Theoretical Mechanics
理论力学

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Syllabus

- Chapter 0 Preface
- Chapter 1 Survey of the Elementary Principles
- Chapter 2 Variational Principle and Lagrange's Equations
- Chapter 3 The Central Force Problem
- Chapter 4 The Kinematics of Rigid Body Motion

Mid-term exam

- Chapter 5 The Rigid Body Equations of Motion
- Chapter 6 Oscillations
- Chapter 7 The Hamilton Equations of Motion
- Chapter 8 Canonical Transformations

Final term exam

Please express your feelings directly!

- Syllogism（三段论）
  - A major premise
  - A minor premise
  - A conclusion
- Students occasionally lost their minds in lectures, especially when a teacher said “obviously…" or “because …”.
- In the above contexts, the teacher took it for grant that students had already known the major premise. However, the teacher and students might not share the same major premises.
- No one can read others’ minds. Please express your feelings directly!

Chapter 1 Survey of the Elementary Principles

The motion of material bodies formed the subject of some of the earliest research pursued by the pioneers of physics. From their efforts there has evolved a vast field known as analytical mechanics or dynamics, or simply, mechanics.

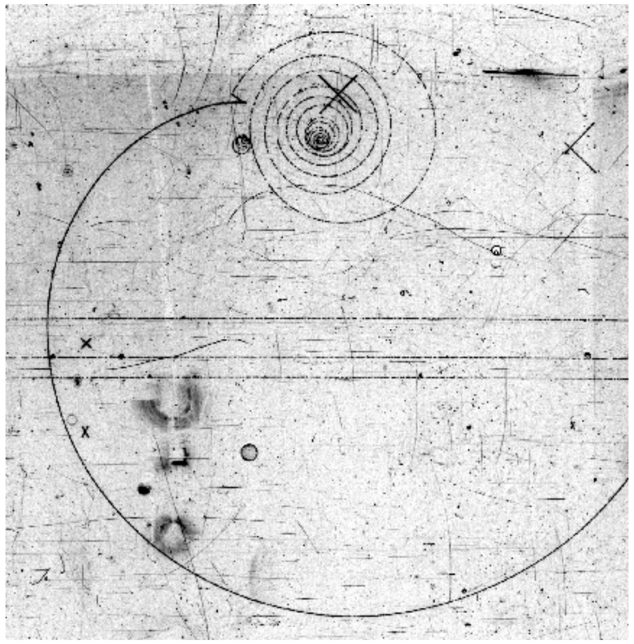
In the present century the term "classical mechanics" has come into wide use to denote this branch of physics in contradistinction to the newer physical theories, especially quantum mechanics.

We shall follow this usage, interpreting the name to include the type of mechanics arising out of the special theory of relativity. It is the purpose of this book to develop the **structure** of classical mechanics and to outline some of its applications of present-day interest in pure physics.

Basic to any presentation of mechanics are a number of fundamental physical concepts, such as **space**, **time**, **simultaneity**, **mass** and **force**. For the most part, however, these concepts will not be analyzed critically here; rather, they will be assumed as undefined terms whose meanings are familiar to the reader.

1.1 Mechanics of a particle

Let **r** be the radius vector of a particle from some given origin and **v** its



vector velocity: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$

Linear momentum: $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$

Newton's second law of motion: $\mathbf{F} = \dfrac{d\mathbf{p}}{dt} = \dot{\mathbf{p}} = m\ddot{\mathbf{r}}$

Inertial or Galilean system = a reference frame where $\mathbf{F} = \dot{\mathbf{p}}$ holds

1.1 Mechanics of a particle

Newton's second law can be expressed as

There exist reference frames where the time derivation of the linear momentum equals to the force.

Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

Momentum of force: $\mathbf{N} = \mathbf{r} \times \mathbf{F}$

$$\mathbf{N} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{L}}{dt} \equiv \dot{\mathbf{L}}$$

1.1 Mechanics of a particle

Conservation Theorems for the linear and angular momentums

From $\mathbf{F} = \dot{\mathbf{p}}$

If the total force $\mathbf{F}$ is zero, the linear momentum $\mathbf{p}$ is conserved.

From $\mathbf{N} = \dot{\mathbf{L}}$

If the total torque $\mathbf{N}$ is zero, the angular momentum $\mathbf{L}$ is conserved.

