Theoretical Mechanics
理论力学

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Syllabus

- Chapter 0 Preface
- Chapter 1 Survey of the Elementary Principles
- Chapter 2 Variational Principle and Lagrange's Equations
- Chapter 3 The Central Force Problem
- Chapter 4 The Kinematics of Rigid Body Motion

Mid-term exam

- Chapter 5 The Rigid Body Equations of Motion
- Chapter 6 Oscillations
- Chapter 7 The Hamilton Equations of Motion
- Chapter 8 Canonical Transformations

Final term exam

Angular Momentum

Consider a multi-particle system $\mathbf{r}_i = \mathbf{r}_i\left(q_1, \ldots, q_n, t\right)$

Suppose q_j turns the whole system around

Assume V does not depend on $\dot{q}_j$

Conjugate momentum is

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

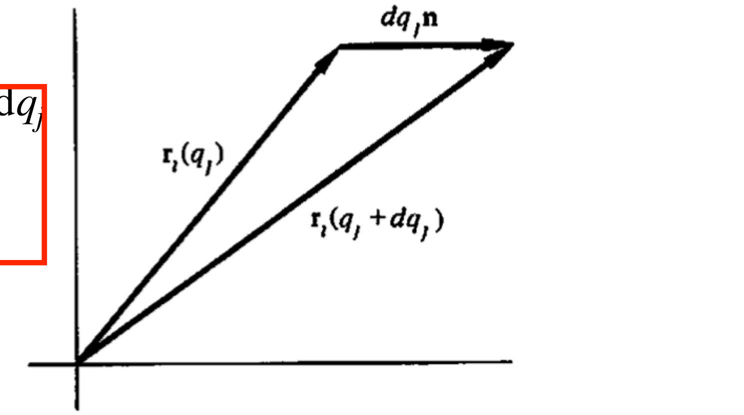
$$= \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

$$= \sum_i m_i \mathbf{v}_i \cdot \left(\mathbf{n} \times \mathbf{r}_i\right) = \sum_i \mathbf{n} \cdot \left(\mathbf{r}_i \times m_i \mathbf{v}_i\right)$$

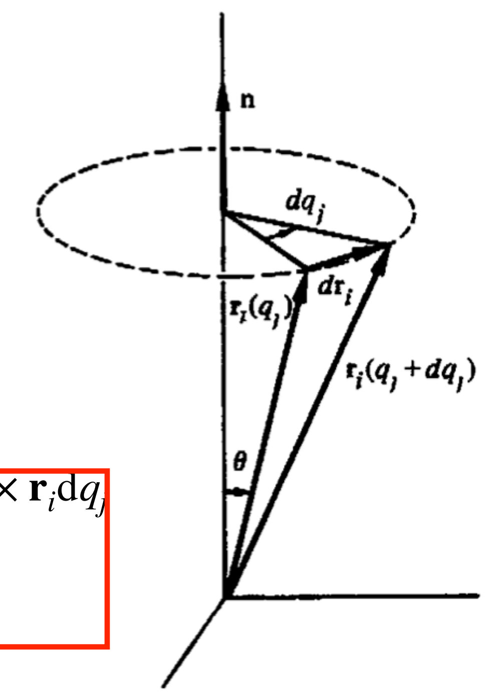
$$= \mathbf{n} \cdot \sum_i \mathbf{L}_i = \mathbf{n} \cdot \mathbf{L}$$

Axis of rotation **Total angular momentum**

$$\boxed{\begin{aligned} &\mathbf{r}_i(q_j + \mathrm{d}q_j) = \mathbf{r}(q_j) + \mathbf{n}\,\mathrm{d}q_j \\ &\Longrightarrow \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n} \end{aligned}}$$



Change in a position vector under translation of the system.



$$\boxed{\begin{aligned} &\mathbf{r}_i(q_j + \mathrm{d}q_j) = \mathbf{r}_i(q_j) + \mathbf{n} \times \mathbf{r}_i\,\mathrm{d}q_j \\ &\Longrightarrow \frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n} \times \mathbf{r}_i \end{aligned}}$$

Change of a position vector under rotation of the system.

