# Problem Set 5

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1. (a) $\forall z \in U, \exists x \in S, \exists y \in T,\ z = x + y$
   (b) $\forall x \in S, \forall y \in T, \exists z \in S,\ z = xy$
   (c) $\forall x \in S, \exists y \in T,\ y > x$

2. (a) True
   (b) True
   (c) False
   (d) True
   (e) True
   (f) False
   (g) True

3. Option (d) implies that $(\sim P(x)) \implies Q(x)$ is false for some $x \in \mathbb{Q}$. The others do not.

4. (a) For all circles, $C_1$, in $\mathcal{A}$ there is at least one circle, $C_2$, in $\mathcal{B}$, such that $C_1$ and $C_2$ have exactly two points in common.
   (b) $\exists C_1 \in \mathcal{A}, \forall C_2 \in \mathcal{B}, \sim P(C_1, C_2)$.
   (c) There exists a circle, $C_1$, in $\mathcal{A}$, such that there is no circle, $C_2$, in $\mathcal{B}$, for which $C_1$ and $C_2$ share exactly two points.
   (d) The statement in (a) is true. The statements in (b) and (c) are false.

5. (a) This is true. If two lines are perpendicular, all angles between them are $90°$. Since $\ell_1$ and $\ell_2$ are perpendicular to $\ell_3$, the angles between $\ell_1$ and $\ell_3$ and the angles between $\ell_2$ and $\ell_3$ will all be $90°$. Thus, the corresponding angles will be congruent, and by the corresponding angles theorem, $\ell_1$ and $\ell_2$ are parallel.

   (b) This is true. Two lines are parallel if they never intersect. Any third line that does intersect $\ell_1$ is parallel to $\ell_1$, and if that line also doesn't intersect $\ell_2$ it is parallel to $\ell_2$. Since $\ell_1$ and $\ell_2$ are parallel to the same line, they are parallel by the parallel transitive theorem.

   (c) This is true by the corresponding angles theorem.

   (d) This is false. Parallel lines do not cover the entire space that they exist in.

6. $\forall a, b, c \in S, a+b+c = 0 \implies abc < 0$, where $S = \{x | x = 2k+1, k \in \mathbb{Z}\}$.
   Let $a = 2k + 1$, $b = 2l + 1$, $c = 2m + 1$, where $k, l, m \in \mathbb{Z}$.
   $$\begin{aligned} a + b + c &= 2k + 1 + 2l + 1 + 2m + 1 \\ &= 2(k + l + m + 1) + 1 \end{aligned}$$
   Therefore $\forall a, b, c \in S$, $a + b + c \neq 0$. Thus, the implication is true for all $a, b, c \in S$ since the premise is always false.

7. Prove: $\forall k \in \mathbb{Z}, \exists x \in \mathbb{Z}, x = 2k \implies \exists l \in \mathbb{Z}, 7x - 3 = 2l + 1$.
   $$\begin{aligned} 7x - 3 &= 7(2k) - 3 \\ &= 2(7k) - 3 \\ &= 2(7k - 2) + 1 \end{aligned}$$
   Let $l = 7k - 2$.
   Therefore, $7x - 3 = 2l + 1$.
   Prove: $\forall l \in \mathbb{Z}, \exists x \in \mathbb{Z}, 7x - 3 = 2l + 1 \implies \exists k \in \mathbb{Z}, x = 2k$.
   $7x - 3 = 2l - 1$
   $7x = 2l + 2$
   $x = \frac{2}{7}(l + 1)$
   Since $x \in \mathbb{Z}$, it follows that $\frac{l+1}{7} \in \mathbb{Z}$.
   Let $k = (l + 1)/7$.
   Therefore $x = 2k$.

8. Prove: $\forall k \in \mathbb{Z}, \exists x \in \mathbb{Z}, 3x - 1 = 2k \implies \exists l \in \mathbb{Z}, 5x + 2 = 2l + 1$.
   Suppose x is even.
   $x = 2m$, $m \in \mathbb{Z}$.
   $6m - 1 = 2k$.
   $m = \frac{1}{3}k - \frac{1}{6}\#$. Therefore, $x$ must be odd.
   $x = 2m + 1$
   $5(2m + 1) + 2 = 2l + 1$
   $10m + 7 = 2l + 1$
   $2(5m + 3) + 1 = 2l + 1$
   Therefore, if $3x - 1$ is even, $5x + 2$ must be odd.
   Prove: $\forall l \in \mathbb{Z}, \exists x \in \mathbb{Z}, 5x + 2 = 2l + 1 \implies \exists k \in \mathbb{Z}, 3x - 1 = 2k$.
   Suppose $x$ is even.
   $x = 2m$, $m \in \mathbb{Z}$
   $5(2m) + 2 = 2l + 1$
   $10m + 2 = 2l + 1$
   $m = \frac{1}{5}l + \frac{1}{10}\#$. Therefore, $x$ is odd.
   $x = 2m + 1$
   $3(2m + 1) - 1 = 2k$
   $6m + 2 = 2k$
   $2(3m + 1) = 2k$
   Therefore, if $5x + 2$ is odd, $3x - 1$ must bee even.