# Theoretical Mechanics
# 理论力学

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# Syllabus

- Chapter 0 Preface
- Chapter 1 Survey of the Elementary Principles
- Chapter 2 Variational Principle and Lagrange's Equations
- Chapter 3 The Central Force Problem
- Chapter 4 The Kinematics of Rigid Body Motion

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Mid-term exam

- Chapter 5 The Rigid Body Equations of Motion
- Chapter 6 Oscillations
- Chapter 7 The Classical Mechanics of the Special Theory of Relativity
- Chapter 8 The Hamilton Equations of Motion
- Chapter 9 Canonical Transformations

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Final term exam

- Chapter 10 Introduction to the Lagrangian and Hamiltonian Formulations for Continuous Systems and Fields

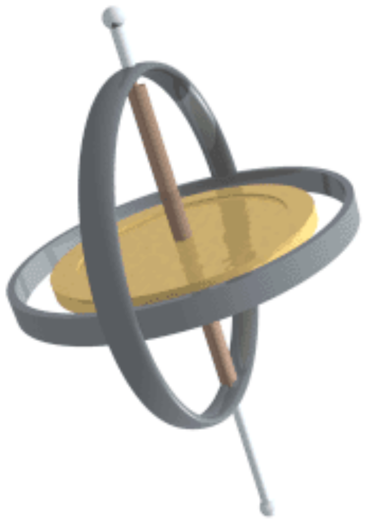
# 5.7 The Heavy Symmetrical Top with One Point Fixed

We introduce torque which make things messy.

$$I_1\dot{\omega}_1 - \omega_2\omega_3\left(I_2 - I_3\right) = N_1$$

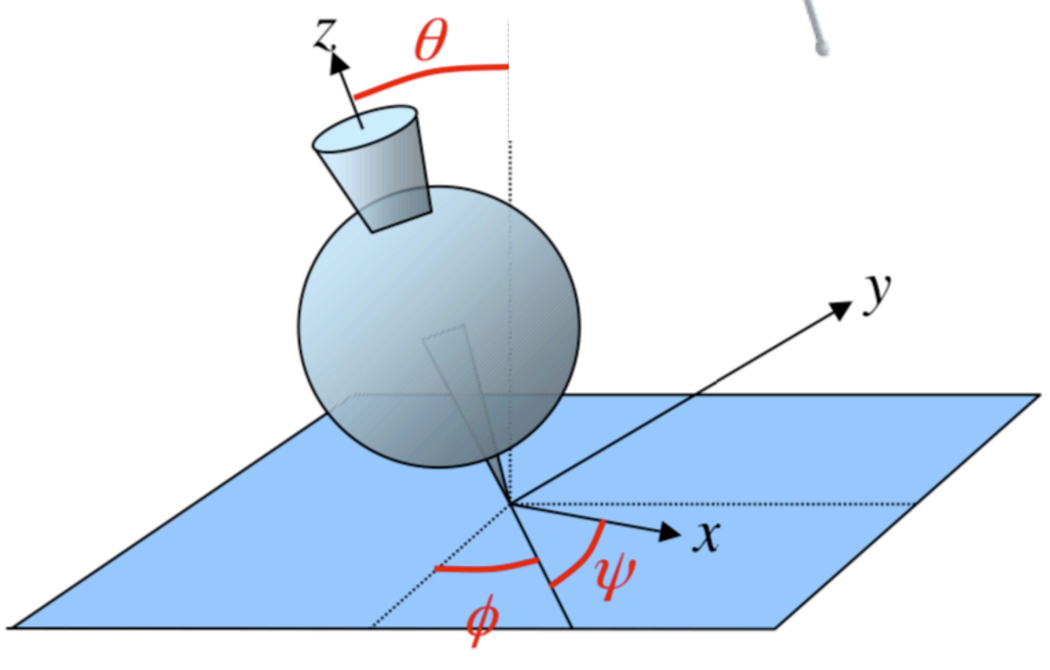
$$I_2\dot{\omega}_2 - \omega_3\omega_1\left(I_3 - I_1\right) = N_2$$

