Green's Functions

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Contents

1 Introduction **2**

2 The Adjoint Operator **2**

3 The Delta Function **4**

1 Introduction

The first part of this text is primarily concerned with solutions to differential equations of the form

$$Lu = \phi \tag{1.1}$$

over an interval $a \leq x \leq b$ and subject to boundary conditions $\{B_1, \ldots, B_n\}$, where L is an nth order linear ordinary differential operator. For L to be linear it must satisfy the condititon

$$L(\alpha v + \beta w) = \alpha Lv + \beta Lw \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant. For this condition to be met L must be of the form

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x) \tag{1.3}$$

The boundary conditions are linear functionals of the form

$$B_j(u) = c_j; \qquad j = 1, 2, \ldots, n \tag{1.4}$$

where c_j is an arbitrary constant.

Here, functional, refers to a transformation which has a set of functions as its domain and a set of numbers. As an example

$$B(u) = u(0) = 0 \tag{1.5}$$

is a simple boundary condition for a 1st order differential operator. Specifically, our B_j's will be limited to linear combinations of u and it's derivatives up to order $n-1$. These boundary conditions have the same linearity constraints as the differential operator L.

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and it's boundary conditions we will need the formal adjoint operator. This operator, which we will call L^*, can be found via repeated integration by parts. In general

$$\int_a^b vLudx = [\cdots]\Big|_a^b + \int_a^b uL^*vdx \tag{2.1}$$

Here, u and v are completely arbitrary while being sufficiently differentiable for L and $L*$ to exist.

As an example, consider

$$L = A(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + C(x) \tag{2.2}$$

To find L^* perform integration by parts on the on each term of the product vLu a number of times equal to the order of the derivative that is a part of the term. That is to say, twice on the first term, once, on the second, and not at all on the third. Doing this, we are left with

$$\begin{aligned}
\int_a^b vLudx &= \int_a^b (vAu'' + vBu' + vC)dx \\
&= (vAu' + vBu)\Big|_a^b + \int_a^b (-(vA)'u' - (vB)'u + vCu)dx \\
&= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b ((vA)''u - (vB)'u + vCu)dx \\
&= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b u((vA)'' - (Bv)' + Cv)dx
\end{aligned} \tag{2.3}$$

From this it is clear that

$$\begin{aligned}
L^*v &= (Av)'' - (Bv)' + Cv \\
&= (A'v + Av')' - B'v - Bv' + Cv \\
&= Av'' + (2A' - B)v' + (A'' - B' + C)
\end{aligned} \tag{2.4}$$

and so the formal adjoint of a second order linear differential operator L must be of the form

$$L^* = A\frac{d^2}{dx^2} + (2A' - B)\frac{d}{dx} + (A'' - B' + C) \tag{2.5}$$

If L^* is found to be equal to L then L is called formally self adjoint. By comparing equations (2.2) and (2.5) we can see that for a second order linear differentiable operator to be formally self adjoint, A' must be equal to B. You may notice that $A'' - B' + C$ must also be equal to C but this is always true given that A' equals B.

If the boundary conditions on L are homogenous then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(Lu, v) = (L^*v, u) \tag{2.6}$$

where (f, g) is the inner produce of f and g

$$(f, g) = \int_a^b f(x)g(x)dx \tag{2.7}$$

We now can understand that the adjoint operator $\mathscr{L}^*$ must consist of L^* and boundary conditions to force the terms that come about from integrating by parts to be zero.

As an example of $\mathscr{L}$ and $\mathscr{L}^*$, consider $\mathscr{L}$ to consist of $L = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \leq x \leq 1$. Then

$$\begin{aligned}
(Lu, v) &= \int_0^1 u'vdx \\
&= (uv)\Big|_0^1 - \int_0^1 uv'dx \\
&= u(1)v(1) - u(0)v(0) + \int_0^1 uL^*vdx \\
&= u(1)(v(1) - 3v(0)) + \int_0^1 uL^*vdx
\end{aligned} \tag{2.8}$$

Since the particular value of $u(1)$ is not given, we must make $v(1) - 3v(0)$ equal zero, because choosing $u(1) = 0$ would undully restrict our solution. Therefore $\mathscr{L}^*$ consists of L^* which is $-\frac{d}{dx}$ and the boundary condition $v(1) - 3v(0) = 0$. As a final note, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called self-adjoint.

3 The Delta Function

The Dirac delta function is integral to the method of green's functions. It is typically represented by the symbol δ and has the properties

$$\delta(x) = 0 \quad \text{for } t \neq 0 \tag{3.1}$$

and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \tag{3.2}$$

but its most important property is

$$\int_a^b f(x)\delta(x - t)dt = f(t), \quad a \leq t \leq b \tag{3.3}$$

If, in the previous integral, t is not between a and b, then the integral is equal to 0.

To attempt to find an intuition for the delta function, we can make use of the idea of "point actions", or actions which are highly localized in space or time. As an example, consider a sharp hammer blow, with force $f(t)$. We do not necessarily know the shape of the force function, but

While commonly referred to as a function, δ is more appropriately defined as a generalized function or a distribution.