Theoretical Mechanics

理论力学

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Syllabus

- Chapter 0 Preface
- Chapter 1 Survey of the Elementary Principles
- Chapter 2 Variational Principle and Lagrange's Equations
- Chapter 3 The Central Force Problem
- Chapter 4 The Kinematics of Rigid Body Motion

Mid-term exam

- Chapter 5 The Rigid Body Equations of Motion
- Chapter 6 Oscillations
- Chapter 7 The Classical Mechanics of the Special Theory of Relativity
- Chapter 8 The Hamilton Equations of Motion
- Chapter 9 Canonical Transformations

Final term exam

- Chapter 10 Introduction to the Lagrangian and Hamiltonian Formulations for Continuous Systems and Fields

Axial Vector

$d\Omega$ behaves pretty much like a vector

$d\Omega$ rotates the same way as $\mathbf{r}$ with coordinate rotations

Space inversion P reveals difference

Ordinary vector flips $\mathbf{r}' = \mathrm{P}(\mathbf{r}) = -\,\mathbf{r}$

dΩ doesn't, because

$$\mathrm{P}(d\mathbf{r}) = -\,d\mathbf{r} = -\,(\mathbf{r} \times d\boldsymbol{\Omega})$$
$$\mathrm{P}(d\mathbf{r}) = \mathrm{P}(\mathbf{r} \times d\boldsymbol{\Omega}) = \mathrm{P}(\mathbf{r}) \times \mathrm{P}(d\boldsymbol{\Omega}) = (-\mathbf{r}) \times \mathrm{P}(d\boldsymbol{\Omega})$$
$$\implies \mathrm{P}(d\boldsymbol{\Omega}) = d\boldsymbol{\Omega}$$

Such a "vector" is called an axial vector (pseudo-vector)

Examples: angular momentum, magnetic field

Parity

Parity operator P represents space inversion

$$(x, y, z) \xrightarrow{\mathbf{P}} (-x, -y, -z)$$

We can control only specific vector spaces.

Quantity	Parity	Eigenvalue
Scalar	$\mathbf{P}S = S$	+1
Pseudoscalar	$\mathbf{P}S^* = -S^*$	−1
Vector	$\mathbf{PV} = -\mathbf{V}$	−1
Axial vector	$\mathbf{PV}^* = \mathbf{V}^*$	+1

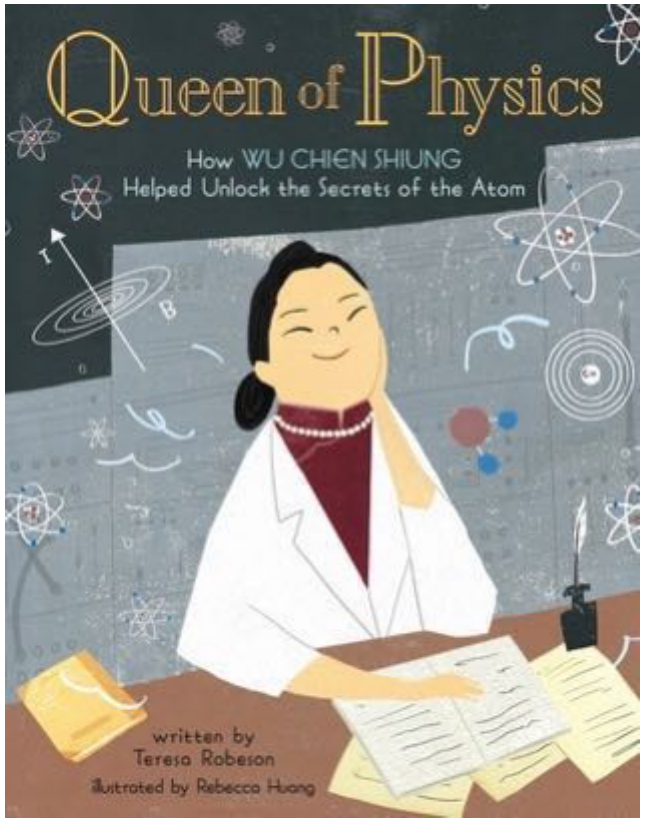
$\mathbf{V} \times \mathbf{V} = \mathbf{V}^*$ $\mathbf{V} \times \mathbf{V}^* = \mathbf{V}$ $S^*\mathbf{V} = \mathbf{V}^*$

