Theoretical Mechanics

理论力学

Zhou Xiang 周详

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- Chapter 10 Introduction to the Lagrangian and Hamiltonian Formulations for Continuous Systems and Fields

Other Types of Generators

Type-1 generator $F = F_1(q, Q, t)$ is still not so general

For example, $F = F_1(q, Q, t)$ can not be a generator for
$Q_i = q_i \quad P_i = p_i$

We need generating functions of different set of independent variables

As we have mentioned, we can have 4 basic types of them

$$F_1(q, Q, t) \quad F_2(q, P, t) \quad F_3(p, Q, t) \quad F_4(p, P, t)$$

We can derive them using the now-familiar rule

i.e. we can add any dF/dt inside the action integral

Type-2 Generator

We can use $F = -\sum_i Q_i P_i$ to convert

$$\delta \int_{t_1}^{t_2} \left[\sum_i P_i \dot{Q}_i - K(Q, P, t) + \frac{\mathrm{d}F}{\mathrm{d}t} \right] dt = 0 \Longrightarrow \delta \int_{t_1}^{t_2} \left[- \sum_i \dot{P}_i Q_i - K(Q, P, t) \right] dt = 0$$

Switch the definition of canonical transformations

$$\sum_i P_i \dot{Q}_i - K + \frac{\mathrm{d}F_1}{\mathrm{d}t} = \sum_i p_i \dot{q}_i - H \Longrightarrow - \sum_i \dot{P}_i Q_i - K + \frac{\mathrm{d}F_2}{\mathrm{d}t} = \sum_i p_i \dot{q}_i - H$$

$$\Longrightarrow \frac{\mathrm{d}F_2}{\mathrm{d}t} = \sum_i p_i \dot{q}_i + \sum_i Q_i \dot{P}_i + K - H$$

To satisfy this

$$F_2 = F_2(q, P, t) \quad \frac{\partial F_2}{\partial q_i} = p_i \quad \frac{\partial F_2}{\partial P_i} = Q_i \quad K = H + \frac{\partial F_2}{\partial t}$$

Type-2 Generator

If we go back to the original definition of generating function $\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = \sum_i p_i \dot{q}_i - H$

$$F = F_2(q, P, t) - \sum_i Q_i P_i, \quad \frac{\partial F_2}{\partial q_i} = p_i, \quad \frac{\partial F_2}{\partial P_i} = Q_i, \quad K = H + \frac{\partial F_2}{\partial t}$$

Trivial case: $F_2 = \sum_i q_i P_i$

$\Longrightarrow p_i = P_i \quad Q_i = q_i$ Identity transformation

We push the same idea to define the other 2 types

Four Basic Generators

Generator	Derivatives	Trivial Case
$F_1(q,Q,t)$	$p_i = \frac{\partial F_1}{\partial q_i}$ $\quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$ $\quad Q_i = p_i$, $P_i = -q_i$
$F_2(q,P,t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$ $\quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$ $\quad Q_i = q_i$, $P_i = p_i$
$F_3(p,Q,t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$ $\quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$ $\quad Q_i = -q_i$, $P_i = -p_i$
$F_4(p,P,t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$ $\quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$ $\quad Q_i = p_i$, $P_i = -q_i$

Just Legendre transformations!

You will learn it again in thermodynamics

$$
\begin{aligned}
E(S,V) &= E \\
H(S,p) &= E + pV \\
F(T,V) &= E - TS \\
G(T,p) &= E - TS + pV
\end{aligned}
\quad \Longleftrightarrow \quad
\begin{aligned}
E &= E(S,V) \\
E &= H(S,p) - pV \\
E &= F(T,V) + TS \\
E &= G(T,p) + TS - pV
\end{aligned}
$$

Simple Example: Identity transformation

$$\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = \sum_i p_i \dot{q}_i - H$$

Try a generating function: $F_2 = \sum_i q_i P_i \Longrightarrow F = \sum_i \left(q_i P_i - Q_i P_i\right)$

Canonical transformation generated by F is

$$\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = -K + \sum_i \left(q_i - Q_i\right)\dot{P}_i + \sum_i P_i \dot{q}_i = \sum_i p_i \dot{q}_i - H$$

$\Longrightarrow Q_i = q_i \quad P_i = p_i \leftarrow$Identity transformation

$K = H$

OK, that was too simple

Let's push this one step further...

