Theoretical Mechanics
理论力学

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Syllabus

- Chapter 0 Preface
- Chapter 1 Survey of the Elementary Principles
- Chapter 2 Variational Principle and Lagrange's Equations
- Chapter 3 The Central Force Problem
- Chapter 4 The Kinematics of Rigid Body Motion

Mid-term exam

- Chapter 5 The Rigid Body Equations of Motion
- Chapter 6 Oscillations
- Chapter 7 The Classical Mechanics of the Special Theory of Relativity
- Chapter 8 The Hamilton Equations of Motion
- Chapter 9 Canonical Transformations

Final term exam

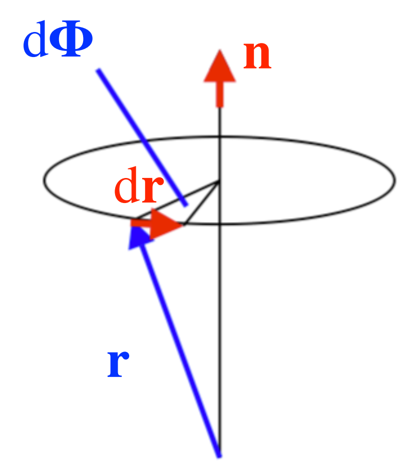
- Chapter 10 Introduction to the Lagrangian and Hamiltonian Formulations for Continuous Systems and Fields

Infinitesimal Rotation

ICT for rotation is generated by $G = \mathbf{L} \cdot \mathbf{n}$

We've studied infinitesimal rotation in Ch. 4

Infinitesimal rotation of $d\Phi$ about $\mathbf{n}$ moves a vector $\mathbf{r}$ as



$$d\mathbf{r} = \mathbf{n} d\Phi \times \mathbf{r}$$

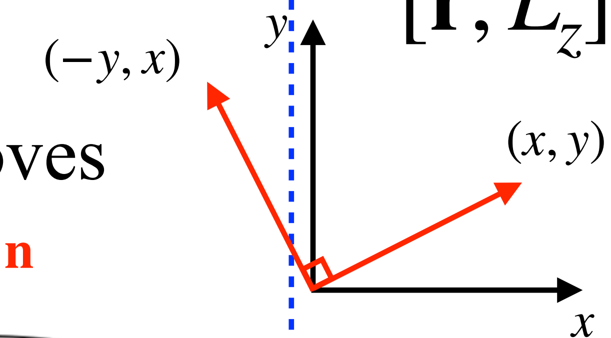
Compare the two expressions

$$d\mathbf{r} = d\Phi[\mathbf{r}, \mathbf{L} \cdot \mathbf{n}] = d\Phi \mathbf{n} \times \mathbf{r} \Rightarrow [\mathbf{r}, \mathbf{L} \cdot \mathbf{n}] = \mathbf{n} \times \mathbf{r}$$

Equation $[\mathbf{r}, \mathbf{L} \cdot \mathbf{n}] = \mathbf{n} \times \mathbf{r}$ holds for any $\mathbf{r}$ that rotates together with the system

Several useful rules can be derived from this

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



$$\begin{aligned}[\mathbf{r}, L_z] &= [\mathbf{r}, \mathbf{L} \cdot \mathbf{k}] \\ &= [x, xp_y - yp_x]\mathbf{i} \\ &\quad + [y, xp_y - yp_x]\mathbf{j} \\ &\quad + [z, xp_y - yp_x]\mathbf{k} \\ &= -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}\end{aligned}$$

$$\mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$$

$$[y, L_z] = (\mathbf{k} \times \mathbf{r})_y = x$$

Scalar Products

Consider a scalar product $\mathbf{a} \cdot \mathbf{b}$ of two vectors

Try to rotate it

$$
\begin{aligned}
[\mathbf{a} \cdot \mathbf{b}, \mathbf{L} \cdot \mathbf{n}] &= \mathbf{a} \cdot [\mathbf{b}, \mathbf{L} \cdot \mathbf{n}] + \mathbf{b} \cdot [\mathbf{a}, \mathbf{L} \cdot \mathbf{n}] \\
&= \mathbf{a} \cdot (\mathbf{n} \times \mathbf{b}) + \mathbf{b} \cdot (\mathbf{n} \times \mathbf{a}) \\
&= \mathbf{a} \cdot (\mathbf{n} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{n}) \\
&= 0
\end{aligned}
$$

