理论力学第 10 次作业

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(a)

洛伦兹力为

$$\boldsymbol{F} = q[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}]$$

粒子所受合力为

$$\boldsymbol{F}(r) = -\frac{\partial V(r)}{\partial r}\frac{\boldsymbol{r}}{r} + q[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}]$$

$$= -\frac{k}{r^3}\boldsymbol{r} + q\boldsymbol{v} \times \frac{b}{r^3}\boldsymbol{r}$$

由牛顿运动定律可得

$$\dot{\boldsymbol{p}} = \boldsymbol{F}(r)$$

令 $c = 1$，则 $\boldsymbol{D}$ 对时间 t 的导数为

$$\frac{d}{dt}\boldsymbol{D} = \frac{d}{dt}\left(\boldsymbol{r} \times \boldsymbol{p} - qb\frac{\boldsymbol{r}}{r}\right)$$

$$= \boldsymbol{r} \times \dot{\boldsymbol{p}} - qb\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right)$$

$$= \boldsymbol{r} \times \left[-\frac{k}{r^3}\boldsymbol{r} + q\boldsymbol{v} \times \frac{b}{r^3}\boldsymbol{r}\right] - qb\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right)$$

$$= \boldsymbol{r} \times q\left(\dot{\boldsymbol{r}} \times \frac{b}{r^3}\boldsymbol{r}\right) - qb\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right)$$

$$= \frac{qb}{r^3}\boldsymbol{r} \times (\dot{\boldsymbol{r}} \times \boldsymbol{r}) - qb\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right)$$

$$= \frac{qb}{r^3}(r^2\dot{\boldsymbol{r}} - \boldsymbol{r}(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})) - qb\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right)$$

$$= qb\left(\frac{\dot{\boldsymbol{r}}}{r} - \frac{\boldsymbol{r}\dot{r}}{r^2}\right) - qb\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = 0$$

所以 $\boldsymbol{D} = \boldsymbol{r} \times \boldsymbol{p} - qb\frac{\boldsymbol{r}}{r}$是守恒的。

(b)

$$\dot{\boldsymbol{p}} \times \boldsymbol{D} = \dot{\boldsymbol{p}} \times \left(\boldsymbol{r} \times \boldsymbol{p} - qb\frac{\boldsymbol{r}}{r}\right)$$
$$= \left[-\frac{k}{r^3}\boldsymbol{r} + q\boldsymbol{v} \times \frac{b}{r^3}\boldsymbol{r}\right] \times \left(\boldsymbol{r} \times \boldsymbol{p} - qb\frac{\boldsymbol{r}}{r}\right)$$
$$=-\frac{k}{r^3}\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{p}) + qb\frac{\boldsymbol{r}}{r} \times q\left(\dot{\boldsymbol{r}} \times \frac{b}{r^3}\boldsymbol{r}\right)$$
$$=-\frac{mk}{r^3}\boldsymbol{r} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) - qb\frac{\boldsymbol{r}}{r} \times q\left(\frac{b}{r^3}\boldsymbol{r} \times \dot{\boldsymbol{r}}\right)$$
$$=-\left(\frac{mk}{r^3} + \frac{q^2b^2}{r^4}\right)\boldsymbol{r} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}})$$
$$= \left(mk + \frac{q^2b^2}{r}\right)\left(\frac{\dot{\boldsymbol{r}}}{r} - \frac{\boldsymbol{r}\dot{r}}{r^2}\right)$$

因为 $\boldsymbol{D}$ 为常数，所以

$$\frac{d}{dt}(\boldsymbol{p} \times \boldsymbol{D}) = \dot{\boldsymbol{p}} \times \boldsymbol{D} = \left(mk + \frac{q^2b^2}{r}\right)\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right)$$

这表明存在一个守恒的矢量$\boldsymbol{A}^{'}$

$$\boldsymbol{A}^{'} = \boldsymbol{p} \times \boldsymbol{D} - \left(mk + \frac{q^2b^2}{r}\right)\frac{\boldsymbol{r}}{r}$$

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势能为

$$V =-\int f(r)dr =-\int \frac{k}{r^3}dr = \frac{k}{2r^2}$$

利用公式(3.97)可得

$$\Theta(s) = \pi - 2\int_0^{u_m} \frac{sdu}{\sqrt{1 - \frac{V(u)}{E} - s^2u^2}} = \pi - 2\int_0^{u_m} \frac{sdu}{\sqrt{1 - \left(\frac{k}{2E} + s^2\right)u^2}}$$

