Homework 4

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Question 1

a) Prove the following theorem

Theorem 1 *Let $(G, \circ)$ be a group, and $H_1, H_2 \subseteq G$ two subgroups of G. Then $H_1 \cap H_2$ is a subgroup of G.*

Proof We begin by showing that $H_1 \cap H_2$ is non-empty. Since H_1 is a subgroup of G, $e \in H_1$ and because H_2 is also a subgroup of G, $e \in H_2$. It then follows that $e \in H_1 \cap H_2$.

Next we show that $h_1 \circ h_2^{-1} \in H_1 \cap H_2$ for all $h_1, h_2 \in H_1 \cap H_2$. Let $h_1, h_2 \in H_1 \cap H_2$. Now, $h_1, h_2 \in H_1$ and $h_1, h_2 \in H_2$. Because, H_1 and H_2 are subgroups of G, we have that $h_1 \circ h_2^{-1} \in H_1$ and $h_1 \circ h_2^{-1} \in H_2$ and so $h_1 \circ h_2^{-1} \in H_1 \cap H_2$. Therefore $H_1 \cap H_2$ is a subgroup of G. $\blacksquare$

b) Give an example of a group $(G, \circ)$, and two subgroups $H_1, H_2 \subseteq G$ such that $H_1 \cup H_2$ is **n**ot as subgroup of G.

One such group is S_3, with the subgroups $H_1 = \{(), (12)\}$ and $H_2 = \{(), (23)\}$. Now, $H_1 \cup H_2 = \{(), (12), (23)\}$ is not a subgroup becuase $(12) \circ (23) = (13) \notin H_1 \cup H_2$.

Question 2

Let $N \in \mathbb{N}_{\geq 1}$, and consider the group $GL(N), \cdot$. We define

$$O(N) := \left\{ M \in GL(N) | M^T \cdot M = I_N \right\}$$

where M^T denotes the matrix transpose of M. Show that $O(N)$ is a subgroup of $GL(N)$.

Proof We begin by showing that the identity element, I_N, is in $O(N)$. The identity matrix is always its own transpose, that is to say, $I_N = I_N^T$. Thus,

$$\begin{aligned} I_N^T \cdot I_N &= I_N \cdot I_N \\ &= I_N. \end{aligned}$$

The identity element is therefore in $O(N)$.

Next, we show that for all $M \in O(N)$ there exists $M^{-1} \in O(N)$. Let $M \in O(N)$, and thus, $M^T \cdot M = I_N$. It follows that $M^T = M^{-1}$. Now,

$$\begin{aligned}(M^T)^T \cdot M^T &= M \cdot M^T \\ &= M \cdot M^{-1} \\ &= I_N.\end{aligned}$$

Thus, $M^T \in O(N)$. Since $M^T = M^{-1}$, we have that $M^{-1} \in O(N)$ for all $M \in O(N)$.

Lastly, we show that $M_1 \cdot M_2 \in O(N)$ for all $M_1, M_2 \in O(N)$. Let $M_1, M_2 \in O(N)$. Now, because $M_1^T = M_1^{-1}$ and $M_2^T = M_2^{-1}$, M_1 and M_2 are orthonormal matrices, and all orthonomal matrices in $GL(N)$ are in $O(N)$ and so $M_1 \cdot M_2$ is also orthonormal. Now, $(M_1 \cdot M_2)^T \cdot (M_1 \cdot M_2) = I_N$, and thus $M_1 \cdot M_2 \in O(N)$.

Therefore, $O(N)$ satisfies all of the conditions for being a subgroup of $GL(N)$. ■