Obvious: scalar product doesn't change by rotation

Also obvious: length of any vector is conserved

Angular Momentum

Try with $\mathbf{L}$ itself $[\mathbf{L}, \mathbf{L}\cdot\mathbf{n}] = \mathbf{n}\times\mathbf{L}$

$x - y - z$ components are

$$[\mathbf{L}, L_y] = \mathbf{n}_y \times \mathbf{L} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ L_x & L_y & L_z \end{vmatrix}$$

$$\Downarrow$$

$$\left[L_x, L_x\right] = 0 \qquad \left[L_x, L_y\right] = L_z \qquad \left[L_x, L_z\right] = -L_y$$

$$\left[L_y, L_x\right] = -L_z \qquad \left[L_y, L_y\right] = 0 \qquad \left[L_y, L_z\right] = L_x$$

$$\left[L_z, L_x\right] = L_y \qquad \left[L_z, L_y\right] = -L_x \qquad \left[L_z, L_z\right] = 0$$

$$\Longrightarrow \left[L_i, L_j\right] = \varepsilon_{ijk} L_k$$

These relationships are well-known in QM

They tell us **two** rather interesting things...

Angular Momentum

Imagine two conserved quantities A and B

$[A, H] = [B, H] = 0$

How does $[A, B]$ change with time?

Jacobi's identity: $[[A, B], H] = -[[B, H], A] - [[H, A], B] = 0$

Have you learnt Jacobi's identity?

Poisson bracket of two conserved quantities is conserved

Now consider $\left[L_i, L_j\right] = \varepsilon_{ijk} L_k$

If 2 components of $\mathbf{L}$ are conserved, the 3rd component must →
Total vector $\mathbf{L}$ is conserved

Angular Momentum

Remember the Fundamental Poisson Brackets?

$$\left[q_i, q_j\right] = \left[p_i, p_j\right] = 0 \qquad \left[q_i, p_j\right] = -\left[p_i, q_j\right] = \delta_{ij}$$

PB of two canonical momenta is 0

Now we know $\left[L_i, L_j\right] = \varepsilon_{ijk} L_k$

Poisson brackets between L_x, L_y, L_z are non-zero

Only 1 of the 3 components of the angular momentum can be a canonical momentum

On the other hand, $\left[L^2, L_i\right] = 0$, so $|\mathbf{L}|$ may be a canonical momentum