$$I_3\dot{\omega}_3 - \omega_1\omega_2\left(I_1 - I_2\right) = N_3$$



Consider a spinning top

Define Euler angles



# 5.7 The Heavy Symmetrical Top with One Point Fixed

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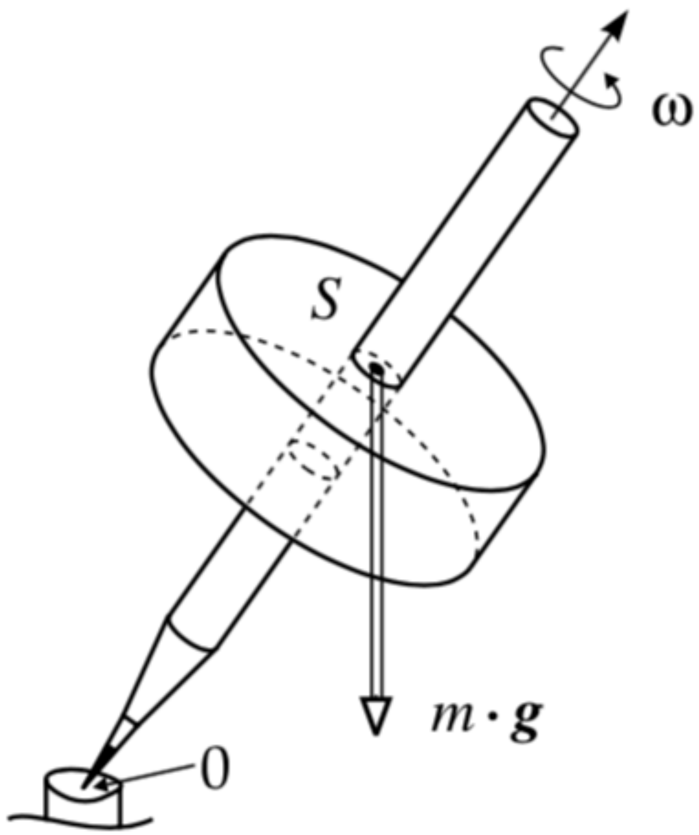
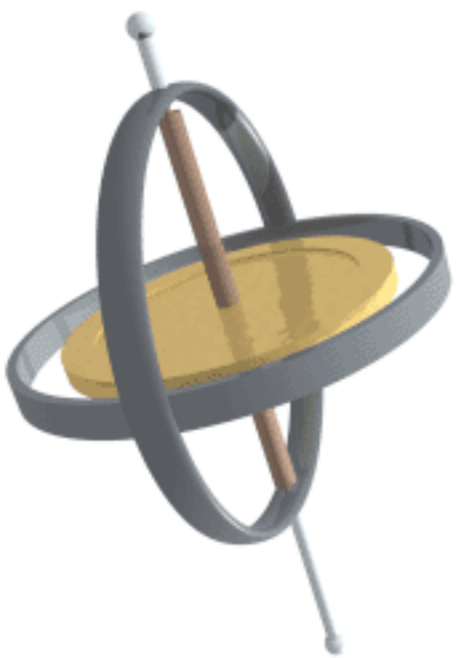


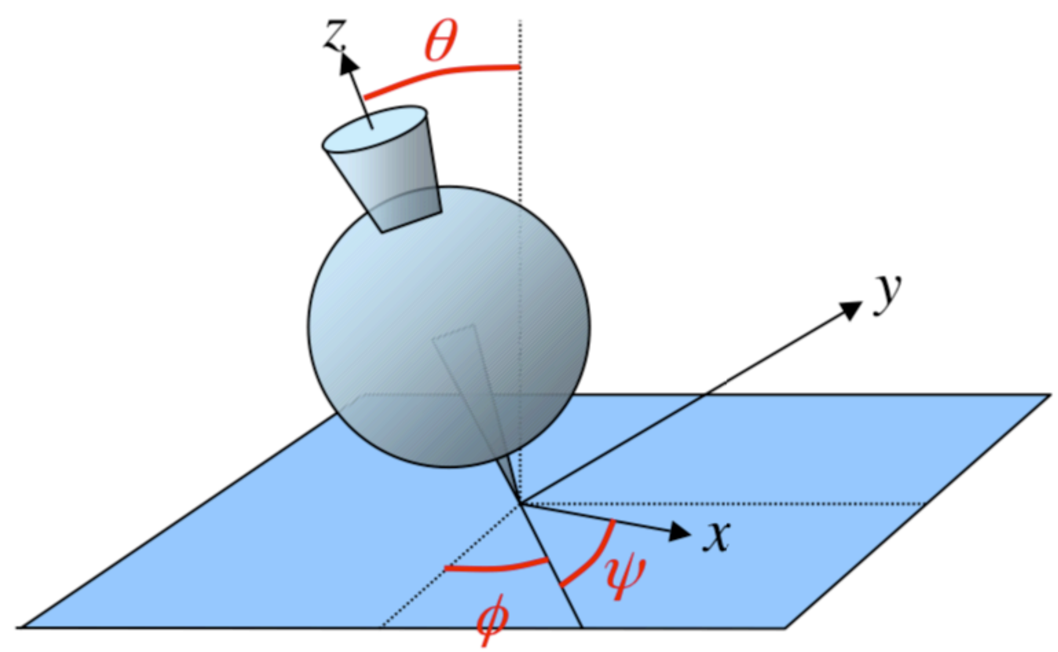



# Lagrangian for a Symmetrical Top

Assume $I_1 = I_2 \neq I_3$

Kinetic energy given by $T = \frac{1}{2} I_1 \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2$

