrightarrow$
Let

$$\begin{aligned}
\mathbf{L}u &= f(x, \underbrace{u, u', u'', \ldots, u^{(n)}}_{\mathbf{u}}) \\
&= f(x, \mathbf{u}).
\end{aligned}$$

[1]For $\mathbf{L}$ to be linear, it must satisfy the condition

$$\mathbf{L}(\alpha v + \beta w) = \alpha\mathbf{L}v + \beta\mathbf{L}w \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant.

Then,

$$\mathbf{L}(\alpha v + \beta w) = f(x, \alpha\mathbf{v} + \beta\mathbf{w})$$

and

$$\alpha\mathbf{L}v + \beta\mathbf{L}w = \alpha f(x, \mathbf{v}) + \beta f(x, \mathbf{w}).$$

If $\mathbf{L}$ is linear then

$$\mathbf{L}(\alpha v + \beta w) = \alpha\mathbf{L}v + \beta\mathbf{L}w$$

or

$$f(x, \alpha\mathbf{v} + \beta\mathbf{w}) = \alpha f(x, \mathbf{v}) + \beta f(x, \mathbf{u})$$

It follows that

$$\begin{aligned} f(x, \mathbf{u} + \epsilon\mathbf{v}) &= f(x, \mathbf{u}) + \epsilon f(x, \mathbf{v}) \\ \frac{f(x, \mathbf{u} + \epsilon\mathbf{v}) - f(x, \mathbf{u})}{\epsilon} &= f(x, \mathbf{v}) \end{aligned}$$

Next, we take the limit as $\epsilon \to 0$.

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{f(x, \mathbf{u} + \epsilon\mathbf{v}) - f(x, \mathbf{u})}{\epsilon} &= \mathbf{D}_{\mathbf{v}} f(x, \mathbf{u}) \\ &= \mathbf{v} \cdot \nabla f(x, \mathbf{u}) \\ &= f(x, \mathbf{v}). \end{aligned} \tag{1.4}$$

Let

$$\mathbf{u} = \mathbf{v} = \langle 0, \ldots, u^{(i)}, 0, \ldots, 0 \rangle$$

and

$$f(x, u^{(i)}) = f(x, \langle 0, \ldots, u^{(i)}, 0, \ldots, 0 \rangle).$$

Then it follows from equation (1.4) that,

$$u^{(i)} \frac{\partial f(x, u^{(i)})}{\partial u^{(i)}} = f(x, u^{(i)}).$$

Thus,

$$f(x, u^{(i)}) = a_i(x) u^{(i)}$$

and so

$$\begin{aligned} f(x, \mathbf{u}) &= f(x, \sum_i \langle 0, 0, \ldots, u^(i), \ldots, 0, 0 \rangle) \\ &= \sum_i f(x, u^{(i)}) \\ &= \sum_i a_i(x) u^{(i)}(x) \\ &= a_n(x) \frac{d^n}{dx^n} u + a_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} u + \cdots + a_0(x) u. \end{aligned}$$

∎

Since $\mathbf{L}$ is of order n, there will be n boundary conditions of the general form

$$\mathbf{B}_j(u) = c_j; \quad j = 1, 2, \ldots, n, \tag{1.5}$$

where the $\mathbf{B}_j$'s are prescribed functionals [2] and c_j's are prescribed constants. We will only consider $\mathbf{B}_j$'s that are linear combinations of u and its derivatives through order $n-1$ and evaluated at the endpoints, a and b.

For $\mathbf{B}_j$ to be **linear**, it must satisfy the condition

$$\mathbf{B}_j(\alpha v + \beta w) = \alpha \mathbf{B}_j(v) + \beta \mathbf{B}_j(w). \tag{1.7}$$

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and its boundary conditions, begin by finding the **adjoint operator**, denoted $\mathscr{L}^*$. The adjoint operator consists of the formal adjoint, $\mathbf{L}^*$, and the boundary conditions associated with the Green's function. To determine these, first form the product, $v\mathbf{L}u$, and integrate it over the interval of interest. By repeated integration by parts, we can express the integral in the form (see appendix A.1 for information about integration by parts)

$$\int_{\mathrm{a}}^{\mathrm{b}} v\mathbf{L}u dx = [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u\mathbf{L}^* v dx, \tag{2.1}$$

where $[\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}}$ represents the boundary terms resulting from successive integration by parts. Here, u and v must be sufficiently differentiable functions so that the left and right sides are well defined.

Example 2.1 Consider the general second-order linear differentiable operator

$$\mathbf{L} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x). \tag{2.2}$$

To find $\mathbf{L}^*$, perform integration by parts on each term of the product $v\mathbf{L}u$ until the integrand has the form $u\mathbf{L}^*v$. That is to say, integrate by parts twice on the first term, once on the second, and

[2] A **functional** is a transformation with a set of functions as its domain and a set of numbers as its range. To illustrate, consider the functional

$$\mathscr{F}(u) = \int_0^1 u^2(x) dx. \tag{1.6}$$

The domain of this functional might be the set of functions defined over the interval $[0, 1]$ and for which the integral of u^2 from 0 to 1 exists. The range is $[0, \infty)$.

not at all on the third. Doing this, we are left with

$$
\begin{aligned}
\int_a^b v\mathbf{L}u\,dx &= \int_a^b (vau'' + vbu' + vc)dx \\
&= (vau' + vbu)\Big|_a^b + \int_a^b (-(va)'u' - (vb)'u + vcu)dx \\
&= (vau' + vbu - (va)'u)\Big|_a^b + \int_a^b ((va)''u - (vb)'u + vcu)dx \\
&= (vau' + vbu - (va)'u)\Big|_a^b + \int_a^b u((va)'' - (bv)' + cv)dx.
\end{aligned}
\tag{2.3}
$$

From this, it is clear that

$$
\begin{aligned}
\mathbf{L}^*v &= (av)'' - (bv)' + cv \\
&= (a'v + av')' - b'v - bv' + cv \\
&= av'' + (2a' - b)v' + (a'' - b' + c)
\end{aligned}
\tag{2.4}
$$

and so the formal adjoint of the second-order linear differential operator L must be of the form

$$
\mathbf{L}^* = a\frac{d^2}{dx^2} + (2a' - b)\frac{d}{dx} + (a'' - b' + c). \tag{2.5}
$$

If $\mathbf{L}^* = \mathbf{L}$, then $\mathbf{L}$ is called **formally self-adjoint**. By comparing equations (2.2) and (2.5), we can see that for a second-order linear differentiable operator to be formally self-adjoint, it is sufficient that $a' = b$ since this implies $2a' - b = a'$ and $a'' - b' + c = a'' - a'' + c = c$.