$\mathbf{V} \cdot \mathbf{V}^* = S^*$ $\mathbf{V}^* \cdot \mathbf{V}^* = S$ $S^*\mathbf{V}^* = \mathbf{V}$ etc.

Parity

- 李政道：讨论中，我忽生灵感，突然很清楚地明白了，要解决“$\theta-\tau$之谜”，必须先离开这个系统，假定“$\theta-\tau$”之外的粒子也可能产生宇称不守恒的新现象。我发现，用斯坦伯格实验中重粒子产生和衰变的几个动量，便能去组织一个新的赝标量。用了这个$\theta-\tau$以外的赝标量，就可试验$\theta-\tau$以外的系统宇称是否不守恒。而这些赝标量，很显然的，没有被以前任何实验测量过。用了这些新的赝标量，就可以系统地研究宇称是否不守恒那个大问题。（http://zhishifenzi.com/depth/character/10913.html）

- 吴健雄：1956年早春的一天，李政道教授来到浦品物理实验室第13层楼我的小办公室……他先向我解释了$\tau-\theta$之谜，以及它如何引起在弱衰变中宇称是否守恒的问题。他继续说，如果$\tau-\theta$之谜的答案是宇称不守恒，那么这种破坏在极化核的β衰变的空间分布中也应该观察到；我们必须去测量赝标量$\langle\boldsymbol{\sigma}\cdot\mathbf{p}\rangle$，这里$\mathbf{p}$是电子的动量，$\boldsymbol{\sigma}$是核的自旋……（http://www.ihep.cas.cn/xh/gnwlxh/cgb/201604/t20160430_4593956.html）



Axial Vector and pseudo-scalar

If $\mathbf{A}$ and $\mathbf{B}$ are vectors, then $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is an axial vector or pseudo-vector. $C_i = \epsilon_{ijk} A_j B_k$, where ϵ_{ijk} is the antisymmetric tensor (of 3-rank, 全反对称张量) or Levi-Civita symbol (density, Levi-Civita符号).

$$\mathrm{P}\left(\mathbf{C}\right) = \mathrm{P}\left(\mathbf{A} \times \mathbf{B}\right) = \mathrm{P}\left(\mathbf{A}\right) \times \mathrm{P}\left(\mathbf{B}\right) = \left(-\mathbf{A}\right) \times \left(-\mathbf{B}\right) = \mathbf{C}$$

However, if $\mathrm{P}(A_1, A_2, A_3) = (-A_1, -A_2, -A_3)$, why $\mathrm{P}(C_1, C_2, C_3) \neq (-C_1, -C_2, -C_3)$?

Because P is a "physical" operator which "directly" exerts only on the vector spaces of $\mathbf{A}$ and $\mathbf{B}$, not the one of $\mathbf{C}$.

In the vector space of $\mathbf{C}$, the "mathematical" inverse operator changes $\mathbf{C}$ to $-\mathbf{C}$.

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are vectors, $\mathbf{A} \cdot \left(\mathbf{B} \times \mathbf{C}\right)$ is a pseudo-scalar.