Use Euler angles




$$\omega = \begin{bmatrix} \dot\phi \sin\psi \sin\theta + \dot\theta \cos\psi \\ \dot\phi \cos\psi \sin\theta - \dot\theta \sin\psi \\ \dot\phi \cos\theta + \dot\psi \end{bmatrix}$$

$$\implies T = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + \frac{I_3}{2}(\dot\phi \cos\theta + \psi)^2$$

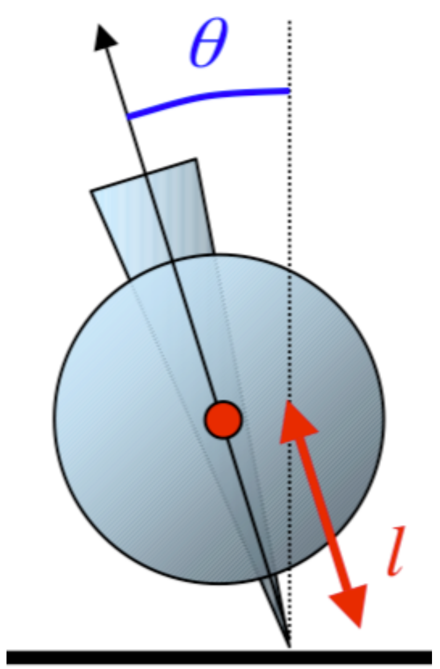
# Lagrangian for a Symmetrical Top

Potential energy is given by the height of the CoM

$$V = Mgl\cos\theta$$

Lagrangian is

$$L = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta$$




Finally we are in real business!

How we solve this?

Note φ and ψ are cyclic

Can define conserved conjugate momenta

# Conserved Momenta

$$L = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = I_3\omega_3 = \text{const.} \equiv I_1 a$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3\cos\theta(\dot{\phi}\cos\theta + \dot{\psi}) = \text{const.} \equiv I_1 b$$

Solve them for $\dot{\phi}$ and $\dot{\psi}$

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \qquad \dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta\frac{b - a\cos\theta}{\sin^2\theta}$$

We need $\theta(t)$ to get $\phi(t)$ and $\psi(t)$←Got rid of 2 degrees of freedom

# Energy Conservation

$$E = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \underline{\frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2} + Mgl\cos\theta$$

$$\frac{1}{2}I_3\omega_3^2 \text{ is constant and } \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

$$E' = E - \frac{I_3\omega_3^2}{2} = \frac{I_1\dot{\theta}^2}{2} + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

We've got a 1-dim equation of motion of θ

It looks like a particle of "mass" $I_1$ under a potential

$$V(\theta) = \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta$$

# 1-D Equation of Motion

Simplify the equation of motion by defining

$$\alpha \equiv \frac{2E - I_3\omega_3^2}{I_1} \text{ and } \beta \equiv \frac{2Mgl}{I_1}$$

EoM becomes $\alpha = \dot{\theta}^2 + \left(\dfrac{b - a\cos\theta}{\sin\theta}\right)^2 + \beta\cos\theta$

Switch variable from $\theta$ to $u = \cos\theta$

EoM $\rightarrow \dot{u}^2 = \left(1 - u^2\right)(\alpha - \beta u) - (b - au)^2$

Integrate $t = \displaystyle\int_{u(0)}^{u(t)} \frac{du}{\sqrt{\left(1 - u^2\right)(\alpha - \beta u) - (b - au)^2}}$

which is Elliptic integral

# Qualitative Behavior

Try to extract qualitative behavior

Same way as we did with central force problem

Consider the RHS of the last equation

$$\dot{u}^2 = f(u) \equiv \left(1 - u^2\right)(\alpha - \beta u) - (b - au)^2$$
$$= \beta u^3 - \left(\alpha + a^2\right) u^2 + (2ab - \beta)u + \left(\alpha - b^2\right)$$

Physical range is $f(u) = \dot{u}^2 \geq 0$ and $-1 \leq u \leq 1 (u = \cos\theta)$

