Problem Set 7

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1. Prove: the product of an irrational number and a nonzero rational number is irrational.
   Proof: Suppose that $xy = z$, for some $x, z \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$. By definition, $x = \frac{a}{b}$ and $z = \frac{a'}{b'}$, for some $a, a', b, b' \in \mathbb{Z}$, with $a, a', b, b' \neq 0$. Then, $\frac{a}{b}y = \frac{a'}{b'}$, and so, $y = \frac{a'b}{b'a}$. Now, $a'b$ and $b'a$, are integers, and so y must be rational by definition. This contradicts the initial assumption that y is irrational. Thus, the product of an irrational and a nonzero rational number, cannot be rational and must therefore be rational. ■

2. Prove: $\sqrt{2} + \sqrt{3}$ is an irrational number.
   Proof: Suppose that $\sqrt{2} + \sqrt{3}$ is rational, then $(\sqrt{2} + \sqrt{3})^2$ must also be rational. Now, $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$. If $\sqrt{6}$ is irrational, then $5 + 2\sqrt{6}$ is also irrational. Suppose that $\sqrt{6}$ is rational. Let $\sqrt{6} = \frac{a}{b}$, for some $a, b \in \mathbb{Z}$, with a and b being coprime. Then, $6 = \frac{a^2}{b^2}$, and so $b^2 = \frac{a^2}{6}$. Since, b is an integer, b^2 is also an integer. Thus, $6|a^2$. It follows that $6|a$ since $\sqrt{6} \notin \mathbb{Z}$. By definition, $a =$ Now, return to the equation $6 = \frac{a^2}{b^2}$.