1.1 Mechanics of a particle

Work done by the external **F** is

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = \frac{m}{2} \int \frac{d}{dt} \left(v^2\right) dt = \frac{m}{2} \left(v_2^2 - v_1^2\right)$$

Kinetic energy: $T = \dfrac{1}{2} m v^2$

$$W_{12} = T_2 - T_1$$

1.1 Mechanics of a particle

Conservation Force

If W_{12} is the same for any physically possible path between points 1 and 2, the force (and the system) is said to be conservative.

Equivalently, the total work done around a closed circuit is zero: $\oint \mathbf{F} \cdot d\mathbf{s} = 0$

$\mathbf{F}$ is conservative $\Longleftrightarrow \mathbf{F} = -\nabla V(\mathbf{r})$

1.1 Mechanics of a particle

For a conservative system, the work done by the forces is $W_{12} = V_1 - V_2$

So that $T_1 + V_1 = T_2 + V_2$

Energy Conservation Theorem for a particle :

If the force acting on a particle are conservative, then the total energy of the particle, $T + V$, is conserved.

1.2 Mechanics of a system of particles

What's the difference of mechanics for one particle and particles? **Just add indices!**

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \quad \mathbf{N}_i = \dot{\mathbf{L}}_i$$

The Newton's second law for particles:

$$\mathbf{F}_i = \sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i$$

An external force on *i* particle: $\mathbf{F}_i^{(e)}$

The internal force on the *i*th particle due to the *j*th particle: $\mathbf{F}_{ij}$

In the lecture, I do **NOT** use the so-called Einstein summation convention as possible as I can!

1.2 Mechanics of a system of particles

If $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, $\sum_i \mathbf{F}_i = \sum_i \mathbf{F}_i^{(e)}$

Weak law of action and reaction

$$\sum_i \mathbf{F}_i = \sum_{\substack{i,j \\ i \neq j}} \mathbf{F}_{ji} + \sum_i \mathbf{F}_i^{(e)} = \sum_{i<j} (\mathbf{F}_{ji} + \mathbf{F}_{ij}) + \sum_i \mathbf{F}_i^{(e)}$$

1.2 Mechanics of a system of particles

The equations of motion for particles

$$\sum_i \mathbf{F}_i = \sum_i \mathbf{F}_i^{(e)} = \sum_i \dot{\mathbf{p}}_i = \frac{d^2}{dt^2}\sum_i m_i \mathbf{r}_i$$

Define the center of mass: $\mathbf{R} \equiv \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}$

So that $M\ddot{\mathbf{R}} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}$

1.2 Mechanics of a system of particles

Total linear momentum

$$\mathbf{P} = \sum_i \mathbf{p}_i = \sum_i m_i \dot{\mathbf{r}}_i = M\dot{\mathbf{R}}$$

Newton's equation of motion for the center of mass

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F}^{(e)}$$

Conservation Theorem for the Linear Momentum of a System of Particles: The total linear momentum is conserved if the total external force is zero

1.2 Mechanics of a system of particles

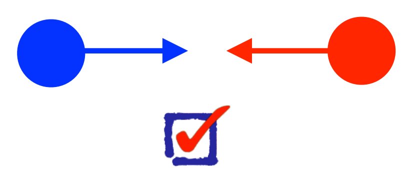
Total angular momentum

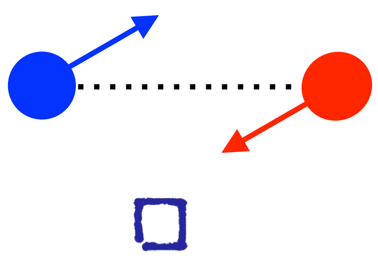
$$\mathbf{L} = \sum_i \mathbf{L}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

Newton's equation of motion for the center of mass

$$\dot{\mathbf{L}} = \sum_{\substack{i,j \\ i \neq j}} \mathbf{r}_i \times \mathbf{F}_{ji} + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)}$$

Strong law of action and reaction:





$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = \left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{ji}$$

$$\dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \sum_i \mathbf{N}_i^{(e)} = \mathbf{N}^{(e)}$$

1.2 Mechanics of a system of particles

Conservation Theorem for Total Angular Momentum: The total angular momentum is conserved if the external torque is zero.

↓

A multi-particle system can be treated as if it were a single particle if the internal forces obey the strong law of action and reaction.

Most forces, such as gravity, electrostatic force, obey strong law of action and reaction.

1.2 Mechanics of a system of particles

There are rare exceptions, for example, Lorentz force

However, if we take into account the EM field, conservation laws will be restored.

EM field has linear and angular momenta!