Energy Conservation

Energy function $h(q, \dot{q}, t) \equiv \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$ equals to the total energy if

Constraints are time-independent → Kinetic energy T is a 2nd order homogeneous function of the velocities

Potential V is velocity-independent

Energy function is conserved if

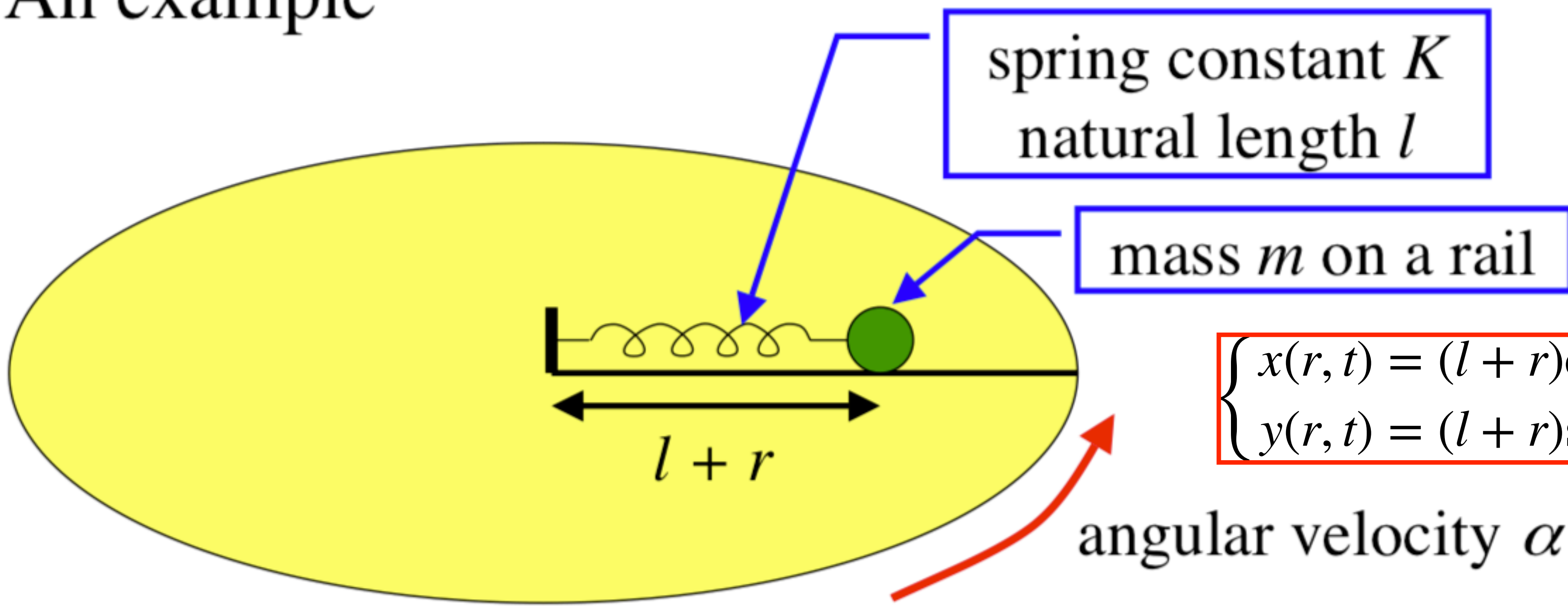
Lagrangian does not depend explicitly on time

Lagrangian is invariant under time translation $t \rightarrow t + \Delta t$

These are restatements of the energy conservation theorem in a more general framework because conditions are clearly defined.