Simple Example: Point transformation

$$\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = \sum_i p_i \dot{q}_i - H$$

Let's try this one : $F_2 = f_i\left(q_1, \ldots, q_n, t\right) P_i \Longrightarrow F = f_i\left(q_1, \ldots, q_n, t\right) P_i - Q_i P_i$

f_i are arbitrary functions of $q_1 \ldots q_n$ and t

$$\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = -K + \sum_i \left(f_i - Q_i\right) \dot{P}_i + \sum_i P_i \frac{\partial f_i}{\partial q_j} \dot{q}_j + \sum_i \frac{\partial f_i}{\partial t} P_i = \sum_i p_i \dot{q}_i - H$$

$\Longrightarrow Q_i = f_i\left(q_1, \ldots, q_n, t\right) \leftarrow$ All "point transformations" of generalized coordinates are covered

$$p_i = \sum_j \frac{\partial f_j}{\partial q_i} P_j$$

$$K = H + \sum_i \frac{\partial f_i}{\partial t} P_i$$

They are all the same!

$$P_\mu = \sum_i p_i \left\{ \frac{\partial f_i(\mathbf{Q}, t)}{\partial Q_\mu} \right\}_{i\mu} \quad \text{with } q_i = f(\mathbf{Q}, t)$$

We can do all what we could do before

Arbitrarity

Generating function F→a canonical transformation

Opposite mapping is not unique

There are many possible Fs for each transformation

e.g. add an arbitrary function of time $g(t)$ to F

$$\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} \rightarrow \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} + \frac{dg(t)}{dt}$$

Does not affect the action integral

$\Longrightarrow K \rightarrow K + \dfrac{dg(t)}{dt}$ Just modifies the Hamiltonian without affecting physics

F is arbitrary up to any function of time only

So is the Hamiltonian

Four Basic Generators

The 4 types of generators are almost equivalent

It may look as if F_1 is special, but it isn't

$$\sum_i P_i\dot{Q}_i - K + \frac{dF_1}{dt} = \sum_i p_i\dot{q}_i - H$$

$$-\sum_i \dot{P}_iQ_i - K + \frac{dF_2}{dt} = \sum_i p_i\dot{q}_i - H$$

$$\sum_i P_i\dot{Q}_i - K + \frac{dF_3}{dt} = -\sum_i \dot{p}_iq_i - H$$

$$-\sum_i \dot{P}_iQ_i - K + \frac{dF_4}{dt} = -\sum_i \dot{p}_iq_i - H$$

There is **NO** reason to consider any of these 4 definitions to be more fundamental than the others!

We **arbitrarily** chose the first form (which happens to be the *Lagrangian* form) to write the generating functions in the table.

Four Basic Generators

Some canonical transformations cannot be generated by all 4 types

e.g. identity transf. is generated only by F_2 or F_3

This does not present a fundamental problem

One can always swap coordinate and momentum

$$Q_i = p_i \quad P_i = -q_i$$

One can always change sign by scale transformation

$$Q_i = \pm q_i \quad P_i = \pm p_i$$

These transformations make the 4 types practically equivalent

Infinitesimal CT

Consider a CT in which q, p are changed by small (infinitesimal) amounts

$$Q_i = q_i + \delta q_i \quad P_i = p_i + \delta p_i$$

Infinitesimal Canonical Transformation (ICT)

ICT is close to identity transf.

Generating function should be $F_2(q, P, t) = q_i P_i + \varepsilon G(q, P, t)$

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i + \varepsilon \frac{\partial G}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} = q_i + \varepsilon \frac{\partial G}{\partial P_i}$$

$$\delta q_i = \varepsilon \frac{\partial G}{\partial P_i} \approx \varepsilon \frac{\partial G}{\partial p_i}, \quad \delta p_i = -\varepsilon \frac{\partial G}{\partial q_i}$$

Generator of ICT

An ICT is generated by $F_2(q, P, t) = q_i P_i + \varepsilon G(q, P, t)$

$$Q_i = q_i + \varepsilon \frac{\partial G}{\partial P_i} \quad P_i = p_i - \varepsilon \frac{\partial G}{\partial q_i}$$