4.9 Rate of Change of A Vector

Consider a rotating rigid body

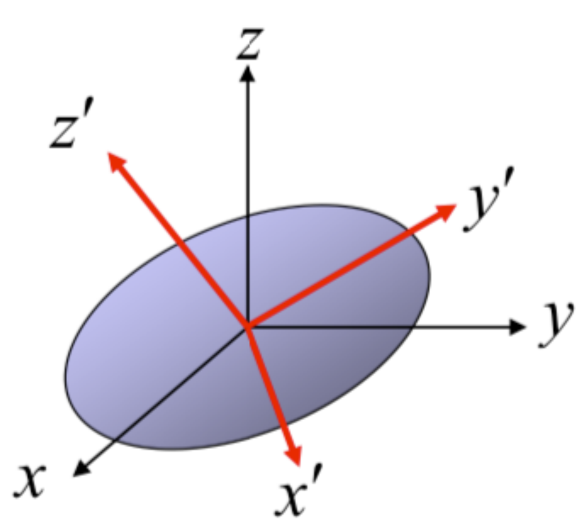
Define body coordinates (x', y', z')

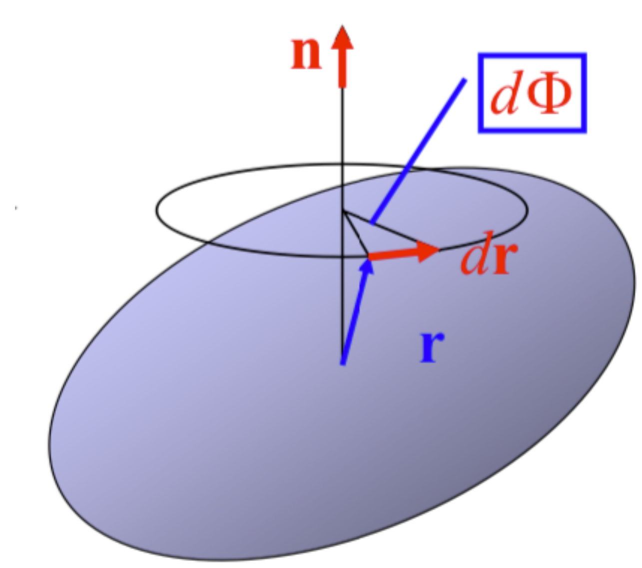
Between t and $t + dt$, the body coordinates rotate by $d\mathbf{\Omega} = \mathbf{n}d\Phi$

The direction is **CCW** because we are talking about the coordinate axes

Observed in the space coordinates, any point $\mathbf{r}$ on the body moves by
$d\mathbf{r} = d\Phi\mathbf{n} \times \mathbf{r} = d\mathbf{\Omega} \times \mathbf{r}$

Sign opposite from previous notation because the rotation is CCW





General Vectors

Now consider a general vector $\mathbf{G}$

How does it move in space/body coordinates?

i.e. what's the time derivative $d\mathbf{G}/dt$?

Movement $d\mathbf{G}$ differs in space and body coordinates because of the rotation of the latter

$$(d\mathbf{G})_{\text{space}} = (d\mathbf{G})_{\text{body}} + (d\mathbf{G})_{\text{rot}}$$

If G is fixed to the body

$$(d\mathbf{G})_{\text{body}} = 0 \quad \text{and} \quad (d\mathbf{G})_{\text{space}} = d\boldsymbol{\Omega} \times \mathbf{G}$$

$$\Longrightarrow (d\mathbf{G})_{\text{rot}} = d\boldsymbol{\Omega} \times \mathbf{G}$$

Angular Velocity

For any vector $\mathbf{G}$ $\quad (d\mathbf{G})_{\mathbf{space}} = (d\mathbf{G})_{\mathbf{body}} + d\mathbf{\Omega} \times \mathbf{G}$

$$\Longrightarrow \left(\frac{d\mathbf{G}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{G}}{dt}\right)_{\text{body}} + \omega \times \mathbf{G} \qquad \boldsymbol{\omega} dt = d\mathbf{\Omega} = d\Phi\mathbf{n}$$

i.e. what's the time derivative $d\mathbf{G}/dt$?