$f(u)$ is a cubic function of $u$ with $\beta \equiv \dfrac{2Mgl}{I_1} > 0$

$f(\pm 1) = -(b - au)^2 \leq 0$

# Shape of $f(u)$

$f(u) = 0$ has 3 roots $-1 \le u_1 \le u_2 \le 1 \le u_3$

Solution for $\dot{u}^2 = f(u)$ is bounded inside $u_1 \le u \le u_2$

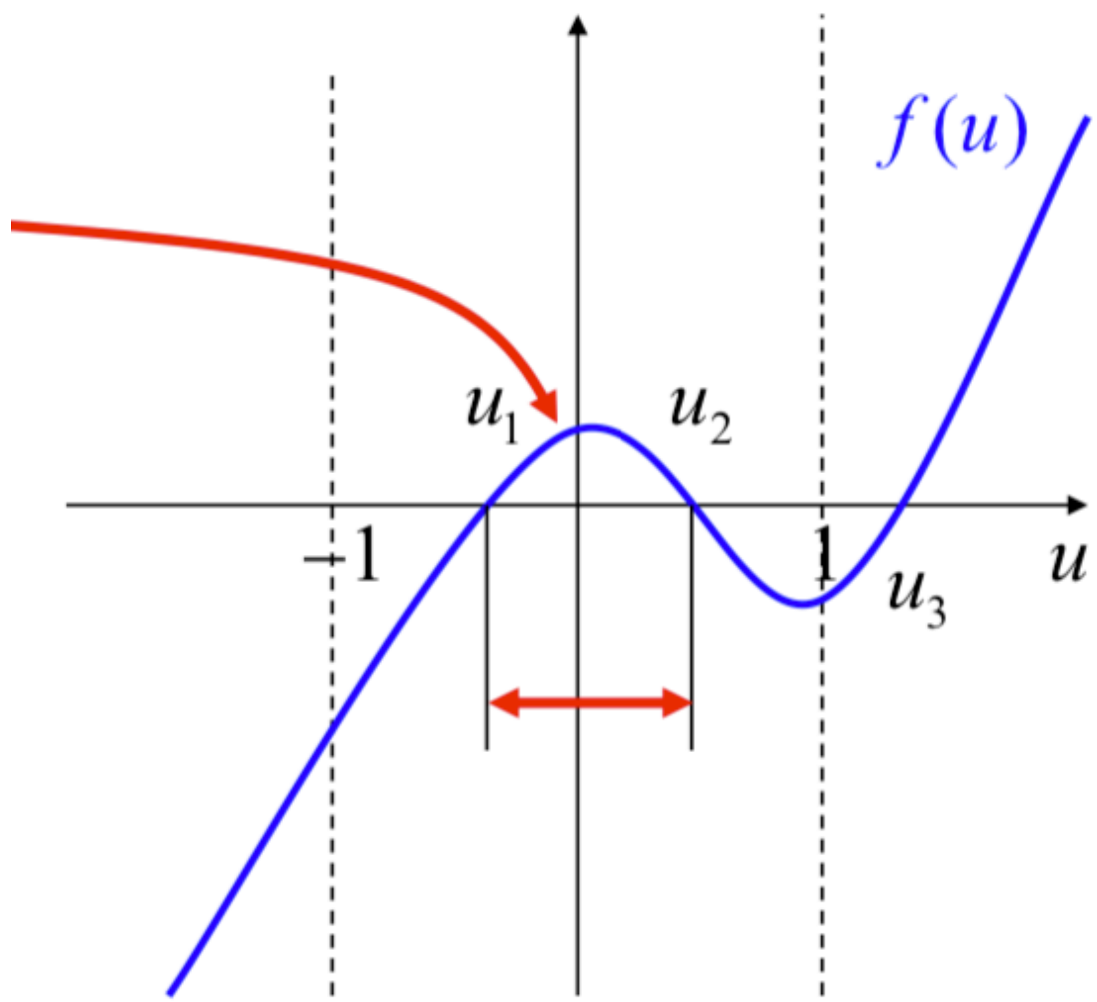
$\theta$ oscillates between

$\arccos(u_1)$ and $\arccos(u_2)$

$\phi$ and $\psi$ determined by

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

$$\dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta \frac{b - a\cos\theta}{\sin^2\theta}$$