1.2 Mechanics of a system of particles

Total angular momentum again!

$$\mathbf{L} = \sum_i \mathbf{L}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R}\ ,\ \mathbf{v}_i = \mathbf{v}_i' + \mathbf{v} \text{ and } \ \mathbf{v} = \frac{d\mathbf{R}}{dt}$$

$$\mathbf{L} = \sum_i \mathbf{R} \times m_i \mathbf{v} + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' + \left(\sum_i m_i \mathbf{r}_i' \right) \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} \sum_i m_i \mathbf{r}_i'$$

$$\mathbf{L} = \underbrace{\mathbf{R} \times M\mathbf{v}}_{\substack{\textbf{Angular momentum}\\ \textbf{of motion concentrated}\\ \textbf{at the center of mass}}} + \underbrace{\sum_i \mathbf{r}_i' \times \mathbf{p}_i'}_{\substack{\textbf{Angular momentum}\\ \textbf{of motion around}\\ \textbf{the center of mass}}}$$

1.2 Mechanics of a system of particles

Work: $W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i$

Position 1 and 2 are configurations (sets of positions)

Work: $W_{12} = T_2 - T_1$ where $T = \sum_i \frac{1}{2} m_i v_i^2$

Kinetic Energy T can be split into two pieces

$$T = \frac{1}{2}\sum_i m_i \left(\mathbf{v} + \mathbf{v}_i'\right) \cdot \left(\mathbf{v} + \mathbf{v}_i'\right) = \frac{1}{2}\sum_i m_i v^2 + \frac{1}{2}\sum_i m_i v_i'^2 + \mathbf{v} \cdot \frac{d}{dt}\left(\sum_i m_i \mathbf{r}_i'\right)$$

$$T = \frac{1}{2} M v^2 + \frac{1}{2}\sum_i m_i v_i'^2$$

1.2 Mechanics of a system of particles

Work: $W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i$

If external forces are conservative: $\mathbf{F}_i^{(e)} = -\nabla_i V_i$

$$\sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i = -\sum_i \int_1^2 \nabla_i V_i \cdot d\mathbf{s}_i = -\sum_i V_i \bigg|_1^2$$

If internal forces are conservative: $\mathbf{F}_{ij} = -\nabla_i V_{ij}$ and forces satisfy the strong law of action and reaction because of $V_{ij} = V_{ij}\left(\left|\mathbf{r}_i - \mathbf{r}_j\right|\right)$

$$\sum_{\substack{i,j \\ i \neq j}} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i = -\sum_{\substack{i,j \\ i \neq j}} \int_1^2 \nabla_i V_{ij} \cdot d\mathbf{s}_i = -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij} \bigg|_1^2$$

The above derivation is so important that you have to re-derive it by yourself!

1.2 Mechanics of a system of particles

If all forces are conservative, total energy $T + V$ is conserved.

Total Potential Energy: $V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij}$

The second term is internal potential energy.

Generally, the internal potential energy need not be zero. In fact it may be vary as the system changes with time.

In a rigid body the internal force do no work. The internal potential must be constant.

1.3 Constraints

From the previous sections one might obtain the impression that all problem in mechanics have been reduced to solving the set of differential equation:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ji}$$

Even form a purely physical standpoint, however, this view is **oversimplified**.