$\boldsymbol{\omega}$ = instantaneous angular velocity

Direction = $\mathbf{n}$ = instantaneous axis of rotation

Magnitude = $d\Phi/dt$ = instantaneous rate of rotation

Since this works for any vector, we can say

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \boldsymbol{\omega} \times$$

s for space coordinates, r for rotating coordinates

Summary

Discussed 3-d rotation

- Preparation for rigid body motion
  - Movement in 3-d + Rotation in 3-d = 6 coordinates

Looked for ways to describe 3-d rotation

- Euler angles one of the many possiblilites
- Euler's theorem

Defined infinitesimal rotation $d\Omega$

- Commutative (unlike finite rotation)
- Behaves as an axial vector (like angular momentum)

Express $\boldsymbol{\omega}$ by the Euler angles

Chapter 4 presents all the kinematical tools needed in the discussion of rigid body motion… This tools will now be applied to obtain the Euler dynamical equations of motion of the rigid body in their most convenient form. With the help of the equations of motion, some simple but highly important problems of rigid body motion can be discussed.

Prerequisites: Tensor

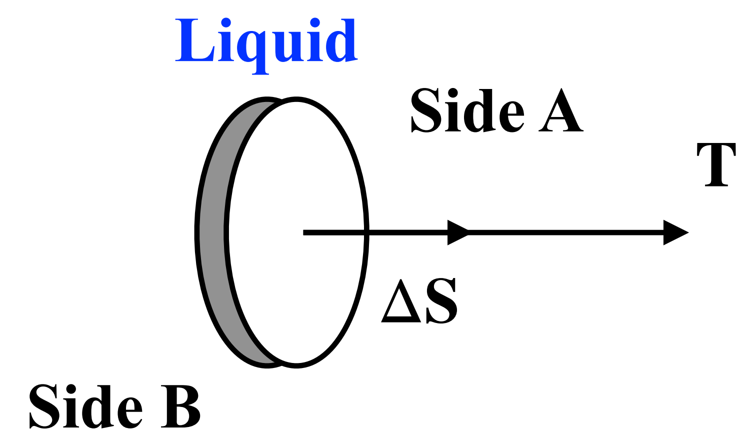
The word "tensor" has a root in Latin, "tensus", meaning stretch or tension. Literally "tensor" means "something that stretches".

There are two major threads which led to the modern theory of tensors. The dyadic developed by Gibbs and the contravariant and covariant tensors developed by Ricci in the context of absolute differential calculus.

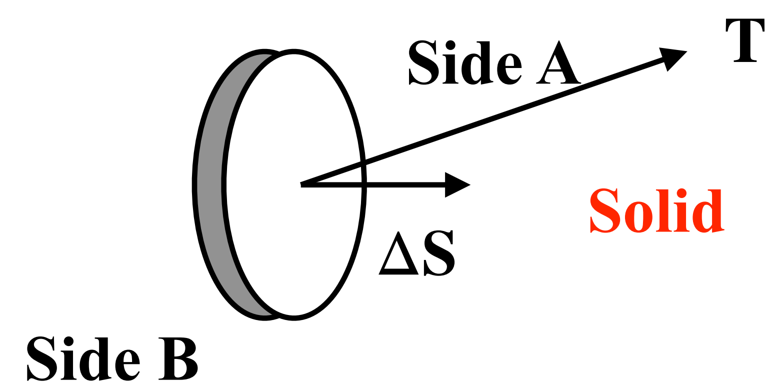
We follow the (pseudo-)thread of Gibbs which is easy for physicists.