令

$$\xi = \sqrt{\frac{k}{2E} + s^2 u},\ d\xi = \sqrt{\frac{k}{2E} + s^2 du}$$

则

$$\Theta(s)=\pi-\frac{2s}{\sqrt{\frac{k}{2E}+s^2}}\int_0^{u_m\sqrt{\frac{k}{2E}+s^2}}\frac{d\xi}{\sqrt{1-\xi^2}}=\pi-\frac{2s}{\sqrt{\frac{k}{2E}+s^2}}\arcsin u_m\sqrt{\frac{k}{2E}+s^2}$$

边界条件为

$$u_m=\frac{1}{\sqrt{\frac{k}{2E}+s^2}}$$

回代可得

$$\Theta(s)=\pi\left(1-\frac{s}{\sqrt{\frac{k}{2E}+s^2}}\right)$$

所以

$$x=\frac{\Theta}{\pi}=1-\frac{s}{\sqrt{\frac{k}{2E}+s^2}}$$

$$s^2=\frac{k}{2E}\frac{(1-x)^2}{x(2-x)}$$

$$sds=\frac{1}{2}d(s^2)=\frac{k}{2E}\frac{x-1}{(x-2)^2x^2}dx$$

所以

$$\sigma(\Theta)d\Theta=\left|\frac{s}{\sin\Theta}ds\right|=\frac{k}{2E}\frac{(1-x)}{x^2(2-x)^2\sin\pi x}dx$$

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$$1-\frac{V(r_m<a)}{E}-s^2u_m^2=0$$

即

$$1+\frac{V_0}{E}-s^2u_m^2=0$$

可得

$$u_m = \frac{1}{s}\sqrt{\frac{E+V_0}{E}}$$

散射角为

$$\Theta(s) = \pi - 2\int_0^{u_m} \frac{s du}{\sqrt{1-\frac{V(u)}{E} - s^2u^2}}$$

当 $u \in \left(0, \frac{1}{a}\right)$时，有

$$\int_0^{\frac{1}{a}} \frac{s da}{\sqrt{1-s^2u^2}} = \arcsin\frac{s}{a}$$

当 $u \in \left(\frac{1}{a}, u_m\right)$时

$$\int_{\frac{1}{a}}^{u_m} \frac{s du}{\sqrt{1+\frac{V_0}{E} - s^2u^2}} = \frac{s}{\sqrt{1+\frac{V_0}{E}}}\int_{\frac{1}{a}}^{u_m} \frac{du}{\sqrt{1-\frac{s^2}{\frac{E+V_0}{E}}}} = \arcsin u_m \frac{s}{\sqrt{\frac{E+V_0}{E}}} - \arcsin\frac{1}{a}\frac{s}{\sqrt{\frac{E+V_0}{E}}}$$

$$= \arcsin 1 - \arcsin\frac{1}{a}\frac{s}{\sqrt{\frac{E+V_0}{E}}}$$

$$= \frac{\pi}{2} - \arcsin\frac{1}{a}\frac{s}{\sqrt{\frac{E+V_0}{E}}}$$

所以

$$\frac{\Theta(s)}{2} = \arcsin\frac{s}{a} - \arcsin\frac{s}{a}\sqrt{\frac{E}{E+V_0}}$$

$$s = a\sin\alpha$$

所以

$$\frac{\Theta}{2} = \alpha - \arcsin\left(\frac{1}{n}\sin\alpha\right)$$

即

$$\Theta=2\left[\alpha-\arcsin\left(\frac{1}{n}\sin\alpha\right)\right]$$

$$\Theta=2\left(\alpha-\alpha^{'}\right)$$

由斯涅耳定律得

$$\sin\alpha=n\sin\alpha^{'}\rightarrow\alpha^{'}=\arcsin\left(\frac{1}{n}\sin\alpha\right)$$

因为

$$\sigma(\Theta)=\frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right|$$

