Theoretical Mechanics

理论力学

Zhou Xiang 周详

Syllabus

- Chapter 0 Preface
- Chapter 1 Survey of the Elementary Principles
- Chapter 2 Variational Principle and Lagrange's Equations
- Chapter 3 The Central Force Problem
- Chapter 4 The Kinematics of Rigid Body Motion

Mid-term exam

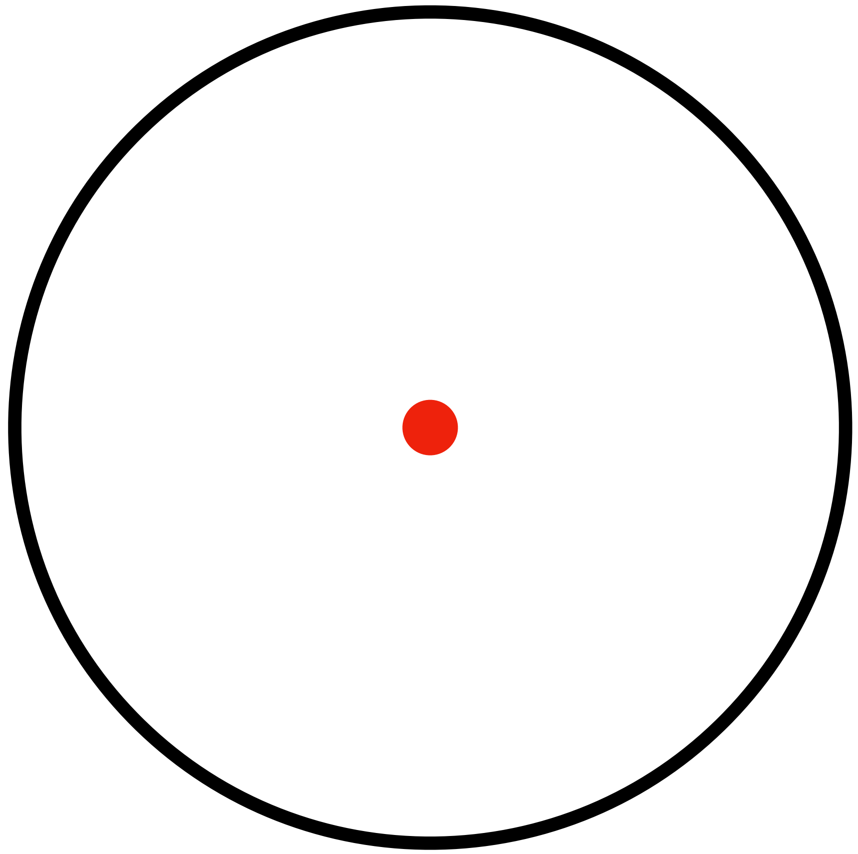
- Chapter 5 The Rigid Body Equations of Motion
- Chapter 6 Oscillations
- Chapter 7 The Hamilton Equations of Motion
- Chapter 8 Canonical Transformations