Theorem 2.1. *For any such* $\mathbf{L}$, $\sigma\mathbf{L}$ *is self adjoint if*

$$
\sigma = \exp\left(\int \frac{b - a'}{a}dx\right). \tag{2.6}
$$

Proof It follows from the previous example that $\sigma\mathbf{L}$ is self adjoint if

$$
\frac{d}{dx}\left(a(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx}\right) = b(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx}.
$$

Evaluating $\frac{d}{dx}\left(a(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx}\right)$, we find

$$
\begin{aligned}
\frac{d}{dx}(a(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx}) &= a'(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx} + a(x)\frac{b(x) - a'(x)}{a(x)}e^{\int \frac{b(x)-a'(x)}{a(x)}dx} \\
&= a'(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx} + b(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx} - a'(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx} \\
&= b(x)e^{\int \frac{b(x)-a'(x)}{a(x)}dx}.
\end{aligned}
$$

∎

Definition If the boundary conditions on $\mathbf{L}$ are homogeneous[3], then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(\mathbf{L}u, v) = (u, \mathbf{L}^* v) \tag{2.7}$$

where (f, g) is the **inner product** of f and g,

$$(f, g) = \int_a^b f(x)g(x)dx. \tag{2.8}$$

This means that the adjoint operator $\mathscr{L}^*$ consists of $\mathbf{L}^*$ and boundary conditions for which the boundary terms of the integral are zero.

Example 2.2 Consider $\mathscr{L}$ to consist of $\mathbf{L} = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \leq x \leq 1$. Then

$$\begin{aligned}
(\mathbf{L}u, v) &= \int_0^1 u'v dx \\
&= (uv)\Big|_0^1 - \int_0^1 uv' dx \\
&= u(1)v(1) - u(0)v(0) + \int_0^1 u\mathbf{L}^* v dx \\
&= u(1)(v(1) - 3v(0)) + \int_0^1 u\mathbf{L}^* v dx
\end{aligned} \tag{2.9}$$

Since the particular value of $u(1)$ is not given, we must impose the condition $v(1) - 3v(0) = 0$, because choosing $u(1) = 0$ would unduly restrict our solution. Therefore $\mathscr{L}^*$ consists of $\mathbf{L}^* = -\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$.

As a final note, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called **self-adjoint**.

3 The Dirac delta function

3.1 Delta Sequences

In physics, we often consider the idea of a point mass. Suppose we have a unit point mass at $x = 0$ with mass density given by $w(x)$. We are interested in the mass but do not know the details of its density. We do, however, know that the $w(x)$ will be highly localized in space and that

$$\int_{-\infty}^{\infty} w(x)dx = 1, \tag{3.1}$$

so that the net mass is unity.

[3]By **homogeneous**, we mean that each boundary condition is of the form $\mathbf{B}_j(u) = 0$.

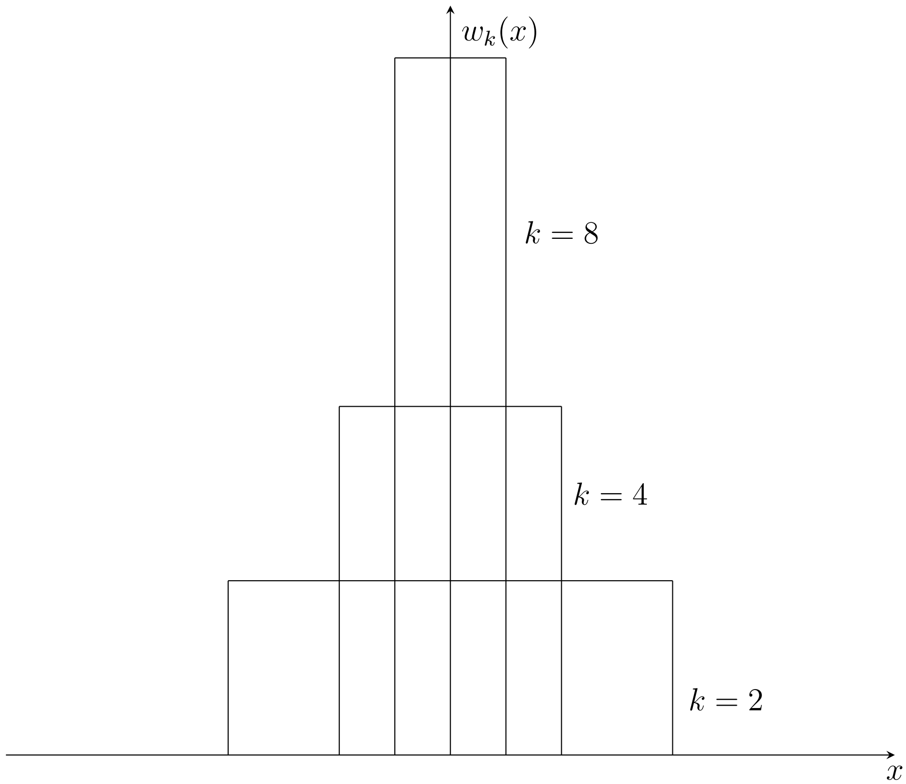

Figure 3.1: Mass Density; eq 3.2

We expect two highly concentrated unit mass densities to produce masses with nearly identical physical effects. As such, we might simplify the problem by deciding, a priori, on a definite form for w, such as

$$w_k(x) = \begin{cases} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases} \tag{3.2}$$

or

$$w_k(x) = \frac{k}{\pi(1 + k^2x^2)}, \tag{3.3}$$

where k is some larger natural number. In Fig 3.2, we see that w becomes highly concentrated at $x = 0$ when k is large.

If we let $k \to \infty$, then the mass distribution approaches our idea of a point mass at $x = 0$. We would like to write

$$\delta(x) \stackrel{?}{=} \lim_{k\to\infty} w_k(x) \tag{3.4}$$

where, $\delta(x)$ is the **Dirac delta function**. This "definition" feels intuitive, but it is not a rigorous definition of the Dirac delta function because the limit is infinite for $x = 0$. That is, the right-hand side is not a function. We instead define the Dirac delta function, $\delta(x)$, in the following way

$$\begin{aligned} \lim_{k\to\infty} \int_{-\infty}^{\infty} h(x)w_k(x)dx &= \int_{-\infty}^{\infty} h(x)\delta(x)dx \\ &= h(0) \end{aligned} \tag{3.5}$$

