理论力学第 4 次作业

1.19

设质量为 m，杆长为 l，杆与 z 轴夹角为ϕ，杆绕 z 轴旋转的角度为θ，取球心为零势能处

动能为

$$T=\frac{1}{2}mv^2=\frac{ml^2}{2}\left(\sin^2\phi\,\dot{\theta}^2+\dot{\phi}^2\right)$$

$$L=T-V=\frac{ml^2}{2}\left(\sin^2\phi\,\dot{\theta}^2+\dot{\phi}^2\right)-mgl\cos\phi$$

对于θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0$$

$$\frac{d}{dt}\left(\frac{ml^2}{2}\left(2\sin^2\phi\,\dot{\theta}\right)\right)=0$$

$$\mathcal{L}=ml^2\left(4\sin\phi\cos\phi\,\dot{\theta}\dot{\phi}+2\sin^2\phi\,\ddot{\theta}\right)=0$$

对于ϕ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\frac{\partial L}{\partial\phi}=0$$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right)-\left(\frac{ml^2}{2}\left(2\sin\phi\cos\phi\,\dot{\theta}^2\right)+mgl\sin\phi\right)=0$$

$$\mathcal{L}=ml^2\ddot{\phi}-\left(\frac{ml^2}{2}\left(2\sin\phi\cos\phi\,\dot{\theta}^2\right)+mgl\sin\phi\right)=0$$

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$$T_1=\frac{1}{2}m_1v_1^2=\frac{1}{2}m_1\left(\dot{l}_1^2+l_1^2\dot{\phi}^2\right)$$

$$T_2=\frac{1}{2}m_2v_2^2=\frac{1}{2}m_2\dot{l}_2^2$$

$$l_2=l-l_1,\ \dot{l}_2=-\dot{l}_1,\ \dot{l}_2^2=\dot{l}_1^2$$

$$\therefore T_2=\frac{1}{2}m_2v_2^2=\frac{1}{2}m_2\dot{l}_1^2$$

$$U=-m_2gl_2=-m_2g(l-l_1)$$

$$
\begin{aligned}
L &= T_1 + T_2 - U \\
&= \frac{1}{2}m_1\left(\dot{l}_1^2 + l_1^2\dot{\phi}^2\right) + \frac{1}{2}m_2\dot{l}_1^2 - m_2 g l_1 + m_2 g l
\end{aligned}
$$

对于l_1

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}_1}\right) - \frac{\partial l}{\partial l_1} = 0$$

得

$$(m_1 + m_2)\ddot{l}_1 - m_1 l_1 \dot{\phi}^2 + m_2 g = 0$$

对于ϕ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial l}{\partial \phi} = 0$$

有

$$\frac{\partial L}{\partial \phi} = 0$$

所以

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0,\ \frac{\partial L}{\partial \dot{\phi}} = C$$

而

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}m_1\left(\dot{l}_1^2 + l_1^2\dot{\phi}^2\right) + \frac{1}{2}m_2\dot{l}_1^2 - m_2 g l_1 + m_2 g l\right) = m_1 l_1^2 \dot{\phi} = l_z$$

所以 z 方向上角动量守恒。

$$\phi = \frac{l_z}{m_1 l_1^2}$$

回代得

$$(m_1 + m_2)\ddot{l}_1 - \frac{l_z^2}{m_1 l_1^3} + m_2 g = 0$$

两边同乘$\dot{l}_1$得

$$(m_1 + m_2)\dot{l}_1\ddot{l}_1 - \frac{l_z^2}{m_1 l_1^3}\dot{l}_1 + m_2 g \dot{l}_1 = 0$$

即

$$\frac{d}{dt}\left(\frac{1}{2}(m_1 + m_2)\dot{l}_1^2 + \frac{l_z^2}{m_1 l_1^2} + m_2 g l_1\right) = 0$$