Final term exam

Circular Motion

$$E = E_4 \rightarrow r = r_1 \text{ (fixed)}$$

Only one radius is allowed

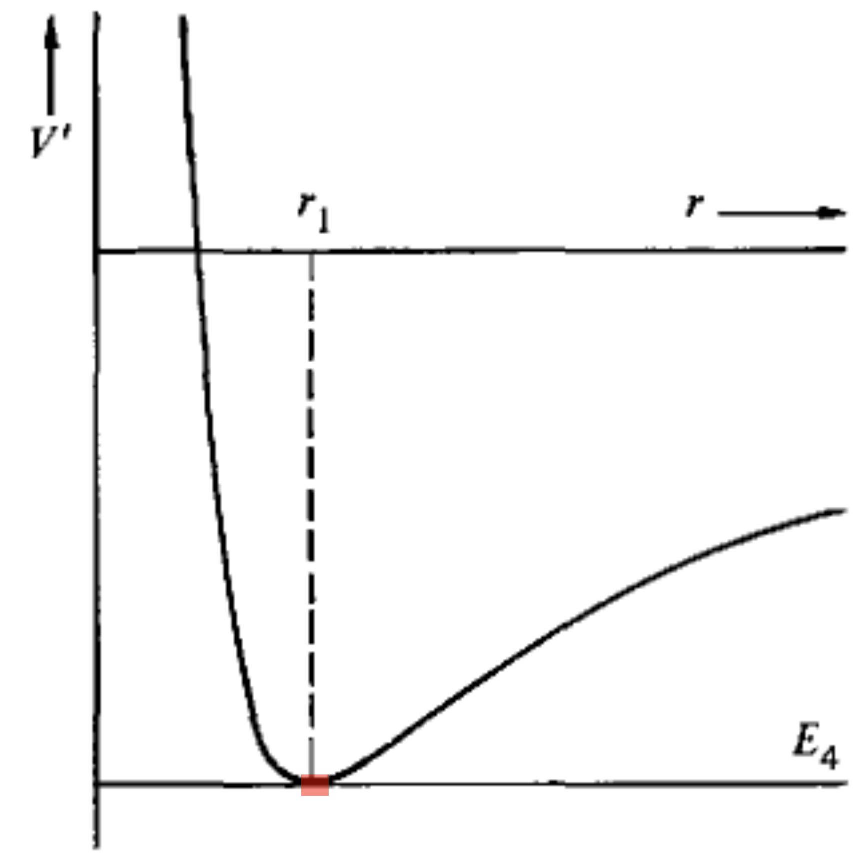


Stays on a circle

$E = V'(r_1)$

$\dot{r} = 0$

$r = \text{const} = r_1$

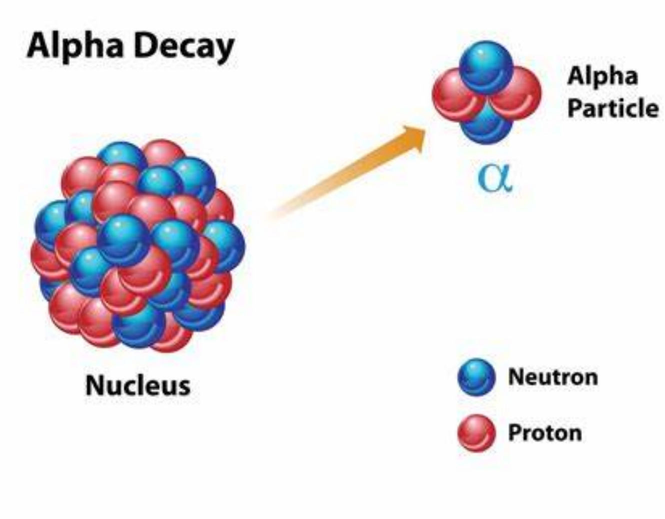


A $1/r^2$ force would make a circle

Classification into unbounded, bounded and circular motion depends only on the general shape of V', not on the details ($1/r^2$ or otherwise)

Another Example

$$V = -\frac{a}{r^3} \leftrightarrow f = -\frac{3a}{r^4} \Longrightarrow V' = -\frac{a}{r^3} + \frac{l^2}{2mr^2}$$

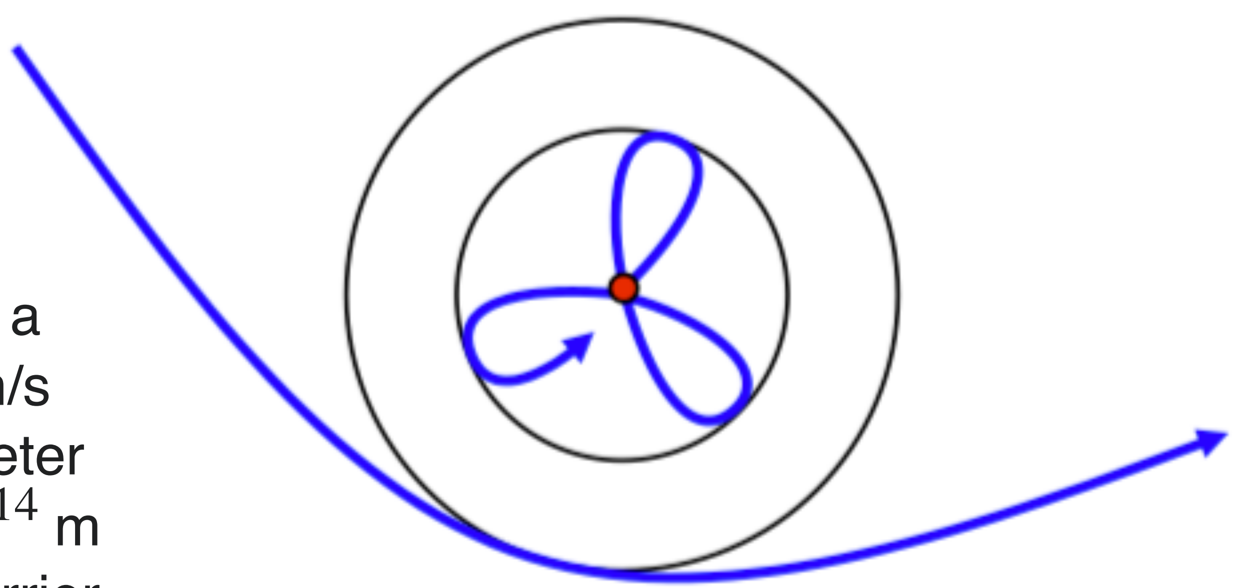


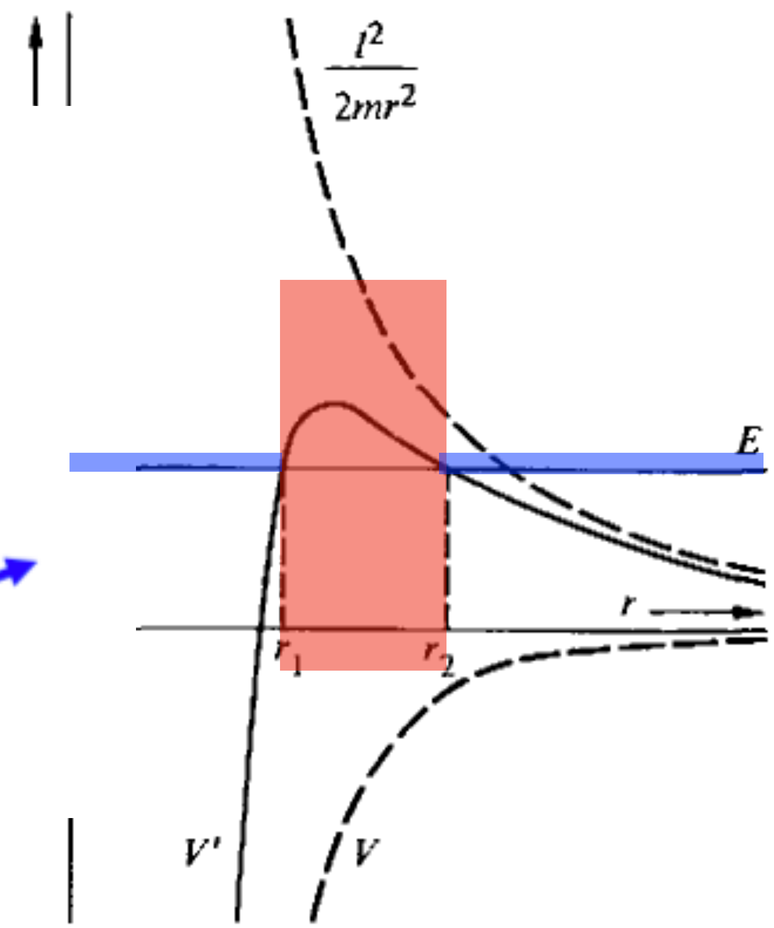
Attractive r^{-4} force

V' has a bump

Particle with energy E may be either bounded or unbounded depending on the initial r

An alpha particle with a speed of 1.5×10^7 m/s within a nuclear diameter of approximately 10^{-14} m will collide with the barrier more than 10^{21} times per second.