再次得到了

$$\frac{\Theta}{2}=\arcsin\frac{s}{a}-\arcsin\frac{s}{na}$$

所以

$$\begin{aligned}&\sin\frac{\Theta}{2}=\sin\left(\arcsin\frac{s}{a}-\arcsin\frac{s}{na}\right)\\&=\sin\arcsin\frac{s}{a}\cos\arcsin\frac{s}{na}-\cos\arcsin\frac{s}{a}\sin\arcsin\frac{s}{na}\\&=\frac{s}{a}\sqrt{1-\frac{s^2}{n^2a^2}}-\frac{s}{na}\sqrt{1-\frac{s^2}{a^2}}\end{aligned}$$

$$\cos\frac{\Theta}{2}=\sqrt{1-\frac{s^2}{a^2}}\sqrt{1-\frac{s^2}{n^2a^2}}+\frac{s^2}{na^2}$$

令

$$\mu=\sqrt{1-\frac{s^2}{n^2a^2}},\ \lambda=\sqrt{1-\frac{s^2}{a^2}}$$

则

$$\sin\frac{\Theta}{2}=\frac{s}{a}\mu-\frac{s}{na}\lambda,\ \ \cos\frac{\Theta}{2}=\mu\lambda+\frac{s^2}{na^2}$$

所以

$$\sin^2\frac{\Theta}{2}=\frac{s^2}{a^2}\left(\mu^2+\frac{1}{n^2}\lambda^2-\frac{2}{n}\cos\frac{\Theta}{2}+\frac{2s^2}{n^2a^2}\right)$$

又因为

$$\mu^2+\frac{1}{n^2}\lambda^2=1+\frac{1}{n^2}-\frac{2s^2}{n^2a^2}$$

所以

$$\sin^2\frac{\Theta}{2}=\frac{s^2}{n^2a^2}\left(1+n^2-2n\cos\frac{\Theta}{2}\right)$$

$$s^2=\frac{n^2a^2}{1+n^2-2n\cos\frac{\Theta}{2}}$$

$$2s\frac{ds}{d\Theta}=\frac{d(s^2)}{d\Theta}=\frac{a^2n^2}{\left(n^2+1-2n\cos\frac{\Theta}{2}\right)^2}\left(n\cos\frac{\Theta}{2}-1\right)\left(n-\cos\frac{\Theta}{2}\right)$$

所以

$$\sigma(\Theta)=\frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right|=\frac{n^2a^2}{4\cos\frac{\Theta}{2}}\frac{\left(n\cos\frac{\Theta}{2}-1\right)\left(n-\cos\frac{\Theta}{2}\right)}{\left(1+n^2-2n\cos\frac{\Theta}{2}\right)^2}$$

所以

$$\sigma_{tot}=\frac{n^2a^2}{4}\int_0^{2\pi}d\varphi\int_0^{\pi}\frac{\left(n\cos\frac{\Theta}{2}-1\right)\left(n-\cos\frac{\Theta}{2}\right)}{\left(1+n^2-2n\cos\frac{\Theta}{2}\right)^2}\frac{\sin\Theta}{\cos\frac{\Theta}{2}}d\Theta$$

$$=2\pi n^2a^2\int_0^{\pi}\frac{\left(n\cos\frac{\Theta}{2}-1\right)\left(n-\cos\frac{\Theta}{2}\right)}{\left(1+n^2-2n\cos\frac{\Theta}{2}\right)^2}\sin\frac{\Theta}{2}d\frac{\Theta}{2}$$

令

$$\xi=1+n^2-2n\cos\frac{\Theta}{2}$$

则

$$d\xi=n\sin\frac{\Theta}{2}d\Theta,\ \ \sin\frac{\Theta}{2}d\Theta=\frac{1}{n}d\xi$$

$$\cos\frac{\Theta}{2}=\frac{1+n^2-\xi}{2n}$$

所以

$$\frac{\left(n\cos\frac{\Theta}{2}-1\right)\left(n-\cos\frac{\Theta}{2}\right)}{\left(1+n^2-2n\cos\frac{\Theta}{2}\right)^2}=\frac{1}{\xi^2}\frac{(n^2-1)^2-\xi^2}{4n}=\frac{(n^2-1)^2}{4n\xi^2}-\frac{1}{4n}$$

$$\sigma_{tot}=2\pi n^2a^2\int_{(1-n)^2}^{1+n^2}\left(\frac{(n^2-1)^2}{4n\xi^2}-\frac{1}{4n}\right)d\xi=2\pi a^2\frac{n^2}{1+n^2}$$