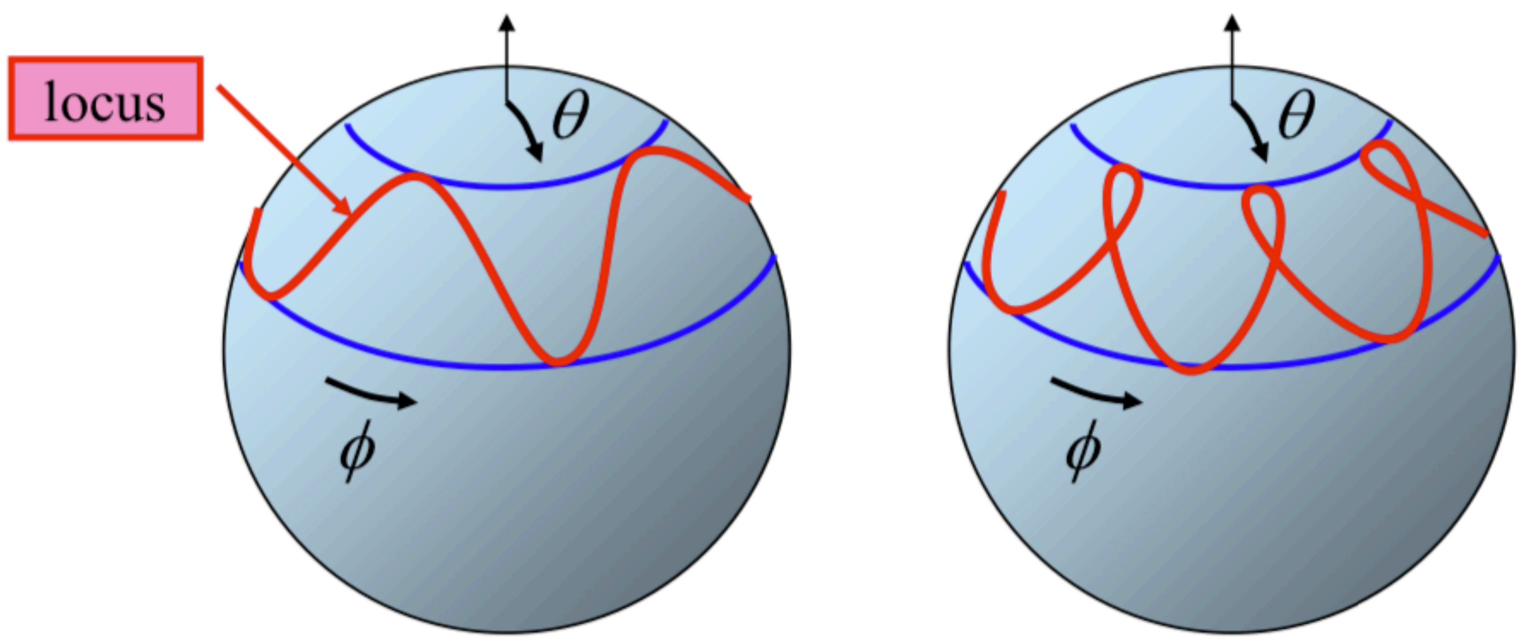
# Nutation

Consider the sign of $\dot{\phi} = \dfrac{b - a\cos\theta}{\sin^2\theta} = \dfrac{b - au}{1 - u^2}$

$\dot{\phi}$ changes sign at $u = u' = b/a$

$u' < u_1$ or $u' > u_2 \Longrightarrow \phi$ is monotonous

$u_1 < u' < u_2 \Longrightarrow \phi$ switches direction



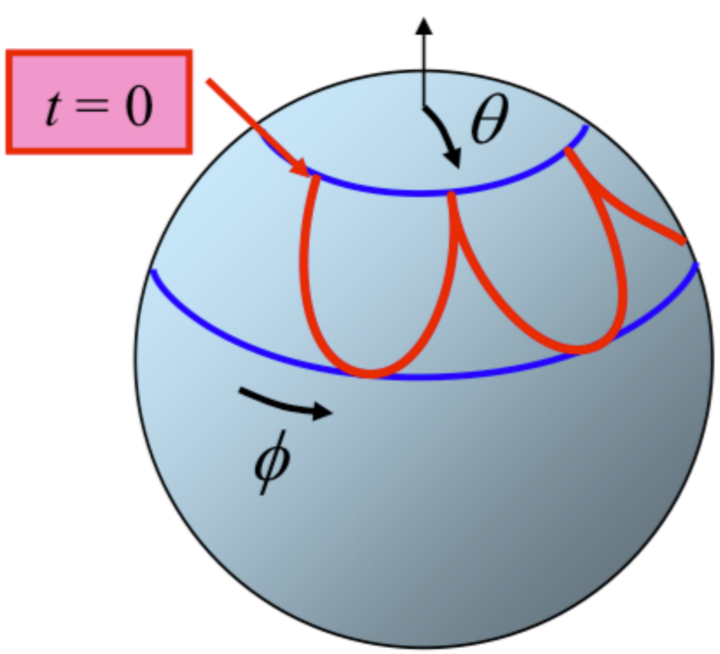
# Initial Condition

Suppose the figure axis is initially at rest

Spin the top, then release it "quietly"

$$\dot{\theta}_{t=0} = 0 \Longrightarrow f\left(u_{t=0}\right) = 0 \Longrightarrow u_{t=0} = u_1 \text{ or } u_2$$

$$\dot{\phi}_{t=0} = 0 \Longrightarrow b - au_{t=0} = 0 \Longrightarrow u_{t=0} = u'$$