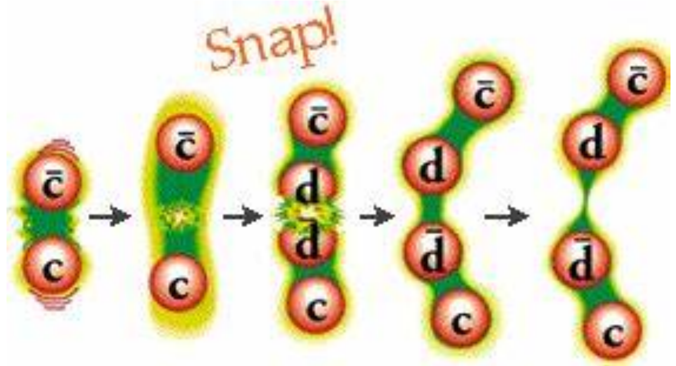


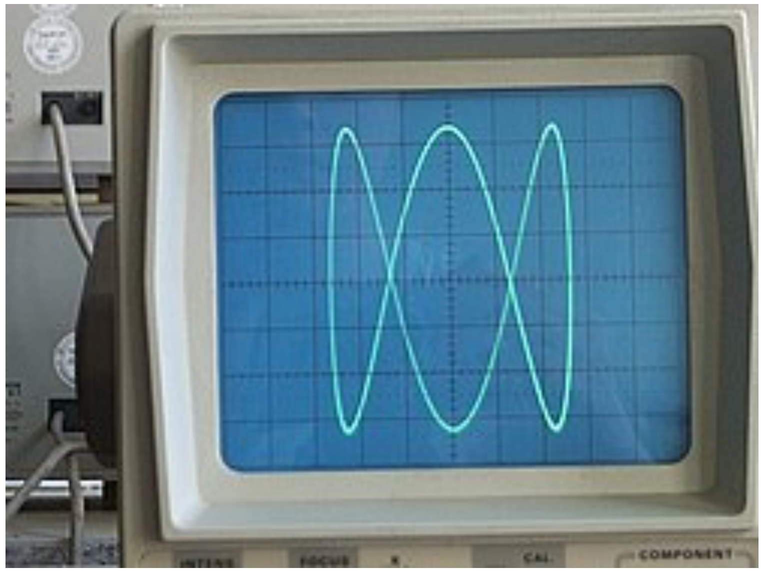
Another Example

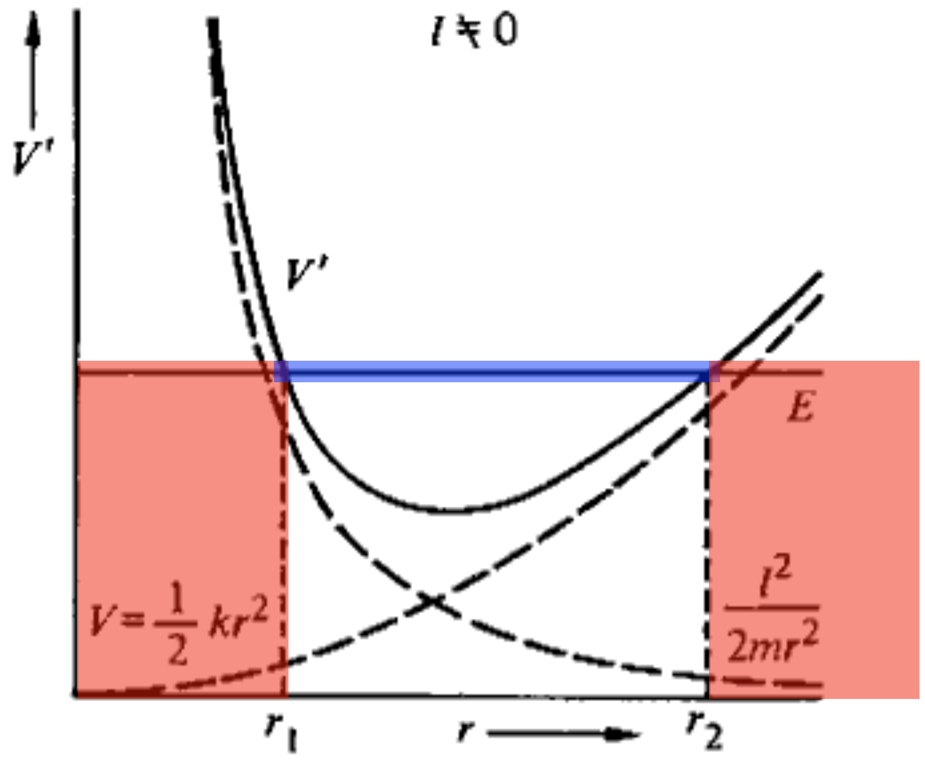
$$V = \frac{1}{2}kr^2 \leftrightarrow f = -kr \Longrightarrow V' = \frac{1}{2}kr + \frac{l^2}{2mr^2}$$

The motion is always bounded for all physically possible.