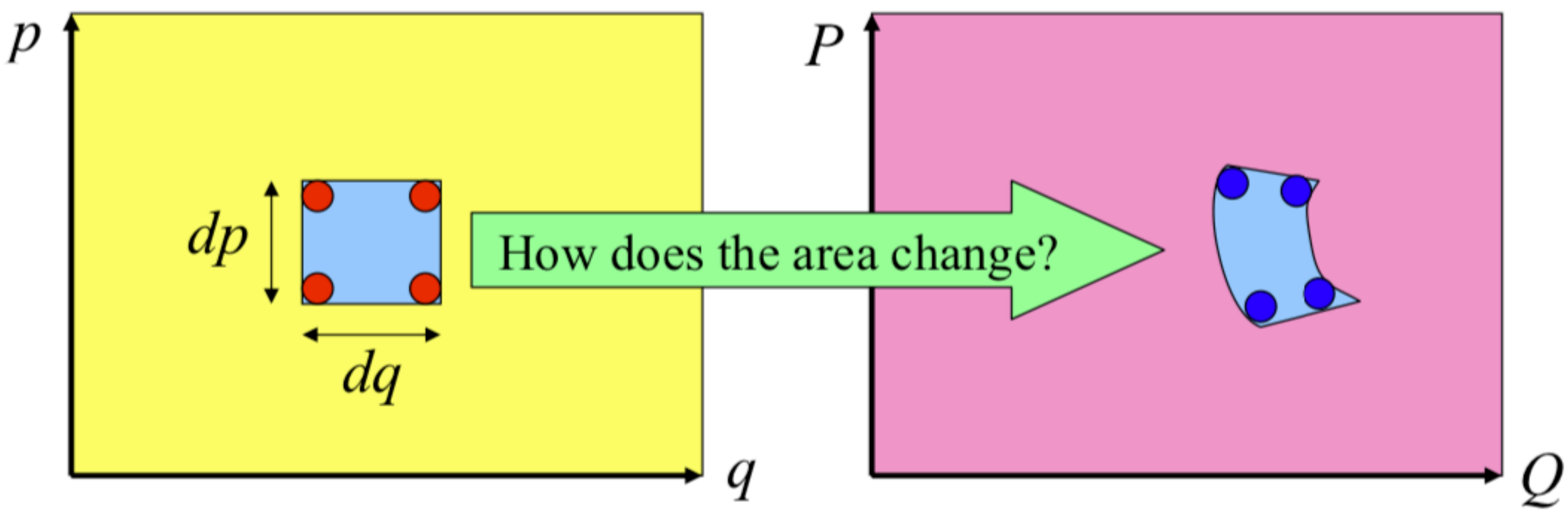
QM: You may measure $|\mathbf{L}|$ and, e.g., L_z simultaneously, but not L_x and L_y, etc.

Phase Volume

Static view: CT moves a point in one phase space to a point in another phase space

Dynamic view: CT moves a point in one phase space to another point in the same space

If you consider a set of points, CT moves a volume to another volume, e.g.



Phase Volume

Easy to calculate the Jacobian for 1-dimension

$$dQdP = |\mathbf{M}|\, dqdp \text{ where } \mathbf{M} = \begin{bmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{bmatrix}$$

$$|\mathbf{M}| = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = [Q, P] = 1 \rightarrow$$

$$dQdP = dqdp$$

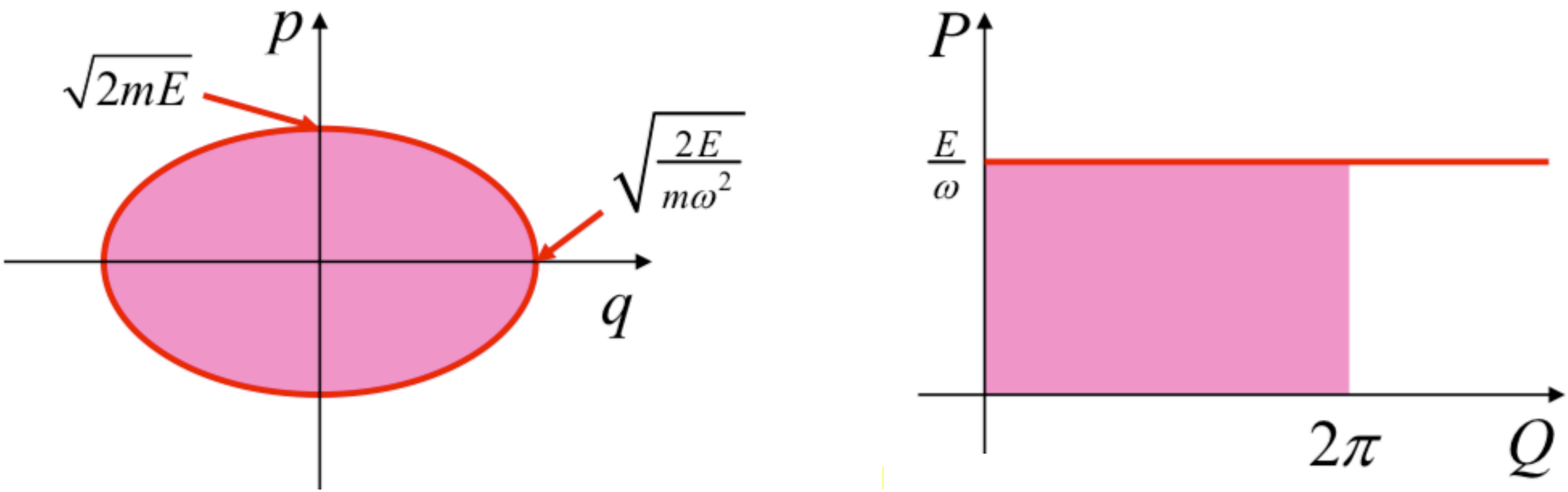
i.e., volume in 1-dim. phase space is invariant

