理论力学第 17 次作业

8.7

$$H=\frac{1}{2m}p^2+\frac{1}{2}k(x-v_0t)^2$$

$$H^{'}=\frac{1}{2m}\left(p^{'}-mv_0\right)^2+\frac{1}{2}kx^{'2}-\frac{1}{2}mv_0^2$$

其中

$$x^{'}=x-v_0t$$

计算得

$$\dot{x}=\frac{\partial H}{\partial p}=\frac{p}{m}$$

$$\dot{p}=-\frac{\partial H}{\partial x}=-k(x-v_0t)$$

$$\dot{x^{'}}=\frac{\partial H^{'}}{\partial p^{'}}=\frac{1}{m}\left(p^{'}-mv_0\right)$$

$$\dot{p^{'}}=-\frac{\partial H^{'}}{\partial x^{'}}=-kx^{'}$$

$\dot{x}$与$\dot{x^{'}}$对时间求导可得

$$m\ddot{x}=-k(x-v_0t)$$
$$m\ddot{x^{'}}=-kx^{'}$$

可见，确实得到了相同的结果。

8.14

$$L=a\dot{x}^2+b\frac{\dot{y}}{x}+c\dot{x}\dot{y}+fy^2\dot{x}\dot{z}+g\dot{y}^2-k\sqrt{x^2+y^2}$$

拉格朗日量可以泰勒展开为

$$L=L_0+\widetilde{\dot{\boldsymbol{q}}}\begin{bmatrix}0\\ \frac{b}{x}\\ 0\end{bmatrix}+\frac{1}{2}\widetilde{\dot{\boldsymbol{q}}}\begin{bmatrix}2a & c & fy^2\\ c & 2g & 0\\ fy^2 & 0 & 0\end{bmatrix}\dot{\boldsymbol{q}}$$

其中

$$\tilde{\boldsymbol{a}} = \begin{bmatrix} 0 & \frac{b}{x} & 0 \end{bmatrix},\ \boldsymbol{T} = \begin{bmatrix} 2a & c & fy^2 \\ c & 2g & 0 \\ fy^2 & 0 & 0 \end{bmatrix}$$

哈密顿量为

$$H = \frac{1}{2}(\tilde{\boldsymbol{p}} - \tilde{\boldsymbol{a}})\boldsymbol{T}^{-1}(\boldsymbol{p} - \boldsymbol{a}) - L_0$$

其中

$$\boldsymbol{T}^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{fy^2} \\ 0 & \frac{1}{2g} & -\frac{c}{2fy^2 g} \\ \frac{1}{fy^2} & -\frac{c}{2fy^2 g} & \frac{c^2 - 4ga}{2f^2 gy^4} \end{bmatrix}$$

所以

$$H = \frac{1}{fy^2}p_x p_z + \frac{1}{4g}\left(p_y - \frac{b}{x}\right)^2 - \frac{c}{2fgy^2}\left(p_y - \frac{b}{x}\right) + \frac{c^2 - 4ga}{4f^2 gy^4}p_z^2 + k\sqrt{x^2 + y^2}$$

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial z} = 0$$

所以 H 是常量，p_z是常量。

8.27

(a)

$$L = \frac{1}{2}m(\dot{q}^2 \sin^2 \omega t + q\dot{q}\omega \sin 2\omega t + \omega^2 q^2)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\sin^2 \omega t + \frac{1}{2}m\omega q \sin 2\omega t$$

$$H = p\dot{q} - L = \frac{p^2}{2m} - \omega \cot \omega t\, qp + m\omega^2 \cos^2 \omega t\, q^2 - \frac{1}{2}m\omega^2 q^2$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0$$

(b)

$$Q = q\sin\omega t$$

$$\dot{q} = \frac{d}{dt}\frac{Q}{\sin\omega t} = \frac{\dot{Q}\sin\omega t - Q\omega\cos\omega t}{\sin^2\omega t}$$

$$L = \frac{1}{2}m\left(\frac{\left(\dot{Q}\sin\omega t - Q\omega\cos\omega t\right)^2}{\sin^2\omega t} + \frac{Q\omega\sin 2\omega t}{\sin\omega t}\frac{\dot{Q}\sin\omega t - Q\omega\cos\omega t}{\sin^2\omega t} + \frac{\omega^2 Q^2}{\sin^2\omega t}\right)$$

$$= \frac{1}{2}m\left(\dot{Q}^2 + \omega^2 Q^2\right)$$

$$\widetilde{H} = \frac{P^2}{2m} - \frac{1}{2}m\omega^2 Q^2$$

其中

$$P = m\dot{Q}$$