Lissajous figures







Stable Circular Orbit

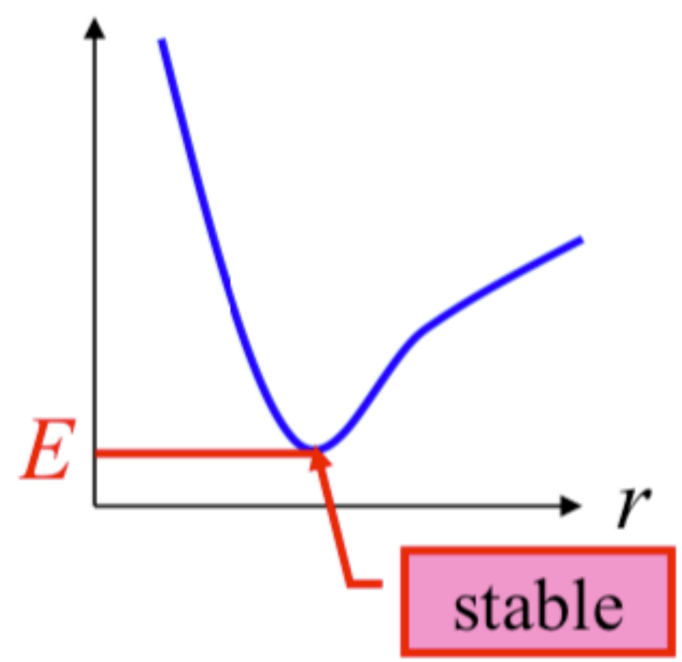
Circular orbit occurs at the bottom of a dip of V'

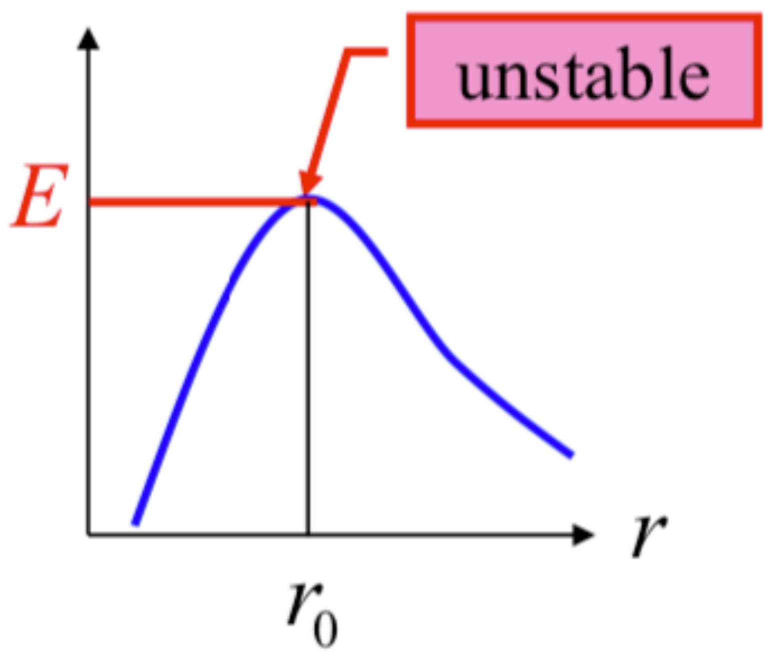
$$\frac{m\dot{r}^2}{2} = E - V' = 0 \rightarrow m\ddot{r} = -\frac{dV'}{dr} = 0 \rightarrow r = \text{const}$$

Top of a bump works in theory but it is unstable

Initial condition must be exactly $\dot{r} = 0$ and $r = r_0$

Stable circular orbit requires $\dfrac{d^2V'}{dr^2} > 0$





3.4 The Virial Theorem

The virial theorem is a general theorem valid for a large variety of systems. It is concerned with the time averages of various mechanical quantities and is useful in Statistical Mechanics.

For a system, position vectors $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$ (including any forces of constraint)

$$\dot{\mathbf{p}}_i = \mathbf{F}_i$$

Consider a quantity: $G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i$

3.4 The Virial Theorem

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i$$

It's total time derivative: $\dfrac{dG}{dt} = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i + \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i$

The first term: $\sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i = \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T$

The Second term: $\sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$

Thus $\qquad \dfrac{d}{dt} \sum_i \mathbf{p}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$

3.4 The Virial Theorem

$$\frac{d}{dt}\sum_i \mathbf{p}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$$

The time average of the above equation over a time interval τ is:

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt \equiv \overline{\frac{dG}{dt}} = \overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$$

Then $\overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau}[G(\tau) - G(0)]$

If motion is periodic $[G(\tau) - G(0)]$ is zero

If motion is not periodic $\lim_{\tau\to\infty} \frac{1}{\tau}[G(\tau) - G(0)] = 0$

3.4 The Virial Theorem

$$\overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau}[G(\tau) - G(0)] = 0 \rightarrow$$

Virial theorem: $\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$

$\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$ is called the virial (维里/位力) of Clausius

Ideal Gas Pressure Equation

Consider a gas consisting of N monoatoms with volume V. According to the equipartition theorem, the average kinetic energy of each atom is $\frac{3}{2}k_BT$, thus $\overline{T} = \frac{3}{2}Nk_BT$

The pressure is $P\mathbf{n} = -\frac{d\mathbf{F}_i}{dA}$, then

$$\frac{1}{2}\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -\frac{P}{2}\int \mathbf{n} \cdot \mathbf{r} dA$$

According to Gauss's theorem $\int \mathbf{n} \cdot \mathbf{r} dA = \int \nabla \cdot \mathbf{r} dV = 3\boldsymbol{V}$

Then $$\frac{3}{2}Nk_BT = \frac{3}{2}PV \Longrightarrow PV = Nk_BT$$

Central force

If the forces are conserved and $\mathbf{F}_i = -\nabla V_i$, then

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{2}\overline{\sum_i \nabla V_i \cdot \mathbf{r}_i}$$

For a single particle moving under a central force

$$\overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r}r}$$

If V is a power-law function of r:

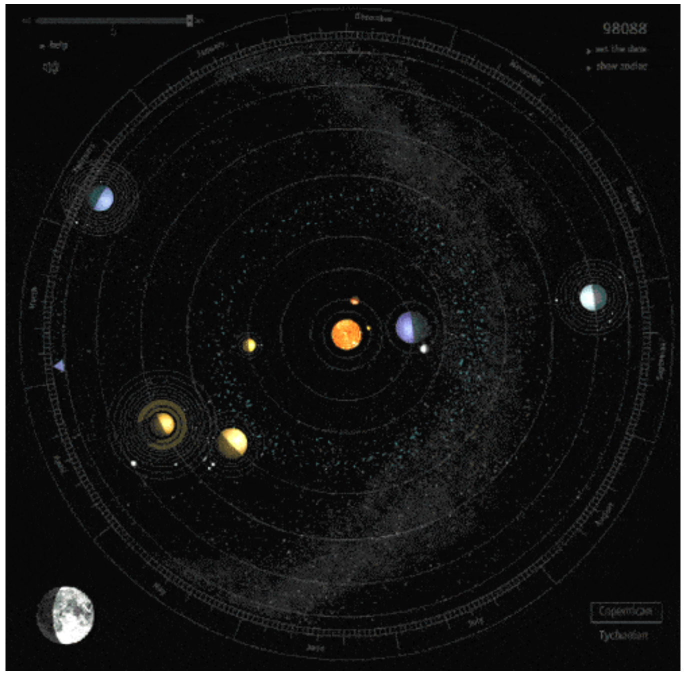
$$V = ar^{n+1} \text{ (Why } n+1\text{?)} \qquad \frac{\partial V}{\partial r}r = (n+1)V$$

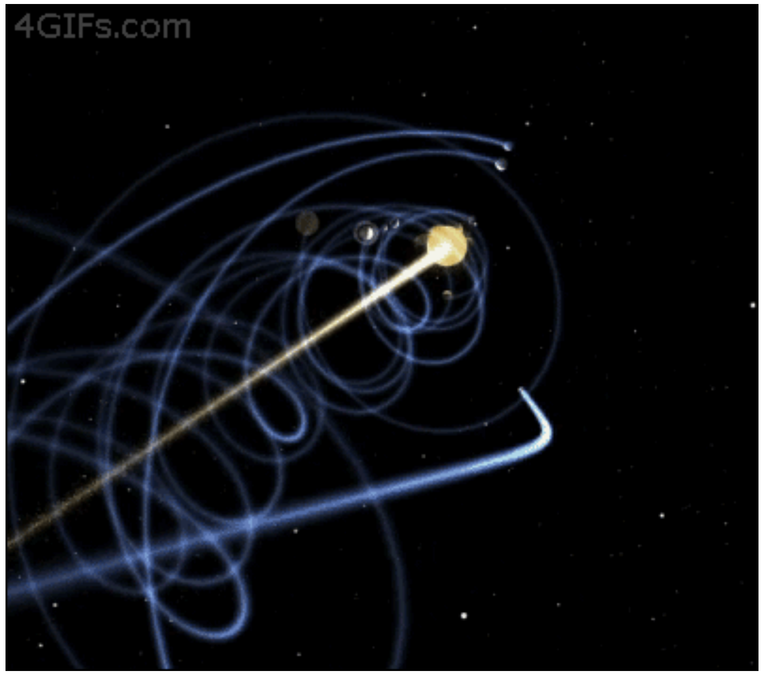
We have $\overline{T} = \dfrac{n+1}{2}\overline{V}$. For $n = -2$, $\overline{T} = -\dfrac{1}{2}\overline{V}$.

3.5 The Differential Equation for the Orbit, and Integrable Power-Law Potentials

Hitherto solving a problem has meant finding r and θ as functions of time with E, l, etc., as constants of integration.

But most what we really seek is the equation of the orbit, i.e., the dependence of r upon θ, eliminating the parameter t.





3.5 The Differential Equation for the Orbit, and Integrable Power-Law Potentials

We have been trying to solve $r = r(t)$ and $\theta = \theta(t)$.
Now we are interested in the shape of orbit $r = r(\theta)$ and switch from dt to $d\theta$

$$l = mr^2\dot{\theta} \Longrightarrow \frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{dV}{dr} = 0 \Longrightarrow \frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} + \frac{dV}{dr} = 0$$

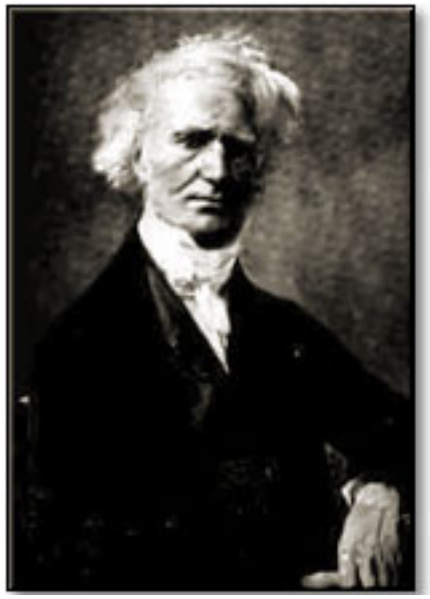
Switch from r to $u \equiv 1/r$

$$\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} \Longrightarrow \frac{d}{dr} = -u^2\frac{d}{du}$$

3.5 The Differential Equation for the Orbit, and Integrable Power-Law Potentials

$$\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} \Longrightarrow \frac{d}{dr} = -u^2\frac{d}{du}$$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} + \frac{dV}{dr} = 0$$



Jacques Philippe Marie Binet
1786-1856

$$\Longrightarrow \frac{d^2u}{d\theta^2} + u + \frac{m}{l^2}\frac{dV\left(\frac{1}{u}\right)}{du} = 0 \quad \text{(Binet equation)}$$

Solving this equation gives the shape of the orbit

3.5 The Differential Equation for the Orbit, and Integrable Power-Law Potentials

$$\frac{d^2u}{d\theta^2} + u + \frac{m}{l^2}\frac{dV\left(\frac{1}{u}\right)}{du} = 0$$

Why we do it? Because it is easier?