This is true for n-dimensions

Volume in Phase Space is a Canonical Invariant

Harmonic Oscillator

We’ve seen it in the oscillator example



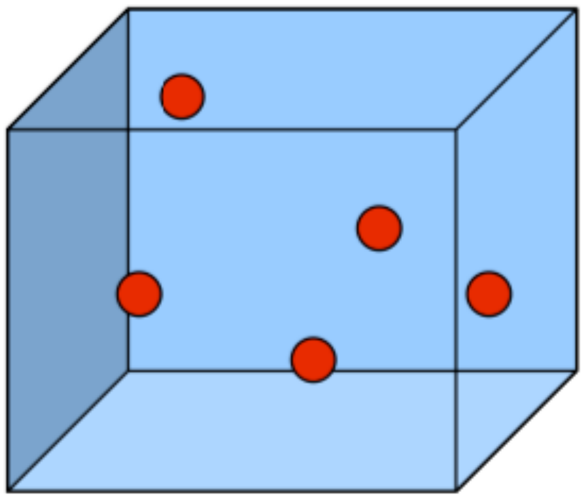
One cycle draws the same area $\dfrac{2\pi E}{\omega}$ in both spaces

That’s static view

Dynamic View

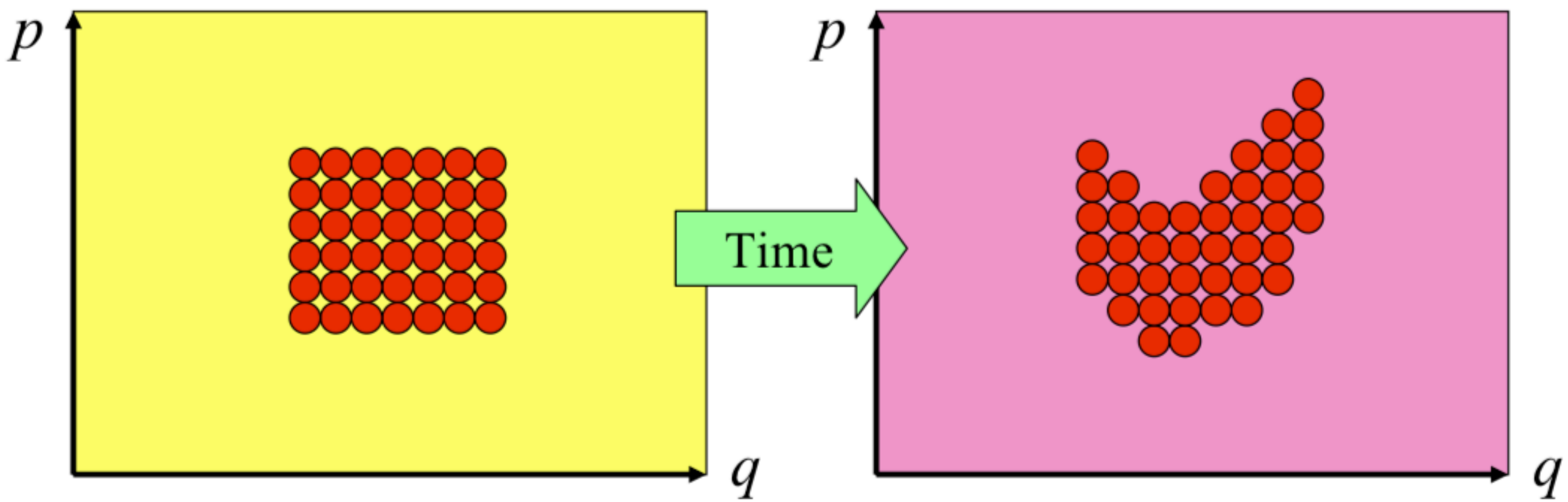
Consider many particles moving independently