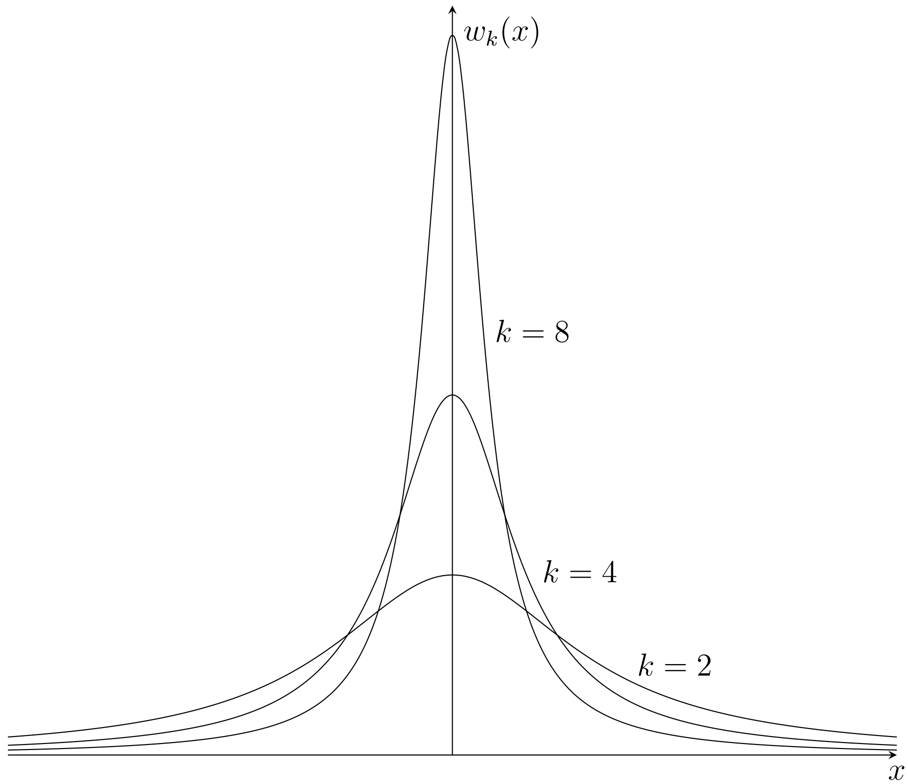

Figure 3.2: Mass Density; eq. 3.3

and call any $w_k(x)$ which has this property a **δ-sequence**. This way of defining the Dirac delta function is more rigorous while still being as intuitive as equation (3.4). However, keep in mind that the delta function is not a function.

We would like to be able to check that a particular sequence $w_k(x)$, is a $\delta - sequence$. We can do this by showing that equation (3.5), holds for $w_k(x)$. However, for certain, $w_k(x)$, it is sufficient to show that $\int_{-\infty}^{\infty} w_k(x)dx = 1$.

Definition A function, f, is **uniformly continuous** if for all $\epsilon > 0$, a $\delta > 0$ exists such that, if $|a - b| < \delta$, then $|f(a) - f(b)| < \epsilon$, for all $a, b \in X$.

Theorem 3.1. *If $w(x)$ is non-negative $\int_{-\infty}^{\infty} w(x)dx = 1$, and $w(x) = O(1/x^{1+\alpha})$ (see appendix A.3 for a description of big O notation) as $|x| \to \infty$ with $\alpha > 0$, then $w_k(x) \equiv kw(kx)$ is a δ-sequence.*

Proof We have

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(x)dx = \underbrace{\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)[h(x) - h(0)]dx}_{I} + \underbrace{\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(0)dx}_{J}. \tag{3.6}$$

Consider J,

$$
\begin{aligned}
J &= h(0) \lim_{k\to\infty} \int_{-\infty}^{\infty} kw(kx)dx \text{ \& Let } \xi = kx \\
&= h(0)k \to \infty \int_{-\infty}^{\infty} w(\xi)d\xi \\
&= h(0).
\end{aligned}
\tag{3.7}
$$

Next, we wish to show that $I = 0$, so that the right hand side of equation (3.6) is $h(0)$. Let $\epsilon > 0$, and since h is continuous at $x = 0$, there must exist a number $\delta > 0$ such that $|h(x) - h(0)| < \epsilon$ whenever $|x - 0| = |x| < \delta$. Breaking up the integral I,

$$
\begin{aligned}
I = &\underbrace{\lim_{k\to\infty} \int_{-\infty}^{-\delta} w_k(x)(h(x) - h(0))dx}_{I_1} + \underbrace{\lim_{k\to\infty} \int_{-\delta}^{\delta} w_k(x)(h(x) - h(0))dx}_{I_2} \\
&+ \underbrace{\lim_{k\to\infty} \int_{\delta}^{\infty} w_k(x)(h(x) - h(0))dx}_{I_3}.
\end{aligned}
\tag{3.8}
$$

Clearly, $\lim_{k\to\infty} w_k(x) = 0$, for each fixed $x \neq 0$, because $w_k(x) = kw(kx) = O(k \cdot k^{-1-\alpha}|x|^{-1-\alpha}) = O(k^{-\alpha}) \to 0$ as $k \to \infty$. Since $w_k(x) \to 0$ uniformly, over $-\infty < x < -\delta$ and $\delta < x < \infty$, then $I_1 = I_3 = 0$. By the definition of uniform continuity $|h(x) - h(0)| < \epsilon$ whenever $|x - 0| = |x| < \delta$. Because ϵ can be selected as small as we like, $|h(x) - h(0)| = 0$. As a result,

$$
\begin{aligned}
I_2 &= \lim_{k\to\infty} \int_{-\delta}^{\delta} w_k(x)(0)dx \\
&= 0
\end{aligned}
$$

Thus, $I = 0$ and so

$$
\lim_{k\to\infty} \int_{-\infty}^{\infty} w_k(x)h(x)dx = h(0). \tag{3.9}
$$

3.2 The Dirac Delta Function as a Generalized Function

The Dirac delta function can also be defined as a generalized function. To understand this way of defining δ, we will begin by defining some terms.

Definition A **closed region** is one that includes its endpoints.

Definition The **support** of a function, f, is the subset, $\mathscr{S}$, of the domain of f such that for all $x \in \mathscr{S}$, $f(x) \neq 0$.

Definition A function has **compact support** if the subset of its domain for which its range is non-zero is closed and bounded.

We will call the space of infinitely differentiable functions with compact support $\mathscr{D}$.

Definition Generalized functions are linear functionals that are uniformly continuous on $\mathscr{D}$, such that all generalized functions have derivatives which are also generalized functions.

We consider the following functional,

$$\mathscr{F}(h) = \int_{-\infty}^{\infty} g(x)h(x)dx. \tag{3.10}$$

This functional assigns a numerical value, $\mathscr{F}(h)$, for each function h within the domain, $\mathscr{D}$, of $\mathscr{F}$.

Example 3.1 Suppose $\mathscr{F}(h)$ is the integral of h from ξ to ∞.

$$\int_{\infty}^{\infty} g(x)h(x)dx = \int_{\xi}^{\infty} h(x)dx \tag{3.11}$$

Then, $g(x)$ is equivalent to the **Heaviside step function**,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ \frac{1}{2}, & x = 0 \end{cases} \tag{3.12}$$

which is a function in the classical sense. [4]

If $\mathscr{F}(h)$ is $h(0)$ so that

$$\int_{-\infty}^{\infty} g(x)h(x)dx = h(0) \tag{3.13}$$

then it can be shown that there is no function, $g(x)$, which exists such that equation (3.13) is true for all functions, $h(x)$, in the domain, $\mathscr{D}$. We call g defined by equation (3.13) a generalized function, and in particular, it is the Dirac delta function. As such, δ is defined in the following way.

$$\int_{-\infty}^{\infty} \delta(x)h(x)dx = h(0) \tag{3.14}$$

for all $h \in \mathscr{D}$.

Although $\delta(x)$ has support at $x = 0$, it can be adjusted to have support at any point by shifting the argument. Thus, $\delta(x - \xi)$ acts at $x = \xi$,

$$\int_{-\infty}^{\infty} \delta(x - \xi)h(x)dx = h(\xi). \tag{3.15}$$

Definition We define the derivative of a generlaized function, g', in relation to g by performing integration by parts on

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = g(x)h(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(x)h'(x)dx. \tag{3.16}$$

[4] We have defined $H(0)$ to be $\frac{1}{2}$, which is a common convention. However, for our purposes, the value at any particular point is not important since we are only ever interested in integrating the function.