G is called (inaccurately) the generator of the ICT

Since the CT is infinitesimal, G may be expressed in terms of q or Q, p or P, interchangeably

For example:

$$G = G(q, p, t) \quad Q_i = q_i + \varepsilon \frac{\partial G}{\partial P_i} \simeq q_i + \varepsilon \frac{\partial G}{\partial p_i} \quad P_i = p_i - \varepsilon \frac{\partial G}{\partial q_i}$$

Hamiltonian

Consider $G = H(q, p, t)$

$$\Longrightarrow \delta q_i = \varepsilon \frac{\partial H}{\partial p_i} = \varepsilon \dot{q}_i \quad \delta p_i = -\varepsilon \frac{\partial H}{\partial q_i} = \varepsilon \dot{p}_i$$

What does ε look like?→Infinitesimal time δt

$$\delta q_i = \dot{q}_i \delta t \quad \delta p_i = \dot{p}_i \delta t$$

Hamiltonian is the generator of infinitesimal time transformation

In QM, you learn that Hamiltonian is the operator that represents advance of time

Direct Conditions

Consider a restricted Canonical Transformation

Generator has no t dependence

$$\frac{\partial F}{\partial t} = 0 \implies K(Q, P) = H(q, p)$$

Hamiltonian is unchanged

Q and P depends only on q and p

$$Q_i = Q_i(q, p) \quad P_i = P_i(q, p)$$

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

$$\dot{P}_i = \frac{\partial P_i}{\partial q_j}\dot{q}_j + \frac{\partial P_i}{\partial p_j}\dot{p}_j = \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

Direct Conditions

On the other hand, Hamilton's eqns say

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial H}{\partial q_j}\boxed{\frac{\partial q_j}{\partial P_i}} + \frac{\partial H}{\partial p_j}\boxed{\frac{\partial p_j}{\partial P_i}} \Longleftrightarrow \dot{Q}_i = \boxed{\frac{\partial Q_i}{\partial q_j}}\frac{\partial H}{\partial p_j} - \boxed{\frac{\partial Q_i}{\partial p_j}}\frac{\partial H}{\partial q_j}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = -\frac{\partial H}{\partial q_j}\boxed{\frac{\partial q_j}{\partial Q_i}} - \frac{\partial H}{\partial p_j}\boxed{\frac{\partial p_j}{\partial Q_i}} \Longleftrightarrow \dot{P}_i = \boxed{\frac{\partial P_i}{\partial q_j}}\frac{\partial H}{\partial p_j} - \boxed{\frac{\partial P_i}{\partial p_j}}\frac{\partial H}{\partial q_j}$$

Direct Conditions for a Canonical Transformation

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

Direct Conditions

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

Direct Conditions are necessary and sufficient for a time-independent transformation to be canonical

You can use them to test a CT

In fact, this applies to all Canonical Transformations

Infinitesimal CT

Does an ICT satisfy the DCs?

$$\delta q_i = \varepsilon \frac{\partial G}{\partial P_i} \approx \varepsilon \frac{\partial G}{\partial p_i} \quad \delta p_i = -\varepsilon \frac{\partial G}{\partial q_i} \approx -\varepsilon \frac{\partial G}{\partial Q_i}$$

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial \left(q_i + \delta q_i\right)}{\partial q_j} = \delta_{ij} + \varepsilon \frac{\partial^2 G}{\partial P_i \partial q_j} \quad \frac{\partial p_j}{\partial P_i} = \frac{\partial \left(P_j - \delta p_j\right)}{\partial P_i} = \delta_{ij} + \varepsilon \frac{\partial^2 G}{\partial P_i \partial q_j}$$

$$\frac{\partial Q_i}{\partial p_j} = \frac{\partial \left(q_i + \delta q_i\right)}{\partial p_j} = \varepsilon \frac{\partial^2 G}{\partial P_i \partial p_j} \quad \frac{\partial q_j}{\partial P_i} = \frac{\partial \left(Q_j - \delta q_j\right)}{\partial P_i} = -\varepsilon \frac{\partial^2 G}{\partial P_i \partial p_j}$$