Prerequisites: Tensor

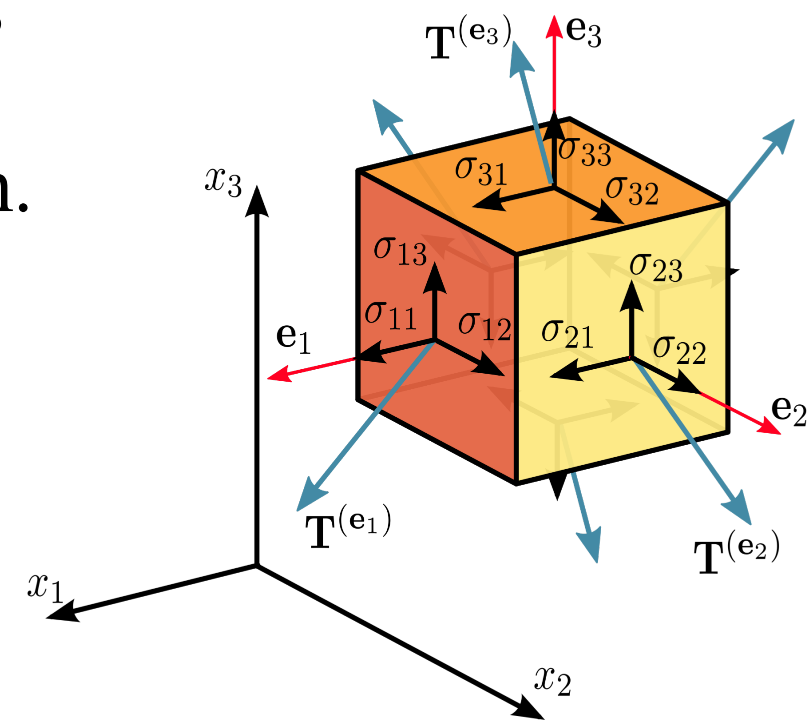
In continuum mechanics, **stress T** is a physical quantity that expresses the internal forces that neighbouring particles of a continuous material exert on each other, while **strain ΔS** is the measure of the deformation of the material.



For liquids, **T** is linearly related to **ΔS**, i.e., $\mathbf{T} = \sigma\Delta\mathbf{S}$, where σ is a scalar coefficient.



For solids, **T** is still linearly related to **ΔS**, i.e., $\mathbf{T} = \Sigma\Delta\mathbf{S}$, where Σ is called Cauchy stress tensor which represents a linear transformation.



$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} \Delta S_1 \\ \Delta S_2 \\ \Delta S_3 \end{pmatrix}$$

Prerequisites: Tensor

Linear map: A Linear map from vector space V to another vector space W (both over $\mathbb{R}$) is a function $T: V \to W$ with the following properties:

additivity: $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$;

homogeneity: $T(\lambda \mathbf{v}) = \lambda(T\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in V$.

The set of all linear maps from V to W is denoted $\mathscr{L}(V, W)$.

Theorem: $\mathscr{L}(V, W)$ is a vector space with dimension $\dim V \cdot \dim W$.

Theorem: Suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of V and $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ a basis of W. Then there exists a **unique linear map** $T: V \to W$ such that $T\mathbf{e}_i = \mathbf{e}'_i$ for each $j = 1, \ldots, n$.

The unique linear map T in $\mathscr{L}(V, W)$ is an abstract but concrete existence. As we know, it can be represented in a matrix form.

The most important and difficult thing in Classical Mechanics is to recognize T in $\mathscr{L}(V, W)$ as a physical (or mathematical) quantity which is called **tensor**. (T是物理量！矩阵不是物理量，矩阵是（数学和）物理量的表现形式。)

It is difficult to human beings, especially young ones, to realize T from its matrix form or anything from its pretty representation (画龙画虎难画骨，知人知面不知心).

Prerequisites: Tensor

In his famous book "Vector Analysis", Josiah Willard Gibbs also developed **tensor algebra** and **tensor analysis**.

The former one is the **nightmare** difficult concept in ***Classical Mechanics*** and the latter one is the **hell** difficult one in ***Classical Electrodynamics***.

However, it is surprising that the original ideal is easily understandable.

What we need is a quantity to make vector **b** to vector **a**. Gibbs said "it is just **ab**!"

