理论力学第 13 次作业

4.9

任何旋转操作可以等效为两次镜像操作。不妨设旋转是绕 z 轴旋转。
第一次以 xz 平面为镜像

$$\boldsymbol{L}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

第一次以 xz 平面旋转$\phi/2$ 为后的平面为镜像

$$\boldsymbol{L}_2 = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & -\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

两次镜像操作合成为

$$\boldsymbol{L}_2\boldsymbol{L}_2 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

恰好对应于一次旋转操作。

4.10

(a)

$$e^{\boldsymbol{B}+\boldsymbol{C}} = \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{B}+\boldsymbol{C})^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \binom{n}{k} \boldsymbol{B}^n \boldsymbol{C}^{n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\boldsymbol{B}^n \boldsymbol{C}^{n-k}}{k!(n-k)!}$$

如果 $\boldsymbol{B}$,$\boldsymbol{C}$ 是可交换的，则

$$e^{\boldsymbol{B}+\boldsymbol{C}} = \sum_{i=0}^{\infty} \frac{\boldsymbol{B}^i}{i!} \sum_{j=0}^{\infty} \frac{\boldsymbol{C}^j}{j!} = e^{\boldsymbol{B}} e^{\boldsymbol{C}}$$

(b)

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{1} = e^{\boldsymbol{B}-\boldsymbol{B}} = e^{\boldsymbol{B}} e^{-\boldsymbol{B}}$$

所以

$$\boldsymbol{A}^{-1} = e^{-\boldsymbol{B}}$$

(c)

$$e^{\boldsymbol{CBC}^{-1}}=\sum_{k=0}^{\infty}\frac{1}{k!}(\boldsymbol{CBC}^{-1})^k=\sum_{k=0}^{\infty}\frac{1}{k!}\boldsymbol{CB}^k\boldsymbol{C}^{-1}=\boldsymbol{C}\sum_{k=0}^{\infty}\frac{1}{k!}\boldsymbol{B}^k\boldsymbol{C}^{-1}=\ \boldsymbol{C}e^{\boldsymbol{B}}\boldsymbol{C}^{-1}=\boldsymbol{CAC}^{-1}$$

(d)

如果

$$\widetilde{\boldsymbol{B}}=-\boldsymbol{B}$$

则

$$\widetilde{\boldsymbol{A}}=e^{\widetilde{\boldsymbol{B}}}=e^{-\boldsymbol{B}}=\boldsymbol{A}^{-1}$$

所以 $\boldsymbol{A}$ 是正交矩阵。

4.14

(a)

$$\begin{aligned}\sum_p \epsilon_{ijp}\epsilon_{rmp}&=\begin{vmatrix}\delta_{1i}&\delta_{2i}&\delta_{3i}\\ \delta_{1j}&\delta_{2j}&\delta_{3j}\\ \delta_{1p}&\delta_{2p}&\delta_{3p}\end{vmatrix}\begin{vmatrix}\delta_{1r}&\delta_{2r}&\delta_{3r}\\ \delta_{1m}&\delta_{2m}&\delta_{3m}\\ \delta_{1p}&\delta_{2p}&\delta_{3p}\end{vmatrix}\\ &=\begin{vmatrix}\delta_{ir}&\delta_{im}&\delta_{ip}\\ \delta_{jr}&\delta_{jm}&\delta_{jp}\\ \delta_{pr}&\delta_{pm}&\delta_{pp}\end{vmatrix}\\ &=\delta_{ir}\delta_{jm}-\delta_{im}\delta_{jr}\end{aligned}$$

(b)

$$\sum_{i,j}\epsilon_{ijp}\epsilon_{ijk}=\sum_j\sum_i\epsilon_{ijp}\epsilon_{ijk}=\sum_j\left(\delta_{jj}\delta_{pk}-\delta_{jk}\delta_{pj}\right)=3\delta_{pk}-\delta_{pk}=2\delta_{pk}$$

4.15

$$\boldsymbol{\omega}=\dot{\phi}\hat{n}_\phi+\dot{\theta}\hat{n}_\theta+\dot{\psi}\hat{n}_\psi$$

其中

$$\hat{n}_\phi=\hat{z}$$

$$\hat{n}_\phi=\widetilde{\boldsymbol{D}}\hat{x}=\hat{x}\cos\phi+\hat{y}\sin\phi$$

$$\hat{n}_\psi=\widetilde{\boldsymbol{D}}\widetilde{\boldsymbol{C}}\hat{z}=\hat{x}\sin\theta\sin\phi-\hat{y}\sin\theta\cos\phi+\hat{z}\cos\theta$$

所以

$$\boldsymbol{\omega}=\left(\dot{\theta}\cos\phi+\dot{\psi}\sin\theta\sin\phi\right)\hat{x}+\left(\dot{\theta}\sin\phi-\dot{\psi}\sin\theta\cos\phi\right)\hat{y}+\left(\dot{\psi}\cos\theta+\dot{\phi}\right)\hat{z}$$

即

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$

4.17

$i = j$ 时

$$[\boldsymbol{M}_i, \boldsymbol{M}_j] = 0$$

$i \neq j$ 时

$$[\boldsymbol{M}_1, \boldsymbol{M}_2] = \boldsymbol{M}_1\boldsymbol{M}_2 - \boldsymbol{M}_2\boldsymbol{M}_1 = \boldsymbol{M}_3$$
$$[\boldsymbol{M}_2, \boldsymbol{M}_1] = \boldsymbol{M}_2\boldsymbol{M}_1 - \boldsymbol{M}_1\boldsymbol{M}_2 = -\boldsymbol{M}_3$$

$$[\boldsymbol{M}_2, \boldsymbol{M}_3] = \boldsymbol{M}_2\boldsymbol{M}_3 - \boldsymbol{M}_3\boldsymbol{M}_2 = \boldsymbol{M}_1$$
$$[\boldsymbol{M}_3, \boldsymbol{M}_2] = \boldsymbol{M}_3\boldsymbol{M}_2 - \boldsymbol{M}_2\boldsymbol{M}_3 = -\boldsymbol{M}_1$$

$$[\boldsymbol{M}_3, \boldsymbol{M}_1] = \boldsymbol{M}_3\boldsymbol{M}_1 - \boldsymbol{M}_1\boldsymbol{M}_3 = \boldsymbol{M}_2$$
$$[\boldsymbol{M}_1, \boldsymbol{M}_3] = \boldsymbol{M}_1\boldsymbol{M}_3 - \boldsymbol{M}_3\boldsymbol{M}_1 = -\boldsymbol{M}_2$$

所以

$$[\boldsymbol{M}_i, \boldsymbol{M}_j] = \epsilon_{ijk}\boldsymbol{M}_k$$