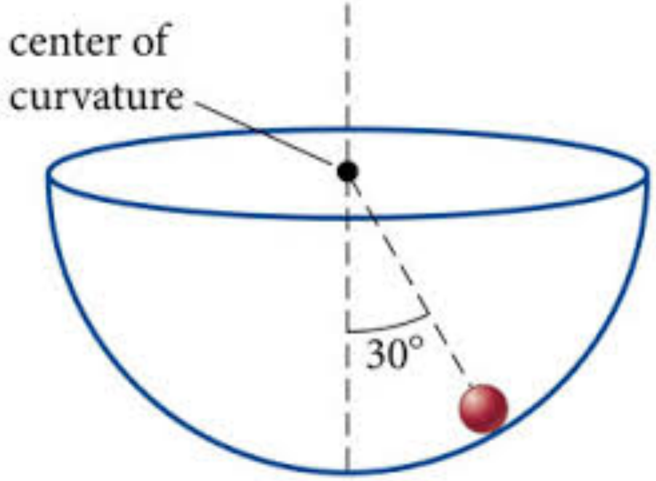
Free space is an idealization

Constraints: limit the motion of the particles

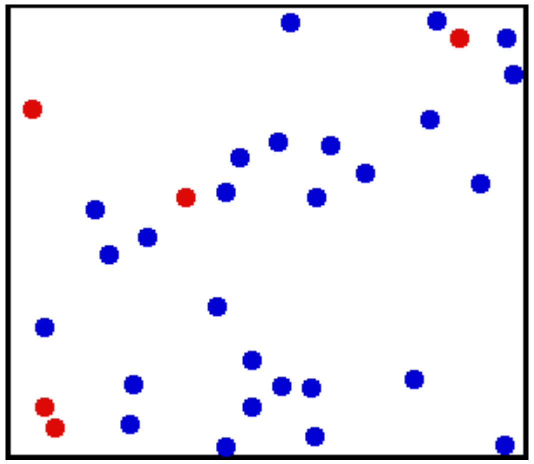
1.3 Constraints

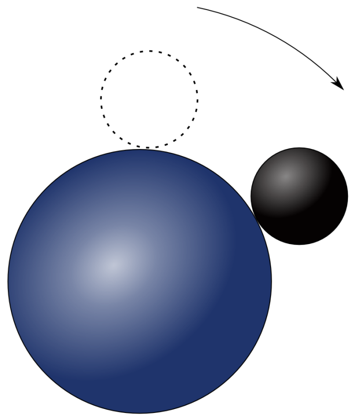
Constraints may be classified in various ways.

Holonomic constraint (完整约束)



Nonholonomic constraint (非完整约束)

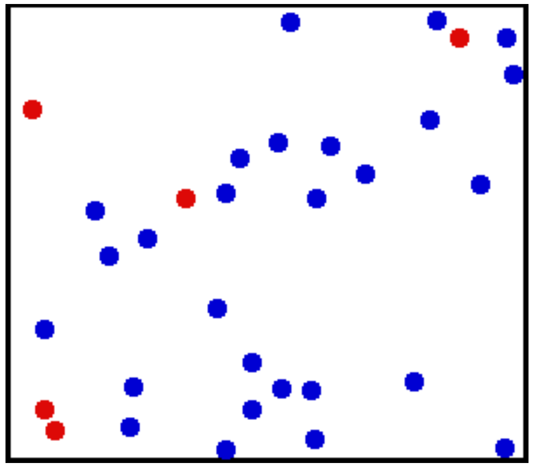




1.3 Constraints

Constraints may be classified in various ways.

Scleronomous constraint (稳定约束)

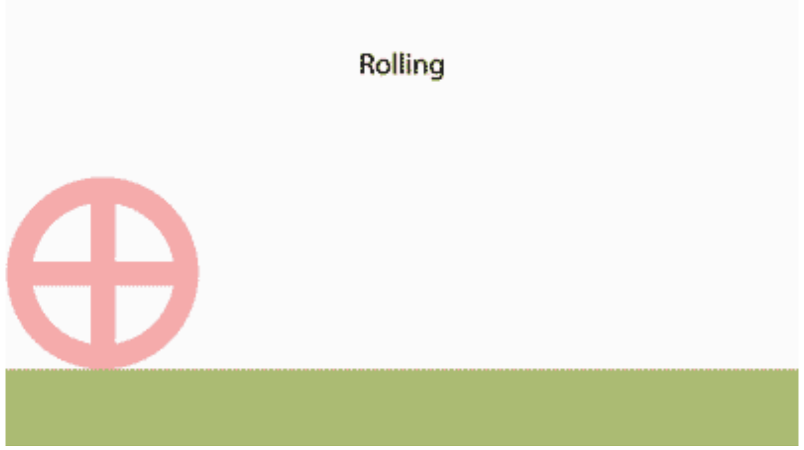


Rheonomous constraint (非稳定约束)

1.3 Constraints

Vertical disk rolling on a line

The constraint is holonomic.



Vertical disk rolling on a horizontal plan

The constraint is nonholonomic.

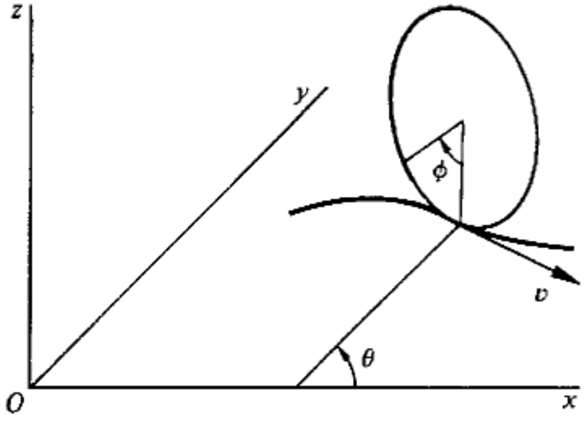


FIGURE 1.5 Vertical disk rolling on a horizontal plane.