We'll do this for inverse-square force later

One more useful knowledge can be extracted without solving the equation

Symmetry of Orbit

$$\frac{d^2u}{d\theta^2} + u + \frac{m}{l^2}\frac{dV\left(\frac{1}{u}\right)}{du} = 0$$

Equation is even, or symmetric, in θ

Replacing θ with $-\theta$ does not change the equation

Solution $u(\theta)$ must be symmetric if the initial condition is

Choosing $\theta = 0$ at $t = 0$, $\theta \to -\theta$ makes $u(0) \to u(0)$ and $\frac{du}{d\theta}(0) \to -\frac{du}{d\theta}(0) \Longrightarrow \frac{du}{d\theta}(0) = 0$

Orbit is symmetric at angles where $\frac{du}{d\theta} = 0$!

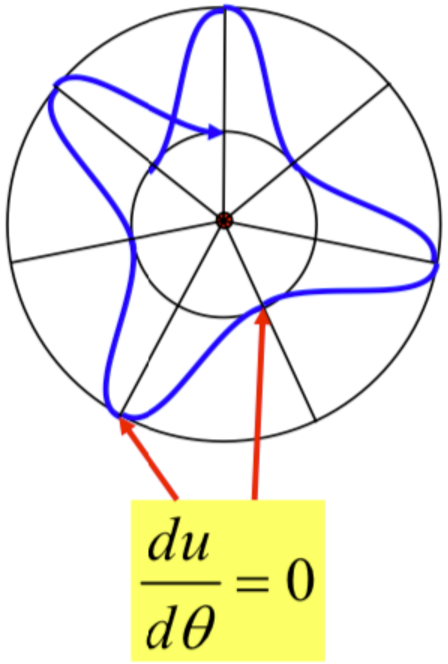
Symmetry of Orbit

Orbit is symmetric about every turning point = apse (拱点)

Orbit is invariant under reflection about apsidal vectors

Solve the orbit between a pair of apsidal points
→ Entire orbit is known

Let’s solve the equation!



3.7 The Kepler Problem: Inverse-Square Law of Force

$$\frac{d^2u}{d\theta^2} + u + \frac{m}{l^2}\frac{dV\left(\frac{1}{u}\right)}{du} = 0$$

Integrating it will give energy conservation

One can use energy conservation to save effort

$$E = \frac{m\dot{r}^2}{2} + \frac{l^2}{2mr^2} + V(r) \rightarrow \dot{r} = \sqrt{\frac{2}{m}\left(E - \frac{l^2}{2mr^2} - V(r)\right)}$$

Switch variables $\dot{r} = -\frac{l}{m}\frac{du}{d\theta}$ (Prove it!)

$$\frac{du}{d\theta} = -\sqrt{\frac{2mE}{l^2} - u^2 - \frac{2mV\left(\frac{1}{u}\right)}{l^2}}$$

3.5 The Differential Equation for the Orbit, and Integrable Power-Law Potentials

Back to $d\theta = \dfrac{ldr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}$

$$\Longrightarrow \theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} + \theta_0 \Longrightarrow \theta = \theta_0 - \int_{\mu_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}$$

If $V = ar^{n+1}$, $\theta = \theta_0 - \displaystyle\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2ma}{l^2}u^{-n-1} - u^2}}$

If $n = 1, -2, -3$, trigonometric functions

If $n = 5,3,0, -4, -5, -7$, elliptic functions

…

3.7 The Kepler Problem: Inverse-Square Law of Force

$$f = -\frac{k}{r^2} \quad V = -\frac{k}{r} \rightarrow \frac{du}{d\theta} = -\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}$$

$$\Longrightarrow \int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}} = -\int d\theta$$

Look it up in a math handbook (if you have) and we can find

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{2}{\sqrt{-\gamma}} \arccos\left(-\frac{\beta + 2\gamma x}{\sqrt{\beta^2 - 4\alpha\gamma}}\right)$$

Just substitute $\alpha, \beta, \gamma\ldots$ or

Working It Out by Yourself

$$\int d\theta = -\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}} = -\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4} - \left(\frac{mk}{l^2} - u\right)^2}}$$

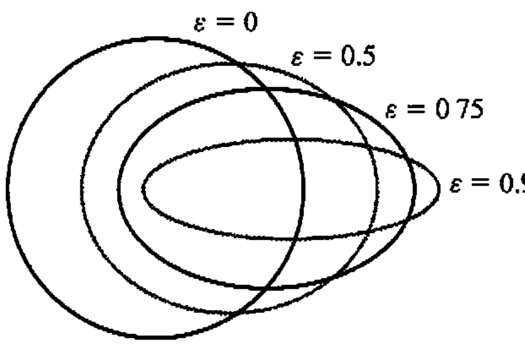
$$= -\frac{1}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}} \int \frac{du}{\sqrt{1 - \left(\frac{\frac{mk}{l^2} - u}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}}\right)^2}} = -\int \frac{\sin\omega}{\sin\omega} d\omega = -\,\omega$$

$$\cos\omega \equiv \frac{\frac{mk}{l^2} - u}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}} \qquad du = \sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}\, \sin\omega\, d\omega$$

$$\cos\omega = \cos(\theta - \theta') = \frac{\frac{mk}{l^2} - u}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}}$$

Working It Out by Yourself

$$\cos\omega = \cos(\theta - \theta') = \frac{\frac{mk}{l^2} - u}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}}$$



Ellipses with the same major axes and eccentricities from 0.0 to 0.9.

$$u = \frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos(\theta - \theta')\right)$$

This matches the general equation of a conic (圆锥截面)

$$\frac{1}{r} = C\left(1 + e\cos(\theta - \theta')\right)$$ One focus is at the origin

e is eccentricity (离心率)

$e > 1,$	$E > 0:$	hyperbola,
$e = 1,$	$E = 0:$	parabola,
$e < 1,$	$E < 0:$	ellipse,
$e = 0,$	$E = -\frac{mk^2}{2l^2}:$	circle.