Initially, the figure axis falls

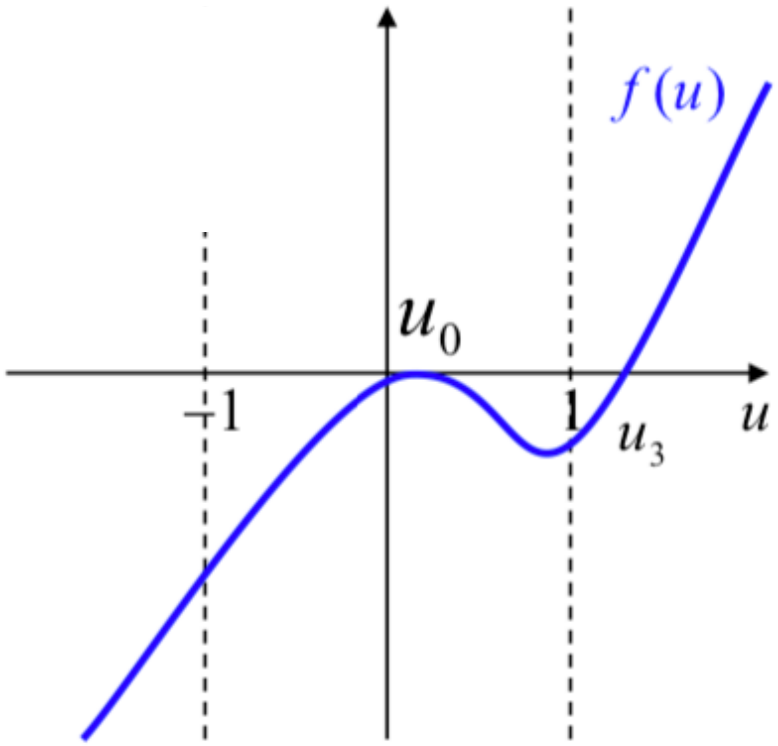
It then picks up procession in $\phi$

How does it know which way to go?

# Uniform Precession

Can we make a top precess without bobbing?

i.e. $\dot{\theta} = 0,\ \dot{\phi} = \text{const} \leftarrow \dot{\phi} = \dfrac{b - a\cos\theta}{\sin^2\theta}$




We need to have a double root for $f(u) = 0$

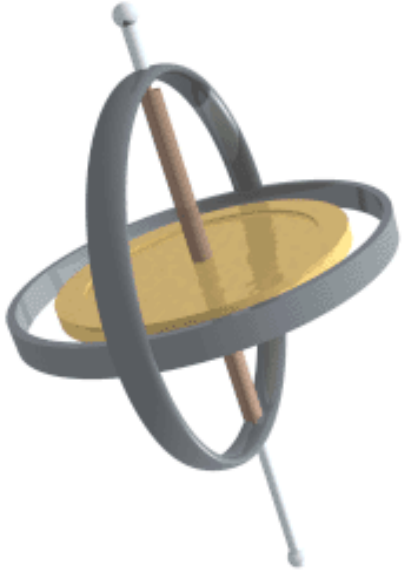
$$f\left(u_0\right) = \left(1 - u_0^2\right)\left(\alpha - \beta u_0\right) - \left(b - a u_0\right)^2 = 0$$

$$f'\left(u_0\right) = -2u_0\left(\alpha - \beta u_0\right) - \beta\left(1 - u_0^2\right) + 2a\left(b - a u_0\right) = 0$$

Combine $\rightarrow \dfrac{\beta}{2} = a\dot{\phi} - \dot{\phi}^2 u_0$

$I_1 a \equiv I_3 \omega_3$

$\beta \equiv \dfrac{2Mgl}{I_1} \rightarrow Mgl = \dot{\phi}\left(I_3\omega_3 - I_1\dot{\phi}\cos\theta_0\right)$



# Uniform Precession

$$Mgl = \dot{\phi}\left(I_3\omega_3 - I_1\dot{\phi}\cos\theta_0\right)$$

For any given value of $\omega_3$ and $\cos\theta_0$, you must give exactly the right "push" in $\phi$ to achieve uniform precession

