Green's Functions

Ryan Coyne

February 12, 2023

Contents

1 Introduction

The first part of this text is primarily concerned with solutions to differential equations of the form

$$Lu = \phi \tag{1.1}$$

over an interval $a \leq x \leq b$ and subject to boundary conditions $\{B_1, \ldots, B_n\}$, where L is an nth order linear ordinary differential operator. For L to be linear it must satisfy the condititon

$$L(\alpha v + \beta w) = \alpha Lv + \beta Lw \tag{1.2}$$

for arbitrary functions v and w, with α and β being constant. For this condition to be met L must be of the form

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x) \tag{1.3}$$

The boundary conditions are linear functionals of the form

$$B_j(u) = c_j; \quad j = 1, 2, \ldots, n \tag{1.4}$$

where c_j is an arbitrary constant.

A functional, as used here, has a set of functions as it's domain and a set of numbers. As an example

$$B(u) = u(0) = 0 \tag{1.5}$$

is a simple boundary condition for a 1st order differential operator. Specifically, our B_j's will be limited to linear combinations of u and it's derivatives up to order $n-1$. These boundary conditions have the same linearity constraints as the differential operator L.

2 The Adjoint Operator

To determine the Green's function for a particular differential equation and it's boundary conditions we will need the formal adjoint operator. This operator, which we will call L^*, can be found via repeated integration by parts. In general

$$\int_a^b vLudx = [\cdots]\Big|_a^b + \int_a^b uL^*vdx \tag{2.1}$$

Here, u and v are completely arbitrary while being sufficiently differentiable for L and $L*$ to exist.

As an example, consider

$$L = A(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + C(x) \tag{2.2}$$

To find L^* perform integration by parts on the on each term of the product vLu a number of times equal to the order of the derivative that is a part of the term. That is to say, twice on the first term, once, on the second, and not at all on the third. Doing this, we are left with

$$\begin{aligned}\int_a^b vLudx &= \int_a^b (vAu'' + vBu' + vC)dx \\ &= (vAu' + vBu)\Big|_a^b + \int_a^b (-(vA)'u' - (vB)'u + vCu)dx \\ &= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b ((vA)''u - (vB)'u + vCu)dx \\ &= (vAu' + vBu - (vA)'u)\Big|_a^b + \int_a^b u((vA)'' - (Bv)' + Cv)dx\end{aligned} \tag{2.3}$$

From this it is clear that

$$\begin{aligned}L^*v &= (Av)'' - (Bv)' + Cv \\ &= (A'v + Av')' - B'v - Bv' + Cv \\ &= Av'' + (2A' - B)v' + (A'' - B' + C)\end{aligned} \tag{2.4}$$

and so the formal adjoint of a second order linear differential operator L must be of the form

$$L^* = A\frac{d^2}{dx^2} + (2A' - B)\frac{d}{dx} + (A'' - B' + C) \tag{2.5}$$

If L^* is found to be equal to L then L is called formally self adjoint. By comparing equations (2.2) and (2.5) we can see that for a second order linear differentiable operator to be formally self adjoint, A' must be equal to B. You may notice that $A'' - B' + C$ must also be equal to C but this is always true given that A' equals B.

If the boundary conditions on L are homogenous then we can also define an adjoint operator, $\mathscr{L}^*$, by the relation

$$(Lu, v) = (L^*v, u) \tag{2.6}$$

where (f, g) is the inner produce of f and g

$$(f, g) = \int_a^b f(x)g(x)dx \tag{2.7}$$

We now can understand that the adjoint operator $\mathscr{L}^*$ must consist of L^* and boundary conditions to force the terms that come about from integrating by parts to be zero.

As an example of $\mathscr{L}$ and $\mathscr{L}^*$, consider $\mathscr{L}$ to consist of $L = \frac{d}{dx}$ and the boundary condition $u(0) = 3u(1)$ over the interval $0 \leq x \leq 1$. Then

$$\begin{aligned}(Lu, v) &= \int_0^1 u'vdx \\ &= (uv)\Big|_0^1 - \int_0^1 uv'dx \\ &= u(1)v(1) - u(0)v(0) + \int_0^1 uL^*vdx \\ &= u(1)(v(1) - 3v(0)) + \int_0^1 uL^*vdx\end{aligned} \tag{2.8}$$

Since the particular value of $u(1)$ is not given, we must make $v(1) - 3v(0)$ equal zero, because choosing $u(1) = 0$ would unduly restrict our solution. Lastly, if $\mathscr{L} = \mathscr{L}^*$, then $\mathscr{L}$ is called self-adjoint.

3 The Delta Function