$$\frac{\partial P_i}{\partial q_j} = \frac{\partial \left(p_i + \delta p_i\right)}{\partial q_j} = -\varepsilon \frac{\partial^2 G}{\partial Q_i \partial q_j} \quad \frac{\partial p_j}{\partial Q_i} = \frac{\partial \left(P_j - \delta p_j\right)}{\partial Q_i} = \varepsilon \frac{\partial^2 G}{\partial Q_i \partial q_j}$$

$$\frac{\partial P_i}{\partial p_j} = \frac{\partial \left(p_i + \delta p_i\right)}{\partial p_j} = \delta_{ij} - \varepsilon \frac{\partial^2 G}{\partial Q_i \partial p_j} \quad \frac{\partial q_j}{\partial Q_i} = \frac{\partial \left(Q_j - \delta q_j\right)}{\partial Q_i} = \delta_{ij} - \varepsilon \frac{\partial^2 G}{\partial Q_i \partial p_j}$$

Successive CTs

Two successive CTs make a CT

$$P_i\dot{Q}_i - K + \frac{dF_1}{dt} = p_i\dot{q}_i - H \quad + \quad Y_i\dot{X}_i - M + \frac{dF_2}{dt} = P_i\dot{Q}_i - K$$

$$\Longrightarrow Y_i\dot{X}_i - M + \frac{d\left(F_1 + F_2\right)}{dt} = p_i\dot{q}_i - H$$ True for unrestricted CTs

Direct Conditions can also be “chained”, e.g.,

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \quad + \quad \left(\frac{\partial X_i}{\partial Q_j}\right)_{Q,P} = \left(\frac{\partial P_j}{\partial Y_i}\right)_{X,Y}$$

$$\Longrightarrow \left(\frac{\partial X_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial Y_i}\right)_{X,Y}$$ Easy to prove

Unrestricted CT

Now we consider a general, time-dependent CT

$$Q_i = Q_i(q, p, t) \quad P_i = P_i(q, p, t) \quad K = H + \frac{\partial F}{\partial t}$$

Let's do it in two steps

$$q, p \mapsto Q\left(q, p, t_0\right), P\left(q, p, t_0\right) \mapsto Q(q, p, t), P(q, p, t)$$

First step is t-independent → Satisfies the DCs

We must show that the second step satisfies the DCs

Unrestricted CT

Concentrate on a time-only CT $Q\left(t_0\right), P\left(t_0\right) \mapsto Q(t), P(t)$

Break $t - t_0$ into pieces of infinitesimal time dt

$$Q\left(t_0\right), P\left(t_0\right) \Rightarrow Q\left(t_0 + dt\right), P\left(t_0 + dt\right) \Rightarrow \Rightarrow Q(t), P(t)$$

Each step is an ICT→Satisfies Direct Conditions

“Integrating” gives us what we needed

All Canonical Transformations satisfies the Direct Conditions, and vice versa

The proof worked because a time-only CT is a continuous transformation, parameterized by t

Poisson Bracket

For u and v expressed in terms of q and p

$$[u, v]_{q,p} \equiv \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} \leftarrow \text{Poisson Bracket}$$

This weird construction has many useful features

If you know QM, this is analogous to the commutator

$$\frac{1}{i\hbar}[u, v] \equiv \frac{1}{i\hbar}(uv - vu) \text{ for two operators } u \text{ and } v$$

Let's start with a few basic rules

Poisson Bracket Identities

$$[u, v]_{q,p} \equiv \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

For quantities u, v, w and constants a, b

$$[u, u] = 0 \quad [u, v] = -[v, u]$$

$$[au + bv, w] = a[u, w] + b[v, w]$$

$$[uv, w] = [u, w]v + u[v, w]$$

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$ **Jacobi's Identity**

Fundamental Poisson Brackets

Consider PBs of q and p themselves

$$\left[q_j, q_k\right] = \frac{\partial q_j}{\partial q_i}\frac{\partial q_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial q_k}{\partial q_i} = 0 \quad \left[p_j, p_k\right] = 0$$

$$\left[q_j, p_k\right] = \frac{\partial q_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial p_k}{\partial q_i} = \delta_{jk} \quad \left[p_j, q_k\right] = -\delta_{jk}$$

Called the Fundamental Poisson Brackets

Now we consider a Canonical Transformation

$$q, p \rightarrow Q, P$$

What happens to the Fundamental PB?