The integral term is fairly simple to interpret since it is of the same form as equation (3.10), but the boundary term is not as nice because it involves knowing the values of g, which are never known. However, our restriction that h has compact support gives the intuition that it must vanish at infinity, and since we are integrating from $-\infty$ to ∞, the boundary term must be zero,

$$\int_{-\infty}^{\infty} g'(x)h(x)dx = -\int_{-\infty}^{\infty} g(x)h'(x)dx. \tag{3.17}$$

For the Dirac delta function, this means

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} \delta(x-\xi)h'(x)dx \\ &= -h'(\xi).\end{aligned} \tag{3.18}$$

Theorem 3.2. *The jth derivative of the Dirac delta function is defined by*

$$\int_{-\infty}^{\infty} \delta^{(j)}(\xi - x)h(\xi)d\xi = (-1)^j h^{(j)}(x). \tag{3.19}$$

Proof We prove the statement by induction.
Base case: Proven to be true for $k = 1$ in equation (3.18).
Induction step: Let $k \in \mathbb{N}$ and suppose the statements holds for some $k \geq 1$. Then,

$$\begin{aligned}\int_{-\infty}^{\infty} \delta^{(k+1)}(\xi - x)h(\xi)d\xi &= \delta^{(k)}(\xi - x)h(\xi)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta^{(k)}(\xi - x)h'(\xi)d\xi \\ &= -\int_{-\infty}^{\infty} \delta^{(k)}(\xi - x)h'(\xi)d\xi \\ &= -(-1)^k (h')^{(k)}(x) \\ &= (-1)^{k+1} h^{(k+1)}(x)\end{aligned}$$

∎

Note that because of the discontinuity in $H(x-\xi)$ at the point $x = \xi$, the derivative of H does not exist as an ordinary function. However, the previous method does allow us to find $H'(x-\xi)$ as a generalized function,

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x-\xi)h(x)dx &= -\int_{-\infty}^{\infty} H(x-\xi)h'(x)dx \\ &= -\int_{\xi}^{\infty} h'(x)dx \\ &= h(\xi).\end{aligned} \tag{3.20}$$

Since

$$\int_{-\infty}^{\infty} \delta(x-\xi)h(x)dx = h(\xi) \tag{3.21}$$

it must be the case that, in the sense of generalized functions,

$$H'(x-\xi) = \delta(x-\xi). \tag{3.22}$$

Such equalities between generalized functions, as seen in (3.18), are understood in the sense that if some h in $\mathscr{D}$ is multiplied through, and we integrate over $(-\infty, \infty)$ then the result will hold. To wit, we consider generalized functions, g_1 and g_2, to be equal if, for all $h \in \mathscr{D}$,

$$\int_{-\infty}^{\infty} g_1(x)h(x)dx = \int_{-\infty}^{\infty} g_2(x)h(x)dx. \tag{3.23}$$

Notice that for all $n > 0$

$$x^n \delta(x) = 0 \tag{3.24}$$

as a result of

$$\int_{-\infty}^{\infty} x^n \delta(x)h(x)dx = [x^n h(x)]|_{x=0} = 0. \tag{3.25}$$

Example 3.2 We would like to show that the sequence

$$w_k(x) = \begin{cases} k, & 0 < x < \frac{1}{k} \\ 0, & x \leq 0 \text{ or } x \geq \frac{1}{k} \end{cases}$$

is a δ-sequence using theorem (3.1). It is clear that $w_k(x) \geq 0$ for all x and $w(x) = O(1/|x|^{1+\alpha})$ as $|x| \to \infty$ with $\alpha > 0$ since it is zero when $x \leq 0$ or $x \geq 1/k$. Lastly, we must show that the area under $w(x)$ is 1. Choosing $w(x)$ so that $kw(kx) = w_k(x)$

$$w(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \leq 0 \text{ or } x \geq 1. \end{cases}$$

Integrating:

$$\begin{aligned} \int_{-\infty}^{\infty} w(x)dx &= \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{\infty} 0dx \\ &= \int_{0}^{1} 1dx \\ &= 1 \end{aligned} \tag{3.26}$$

Example 3.3 We would like to show that the sequence,

$$w_k(x) = \begin{cases} -k, & |x| < \frac{1}{2k} \\ 2k, & \frac{1}{2k} \leq |x| \leq \frac{1}{k} \\ 0, & |x| > \frac{1}{k}, \end{cases} \tag{3.27}$$

is a delta sequence. Theorem (3.1) does not apply in this case because $w_k(x)$ is negative for some values of x so we should instead show

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} w_k(x)h(x)dx = h(0). \tag{3.28}$$

To begin, we break up the integral

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} w_k(x)h(x)dx = \lim_{k\to\infty}\int_{-\frac{1}{k}}^{-\frac{1}{2k}} 2kh(x)dx - \lim_{k\to\infty}\int_{-\frac{1}{2k}}^{\frac{1}{2k}} kh(x)dx \\ + \lim_{k\to\infty}\int_{\frac{1}{2k}}^{\frac{1}{k}} 2kh(x)dx \tag{3.29}$$

Consider each integral. Let $-\frac{1}{k} < \xi < -\frac{1}{2k}$. By the mean value theorem

$$\begin{aligned}\lim_{k\to\infty}\int_{-\frac{1}{k}}^{-\frac{1}{2k}} 2kh(x)dx &= \lim_{k\to\infty}\left(\left(-\frac{1}{2k}+\frac{1}{k}\right)\cdot 2k\cdot h(\xi)\right)\\ &= h(0).\end{aligned}$$

$$\begin{aligned}\lim_{k\to\infty}\int_{-\frac{1}{2k}}^{\frac{1}{2k}} -kh(x)dx &= \lim_{k\to\infty}\left(\left(\frac{1}{2k}+\frac{1}{2k}\right)\cdot -k\cdot h(\xi)\right)\\ &= -h(0).\end{aligned}$$

$$\begin{aligned}\lim_{k\to\infty}\int_{\frac{1}{2k}}^{\frac{1}{k}} 2kh(x)dx &= \lim_{k\to\infty}\left(\left(\frac{1}{k}-\frac{1}{2k}\right)\cdot 2k\cdot h(\xi)\right)\\ &= h(0).\end{aligned}$$

The last step for each integral follows from the fact that as $k\to\infty$ $\xi\to 0$. Adding each integral shows that

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} w_k(x)h(x)dx = h(0).$$

Example 3.4 We would like to show that $e^x\delta(x) = \delta(x)$. To begin with, we integrate $e^x\delta(x)f(x)$ and would like to show that this is equal to $f(0)$ to satisfy the generalized function definition of the delta function.

