Homework 5

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Question 1

Prove the following theorem:

Theorem 1 *Let H_1 and H_2 be groups, and define*

$$H_1 \times \{e\} := \{(h_1, e) | h_1 \in H_1\} \subset H_1 \times H_2$$

and

$$\{e\} \times H_2 := \{(e, h_2) | h_2 \in H_2\} \subset H_1 \times H_2.$$

Then

a) *$H_1 \times \{e\}$ and $\{e\} \times H_2$ are normal subgroups of $H_1 \times H_2$,*

b) *$(H_1 \times \{e\}) \cap (\{e\} \times H_2) = \{(e, e)\}$, and*

c) *$(H_1 \times \{e\})(\{e\} \times H_2) = H_1 \times H_2$.*

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a) **Proof** Let $(h_1, h_2) \in H_1 \times H_2$ so $h_1 \in H_1$ and $h_2 \in H_2$ and let $(x, e) \in H_1 \times \{e\}$ so $x \in H_1$. Now,

$$\begin{aligned}(h_1, h_2) \circ (x, e) \circ (h_1, h_2)^{-1} &= (h_1, h_2) \circ (x, e) \circ (h_1^{-1}, h_2^{-1}) \\ &= (h_1 x h_1^{-1}, h_2 e h_2^{-1}) \\ &= (h_1 x h_1^{-1}, h_2 h_2^{-1}) \\ &= (h_1 x h_1^{-1}, e).\end{aligned}$$

Since, $h_1, h_1^{-1}, x \in H_1$, it follows that $h_1 x h_1^{-1} \in H_1$. Thus, $(h_1, h_2) \circ (x, e) \circ (h_1, h_2)^{-1} = (h_1 x h_1^{-1}, e) \in H_1 \times H_2$ and therefore $H_1 \times \{e\}$ is normal.
Now, let $y \in H_1$. We have that

$$\begin{aligned}(h_1, h_2) \circ (e, y) \circ (h_1, h_2)^{-1} &= (h_1, h_2) \circ (e, y) \circ (h_1^{-1}, h_2^{-1}) \\ &= (h_1 e h_1^{-1}, h_2 y h_2^{-1}) \\ &= (h_1 h_1^{-1}, h_2 y h_2^{-1}) \\ &= (e, h_2 y h_2^{-1}).\end{aligned}$$

Since, $h_2, h_2^{-1}, y \in H_1$, it follows that $h_2 y h_2^{-1} \in H_2$. Thus, $(h_1, h_2) \circ (y, e) \circ (h_1, h_2)^{-1} = (e, h_2 y h_2^{-1}) \in H_1 \times H_2$ and therefore $\{e\} \times H_2$ is normal. ■

b) **Proof** Let $x \in (H_1 \times \{e\}) \cap (\{e\} \times H_2)$. Now, $x \in \{e\} \times H_2$ and $x \in H_1 \times \{e\}$. Thus, $(h_1, e) = x = (e, h_2)$ for some $h_1 \in H_1$ and $h_2 \in H_2$. Since $(h_1, e) = (e, h_2)$ it follows that $h_1 = e$ and $h_2 = e$. Thus $x = (e, e)$ and so, $(H_1 \times \{e\}) \cap (\{e\} \times H_2) = \{(e, e)\}$. ■

c) **Proof** By definition

$$\begin{aligned}(H_1 \times \{e\})(\{e\} \times H_2) &= \{(h_1, e)(e, h_2) | h_1 \in H_1 \text{ and } h_2 \in H_2\} \\ &= \{(h_1 e, e h_2) | h_1 \in H_1 \text{ and } h_2 \in H_2\} \\ &= \{(h_1, h_2) | h_1 \in H_1 \text{ and } h_2 \in H_2\} \\ &= H_1 \times H_2.\end{aligned}$$

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Question 2

Prove the following theorem:

Theorem 2 *Let G be a group, and H a subgroup of G. Then H is normal if and only if $gH = Hg$ for all $g \in G$.*

Proof
($\Leftarrow$) Let $h \in H$ and $g \in G$. Now, $gh = hg$ and so, $ghg^{-1} = h \in H$. Therefore, H is normal.
($\Rightarrow$) Suppose H is normal. Then, $gh_1g^{-1} = h_2 \in H$ for some $h_1 \in H$ and $g \in G$. It follows that $gh_1 = h_2 g$. Now, we define a function, $f : H \to H$, as

$$f(h) = ghg^{-1} \tag{1}$$

and let $x \in \ker(f)$. Then, $gxg^{=1} = e$, and so $x = g^{-1}g = e$. Therefore $\ker(f) = \{e\}$ and f is injective. Since f is a function between two finite sets of the same size, it must also be surjective. Thus, f produces a different $h_2 \in H$ for each $h_1 \in H$, and every h_2 is produced by some h_1. Therefore, $gH = Hg$ for all $g \in G$. ■