Fundamental PB and CT

$$\left[Q_j, Q_k\right]_{q,p} = \frac{\partial Q_j}{\partial q_i}\frac{\partial Q_k}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial Q_k}{\partial q_i} = \frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial P_k} - \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial P_k} = -\frac{\partial Q_j}{\partial P_k} = 0$$

$$\left[P_j, P_k\right]_{q,p} = \frac{\partial P_j}{\partial q_i}\frac{\partial P_k}{\partial p_i} - \frac{\partial P_j}{\partial p_i}\frac{\partial P_k}{\partial q_i} = \frac{\partial P_j}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial P_j}{\partial p_i}\frac{\partial p_i}{\partial Q_k} = \frac{\partial P_j}{\partial Q_k} = 0$$

$$\left[Q_j, P_k\right]_{q,p} = \frac{\partial Q_j}{\partial q_i}\frac{\partial P_k}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial P_k}{\partial q_i} = \frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial Q_k} = \frac{\partial Q_j}{\partial Q_k} = \delta_{jk}$$

$$\left[P_j, Q_k\right]_{q,p} = -\left[Q_k, P_j\right] = -\delta_{jk}$$

Fundamental Poisson Brackets are invariant under CT

Poisson Bracket and CT

What happens to a Poisson Bracket under CT?

For a time-independent CT:

$$
\begin{aligned}
[u,v]_{Q,P} &\equiv \frac{\partial u}{\partial Q_i}\frac{\partial v}{\partial P_i} - \frac{\partial u}{\partial P_i}\frac{\partial v}{\partial Q_i} \\
&= \left(\frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial u}{\partial p_j}\frac{\partial p_j}{\partial Q_i}\right)\left(\frac{\partial v}{\partial q_k}\frac{\partial q_k}{\partial P_i} + \frac{\partial v}{\partial p_k}\frac{\partial p_k}{\partial P_i}\right) - \left(\frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial u}{\partial p_j}\frac{\partial p_j}{\partial P_i}\right)\left(\frac{\partial v}{\partial q_k}\frac{\partial q_k}{\partial Q_i} + \frac{\partial v}{\partial p_k}\frac{\partial p_k}{\partial Q_i}\right) \\
&= \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial q_k}\left[q_j, q_k\right]_{Q,P} + \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_k}\left[q_j, p_k\right]_{Q,P} + \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_k}\left[p_j, q_k\right]_{Q,P} + \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial p_k}\left[p_j, p_k\right]_{Q,P} \\
&= \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_k}\delta_{jk} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_k}\delta_{jk} \\
&= [u,v]_{q,p}
\end{aligned}
$$

Poisson Brackets are invariant under CT.

Invariance of Poisson Bracket

Poisson Brackets are canonical invariants

True for any Canonical Transformations

Goldstein shows this using “symplectic” approach

We don’t have to specify q, p in each PB

$$[u, v]_{q,p} \longrightarrow [u, v]$$

Fundamental Poisson Brackets

Consider a 1-dimensional harmonic oscillator

$$H(q,p) = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \frac{1}{2m}\left(p + im\omega q\right)\left(p - im\omega q\right)$$

Let's define $\hat{a}_{\pm} = \dfrac{1}{\sqrt{2m}}\left(\hat{p} \pm im\omega\hat{q}\right)$ with $\left[\hat{q},\hat{p}\right] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar$.

$$\left.\begin{aligned} \hat{a}_+\hat{a}_- &= \frac{1}{2m}\left(\hat{p} + im\omega\hat{q}\right)\left(\hat{p} - im\omega\hat{q}\right) \\ &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 + \frac{i}{2}\omega\left[\hat{q},\hat{p}\right] \\ &= \hat{H} - \frac{1}{2}\hbar\omega \end{aligned}\right\} \Longrightarrow \hat{H} = \hat{a}_+\hat{a}_- + \frac{1}{2}\hbar\omega$$

In QM, you will learn that $\hat{a}_+\hat{a}_- = n\hbar\omega$ and then $E = \left(n + \dfrac{1}{2}\right)\hbar\omega$