$$\begin{aligned}\int_{-\infty}^{\infty} e^x\delta(x)f(x)dx &= \int_{-\infty}^{\infty}\delta(x)\underbrace{e^xf(x)}_{g(x)}dx\\ &= \int_{-\infty}^{\infty}\delta(x)g(x)dx\\ &= g(0)\\ &= e^0f(0)\\ &= f(0)\end{aligned} \tag{3.30}$$

Therefore

$$\int_{-\infty}^{\infty} e^x\delta(x)f(x)dx = \int_{-\infty}^{\infty}\delta(x)f(x)dx \tag{3.31}$$

Example 3.5 We would like to show that $\frac{d^4}{dx^4}|x|^4 = 12\delta(x)$. Note that $|x| = 2xH(x) - x$. We begin by manipulating $|x|^3$ into a more convenient form.

$$\begin{aligned} |x|^3 &= (2xH(x) - x)^3 \\ &= x^3(2H(x) - 1)^3 \end{aligned} \tag{3.32}$$

Next, it is useful to simplify.
Note: $H^n(x) = H(x)$, $n \in \mathbb{N}$.

$$\begin{aligned} (2H(x) - 1)^3 &= (2H(x))^3 - 3(2H(x))^2 + 3(2H(x)) - 1 \\ &= 8H(x) - 12H(x) + 6H(x) - 1 \\ &= 2H(x) - 1 \end{aligned} \tag{3.33}$$

Finally, we the fourth derivative.

$$\begin{aligned} D^4(x^3(2H(x) - 1)) &= D^3(3x^2(2H(x) - 1) + 2x^3\delta(x)) \\ &= D^3(3x^2(2H(x) - 1)) \\ &= D^2(6x(2H(x) - 1) + 6x^2\delta(x)) \\ &= D^2(6x(2H(x) - 1)) \\ &= D(12(H(x) - 1) + 12x\delta(x)) \\ &= D(12(H(x) - 1)) \\ &= 12\delta(x) \end{aligned} \tag{3.34}$$

4 The Method of Green's Functions

4.1 Introduction

A **Green's function** is the solution to a differential equation of the form

$$\mathbf{L}_\xi^* G(\xi, x) = \delta(\xi - x). \tag{4.1}$$

The subscript ξ on the differential operator indicates that the derivatives are being taken with respect to the variable ξ. By making use of some of the delta function's unusual properties, Green's functions can be used to solve nonhomogeneous linear differential equations.

To find the solution to the linear differential equation

$$\mathbf{L}u = \phi, \tag{4.2}$$

we start by finding the formal adjoint as in equation (2.1). If we replace v with G, and we replace x with a dummy variable ξ, we are left with an equation of the form

$$\int_{\mathrm{a}}^{\mathrm{b}} G(\xi, x)\mathbf{L}u(\xi)d\xi = [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u(\xi)\mathbf{L}^* G(\xi, x)d\xi. \tag{4.3}$$

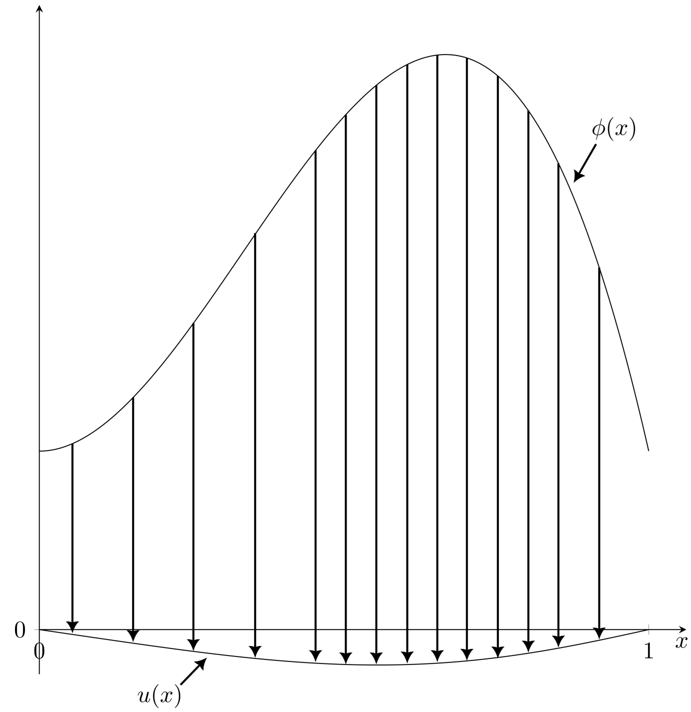

Figure 4.1: Loaded String

It follows from equation (4.1) that we can replace $\mathbf{L}^*G$ with δ, and from equation (4.2) that we can replace $\mathbf{L}u$ with ϕ,

$$\begin{aligned}\int_{\mathrm{a}}^{\mathrm{b}} G(\xi,x)\phi(\xi)d\xi &= [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u(\xi)\delta(\xi-x)d\xi \\ &= [\cdots]\Big|_{\mathrm{a}}^{\mathrm{b}} + u(x).\end{aligned} \tag{4.4}$$

Therefore, if we choose boundary conditions for G such that the boundary terms do not depend on u and we are able to find G, then finding u is reduced to a problem of integrating $G\phi$.

To illustrate the key ideas of the method, we will consider several examples which begin simply and become more complex. Each example will be concerned with a key concept in implementing the method of Green's functions.

4.2 Example 1 *Loaded String*

Consider the boundary value problem

$$u''(x) = \phi(x); \quad u(0) = u(1) = 0 \tag{4.5}$$

where $\phi(x)$ is prescribed. Equation (4.5) can be regarded as describing the static deflection of a string under unit tension between fixed endpoints and subjected to a force distribution $\phi(x)$.

To find a solution to this differential equation, we first find the formal adjoint $\mathbf{L}*$ as in equation

(4.3).

$$\begin{aligned}\int_0^1 G(\xi,x)\mathbf{L}u(\xi)d\xi &= \left(G(\xi,x)u'(\xi) - G_\xi(\xi,x)u(\xi)\right)\Big|_0^1 + \int_0^1 uG_{\xi\xi}d\xi \\ &= G(1,x)u'(1) - G_\xi(1,x)u(1) \\ &\quad - G(0,x)u'(0) + G_\xi(0,x)u(0) + \int_0^1 uG_{\xi\xi}d\xi.\end{aligned} \tag{4.6}$$

Therefore,

$$\mathbf{L}^* = \frac{d^2}{d\xi^2} \tag{4.7}$$

and

$$\mathbf{L}^* G = G_{\xi\xi} = \delta(\xi - x). \tag{4.8}$$

Because of the boundary conditions on u in equation (4.5), two of our boundary terms are zero. Thus,

$$\int_0^1 G(\xi,x)\mathbf{L}u(\xi)d\xi = G(1,x)u'(1) - G(0,x)u'(0) + \int_0^1 uG_{\xi\xi}d\xi. \tag{4.9}$$

Now, we want to remove the u dependency from the boundary terms and, as such, decide that

$$G(1,x) = G(0,x) = 0. \tag{4.10}$$

Then the solution is given by

$$u(x) = \int_0^1 G(\xi,x)\phi(\xi)d\xi. \tag{4.11}$$

To calculate the Green's function, we integrate equation (4.8), regarding x as fixed.