Energy and Eccentricity

$E = 0$ separates unbounded and bounded orbits

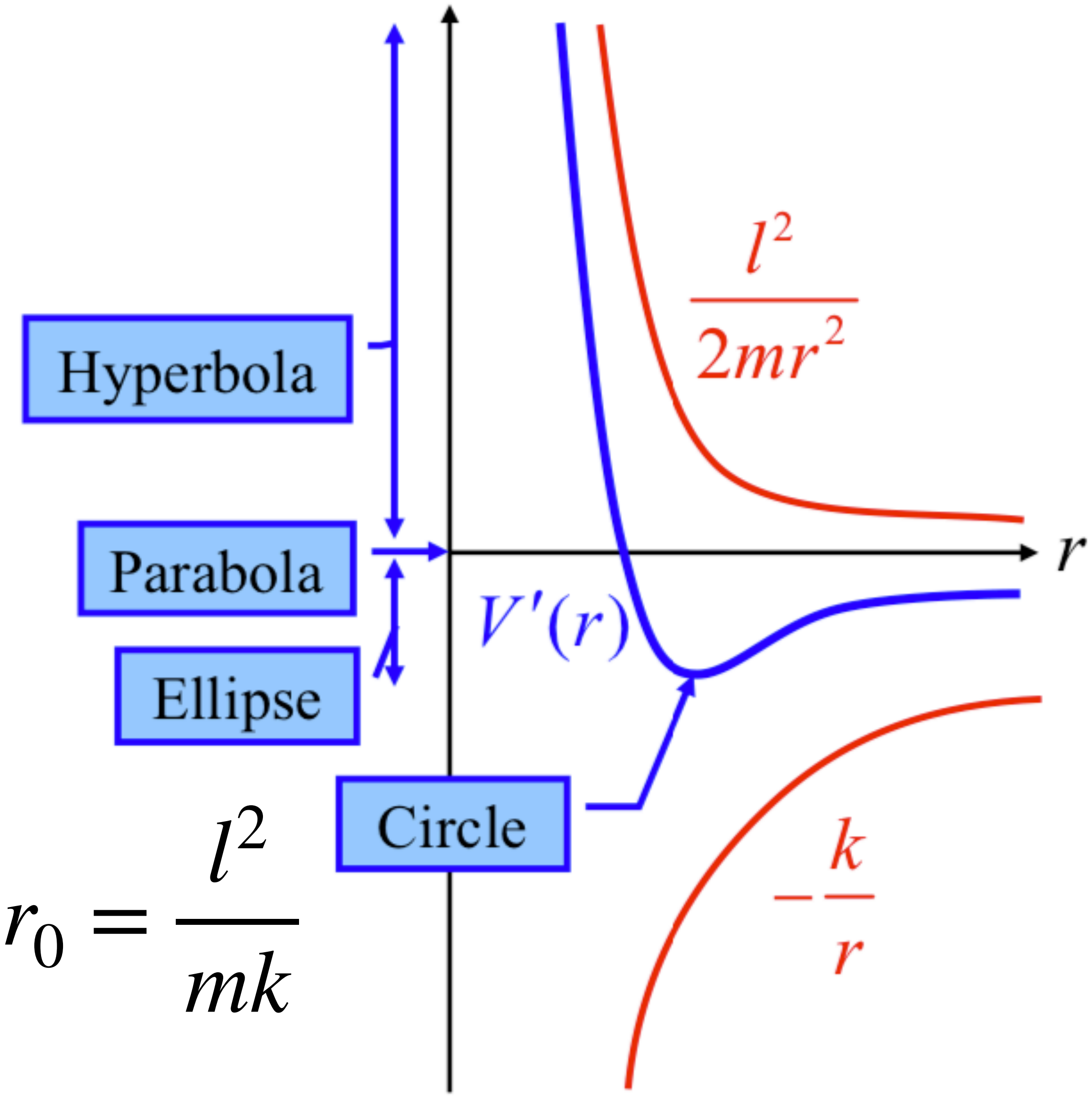
Borderline = Parabola

Circular orbit requires

$$V'(r_0) = -\frac{k}{r_0} + \frac{l^2}{2mr_0^2} = E$$

$$\left.\frac{dV'}{dr}\right|_{r_0} = \frac{k}{r_0^2} - \frac{l^2}{mr_0^3} = 0 \Longrightarrow r_0 = \frac{l^2}{mk}$$

$$\Longrightarrow E = -\frac{mk^2}{2l^2}$$

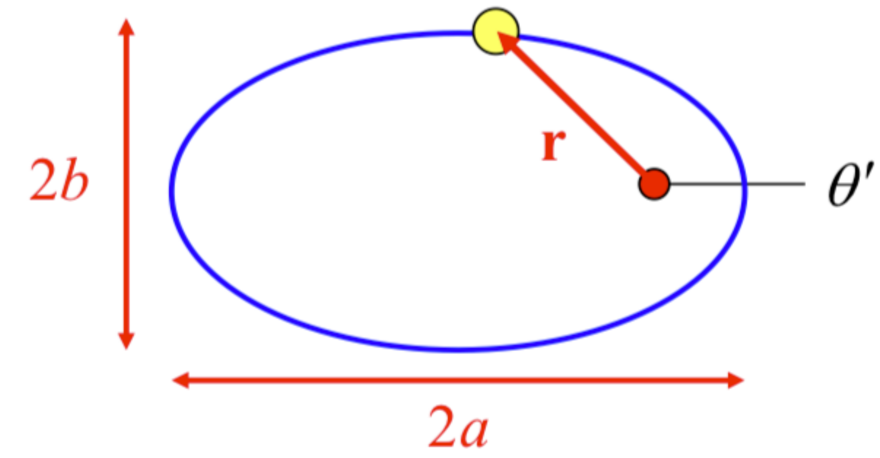


Bound Orbits

$$\frac{1}{r} = C\left(1 + e\cos\left(\theta - \theta'\right)\right),\ C = \frac{mk}{l^2},\ e = \sqrt{1 + \frac{2El^2}{mk^2}}$$