所以

$$\frac{1}{2}(m_1 + m_2)\dot{l}_1^2 + \frac{l_z^2}{m_1 l_1^2} + m_2 g l_1 = E = const$$

$$T_t = \frac{1}{2}(m_1 + m_2)\dot{l}_1^2$$

$$T_{rot} = \frac{l_z^2}{m_1 l_1^2}$$

$$U = mgl_1 = mg(l - l_2) = mgl - mgl_2$$

拉格朗日量方程为

$$(m_1 + m_2)\ddot{l}_1 - m_1 l_1 \dot{\phi}^2 + m_2 g = 0,\ \frac{\partial L}{\partial \dot{\phi}} = l_z$$

可以化简为

$$(m_1 + m_2)\ddot{l}_1 - \frac{l_z^2}{m_1 l_1^3} + m_2 g = 0$$

第一积分为

$$\frac{1}{2}(m_1 + m_2)\dot{l}_1^2 + \frac{l_z^2}{m_1 l_1^2} + m_2 g l_1 = E = const$$

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$$\begin{aligned} x_1 &= l_1 \sin(\theta_1) \\ y_1 &= -l_1 \cos(\theta_1) \\ x_2 &= l_1 \sin(\theta_1) - l_2 \sin(\theta_2) \\ y_2 &= -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \end{aligned}$$

$$v_1^2 = \left(\frac{\partial x_1}{\partial t}\right)^2 + \left(\frac{\partial y_1}{\partial t}\right)^2 = l_1^2 \dot{\theta}_1^2$$

$$v_2^2 = \left(\frac{\partial x_1}{\partial t}\right)^2 + \left(\frac{\partial y_1}{\partial t}\right)^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 - 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2)$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 - 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2)\right)$$

$$U = m_1 g y_1 + m_2 g y_2 = -m_1 g l_1 \cos(\theta_1) + m_2 g[-l_1 \cos(\theta_1) - l_2 \cos(\theta_2)]$$

$$\begin{aligned} \therefore L = T - U &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 - 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2)\right) \\ &+ m_1 g l_1 \cos(\theta_1) + m_2 g(l_1 \cos(\theta_1) + l_2 \cos(\theta_2)) \end{aligned}$$

对于θ_1

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

即

$$\begin{aligned} (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 \left[\ddot{\theta} \cos(\theta_1 + \theta_2) - \sin(\theta_1 + \theta_2) \dot{\theta}_1^2 - \sin(\theta_1 + \theta_2) \dot{\theta}_1 \dot{\theta}_2\right] \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 + \theta_2) + (m_1 + m_2) g l_1 \sin(\theta_1) = 0 \end{aligned}$$

对于θ_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

即

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 + \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 + \theta_2) - \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 + \theta_2)\right] \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 + \theta_2) + m_2 g l_2 \sin(\theta_2)$$

Motion of one particle: Cylindrical coordinate or Spherical coordinate

Cylindrical coordinate

$$L = T = \frac{1}{2} m \left[\dot{r}^2 + \left(r\dot{\theta}\right)^2 + \dot{z}^2\right]$$

对于 r

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

即

$$m\ddot{r} - mr\dot{\theta}^2 = 0$$

对于θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

即

$$m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) = 0$$

对于 z

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

即

$$m\ddot{z} = 0$$

Spherical coordinate

$$L = T = \frac{1}{2} m \left[\dot{r}^2 + \left(r\dot{\theta}\right)^2 + \left(r \sin\theta\, \dot{\phi}\right)^2\right]$$

对于 r

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

即

$$m\ddot{r} - mr\ddot{\theta} - mr\sin\theta\,\dot{\phi}^2 = 0$$

对于θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

即

$$mr^2\sin\theta\cos\theta\,\dot{\phi}^2 - m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) = 0$$

对于ϕ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$

即

$$m\left(2r\dot{r}\sin^2\theta\,\dot{\phi} + 2r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} + r^2\sin^2\theta\,\ddot{\phi}\right) = 0$$