$$\begin{aligned} G_\xi &= H(\xi - x) + A \\ G &= (\xi - x)H(\xi - x) + A\xi + B \end{aligned} \tag{4.12}$$

Imposing the boundary conditions,

$$\begin{aligned} G(0,x) &= 0 = B \\ G(1,x) &= 0 = 1 - x + A + B, \end{aligned} \tag{4.13}$$

shows us that $B = 0$ and $A = x - 1$. Therefore

$$G(\xi,x) = (\xi - x)H(\xi - x) + (x-1)\xi. \tag{4.14}$$

Equation (4.8) can, like equation (4.5), be interpreted as the deflection of a loaded string. Specifically, it is the deflection as a function of ξ due to a point load of unit strength, $\delta(\xi - x)$, instead of a load distribution, ϕ. Rewriting our Green's function as

$$G(\xi,x) = \begin{cases} (x-1)\xi, & \xi \le x \\ (\xi - 1)x, & \xi \ge x \end{cases} \tag{4.15}$$

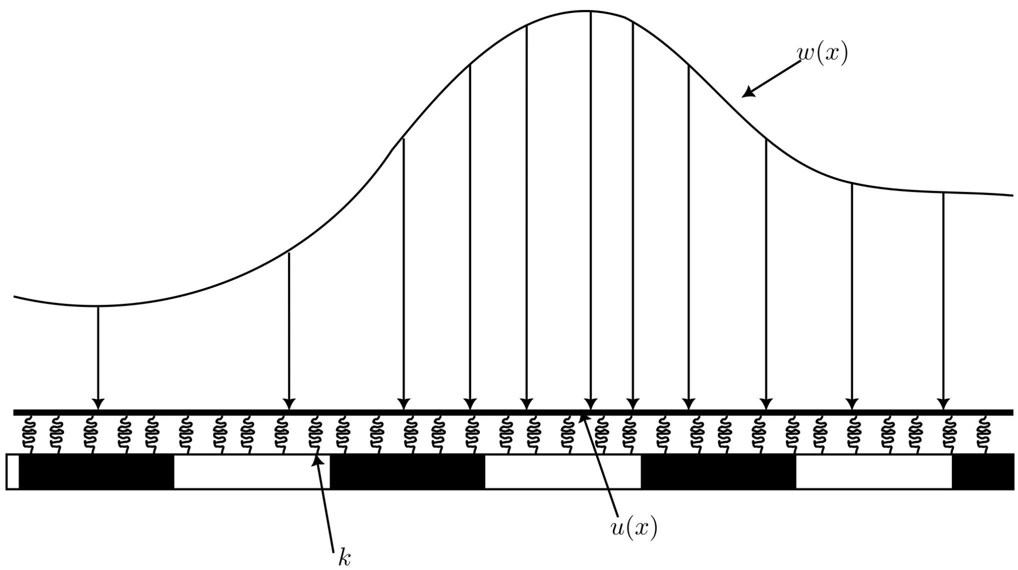

Figure 4.2: Infinite beam on elastic foundation

makes it clear that $G(\xi, x)$ is symemtric, to wit: $G(\xi, x) = G(x, \xi)$. This is often referred to as "Maxwell Reciprocity." Because of this reciprocity, $G(\xi, x)$ is also the deflection, as a function of x, due to a unit load at ξ. Then, $G(\xi, x)\phi(\xi)d\xi$ is the deflection due to an incremental load $\phi(\xi)d\xi$ at ξ. As such, equation (4.11) represents the superposition of these deflections. It is important to recognize that the superposition nature of equation (4.11) is a result of the linearity of $\mathscr{L}$.

Additionally, it should be noted that the boundary terms do not always vanish. For example, if we change the boundary condition $u(1) = 0$ to $u(1) = \alpha$, then

$$u(x) = \alpha x + \int_0^1 G(\xi, x)\phi(\xi)d\xi.$$

4.3 Example 2 *Infinite Beam on an Elastic Foundation*

It is prudent to, next, consider a differential operator defined over an infinite interval. We imagine an infinitely long beam on an elastic foundation. According to the classical Euler beam theory, the deflection, $u(x)$, resulting from a net loading, $p(x)$ force per unit length, satisfies the differential equation

$$(EIu'')'' = p(x).$$

Where the flexural rigidity of the beam, EI, may be a function of x. We also consider the spring constant per unit length,k. For the purposes of this example, we consider both EI and k to be constant and the prescribed loading to be $w(x)$, where $p(x) = w(x) - ku(x)$. By substituting these into equation (4.3) it becomes

$$EIu^{(4)}(x) + ku(x) = w(x).$$

or

$$u^{(4)}(x) + \alpha^4 u(x) = W(x)$$

where

$$\alpha = \frac{k}{EI} \text{ and } W(x) = \frac{w(x)}{EI}.$$

For boundary conditions, we only require that $W(x)$ be such that u and each of its first three derivatives approach a finite value as $|x| \to \infty$.

As before, we integrate by parts:

$$\int_{-\infty}^{\infty} G\mathbf{L}ud\xi = (Gu''' - G_\xi u'' + G_{\xi\xi}u' - G_{\xi\xi\xi}u)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u\mathbf{L}^*Gd\xi.$$

where $\mathbf{L}^* = \mathbf{L}$.

Then,

$$\mathbf{L}G = G_{\xi\xi\xi\xi} + \alpha^4 G = \delta(\xi - x) \tag{4.16}$$

and

$$\lim_{|x|\to\infty} \{G(\xi, x) = G_\xi(\xi, x) = G_{\xi\xi}(\xi, x) = G_{\xi\xi\xi}(\xi, x)\} = 0$$

to remove u, u', u'', and u''' from the boundary terms because they are finite, but not necessarily 0, as $|x| \to \infty$.

To solve equation (4.16), we use the Laplace transform

4.4 Sturm-Liouville Equations

We are interested in finding the Green's function for the general second-order linear differential operator

$$L = D(pD) + q \tag{4.17}$$

with the general unmixed[5] boundary conditions

$$B_1(u) = \alpha_1 u(\mathrm{a}) + \alpha_2 u'(\mathrm{a}) = 0; \quad B_2(u) = \beta_{\mathrm{b}} u(1) + \beta_2 u'(\mathrm{b}) = 0$$

where p and q are functions of the independent variable. Differential operators of this form are **Sturm-Liouville differential operators**. By referring to equation (2.4), it is clear that Sturm-Liouville differential operators are always self-adjoint since $a = p$ and $b = p'$.

To analyze these equations, we define the **conjoint**. Recall from equation (2.4) that for a second order differential operator

$$\mathbf{L}^*v = av'' + (2a' - b)v' + (a'' - b' + c).$$

It follows that

$$\int_{\mathrm{a}}^{\mathrm{b}} (v\mathbf{L}u - u\mathbf{L}^*v)dx = J(v, u)\Big|_{\mathrm{a}}^{\mathrm{b}}$$

where

$$\begin{aligned} J(v, u) &= avu' - a'vu - av'u + bvu \\ &= avu' - u(av)' + bvu. \end{aligned}$$

[5] A boundary condition being unmixed means that all functions it contains are evaluated at a or they are all evaluated at b.