Quadratic equation→2 solutions

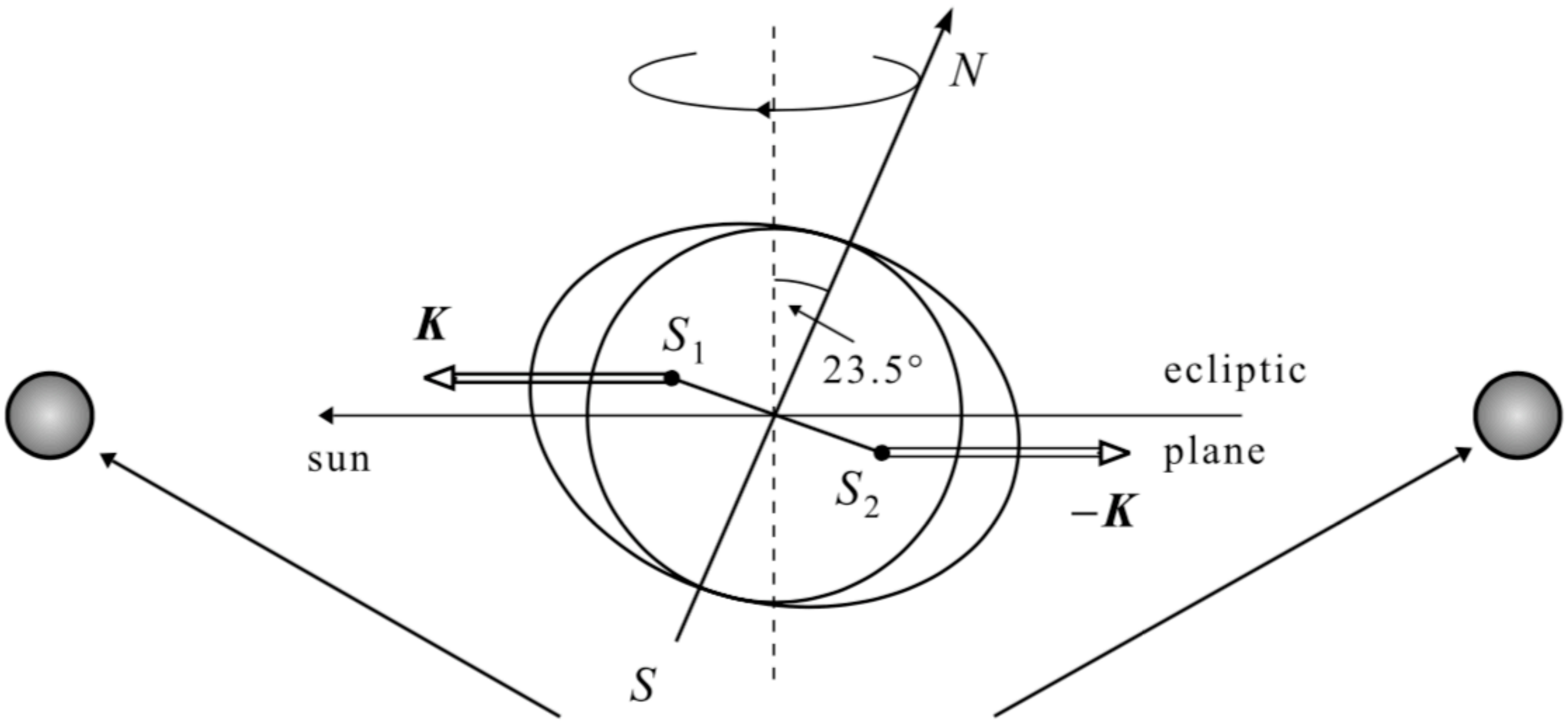
Same top can do "fast" or "slow" precession