e.g., ideal gas molecules in a box



They obey the same EoM independently

Can be represented by multiple points in one phase space

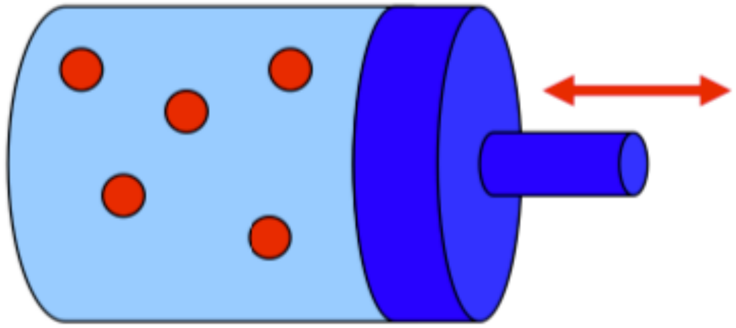


Ideal Gas Dynamics

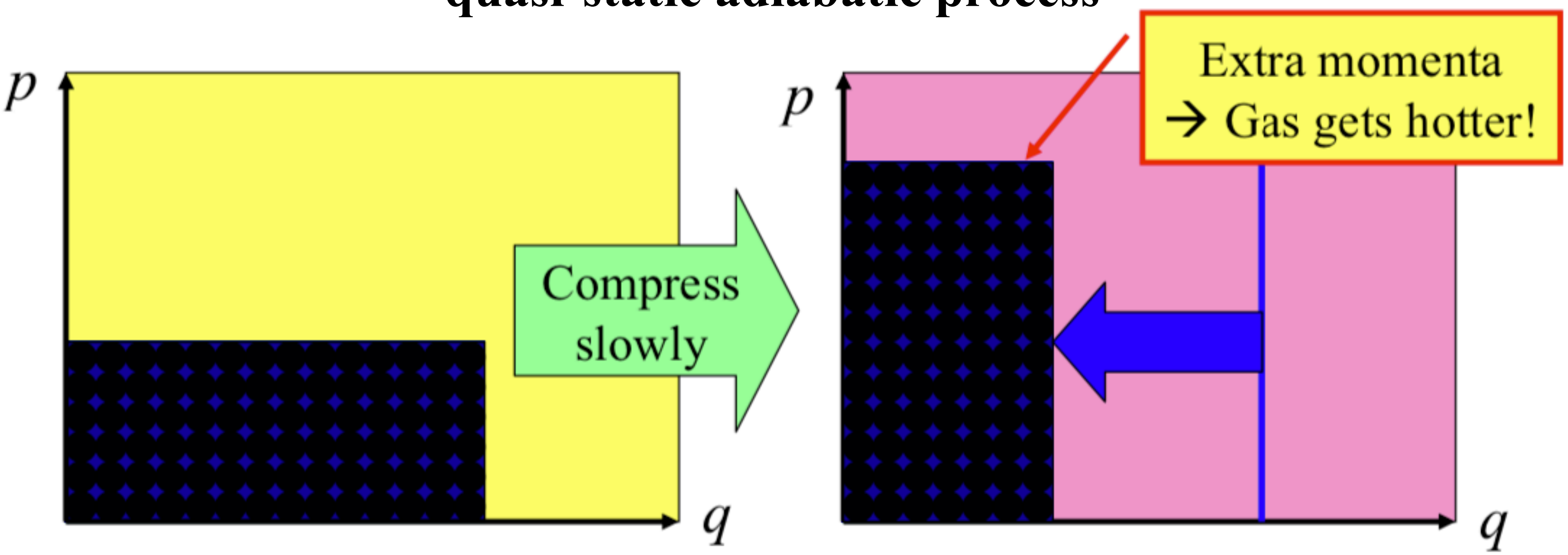
Imagine ideal gas in a cylinder with movable piston