$J(v,u)$ is the **conjunct** of v and u.

We now return to our Sturm-Liouville differential operator, $\mathbf{L}$, with boundary conditions $B_1(u) = B_2(u) = 0$. Because, these boundary conditions are unmixed, we say $B_1(u)$ only involves values at $x = a$, and $B_2(u)$ only invovles values at $x = b$. We also assume that $p(x) \neq 0$ on the interval $0 \leq x \leq 1$.

The Green's function $G(\xi, x)$, is the solution to the equation

$$L_\xi G = (p(\xi)G_\xi(\xi, x))_\xi - q(\xi)G(\xi, x) = -\delta(\xi - x), \tag{4.18}$$

which satisfies the boundary conditions $B_1(G) = B_2(G) = 0$. independent of boundary conditions, such an equation has two linearly independent solutions, $v_1(\xi)$ and $v_2(\xi)$. Now to account for the boundary conditions we let $w_1(\xi)$ be a linear combination of $v_1(\xi)$ and $v_2(\xi)$ which satisfies $B_1(w_1(x)) = 0$ and let $w_2(x)$ be a linear combination of $v_1(\xi)$ and $v_2(\xi)$ which satisfies $B_2(w_2(x)) = 0$. Preliminarily our Green's Function should be

$$G(\xi, x) = \begin{cases} a_1 w_1(\xi), & \xi < x \\ a_2 w_2(\xi), & \xi > x. \end{cases}$$

This satisfies the boundary conditions because we defined it as such, and it almost satisfies equation (4.18); however, we see that $G(\xi, x)$ has a discontinuity at $\xi = x$, which is required of a Green's function, meaning that this Green's function is not yet complete. For G to be continuous it must satisfy

$$G(x^+, x) = G(x^-, x).^6$$

Therefore,

$$a_1(x)w_1(x) = a_2(x)w_2(x)$$

and

$$a_1(x) = c(x)w_2(x); \quad a_2(x) = c(x)w_1(x).$$

Substituting into G results in

$$G(\xi, x) = \begin{cases} c(x)w_2(x)w_1(\xi), & \xi < x \\ c(x)w_1(x)w_2(\xi), & \xi > x. \end{cases}$$

Next, we must find the jump condition by integrating as follows:

$$\int_{x^-}^{x^+} L_\xi G(\xi, x) d\xi = \int_{x^-}^{x^+} \delta(\xi - x) d\xi = 1.$$
$$= p(x)\left[G_\xi(\xi, x)\Big|_{x^-}^{x^+} - G(\xi, x)\Big|_{x^-}^{x^+} \right] + q(x)\int_{x^-}^{x^+} G(\xi, x) d\xi = 1.$$

[6]Here, x^+ and x^- are shorthand for ξ as approached from the right or left side respectively. That is to say

$$G(x^+, x) = \lim_{\epsilon \to 0} G(x + \epsilon, x)$$

and

$$G(x^-, x) = \lim_{\epsilon \to 0} G(x - \epsilon, x)$$

Because we require $G(\xi, x)$ to be continuous at $\xi = x$,

$$G(\xi, x)\Big|_{x^-}^{x^+} = \int_{x^-}^{x^+} G(\xi, x) d\xi = 0.$$

Leaving us with

$$G_\xi(\xi, x)\Big|_{x^-}^{x^+} = 1.$$

Evaluating.

$$\begin{aligned} p(x)[G_\xi(x^+, x) - G_\xi(x^-, x)] &= 1 \\ p(x)[c(x)w_1(x)w_2'(x) - c(x)w_2(x)w_1'(x)] &= 1 \\ c(x)w_1(x)w_2'(x) - c(x)w_2(x)w_1'(x) &= \frac{1}{p(x)} \end{aligned}$$

$$\begin{aligned} c(x) &= \frac{1}{w_1(x)w_2'(x) - w_2(x)w_1'(x)} \left(\frac{1}{p(x)}\right) \\ &= \frac{1}{J(w_2, w_1)} \end{aligned}$$

Therefore the Green's function for the general Sturm-Liouville differential operator is

$$G(\xi, x) = \frac{1}{J(w_2, w_1)} \begin{cases} w_2(x)w_1(\xi), & \xi < x \\ w_1(x)w_2(\xi), & \xi > x. \end{cases}$$

4.5 Example 3 *The Generalized Green's Function*

As a more complex example, consider, the boundary value problem

$$u''(x) + u(x) = \phi(x); \quad u(0) = u(\pi) = 0. \tag{4.19}$$

It is not obvious, but this example will require some special treatment because, surprisingly, the Green's function does not exist if we attempt to find it naïvely in the same way as the previous examples. To illustrate the singular nature of this example, we proceed as before and integrate by parts:

$$\begin{aligned} \int_0^\pi G\mathbf{L}u d\xi &= \int_0^\pi G(u'' + u) d\xi \\ &= G(\pi, x)u'(\pi) - G(0, x)u'(0) + \int_0^\pi u\mathbf{L}^* G d\xi. \end{aligned}$$

This tells us that

$$\mathbf{L}^* G = G_{\xi\xi} + G = \delta(\xi - x) \tag{4.20}$$

and

$$G(0, x) = G(\pi, x) = 0.$$

We split the interval into $0 \leq \xi < x$ and $x < \xi \leq \pi$ and solve equation (4.20) for G on each. Then,

$$G(\xi, x) = \begin{cases} A\sin\xi + B\cos\xi, & 0 \leq \xi < x \\ C\sin\xi + D\cos\xi, & x < \xi \leq \pi. \end{cases}$$

The boundary conditions show that $B = D = 0$ because $\cos(0) \neq 0$ and $\cos(\pi) \neq 0$. Next, we integrate the differential equation from equation (4.20) from x^- to x^+:

$$\int_{x^-}^{x^+} (G_\xi\xi + G)d\xi = \int_{x^-}^{x^+} \delta(\xi - x)d\xi \tag{4.21}$$

Requiring G to be continuous at $\xi = x$ places the condition that

$$A = C, \tag{4.22}$$

and reduces equation (4.21) to the jump condition

$$G_\xi \Big|_{x^-}^{x^+} = 1.$$

or

$$C\cos x - A\cos x = 1.$$

This is a contraction with equation (4.22) because if $A = C$ then

$$C\cos x - A\cos x = 0 \neq 1.$$

Now that we have seen that there is a problem with the naïve approach, we will find the root of the problem by considering the homogeneous version of equation (4.20):

$$v_{\xi\xi} + v = 0; \quad v(0) = v(\pi) = 0 \tag{4.23}$$

Which has the general solution

$$v(\xi) = \alpha\sin\xi.$$

If we modify equation (4.21) to

$$\int_0^\pi v(G_{\xi\xi} + G)d\xi = \int_0^\pi v\delta(\xi - x)d\xi. \tag{4.24}$$

Using integration by parts, we change this into:

$$(vG_\xi - v_\xi G)\Big|_0^\pi + \int_0^\pi G(v_{\xi\xi} + v)d\xi = v(x)$$

Recalling the boundary conditions on v from equation (4.23) reveals the contradiction

$$0 + 0 = v(x).$$

By writing equation (4.21) in terms of the inner product,

$$(v, G_{\xi\xi} + G) = (v, \delta(\xi - x)),$$

we could say that the singularity arises becuase v is not "orthogonal" to $\delta(\xi - x)$, to wit, the right side is not zero. Now, instead of requiring G satisfy equation (4.20), we require

$$\mathbf{L}^* G = \delta(\xi - x) + F$$

where F is a function for which $v(\xi)$ is orthogonal to $\delta(\xi - x) + F$.

