Theorems for Exam 1

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Theorem 1 *Let $(G, \circ)$ be a group, and let $g \in G$. Then*

$$g^{-1} \circ g = e.$$

Proof We define $x = g^{-1} \circ g$. Then we have

$$\begin{aligned} x &= x \circ e \\ &= x \circ x \circ x^{-1} \\ &= g^{-1} \circ g \circ g^{-1} \circ g \circ x^{-1} \\ &= g^{-1} \circ g \circ x^{-1} \\ &= x \circ x^{-1} \\ &= e \end{aligned}$$

Thus $g^-1 \circ g = e$. ■

Theorem 2 *Let $(G, \circ)$ be a group and $H \subseteq G$. Then H is a group if and only if:*

a) $H \neq \emptyset$, and

b) $h_1 \circ h_2^{-1} \in H$ for all $h_1, h_2 \in H$.

Proof First, suppose that H is a subgroup. We get that $e \in H$, so $H \neq 0$. Thus, (a) holds. Now, let $h_1, h_2 \in H$, then we have that $h_2^{-1} \in H$. Thus, $h_1 \circ h_2^{-1} \in H$. Hence (b) holds. Now, suppose that (a) and (b) hold. From (a) we have that $H \neq 0$, so there exists $h \in H$. Thus from (b) we get that

$$e = h \circ h^{-1} \in H$$

and so (1) holds.
Let $h \in H$. We have shown that $e \in H$. So, from (b) we get

$$h^{-1} = e \circ h^{-1} \in H.$$

So, (2) holds.
Let $h_1, h_2 \in H$. We have shown that $h_2^{-1} \in H$. Hence (b) gives that

$$h_1 \circ h_2 = h_1 \circ (h_2^{-1})^{-1} \in H.$$

Thus, (3) holds. Therefore H is a subgroup. ■

Theorem 3 *Let $(G, \circ)$ be a group, and H a subgroup of G. Then for all $g_1, g_2 \in G$ there exists a bijection $g_1H \to g_2H$.*

Proof Let $x \in g_1H$. We define

$$f(x) = g_2 g_1^{-1} x.$$

As $x = g1h$ for some $h \in H$, we have that $f(x) = g_2 g_1^{-1} g_1 h = g_2 h \in g_2 H$. Therefore $f(x) \in g_2H$ and f is a function $g_1H \to g_2H$.
We next claim that f is a bijection.
inj) Let $x_1, x_2 \in g_1H$ such that $f(x_1) = f(x_2)$. Then we have

$$\begin{aligned} g_2 g_2^{-1} x_1 = g_2 g_1^{-1} x_2 &\implies g_1^{-1} x_1 = g_1^{-1} x_2 \\ &\implies x_1 = x_2. \end{aligned}$$

Thus, f is injective.
sur) Let $y \in g_2H$. Then there exists $h \in H$ such that $y = g_2h$. We define

$$x = g_1 h \in g_1 H.$$

We then have

$$f(x) = g_2 g_1^{-1} g_1 h = g_2 h = y.$$

Thus, f is bijective. ■

Theorem 4 *Let $(G_1, \circ)$ and $(G_2, *)$ be groups, and $f : G_1 \to G_2$ a group homomorphism. Then,* $\ker(f)$ *is a subgroup of G_1.*

Proof As $f(e_{G_1}) = e_{G_2}$, we have that $e_{G_1} \in \ker(f)$, and so $\ker(f) \neq \emptyset$.
Let $g_1, g_2 \in ker(f)$. Then,

$$f(g_1) = e_{G_2} = f(g_2).$$

Therefore

$$f(g_1 \circ g_2^{-1}) = f(g_1) * f(g_2)^{-1} = e_{G_2} * e_{G_2}^{-1} = e_{G_2}.$$

Thus, $g_1 \circ g_2^{-1} \in \ker(f)$ and $\ker(f)$ is a subgroup of G_1. ■

Theorem 5 *Let $(G_1, \circ)$ and $(G_2, *)$ be groups, and $f : G_1 \to G_2$ a group homomorphism. Then, f is injective if an only if* $\ker(f) = \{e_{G_1}\}$.

Proof $\implies$) Assume f is injective, and let $g \in \ker(f)$. THen $f(g) = e_{G_2} = f(e_{G_1})$. As f is injective, we get that $g = e_{G_1}$, and so $\ker(f) = \{e_{G_1}\}$.
$\impliedby$) Assume $\ker(f) = \{e_{G_1}\}$ and let $g_1, g_2 \in G_1$ such that $f(g_1) = f(g_2)$. We then have

$$\begin{aligned} &f(g_1) * f(g_2)^{-1} = e_{G_2} \\ \implies &f(g_1 \circ g_2^{-1}) = e_{G_2} \\ \implies &g_1 \circ g_2^{-1} \in \ker(f) \\ \implies &g_1 \circ g_2^{-1} = e_{G_1} \\ \implies &g_1 = g_2. \end{aligned}$$

Hence f is injective. ■

Theorem 6 *Let G and H be groups, then $(G \times H, \circ)$ is a group where*

$$(g_1, h_1) \circ (g_2, h_2) := (g_1 g_2, h_1 h_2).$$