$\mathbf{a} = \alpha \mathbf{ab} \cdot \mathbf{b}$, where $\alpha = \dfrac{1}{b^2}$

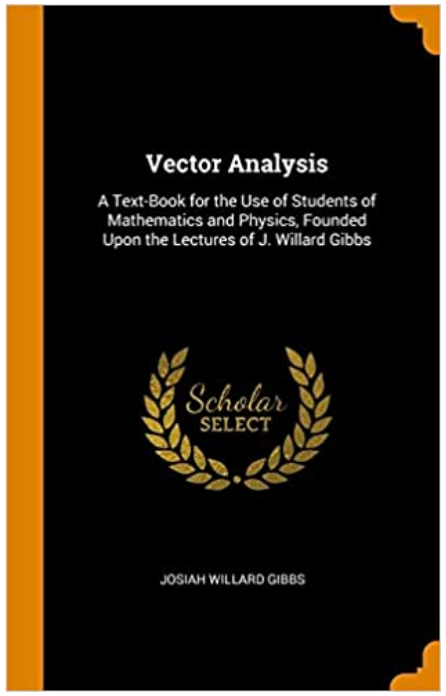
Gibbs called **ab** a dyad (并矢).

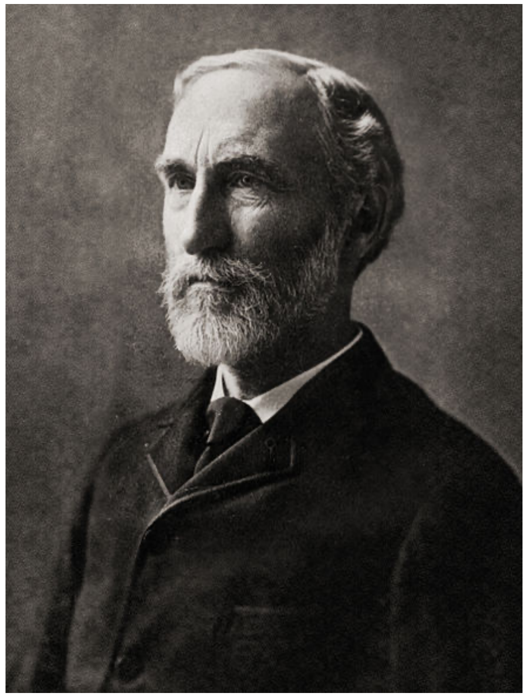
The linear combination of dyads, for example, $\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2$ or $\mathbf{a}_1 \otimes \mathbf{b}_1 + \mathbf{a}_2 \otimes \mathbf{b}_2$, is called dyadic(并矢).

How many independent dyads for 3-dimensional Euclid space?

Obviously, **9**. $\mathbf{e^a}_i\mathbf{e^b}_j$ for $i, j = 1,\ 2,\ 3$ where $\mathbf{e^a}_i$ is the basis of **a**.

In modern language, the operator $\otimes$ is "direct product" and the operator $+$ is "formal sum".





Josiah Willard Gibbs

Prerequisites: Tensor

What type of vector for **a** or **b** in a dyad **ab**?

If one asks $\mathbf{b} \cdot \mathbf{ab} \cdot \mathbf{a}$ to be a scalar and **a** is a contravariant vector, **b** has to be a covariant vector. Tensor **ab** is a bilinear mapping.

If one asks $\mathbf{u} \cdot \mathbf{ab} \cdot \mathbf{u}$ to be a scalar where **u** is a contravariant vector, **a** and **b** are both covariant vectors. Tensor **ab** has a quadratic form.

If one use the "standard basis" for a 3-dimensional Euclid space, the difference of contravariant and covariant vectors **vanishes**.

Therefore, a general tensor (dyadic) **I** can be written as

$$\mathbf{I} = a\mathbf{e}_1\mathbf{e}_1 + b\mathbf{e}_2\mathbf{e}_2 + c\mathbf{e}_3\mathbf{e}_3 + d\mathbf{e}_1\mathbf{e}_2 + e\mathbf{e}_2\mathbf{e}_1 + f\mathbf{e}_2\mathbf{e}_3 + g\mathbf{e}_3\mathbf{e}_2 + h\mathbf{e}_3\mathbf{e}_1 + i\mathbf{e}_1\mathbf{e}_3$$

$$\text{or } \mathbf{I} = \begin{bmatrix} a\mathbf{e}_1\mathbf{e}_1 & d\mathbf{e}_1\mathbf{e}_2 & i\mathbf{e}_1\mathbf{e}_3 \\ e\mathbf{e}_2\mathbf{e}_1 & b\mathbf{e}_2\mathbf{e}_2 & f\mathbf{e}_2\mathbf{e}_3 \\ h\mathbf{e}_3\mathbf{e}_1 & g\mathbf{e}_3\mathbf{e}_2 & c\mathbf{e}_3\mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} a & d & i \\ e & b & f \\ h & g & c \end{bmatrix}$$

