理论力学第 2 次作业

1.4

方程

$$dx - a\sin\theta\, d\phi = 0$$

满足以下形式

$$\sum_i^n g_i(x_1,x_2,\cdots,x_n) = 0$$

其中

$$g_2 = g_3 = 0,\ g_1 dx + g_4 d\phi = 0$$

需要积分因子 $f(x,y,\theta,\phi)$使得

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial\left(fg_j\right)}{\partial x_i}$$

令 $i = 1, j = 3$，则

$$\frac{\partial f}{\partial \theta} = \frac{\partial(f\cdot 0)}{\partial x} = 0$$

令 $i = 3, j = 4$，则

$$\frac{\partial(f\cdot 0)}{\partial \phi} = \frac{\partial[f\cdot(-a\sin\theta)]}{\partial \theta}$$

即

$$0 = \frac{\partial f}{\partial \theta}(-a\sin\theta) + f(-a\cos\theta)$$

要使等式成立，需要有 $f \equiv 0$
显然 $f = 0$ 不是积分因子，所以找不到积分因子 $f(x,y,\theta,\phi)$使得

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial\left(fg_j\right)}{\partial x_i}$$

所以约束方程式非完整约束。

1.5

设两个轮子的中心为$(x_1,y_1),(x_2,y_2)$，则

$$dx_1 - a\sin\theta\, d\phi = 0,\ dx_2 - a\sin\theta\, d\phi' = 0$$
$$dy_1 + a\cos\theta\, d\phi = 0,\ dy_2 + a\sin\theta\, d\phi' = 0$$

$$(dx_1 - a\sin\theta\, d\phi)\cos\theta + (dy_1 + a\cos\theta\, d\phi)\sin\theta = 0$$

即

$$dx_1\cos\theta + dy_1\sin\theta = 0$$

同理可得

$$dx_2\cos\theta + dy_2\sin\theta = 0$$

两式相加得

$$\cos\theta\,(dx_1 + dx_2) + \sin\theta\,(dy_1 + dy_2) = 0$$

令

$$x = \frac{1}{2}(x_1 + x_2),\ y = \frac{1}{2}(y_1 + y_2)$$

有

$$dx = \frac{1}{2}(dx_1 + dx_2),\ dy = \frac{1}{2}(dy_1 + dy_2)$$

所以得到了

$$\cos\theta\, dx + \sin\theta\, dy = 0$$

$$(dx_1 - a\sin\theta\, d\phi)\sin\theta - \left(dy_2 + a\sin\theta\, d\phi'\right)\cos\theta = 0$$

即

$$dx_1\sin\theta - dy_2\cos\theta - a\cos^2\theta\, d\phi' - a\sin^2\theta\, d\phi = 0$$

同理可得

$$dx_2\sin\theta - dy_1\cos\theta - a\cos^2\theta\, d\phi' - a\sin^2\theta\, d\phi = 0$$

两式相加得

$$\sin\theta\,(dx_1 + dx_2) - \cos\theta\,(dy_1 + dy_2) = a\left(d\phi + d\phi'\right)$$

所以

$$\sin\theta\, dx - \cos\theta\, dy = \frac{1}{2}a\left(d\phi + d\phi'\right)$$

又因为

$$\cos\theta = \frac{x_1 - x_2}{b},\ b\cos\theta = x_1 - x_2$$

两边求导可得

$$-b\sin\theta\,\dot{\theta}=\dot{x}_1-\dot{x}_2=a\sin\theta\,\dot{\phi}-a\sin\theta\,\dot{\phi}'$$

即

$$\dot{\theta}=-\frac{a}{b}\left(\dot{\phi}-\dot{\phi}'\right)$$

所以

$$\theta=C-\frac{a}{b}\left(\phi-\phi'\right)$$

1.6

$$\frac{dy}{dx}=\frac{0-y}{f(t)-x}$$

即

$$ydx+\left(f(t)-x\right)dy=0$$

若是完整约束，则有积分因子 $h(x,y,t)$

对于 x,t

$$\frac{\partial(h\cdot y)}{\partial t}=\frac{\partial(h\cdot 0)}{\partial x}=0$$

$$\therefore y\frac{\partial h}{\partial t}=0,\ \frac{\partial h}{\partial t}=0$$

对于 y,t

$$\frac{\partial\{h[f(t)-x]\}}{\partial t}=\frac{\partial(h\cdot 0)}{\partial y}=0$$

即

$$\frac{\partial h}{\partial t}+h\cdot\frac{\partial[f(t)-x]}{\partial t}=0$$

所以需要

$$h\equiv 0$$

所以找不到积分因子。应该是非完整约束。