Each molecule has its own position and momentum
→They fill up a certain volume in the phase space

What happens when we compress it?



quasi-static adiabatic process



Liouville’s Theorem

The phase volume occupied by a group of particles (ensemble in stat. mech.) is conserved

Thus the density in phase space remains constant with time

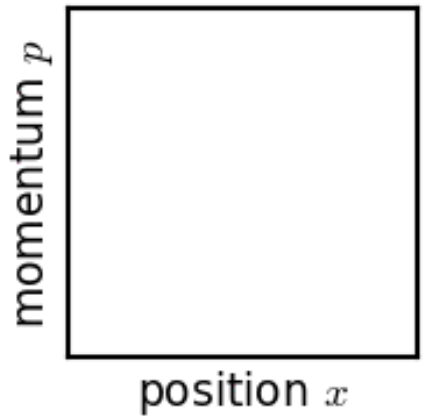
Known as Liouville’s theorem

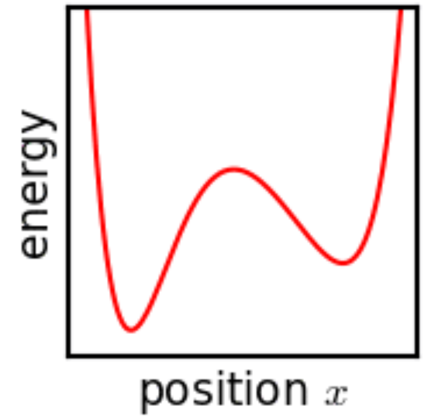
“A mathematician is one to whom that is as obvious as that twice two makes four is to you. Liouville was a mathematician.” -LORD KELVIN

Theoretical basis of equilibrium statistical mechanics

This holds true when there are large enough number of particles so that the distribution may be considered continuous