Prerequisites: Tensor

Tensor $\mathbf{I}$ is symmetric if $\mathbf{I}$ is a quadratic form function so that $\mathbf{I}(n) \equiv \mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n}$.

$$\mathbf{I} = \begin{bmatrix} a\mathbf{e}_1\mathbf{e}_1 & d\mathbf{e}_1\mathbf{e}_2 & e\mathbf{e}_1\mathbf{e}_3 \\ d\mathbf{e}_2\mathbf{e}_1 & b\mathbf{e}_2\mathbf{e}_2 & f\mathbf{e}_2\mathbf{e}_3 \\ e\mathbf{e}_3\mathbf{e}_1 & f\mathbf{e}_3\mathbf{e}_2 & c\mathbf{e}_3\mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

There are only 6 independent dyads.

Cauchy stress tensor is a symmetric tensor.

Metric $[g_{ij}]$ is a symmetric tensor.

As you will learn, inertia tensor (惯量张量) is also a symmetric tensor.

Not all tensor are symmetric tensors, the famous EM field tensor is a anti-symmetric tensor.

It is difficult to young students to realize tensor from its matrix form or anything from its pretty representation (画龙画虎难画骨，知人知面不知心).

Don’t forget: Tensor is a vector in its vector space and has different form when the basis changes.

However, whose basis changes? Basis of $\mathbf{I}$ or bases of $\mathbf{a}$ and $\mathbf{b}$?

5.1 Angular Momentum and Kinetic Energy of Motion About A Point

Chasles' theorem states that any general displacement of a rigid body can be represented by a translation plus a rotation.

The six coordinates needed to describe the motion can be divided into two sets:

Three Cartesian coordinates of a point fixed in the rigid body

Three Euler angles for motion about the point

5.1 Angular Momentum and Kinetic Energy of Motion About A Point

According to Chapter 1, a similar division holds for the kinetic energy of a multi-particle system is

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i v_i'^2$$

If we define the body axis from the center of mass

$$T = \frac{1}{2} M \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + T'(\dot{\phi}, \dot{\theta}, \dot{\psi})$$

T' depends only on the angular velocity

Must be a 2nd order homogeneous function

Potential Energy

Potential energy can often be separated as well

$$V = V_1(x, y, z) + V_2(\phi, \theta, \psi)$$

Lagrangian can be written as

$$L = L_t(x, y, z, \dot{x}, \dot{y}, \dot{z}) + L_r(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})$$

It is often possible to separate the translational and rotational motions by taking the center of mass as the origin of the body coordinate axes.

Rotational Motion

We concentrate on the rotational part

Translational part same as a single particle→Easy

Consider total angular momentum $\mathbf{L} = m_i \mathbf{r}_i \times \mathbf{v}_i$

$\mathbf{V}_i$ is given by the rotation $\boldsymbol{\omega}$ as $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$

$$\mathbf{L} = m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = m_i \left[\boldsymbol{\omega} r_i^2 - \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega}) \right] =$$

$$\boxed{\begin{bmatrix} m_i \left(r_i^2 - x_i^2 \right) & -m_i x_i y_i & -m_i x_i z_i \\ -m_i y_i x_i & m_i \left(r_i^2 - y_i^2 \right) & -m_i y_i z_i \\ -m_i z_i x_i & -m_i z_i y_i & m_i \left(r_i^2 - z_i^2 \right) \end{bmatrix}} \boldsymbol{\omega}$$

Inertia tensor I