Theorem 4.1. *A general form for F which satisfies the orthogonality requirement is*

$$F = -\frac{v(x)v(\xi)}{\int_{\mathrm{a}}^{\mathrm{b}} v^2(\xi)d\xi}$$

Proof We integrate $v(\xi)(\delta(\xi - x) + F)$ over the interval (a,b):

$$\begin{aligned}
\int_{\mathrm{a}}^{\mathrm{b}} v(\xi)\left[\delta(\xi - x) - \frac{v(x)v(\xi)}{\int_{\mathrm{a}}^{\mathrm{b}} v^x(\xi)d\xi}\right] d\xi &= \int_{\mathrm{a}}^{\mathrm{b}} \left(v(\xi)\delta(\xi - x) - v(\xi)\frac{v(x)v(\xi)}{\int_{\mathrm{a}}^{\mathrm{b}} v^2(\xi)d\xi}\right) d\xi \\
&= \int_{\mathrm{a}}^{\mathrm{b}} v(\xi)\delta(\xi - x)d\xi - v(x)\frac{\int_{\mathrm{a}}^{\mathrm{b}} v^2(\xi)d\xi}{\int_{\mathrm{a}}^{\mathrm{b}} v^2(\xi)d\xi} \\
&= v(x) - v(x) \\
&= 0
\end{aligned}$$

∎

For this example, $v(\xi) = \sin\xi$ so $F = -\frac{\pi v(x)v(\xi)}{2}$.

Note that equation (4.19) is of the same form equation (4.20). This means we can apply the same reasoning to show that just as how there is no solution to equation (4.20) when the homogeneous solution $v(\xi)$ is not orthogonal to $d(\xi - x)$, there is no solution to equation (4.19) when $v(x)$ is not orthogonal to $\phi(x)$.

4.6 The Eigenfunction Method

A Appendix

A.1 Tabular Integration by Parts

One can use a table to perform repeated integration by parts quickly. Suppose we would like to use integration by parts to change the integral of $u(x)v(x)$ into boundary terms plus the integral of $u^{(n)}(x)v_{(n)}(x)$. We create a table with alternating signs in the first column, beginning with positive. In the second column, we write $u(x)$ and its derivatives beneath it while alternating the sign. In the third column, we write $v(x)$ and its anti-derivatives beneath it. At the end of this process, the table should look as follows.

$$
\begin{array}{c|rl}
+ & u(x) & v(x) \\
\hline
- & u^{(1)}(x) & v_{(1)}(x) \\
+ & u^{(2)}(x) & v_{(2)}(x) \\
- & u^{(3)}(x) & v_{(3)}(x) \\
\vdots & \vdots & \vdots \\
(-1)^{n-1} & u^{(n-1)}(x) & v_{(n-1)}(x) \\
(-1)^{n} & u^{(n)}(x) & v_{(n)}(x)
\end{array}
$$

Next, we take the i^{th} sign in the left column to be the sign of $u^{(i)}$, multiply diagonal terms, and add each product.

$$
\begin{array}{c|rcl}
+ & u(x) & \searrow & v(x) \\
\hline
- & u^{(1)}(x) & \searrow & v_{(1)}(x) \\
+ & u^{(2)}(x) & \searrow & v_{(2)}(x) \\
- & u^{(3)}(x) & \searrow & v_{(3)}(x) \\
\vdots & \vdots & \searrow & \vdots \\
(-1)^{n-1} & u^{(n-1)}(x) & \searrow & v_{(n-1)}(x) \\
(-1)^{n} & u^{(n)}(x) & & v_{(n)}(x)
\end{array}
$$

These are the boundary terms. Lastly, we multiply the bottom terms together, which becomes the integrand.

$$
\begin{array}{c|rcl}
+ & u(x) & & v(x) \\
\hline
- & u^{(1)}(x) & & v_{(1)}(x) \\
+ & u^{(2)}(x) & & v_{(2)}(x) \\
- & u^{(3)}(x) & & v_{(3)}(x) \\
\vdots & \vdots & & \vdots \\
(-1)^{n-1} & u^{(n-1)}(x) & & v_{(n-1)}(x) \\
(-1)^{n} & u^{(n)}(x) & \longrightarrow & v_{(n)}(x)
\end{array}
$$

At the end of this process, the original integral has become

$$\int_{\mathrm{a}}^{\mathrm{b}} u(x)v(x)dx = \left(\sum_{i=1}^{n}(-1)^{i-1}u^{(i-1)}(x)v_{(i)}(x)\right)\Bigg|_{\mathrm{a}}^{\mathrm{b}} + \int_{\mathrm{a}}^{\mathrm{b}} u^{(n)}(x)v_{(n)}(x)dx.$$

A.2 The Mean Value Theorem

The mean value theorem states that for all $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ such that f is continuous on $[\mathrm{a}, \mathrm{b}]$, and differentiable on (a, b), then

$$\exists c \in (\mathrm{a}, \mathrm{b}) : f'(\mathrm{c}) = \frac{f(\mathrm{b}) - f(\mathrm{a})}{\mathrm{b} - \mathrm{a}}$$

and thus,

$$f'(c)(\mathrm{b} - \mathrm{a}) = f(\mathrm{b}) - f(\mathrm{a})$$

By integrating $f'(x)$, we see that

$$\begin{aligned} \int_{\mathrm{a}}^{\mathrm{b}} f'(x)dx &= f(\mathrm{b}) - f(\mathrm{a}) \\ &= f'(\mathrm{c})(\mathrm{b} - \mathrm{a}) \end{aligned}$$

A.3 Big O Notation

Big O notation is used to describe the limiting behavior of a function as the argument tends to some value or infinity. By $f(x) = O(g(x))$ as $x \to x_0$ we mean that $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$. For example

$$\sin 6x = O(1) \text{ as } x \to \infty.$$

A.4 Alternative Method of Green's Functions

In this text, we approach finding the Green's function by finding the adjoint operator and then defining

$$\mathbf{L}^* G = \delta.$$

Instead, one can find an alternative version of the Green's function, say G^*, using the original differential operator acting on u.

Suppose a differential equation is given by

$$\mathbf{L}_x u = \phi.$$

Then we can define $G^*(\xi, x)$ to be a function that satisfies,

$$\mathbf{L}_x G^* = \delta.$$

Clearly then,

$$\mathbf{L}_x u = \int_{\mathrm{a}}^{\mathrm{b}} L_x G^* \phi d\xi = \phi$$

and

$$u = \int_{\mathrm{a}}^{\mathrm{b}} G^* \phi d\xi.$$

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