Joseph Liouville
1809-1882

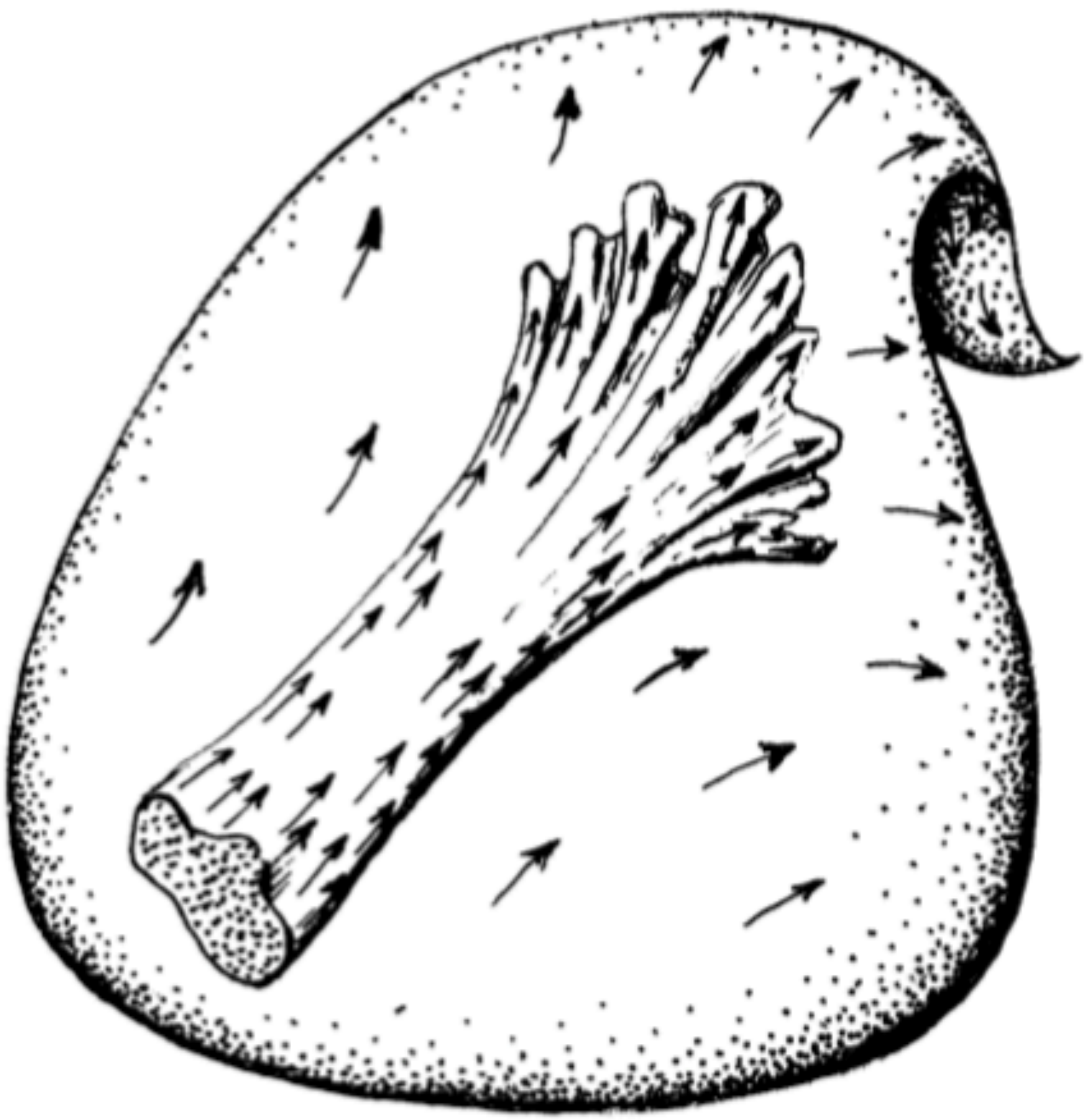




Liouville's Theorem

The Hamiltonian flow preserves the volume of the initial phase-space region (representing a range of possible initial states), even though the shape of this region may become grossly distorted in the time-evolution.

—Roger Penrose



Summary

Canonical transformations $P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$

Hamiltonian formalism is invariant under canonical + scale transformations

Generating functions define canonical transformations

Four basic types of generating functions

$F_1(q, Q, t) \quad F_2(q, P, t) \quad F_3(p, Q, t) \quad F_4(p, P, t)$

They are all practically equivalent

Used it to simplify a harmonic oscillator

Invariance of phase space area

Summary

Direct Conditions: Necessary and sufficient for Canonical Transformation

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \quad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,p} \quad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

Infinitesimal CT

Poisson Bracket $[u, v] \equiv \dfrac{\partial u}{\partial q_i}\dfrac{\partial v}{\partial p_i} - \dfrac{\partial u}{\partial p_i}\dfrac{\partial v}{\partial q_i}$

Canonical invariant

Fundamental PB $\left[q_i, q_j\right] = \left[p_i, p_j\right] = 0 \quad \left[q_i, p_j\right] = -\left[p_i, q_j\right] = \delta_{ij}$

ICT expressed by $\delta u = \varepsilon[u, G] + \dfrac{\partial u}{\partial t}\delta t$

Infinitesimal time transformation generated by Hamiltonian

Summary

Introduced dynamic view of Canonical Transformation

Hamiltonian is the generator of the motion with time

Symmetry of the system→Hamiltonian unaffected by the generatorGenerator is conserved

How to integrate infinitesimal transformations

$$u(\alpha) = u_0 + \alpha[u, G]_0 + \frac{\alpha^2}{2!}[[u, G], G]_0 + \frac{\alpha^3}{3!}[[u, G], G], G\Big]_0 + \cdots$$

Discussed infinitesimal rotation $[\mathbf{r}, \mathbf{L} \cdot \mathbf{n}] = \mathbf{n} \times \mathbf{r}$

Angular momentum→QM

Invariance of the phase volume

Liouville's theorem→Statistical Mechanics

Change Your Mind

Image you are falling in vacuum if you can still survive.

What will you feel?

Nothing!

How do you know your status?

What can you measure?