Theorems for Exam 1

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Theorem 1 *Let $(G, \circ)$ be a group, and let $g \in G$. Then*

$$g^{-1} \circ g = e.$$

Proof We define $x = g^{-1} \circ g$. Then we have

$$\begin{aligned} x &= x \circ e \\ &= x \circ x \circ x^{-1} \\ &= g^{-1} \circ g \circ g^{-1} \circ g \circ x^{-1} \\ &= g^{-1} \circ g \circ x^{-1} \\ &= x \circ x^{-1} \\ &= e \end{aligned}$$

Thus $g^-1 \circ g = e$. ■

Theorem 2 *Let $(G, \circ)$ be a group and $H \subseteq G$. Then H is a group if and only if:*

a) $H \neq \emptyset$, and

b) $h_1 \circ h_2^{-1} \in H$ for all $h_1, h_2 \in H$.

Proof First, suppose that H is a subgroup. We get that $e \in H$, so $H \neq 0$. Thus, (a) holds. Now, let $h_1, h_2 \in H$, then we have that $h_2^{-1} \in H$. Thus, $h_1 \circ h_2^{-1} \in H$. Hence (b) holds.

Now, suppose that (a) and (b) hold. From (a) we have that $H \neq 0$, so there exists $h \in H$. Thus from (b) we get that

$$e = h \circ h^{-1} \in H$$

and so (1) holds.

Let $h \in H$. We have shown that $e \in H$. So, from (b) we get

$$h^{-1} = e \circ h^{-1} \in H.$$

So, (2) holds.

Let $h_1, h_2 \in H$. We have shown that $h_2^{-1} \in H$. Hence (b) gives that

$$h_1 \circ h_2 = h_1 \circ (h_2^{-1})^{-1} \in H.$$

Thus, (3) holds. Therefore H is a subgroup. ■

Theorem 3 *Let $(G, \circ)$ be a group, and H a subgroup of G. Then for all $g_1, g_2 \in G$*