理论力学第 5 次作业

2.1

$$t_{12}=\int_1^2\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}dx$$

其中

$$f(y,\dot{y})=\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$$

有

$$\frac{d}{dt}\left(\frac{\partial f}{\partial\dot{y}}\right)-\frac{\partial f}{\partial y}=0$$

即

$$\frac{2gy\ddot{y}-g\dot{y}^2(1+\dot{y}^2)}{(2gy(1+\dot{y}^2))^{\frac{3}{2}}}+\frac{1}{2}\sqrt{\frac{1+\dot{y}^2}{2gy^3}}=0$$

化简可得

$$y\ddot{y}+\frac{1}{2}(1+\dot{y}^2)=0$$

上式只和 $y,\dot{y},\ddot{y}$有关，令 $z(y)=\dot{y}$，$\ddot{y}=\frac{dz(y)}{dx}=\frac{dz}{dy}\frac{dy}{dx}=\dot{z}(y)\dot{y}=\dot{z}(y)z(y)$

所以上式化为

$$y\dot{z}(y)z(y)=\frac{1}{2}(1+z(y)^2)=0$$

$$\frac{z}{1+z^2}dz=-\frac{1}{2y}dy$$

$$\ln\sqrt{1+z^2}=\ln\sqrt{\frac{C}{y}}$$

$$1+z^2=\frac{C}{y}$$

$$\dot{y}=z=\sqrt{\frac{C-y}{y}}$$

$$\frac{y}{a}=1-\cos\frac{x+\sqrt{y(2a-y)}}{a}$$

1)

无初速度。

$$\dot{y}=\sqrt{\frac{C-y}{y}}=\sqrt{\frac{2a-y}{y}}$$

$$x=\frac{h}{2}(t-\sin t),\ y=\frac{h}{2}(1-\cos t),\ t{:}0\to\pi$$

所以，最速降线是半条摆线。

2)

有初速度。

$$\frac{1}{2}mv^2=mgy+\frac{1}{2}mv_0^2$$

如果把初速度看成是由势能在转化而来的，则

$$\frac{1}{2}mv^2=mgy+\frac{1}{2}mv_0^2=mgu$$

则与无处速度时相同

$$x=\frac{h}{2}(t-\sin t),\ u=\frac{h}{2}(1-\cos t),\ t{:}?\to\pi$$

所以，最速降线是半条摆线的一部分。

2.3

$$ds=\sqrt{dx^2+dy^2+dz^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}dx$$

所以

$$l=\int_1^2 ds=\int_1^2\sqrt{1+\dot{y}^2+\dot{z}^2}dx$$

记 $f=\sqrt{1+\dot{y}^2+\dot{z}^2}$

对于 y 有

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)-\frac{\partial f}{\partial y}=0$$

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2+z^2}}\right)=0$$

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2+z^2}}=c_1$$

同理可得

$$\frac{\dot{z}}{\sqrt{1+\dot{y}^2+z^2}}=c_2$$

联立可得

$$\dot{y}^2=\frac{c_1^2}{1-c_1^2-c_2^2}$$
$$\dot{z}^2=\frac{c_2^2}{1-c_1^2-c_2^2}$$

所以

$$\dot{y}=a,\ \dot{z}=b$$
$$y=ax+c,\ z=bx+d$$

2.4

$$d\boldsymbol{r}=\hat{\boldsymbol{r}}dr+\hat{\boldsymbol{\theta}}rd\theta+\hat{\boldsymbol{\phi}}r\sin\theta\,d\phi$$

$$ds=\sqrt{d\boldsymbol{r}\cdot dr\boldsymbol{r}}=\sqrt{dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2}$$

$$ds=R\sqrt{d\theta^2+\sin^2\theta\,d\phi^2}=R\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}d\theta=R\sqrt{1+\sin^2\theta\,\dot{\phi}^2}d\theta$$

$$l=\int_1^2 ds=R\int_1^2\sqrt{1+\sin^2\theta\,\dot{\phi}}d\theta$$

记

$$f=\sqrt{1+\sin^2\theta\,\dot{\phi}}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\phi}}\right)-\frac{\partial f}{\partial \phi}=0$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\phi}}\right)=0$$

$$\frac{\partial f}{\partial \dot{\phi}}=C$$

即

$$\frac{\sin^2\theta\,\dot{\phi}}{\sqrt{1+\sin^2\theta\,\dot{\phi}}}=C$$

$$\dot{\phi}=\frac{C}{\sin\theta\sqrt{\sin^2\theta-C^2}}$$

积分可得

$$\phi+\phi_0=C\int_1^2\frac{1}{\sin\theta\sqrt{\sin^2\theta-C^2}}d\theta$$

换元：$u=\cot\theta$

$$d\theta=-\frac{1}{1+u^2}du$$

$$\phi+\phi_0=-\int\frac{du}{\sqrt{a^2-u^2}}$$

令$a^2=\frac{1-C^2}{C^2}$，则

$$\phi+\phi_0=-\arccos\frac{u}{a}$$

$$\cos(\phi+\phi_0)=\frac{u}{a}=\frac{\cot\theta}{a}$$

$$a\cos\phi=\cot\theta$$

$$a\cos(\phi+\phi_0)=a\cos\phi\cos\phi_0-a\sin\phi\sin\phi_0=\alpha\cos\phi+\beta\sin\phi$$

$$\alpha=a\cos\phi_0,\ \beta=-a\sin\phi_0$$

$$\alpha\cos\phi+\beta\sin\phi=\frac{\cos\theta}{\sin\theta}$$

$$\alpha\sin\theta\cos\phi+\beta\sin\theta\sin\phi=\cos\theta$$

由笛卡尔坐标系和球坐标系的关系可得

$$\sin\theta\cos\phi=\frac{x}{R},\ \sin\theta\sin\phi=\frac{y}{R},\ \cos\theta=\frac{z}{R}$$

上式即为

$$\alpha x + \beta y = z$$

所以，轨迹在一个过原点的平面上，平面与球面的交线是大圆。

2.5

$$T = \frac{1}{2} m\dot{x}^2$$

$$L = T - V = \frac{1}{2} m\dot{x}^2 + Fx$$

$$S = \int_0^{t_0} L dt$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$F = m\ddot{x}$$

$$x(t) = A + Bt + Ct^2$$
$$\dot{x}(t) = B + 2Ct$$
$$\ddot{x}(t) = 2C$$

$$2mC = F$$
$$C = \frac{F}{2m}$$

$$x(0) = 0$$
$$A = 0$$

$$x(t_0) = a$$
$$0 + Bt_0 + \frac{F}{2m} t_0^2 = a$$
$$B = \frac{a}{t_0} - \frac{Ft_0}{2m}$$

$$\therefore x(t) = \left(\frac{a}{t_0} - \frac{Ft_0}{2m}\right) t + \frac{F}{2m} t^2$$