For the solutions to exist $I_3^2\omega_3^2 > 4MglI_1\cos\theta_0$

$$\implies \omega_3 > \frac{2}{I_3}\sqrt{MglI_1\cos\theta_0}$$

Uniform precession is achieved only by a fast top

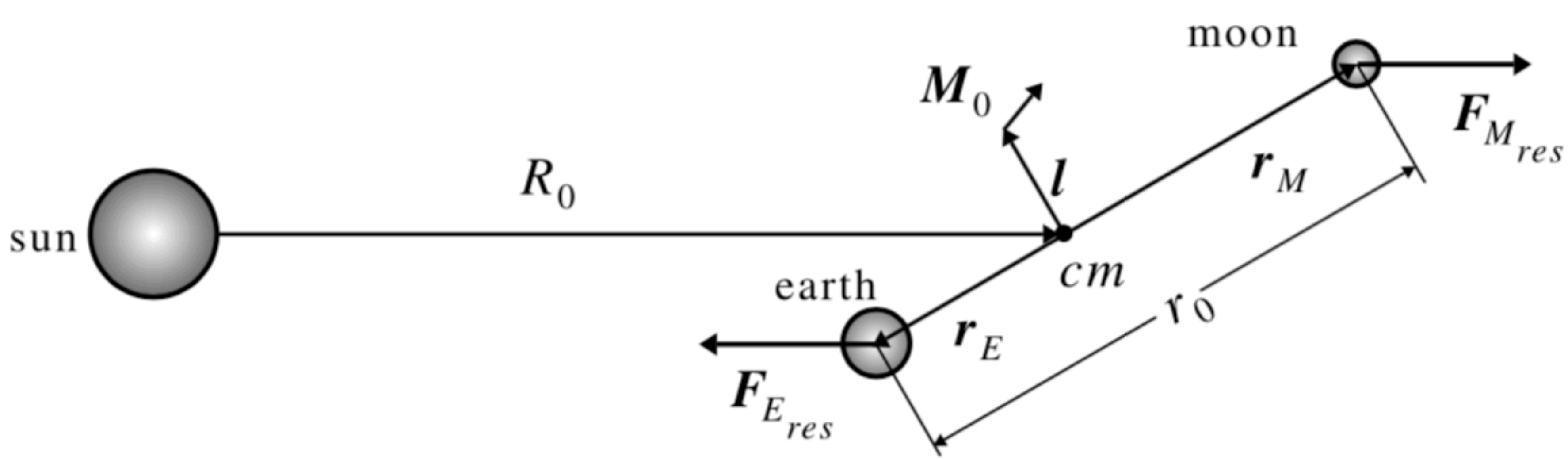
# Platonic year





**Platonic year (柏拉图年) is defined by scientific astronomy as "The period of one complete cycle of the equinoxes around the ecliptic, or about 25,800 years"**

# The Saros Cycle




**The saros (/ˈsɛərɒs/) (沙罗周期, 日蚀和月蚀关系的反复周期) is a period of exactly 223 synodic months (approximately 6585.3211 days, or 18 years, 11 days, 8 hours), that can be used to predict eclipses of the Sun and Moon.**

# Summary

Discussed rotational motion of rigid bodies

- Euler's equation of motion

Analyzed torque-free rotation

- Introduced the inertia ellipsoid
- It rolls on the invariant plane
- Dealt with simple cases

Analyzed the motion of a heavy top

- Reduced into 1-dimensional problem of $\theta$
- Qualitative behavior→Precession + nutation
- Initial condition vs. behavior

# 6.1 Formulation of the Problem

Consider a system with n degrees of freedom

Generalized coordinates $\{q_1, \ldots, q_n\}$

Generalized force at the equilibrium

$$Q_i = -\left(\frac{\partial V}{\partial q_i}\right)_0 = 0 \leftarrow V \text{ is at an extremum}$$

V must be minimum at a stable equilibrium

Taylor expansion of V using $q_i = q_{0i} + \eta_i$

$$V = V_0 + \left(\frac{\partial V}{\partial q_i}\right)_0 \eta_i + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \cdots \approx \frac{1}{2} V_{ij} \eta_i \eta_j$$

**Constant**

**zero**

**Constant symmetric matrix**

# 6.1 Formulation of the Problem

Kinetic energy is a 2nd-order homogeneous function of velocities

$$T = \frac{1}{2} m_{ij}\left(q_1, \ldots, q_n\right) \dot{q}_i \dot{q}_j = \frac{1}{2} m_{ij} \dot{\eta}_i \dot{\eta}_j$$

This requires that the transformation functions do not explicitly depend on time, i.e.

$$q_i = q_i\left(x_1, \ldots, x_N\right)$$

$m_{ij}$ generally depend on $\left\{q_{\mathrm{i}}\right\} \rightarrow$ Taylor expansion

$$m_{ij} = \left(m_{ij}\right)_0 + \cancel{\left(\frac{\partial m_{ij}}{\partial q_k}\right)_0 \eta_k} + \cdots \approx T_{ij} \Longrightarrow T \approx \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j$$

**Constant symmetric matrix**

# Lagrangian

For small deviation $\left\{\eta_i\right\}$ from equilibrium

$$L = T - V = \frac{1}{2}T_{ij}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2}V_{ij}\eta_i\eta_j = \frac{1}{2}\tilde{\dot{\boldsymbol{\eta}}}\,\mathbf{T}\,\dot{\boldsymbol{\eta}} - \frac{1}{2}\tilde{\boldsymbol{\eta}}\mathbf{V}\boldsymbol{\eta}$$

The equations of motion are

$$T_{ij}\ddot{\eta}_j + V_{ij}\eta_j = 0$$

This looks similar to $m\ddot{x} + kx = 0$

Difficulty: $T_{ij}$ and $V_{ij}$ have off-diagonal components

If **T** and **V** were diagonal

$$T_{ij}\ddot{\eta}_j + V_{ij}\eta_j \rightarrow T_{ii}\ddot{v}_i + V_{ii}\eta_i = 0 \leftarrow \text{no sum over } i$$

Can we find a new set of coordinates that diagonalizes **T** and **V**?