理论力学第 14 次作业

5.6

(a)

转动惯量为

$$I_1 = I_2 \neq I_3$$

则，欧拉方程为

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_1 - I_3) \\ I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

令$\Omega = \omega_3\left(\frac{I_3}{I_1} - 1\right)$，则有

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0,\ \ddot{\omega}_2 + \Omega^2\omega_2 = 0$$

能量守恒$\omega_1^2 + \omega_1^2 = const.$

$$\begin{aligned} &\boldsymbol{\omega} = \dot{\phi}\sin\theta\sin\psi\,\hat{\boldsymbol{x}}' + \dot{\phi}\sin\theta\cos\psi\,\hat{\boldsymbol{y}}' + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\hat{\boldsymbol{z}}' \\ &\boldsymbol{\omega} = \omega_0\cos\Omega t\,\hat{\boldsymbol{x}}' + \omega_0\sin\Omega t\,\hat{\boldsymbol{y}}' + \omega_3\hat{\boldsymbol{z}}' \\ &\boldsymbol{L} = \boldsymbol{I\omega} = \omega_0 I_1\cos\Omega t\,\hat{\boldsymbol{x}}' + \omega_0 I_1\sin\Omega t\,\hat{\boldsymbol{y}}' + \omega_3 I_3\hat{\boldsymbol{z}}' \\ &\psi = \frac{\pi}{2} - \Omega t \end{aligned}$$

$$\therefore \dot{\phi} = \frac{\omega_3 - \dot{\psi}}{\cos\theta} = \frac{\omega_3 + \Omega}{\cos\theta} = \frac{\omega_3 + \Omega}{\cos\theta} = \frac{I_3\omega_3}{I_1\cos\theta}$$

(b)

$$\begin{aligned} \omega_x &= \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi = -\Omega\sin\theta\sin\phi \\ \omega_y &= -\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi = \Omega\sin\theta\cos\phi \\ \omega_z &= \dot{\psi}\cos\phi + \dot{\phi} = -\Omega\cos\theta + \dot{\phi} = const. \end{aligned}$$

$$\sin\theta' = \frac{\sqrt{\omega_x^2 + \omega_y^2}}{|\boldsymbol{\omega}|} = \frac{\Omega\sin\theta}{|\boldsymbol{\omega}|}$$

$$\sin\theta'' = \frac{\omega_0}{|\boldsymbol{\omega}|} = \frac{\sqrt{\omega_{x'}^2 + \omega_{y'}^2}}{|\boldsymbol{\omega}|} = \frac{\dot{\phi}\sin\theta}{|\boldsymbol{\omega}|}$$

$$\therefore \sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

$$\therefore d = R\sin\theta' = R\cos\theta\left(1 - \frac{I_1}{I_3}\right)\sin\theta'' \approx 1.5\ cm$$

(c)

假设你把第一个圆锥的尖端放在原点，对称轴沿着 z 轴，把第二个圆锥放在z'轴上，让两个尖端在原点重合。由(a)和(b)可得：当第二个锥体围绕第一个旋转时，注意到角度θ在 z 和z'轴之间是恒定的，与对称自由顶部相同。角速度轴$\boldsymbol{\omega}$以角速度$\dot{\phi}$绕 z 旋转，对称轴z'以相同的角速度$\dot{\phi}$绕 z 轴心旋转。z 和z'之间的角θ是常数，同样的角θ'是常数。因此，我们看到确实有两个视锥的类比。

由 Poisont 构造可知，惯性椭球是绕z'轴对称的，因此在不变平面上的曲线(荷极面)和惯性椭球(荷极面)都是圆。对于体坐标系中的观测者，矢量$\boldsymbol{\rho}$在惯性椭球上跟踪一个圆锥，同样对于空间坐标系中的观测者，$\boldsymbol{\rho}$在不变平面上跟踪一个圆锥。因此，很容易看到有两个锥相互滚动。

5.14

设圆柱体高为 h，半径为 r，直径 $d = 2r$。

$$I_x = I_y = \iiint \frac{m}{\pi r^2 h}(y^2 + z^2)dxdydz$$

$$= \frac{m}{\pi r^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^{2\pi} d\phi \int_0^r (s^2\sin^2\phi + z^2)sds = m\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$$

$$I_z = \frac{m}{\pi r^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^{2\pi} d\phi \int_0^r s^2 \cdot sds = \frac{1}{2}mr^2$$

由题意得

$$m\left(\frac{h^2}{12} + \frac{r^2}{4}\right) = \frac{1}{2}mr^2$$

所以

$$\frac{d}{h} = \frac{2r}{h} = \frac{2}{\sqrt{3}}$$

5.16

$$2 \times \frac{70}{40+50+60} = \frac{14}{15}$$

放在$\left(-\frac{14}{15}, -\frac{14}{15}, -\frac{14}{15}\right)$处

5.27

$$\overline{V}_2(\cos\theta) = -\frac{1}{2}I_1\beta\cos^2\theta\text{，}\ \beta = \frac{3GM(I_3 - I_1)}{2I_1 r^3}$$

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi}^2 + \dot{\phi}^2\cos\theta\right)^2 + \frac{1}{2}I_1\beta\cos^2\theta$$

$$p_\psi = I_3\left(\dot{\psi}^2 + \dot{\phi}^2\cos\theta\right) = I_3\omega_3 = I_1 a$$
$$p_\phi = (I_1\sin^2\theta + I_3\cos^2\theta) + I_3\cos\theta\,\dot{\psi} = I_1 b$$
$$E = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi}^2 + \dot{\phi}^2\cos\theta\right)^2 - \frac{1}{2}I_1\beta\cos^2\theta$$