Holonomic Constraints

Constraints may be expressed by
$$f\left(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, t\right) = 0$$

Particle on the x-y plane $z = 0$

Rigid body $\left(\mathbf{r}_i - \mathbf{r}_j\right)^2 - c_{ij}^2 = 0$

All other cases are called nonholonomic

It means “we don’t really want to mess with it”!

We will deal **only** with holonomic constraints!

Generalized Coordinats

Constraints introduce two types of difficulties:

The coordinates are **no** longer independent.

The forces of constraint is **not** furnished a priori.

N Particles have $3N$ degrees of freedom (obviously?)

k holonomic constraints reduces it to $3N - k$ (obviously?)

In physics, the degrees of freedom (DOF) of a mechanical system is the number of **independent** parameters that define its configuration or state.

In the theory of vector spaces, a set of vectors is said to be **linearly dependent** if at least one of the vectors in the set can be defined as a linear combination of the others; if no vector in the set can be written in this way, then the vectors are said to be **linearly independent**. These concepts are central to the definition of dimension.

Generalized Coordinats

Introducing $3N - k$ generalized coordinates

$q_1, q_2, \ldots, q_{3N-k}$

$$\mathbf{r}_i = \mathbf{r}_i \left(q_1, q_2, \ldots, q_{3N-k}, t \right)$$

$$\Updownarrow$$

$$\mathbf{r}_1 = \mathbf{r}_1 \left(q_1, q_2, \ldots, q_{3N-k}, t \right)$$

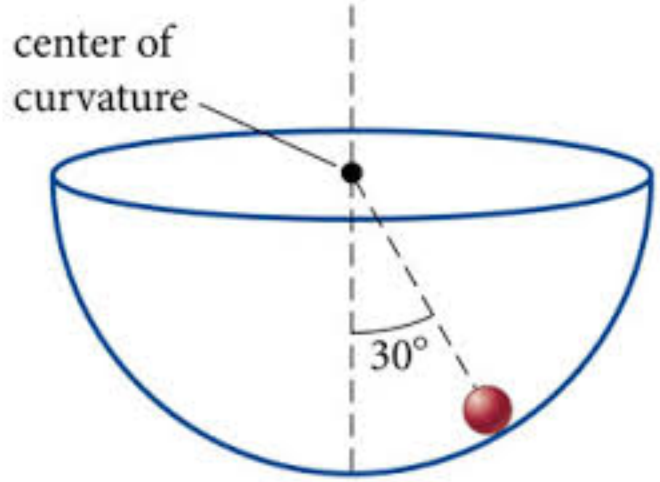
$$\vdots$$

$$\mathbf{r}_N = \mathbf{r}_N \left(q_1, q_2, \ldots, q_3 N - k, t \right)$$

Generalized Coordinates

Example: if a particle is constrained on a sphere with a radius c

$$\begin{cases} x = c\sin\theta\cos\phi \\ y = c\sin\theta\sin\phi \\ z = c\cos\theta \end{cases}$$



Transformation from (x, y, z) to (θ, ϕ)

Example: Double pendulum

Transformation from $(x_1, x_2, x_3, y_1, y_2, y_3)$ to (θ_1, θ_2)

Usually the generalized coordinates, q_l, unlike the Cartesian coordinates, will not divided into convenient groups of three that can be associated together to form vectors.

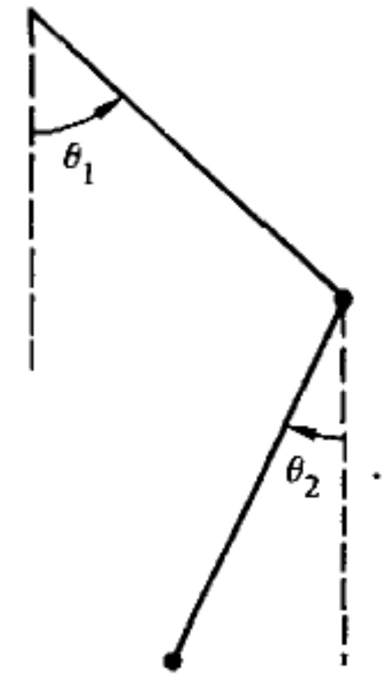


FIGURE 1.4 Double pendulum.