

$$- \quad AAT = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

$$|AAT| = \begin{vmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 4 & 4 & 6 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix} = 0$$

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$r(B) = 5$$

$$- \quad (A - 2I)X = B$$

$$(A - 2I; B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

三. (1). $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 理由如下:

假设线性相关

则存在不全为0的数 k_1, k_2, k_3

$$\text{使 } k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$$

$$\text{故 } k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + 0\alpha_4 = 0$$

故 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 线性相关, 与已知矛盾

故 $\alpha_1, \alpha_2, \alpha_3$ 线性无关

(2). 设存在数 k_1, k_2, k_3

$$\text{使 } k_1(l\alpha_1 + d_2) + k_2(\alpha_2 + d_3) + k_3(m\alpha_3 + \alpha_1) = 0$$

$$\text{即 } (k_1l + k_3)\alpha_1 + (k_1 + k_2)\alpha_2 + (k_2 + k_3m)\alpha_3 = 0$$

又 $\alpha_1, \alpha_2, \alpha_3$ 线性无关

$$\text{故 } \begin{cases} k_1l + k_3 = 0 \\ k_1 + k_2 = 0 \\ k_2 + k_3m = 0 \end{cases} \quad (1)$$

若 $l\alpha_1 + d_2, \alpha_2 + d_3, m\alpha_3 + \alpha_1$ 线性无关

则方程组(1)只有零解

$$\text{故系数行列式 } D = \begin{vmatrix} l & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & m \end{vmatrix} = lm + 1 \neq 0$$

$$10. \quad A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & a & -1 \\ 2 & 3 & 0 & b \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & a-1 & -2 \\ 0 & 0 & -a-1 & b-1 \end{pmatrix}$$

(1). 当 $r(A) \neq r(\bar{A})$ 时, 无解

$$\begin{cases} -a-1=0 \\ b-1 \neq 0 \end{cases} \quad \text{即 } \begin{cases} a=-1 \\ b \neq 1 \end{cases}$$

(2). 当 $r(A) = r(\bar{A}) = n$ 时, 有唯一解

$$-a-1 \neq 0 \quad \text{即 } a \neq -1$$

(3). 当 $r(A) = r(\bar{A}) < n$ 时, 有无穷多解

$$\begin{cases} -a-1=0 \\ b-1=0 \end{cases} \quad \text{即 } \begin{cases} a=-1 \\ b=1 \end{cases}$$

$$\text{此时 } A \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

对应的齐次方程组的基础解系为 $\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ 原方程组的特解为 $\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$

原方程组的通解为 $\eta = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + c \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, c 为任意常数

$$五. (1). \quad A^T = E^T - \frac{2}{\alpha^T \alpha} (\alpha \alpha^T)^T = E - \frac{2}{\alpha^T \alpha} \alpha \alpha^T = A$$

$(kE - A)^T = kE^T - A^T = kE - A$ $(kE - A)$ 为实对称矩阵, 故能相似于一个对角阵

$$(2). \quad A^2 = (E - \frac{2}{\alpha^T \alpha} \alpha \alpha^T) (E - \frac{2}{\alpha^T \alpha} \alpha \alpha^T) = E - \frac{4}{\alpha^T \alpha} \alpha \alpha^T + \frac{4}{(\alpha^T \alpha)^2} (\alpha \alpha^T)^2$$

$$= E - \frac{4}{\alpha^T \alpha} \alpha \alpha^T + \frac{4}{(\alpha^T \alpha)^2} \alpha (\alpha^T \alpha) \alpha^T = E$$

Δ A 为实对称矩阵兼正交矩阵 故 A 的特征值只能为 ± 1

又 $k \neq \pm 1$ 故 $|kE - A| \neq 0$ 故 $(kE - A)$ 可逆

$$\begin{aligned} (3). (E - 2\alpha\alpha^T)^T(E - 2\alpha\alpha^T) &= (E - 2\alpha\alpha^T)(E - 2\alpha\alpha^T) = E - 4\alpha\alpha^T + 4(\alpha\alpha^T)^2 \\ &= E - 4\alpha\alpha^T + 4\alpha(\alpha^T\alpha)\alpha^T = E + 4(\alpha^T\alpha - 1)\alpha\alpha^T = E \end{aligned}$$

因为 $\alpha \neq 0$, 所以 $\alpha\alpha^T \neq 0$, 所以 $\alpha^T\alpha = 1$

7. (1). $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(2). 特征方程为 $|\lambda I - A| = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} = (\lambda + 1)(\lambda - 1)^2 = 0$

特征值为 $\lambda_1 = -1, \lambda_{2,3} = 1$

当 $\lambda_1 = -1$ 时, $(-I - A)x = 0$, 线性无关特征向量为 $x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

单位化, 得 $x_1^* = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

当 $\lambda_{2,3} = 1$ 时, $(I - A)x = 0$, 线性无关特征向量为 $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

规范正交化, 得 $x_2^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x_3^* = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

正交矩阵为 $T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$, 正交变换为 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

(3). $f = -y_1^2 + y_2^2 + y_3^2$ f 为不定二次型

七. (1). 因为 A 正定, 故 A 的特征值均大于 0

设 λ 为 A 的任一特征值

①. 则 $\frac{1}{\lambda}$ 为 A^{-1} 的特征值, 且 $\frac{1}{\lambda} > 0$ ②. 则 $\frac{|A|}{\lambda}$ 为 A^* 的特征值, 又 $|A| > 0$, 故 $\frac{|A|}{\lambda} > 0$

故 A^{-1} 的特征值均大于 0

故 A^* 的特征值均大于 0

因为 $(A^{-1})^T = (A^T)^{-1} = A^{-1}$

因为 $(A^*)^T = (|A|A^{-1})^T = |A|A^{-1} = A^*$

故 A^{-1} 为实对称矩阵

故 A^* 为实对称矩阵

故 A^{-1} 为正定矩阵

故 A^* 为正定矩阵

③. $(A^{-1} + A^*)^T = (A^{-1})^T + (A^*)^T = A^{-1} + A^*$ 故 $A^{-1} + A^*$ 为实对称矩阵

对任一非零列向量 x , $x^T(A^{-1} + A^*)x = x^TA^{-1}x + x^TA^*x > 0$

故 $A^{-1} + A^*$ 为正定矩阵

(2). $C^T = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix}^T = \begin{pmatrix} (A^{-1})^T & 0^T \\ 0^T & (A^*)^T \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix} = C$

故 C 为实对称矩阵

设 A 的特征值为 $\lambda_1, \lambda_2, \dots, \lambda_n$, A^* 的特征值为 $\mu_1, \mu_2, \dots, \mu_n$

因为 A^{-1}, A^* 均为正定矩阵, 故 $\lambda_i > 0, \mu_i > 0$ ($i=1, 2, \dots, n$)

$$|\lambda I - C| = \begin{vmatrix} \lambda I_n - A^{-1} & 0 \\ 0 & \lambda I_n - A^* \end{vmatrix} = |\lambda I_n - A^{-1}| |\lambda I_n - A^*| = 0$$

故 C 的特征值为 $\lambda_1, \lambda_2, \dots, \lambda_n, \mu_1, \mu_2, \dots, \mu_n$, 且均大于 0

故 C 为正定矩阵