令$\alpha = \left(2E - I_3\omega_3^2\right)/I_1$

则

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$
$$\dot{\psi} = \frac{I_1}{I_3}a - \frac{b - a\cos\theta}{\sin^2\theta}\cos\theta$$
$$\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} - \beta\cos^2\theta$$

$$\alpha = \dot{\phi}_0^2\sin^2\theta_0 - \beta\cos^2\theta_0$$
$$b = a\cos\theta_0 + \dot{\phi}_0\sin^2\theta_0$$

令 $u = \cos\theta$

$$\alpha = \dot{u}^2 + \alpha u^2 + (b - au)^2 - \beta u^2(1 - u^2)$$

$$\omega_3 \gg \frac{\beta}{2a}$$

$$V(u) \equiv \alpha u^2 + (b - au)^2 - \beta u^2(1 - u^2)$$

$$u^2(u^2 - 1) \approx u_0^2\left(1 - 3u_0^2\right) + 2u_0\left(2u_0^2 - 1\right)u$$

令$u_0=\cos\theta_0$

$$V(u)\approx\alpha u^2+(b-au)^2+\beta 2u_0\left(2u_0^2-1\right)u+const.$$

$$\because\frac{\partial V}{\partial u}=0,\quad\therefore\overline{u}=\frac{ab-\beta u_0\left(2u_0^2-1\right)}{a^2+\alpha}$$

$$V(u)=(a^2+\alpha)(u-\overline{u})^2+const.$$

$$\because\ddot{u}+a^2\left(1+\frac{\alpha}{a^2}\right)(u-\overline{u})=0,\quad\therefore u(t)=\cos\theta=\overline{u}+A\cos\gamma t$$

其中

$$\gamma=a\sqrt{1+\frac{\dot{\phi}_0^2\sin^2\theta_0}{a^2}+\frac{\beta\cos^2\theta_0}{a^2}}$$

$$A=\cos\theta_0-\frac{\cos\theta_0+\frac{\dot{\phi}_0}{a}\sin^2\theta_0-\frac{\beta}{a^2}\cos\theta_0\left(2\cos^2\theta_0-1\right)}{1+\frac{\dot{\phi}_0^2}{a^2}\sin^2\theta_0-\frac{\beta}{a^2}\cos^2\theta_0}$$

$$=\frac{\frac{\dot{\phi}_0}{a}\left(\frac{\dot{\phi}_0}{a}\cos\theta_0-1\right)-\frac{\beta}{a^2}\cos\theta_0}{1+\frac{\dot{\phi}_0^2}{a^2}\sin^2\theta_0-\frac{\beta}{a^2}\cos^2\theta_0}\sin^2\theta_0$$

$$A=-\frac{\beta}{a^2}\frac{\sin^2\theta_0\cos\theta_0}{\sec^2\theta_0-\frac{\beta}{a^2}\cos^2\theta_0}=-\frac{\beta}{a^2}\sin^2\theta_0\cos^3\theta_0+O\left(\frac{\beta^2}{a^4}\right)$$

$$\cos\theta=\cos\theta_0+\frac{\beta}{a^2}\sin^2\theta_0\cos^3\theta_0\left(1-\cos\gamma t\right)+O\left(\frac{\beta^2}{a^4}\right)$$

$$\theta=\theta_0-\frac{\beta}{a^2}\sin\theta_0\cos^3\theta_0\left(1-\cos\gamma t\right)+O\left(\frac{\beta^2}{a^4}\right)$$

$$\gamma=\frac{a}{\cos\theta_0}-\frac{\beta}{2a}\cos^3\theta_0+O\left(\frac{\beta^2}{a^2}\right)$$

$$x_1=2|A|=\frac{2\beta}{a^2}\sin^2\theta_0\cos^3\theta_0$$

$$\dot{\phi}=\frac{a}{\cos\theta_0}+\frac{\beta}{a}\cos^3\theta_0\left(1-\cos\gamma t\right)+O\left(\frac{\beta^2}{a^3}\right)$$

$$\dot{\psi}=\omega_3-\dot{\phi}\cos\theta=-\omega_3\left(\frac{I_3}{I_1}-1\right)-\frac{\beta}{a}\cos^2\theta_0\left(1-\cos\gamma t\right)+O\left(\frac{\beta^2}{a^3}\right)$$

$$\psi = \psi_0 - \left(\Omega + \frac{\beta \cos^2 \theta_0}{a}\right) t + \frac{\beta}{a^2} \cos^3 \theta_0 \sin \gamma t + O\left(\frac{\beta^2}{a^4}\right)$$

$$\omega_1 = \dot{\phi} \sin \theta \sin \phi + \dot{\theta} \cos \psi$$
$$\omega_2 = \dot{\phi} \sin \theta \cos \phi - \dot{\theta} \sin \psi$$

$$\omega_1 = -\frac{I_3 \omega_3}{I_1 \cos \theta_0} \sin \theta_0 \sin \left[\left(\Omega + \frac{\beta \cos^2 \theta_0}{a}\right) t\right] + O\left(\frac{\beta}{a}\right)$$
$$\omega_2 = \frac{I_3 \omega_3}{I_1 \cos \theta_0} \sin \theta_0 \cos \left[\left(\Omega + \frac{\beta \cos^2 \theta_0}{a}\right) t\right] + O\left(\frac{\beta}{a}\right)$$