Problem Set 12

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1. (a) The relation R_1 is a function since there are no elements in the domain, $\mathbb{R}$, that cannot be mapped to an element in the codomain, $\mathbb{R}$, and $y = 4x - 3$ produces only one y value for each x value.
   (b) The relation R_2 is not a function because each element $a \in A_2$, where $a > 0$, maps to two elements of the codomain. For example, $(4, 0)$ and $(4, -4)$ are both elements of R_2.
   (c) The relation R_3 is not a function because each element $a \in A_3$ to two elements, $b_1, b_2 \in \mathbb{R}$. In particular b_1 and b_2 are related by $b_2 = b_1 + 4$. As an example, $(0, -2)$ and $(0, 2)$ are both in R_3.

2. (a) $f(n) = n$
   (b) $f(n) = n + 1$
   (c) $f(n) = \begin{cases} 1, & x = 1 \\ n - 1, & x > 1 \end{cases}$
   (d) $f(n) = 1$

3. The relation, $\mathcal{F}$, is a set such that, $\mathcal{F} = \{(2, 7), (4, 1), (6, 4)\}$. Thus, this relation is a function since all of the elements in the domain are mapped to exactly one element of the codomain. The relation $\mathcal{F}$ is also one-to-one because no element of the codomain is mapped to by more than one element of the domain.

4. (a) Let $x \in A$. Now, $(f \circ f \circ f)(x) = f(f(f(x)))$. Thus,

$$\begin{aligned}
f(f(f(x))) &= 1 - \frac{1}{1 - \frac{1}{1-\frac{1}{x}}} \\
&= 1 - \frac{1}{\frac{1-\frac{1}{x}-1}{1-\frac{1}{x}}} \\
&= 1 - \frac{1}{\frac{\frac{1}{x}}{\frac{1}{x}-1}} \\
&= 1 - \frac{\frac{1}{x}-1}{\frac{1}{x}} \\
&= 1-(1-x) \\
&= x
\end{aligned}$$

Therefore $f \circ f \circ f$ is the identity function.

(b) Since, $f \circ f \circ f$ is the identity function, it follows that $f \circ f^{-1} = f \circ f \circ f$. Thus,

$$\begin{aligned}
f^{-1}(x) = f \circ f(x) &= 1 - \frac{1}{1-\frac{1}{x}} \\
&= 1 - \frac{1}{\frac{x-1}{x}} \\
&= 1 - \frac{x}{x-1} \\
&= \frac{x-1-x}{x-1} \\
&= \frac{1}{1-x}
\end{aligned}$$

5. (a) The function, F, is not one-to-one because $F(2) = F(4) = 0$. In fact, $F(a) = 0$ for any $a \in (\mathbb{N} \cup \{0\}) \cap \mathbb{E}$, where $\mathbb{E}$ is the set of even integers.

(b) I conjecture that it is onto, but I cannot find a proof of it.

6. Let T, S be sets. Now, $T - S = \{a \in T | a \notin S\} = \{a \in T | a \notin T \cap S\}$. Thus, $T - S = T - T \cap S$. By the same process, it can be shown that $S - T = S - T \cap S$. Assuming the hypothesis that $|T - S| = |S - T|$, it follows that ,$|T - T \cap S| = |S - T \cap S|$. Now since, $T \cap S \subset T$ and $T \cap S \subset S$, it is the case that $|T| - |T \cap S| = |S| - |T \cap S|$. Finally, by algebraic manipulation, $|T| = |S|$.