Ends of the major axis are $1/r = C(1 \pm e)$

Length of the major axis



$$a = \frac{1}{2}\left(\frac{1}{C(1+e)} + \frac{1}{C(1-e)}\right) = -\frac{k}{2E}$$

Major axis is given by the total energy E

Minor axis is $b = a\sqrt{1 - e^2} = \sqrt{-\frac{l^2}{2mE}}$

3.8 The Motion in Time in the Kepler Problem

$$a = -\frac{k}{2E}, b = \sqrt{-\frac{l^2}{2mE}}, A = \pi ab = \pi\sqrt{-\frac{l^2k^2}{8mE^3}}$$

The areal velocity is constant

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m} \Longrightarrow \tau = A/\frac{dA}{dt} = \pi\sqrt{-\frac{mk^2}{2E^3}}$$

Express τ in terms of a

$$\tau = 2\pi\sqrt{\frac{m}{k}}a^{3/2}$$

Kepler's Third Law of Planetary Motion:

Period of ration is proportional to the 3/2 power of the major axis

Kepler's Third Law

Kepler's third law is not exact! $\tau = 2\pi\sqrt{\frac{m}{k}}a^{3/2}$

The reason: reduced mass $\frac{1}{\mu} = \frac{1}{M} + \frac{1}{m}$

k is given by the gravity

$$f = -G\frac{Mm}{r^2} = -\frac{k}{r^2} \Longrightarrow k = GMm$$

Period of rotation becomes

$$\tau = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2} = 2\pi\sqrt{\frac{1}{G(M+m)}}a^{3/2}$$

Coefficient is same for all planets only if $M \gg m$

3.8 The Motion in Time in the Kepler Problems

So far we dealt with the shape of the orbit: $r = r(\theta)$

We don't have the full solutions: $r = r(t)$ and $\theta = \theta(t)$

Why aren't we dong it? Because it's awfully complicated!

Not so bad to get $t = t(\theta)$, learn Sec. 3.8 by yourself

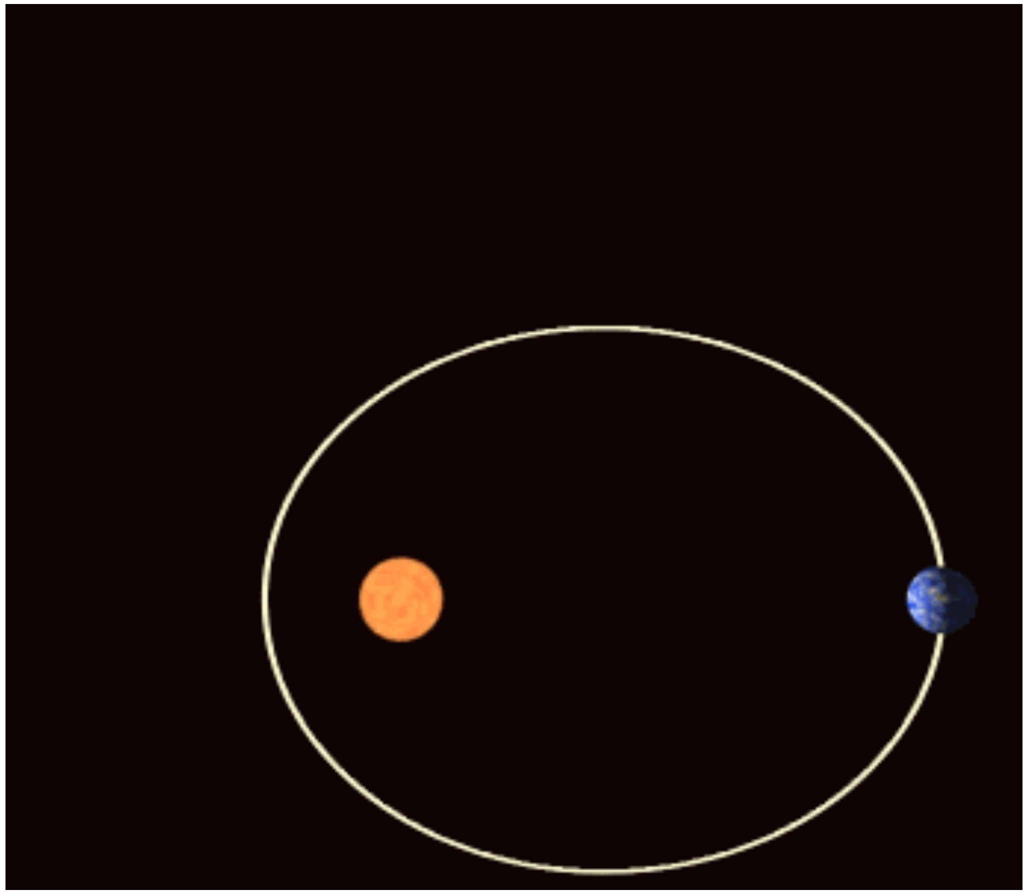
Inverting to $\theta = \theta(t)$ is impossible

Physicists spent **centuries** calculating approximates solutions

We already got physically interesting features of the solution. So we leave it to the computers.

3.6 Conditions for Closed Orbits (Bertrand's Theorem)

Joseph Louis François Bertrand
1822-1900



In classical mechanics, Bertrand's theorem states that among central-force potentials with bound orbits, there are only two types of central force potentials with the property that all bound orbits are also closed orbits: (1) an inverse-square central force such as the gravitational or electrostatic potential; (2) the radial harmonic oscillator potential, i.e., Hooke's law