Example: Time-dependent system

- An example



$$\begin{cases} x(r,t) = (l+r)\cos\alpha t \\ y(r,t) = (l+r)\sin\alpha t \end{cases}$$

The Lagrangian: $L = T - V = \dfrac{m}{2}\left\{\dot{r}^2 + (l+r)^2\alpha^2\right\} - \dfrac{K}{2}r^2$

The total energy is not conserved: $E = T + V = \dfrac{m}{2}\left\{\dot{r}^2 + (l+r)^2\alpha^2\right\} + \dfrac{K}{2}r^2$

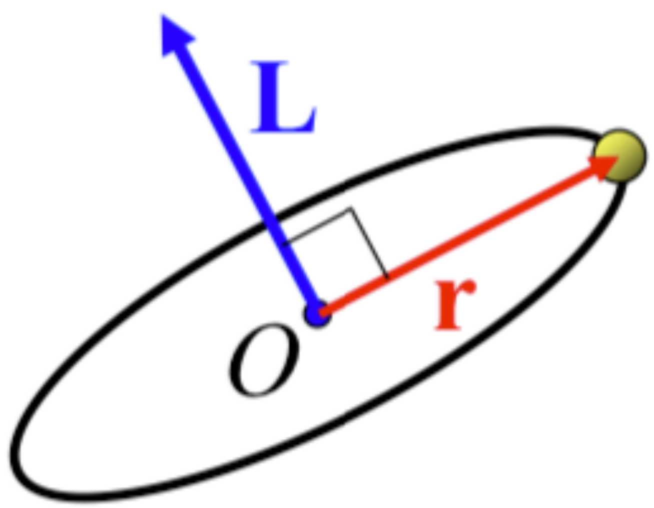
The energy function is conserved: $h = \dot{r}\dfrac{\partial L}{\partial \dot{r}} - L = \dfrac{m}{2}\left\{\dot{r}^2 - (l+r)^2\alpha^2\right\} + \dfrac{K}{2}r^2$

Spherical Symmetry

The 3rd para. of Sec. 3.2 has the "Landau" style. Let's explain it!

Lagrangian in spherical coordinates

$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2 \sin^2\psi\dot{\theta}^2 + r^2\dot{\psi}^2\right) - V(r)$$



θ is cyclic, but ψ is not

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = mr^2\left(\ddot{\psi} - \sin\psi\cos\psi\dot{\theta}^2\right) = 0$$

We choose the polar axis so that the initial condition is

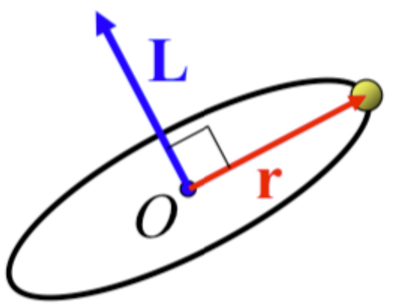
$$\psi(0) = \pi/2 \text{ and } \dot{\psi}(0) = 0 \rightarrow \ddot{\psi} = 0 \text{ and then } \dot{\psi} = 0$$

Now ψ is constant so that we can forget it.

Two constants of L furnishes 2 constants of motion then reduce the problem from 3 to 2 DOF.

Angular Momentum

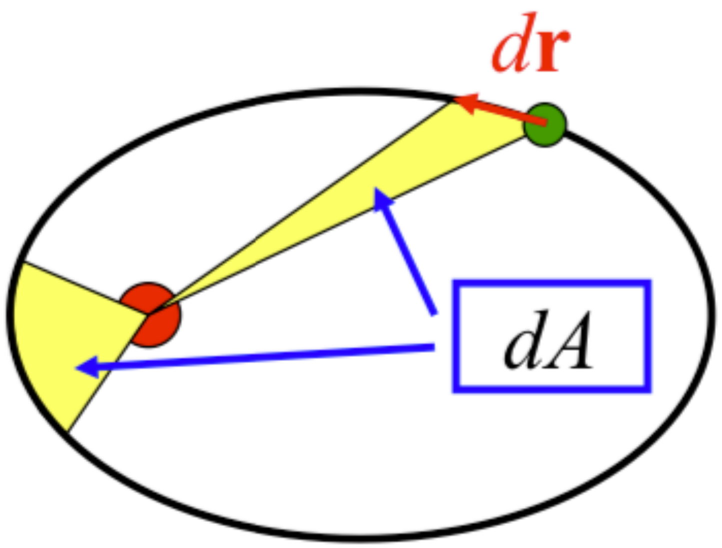
$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$



θ is cyclic. Conjugate momentum p_θ conserves.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{const} \equiv l$$

Areal velocity: $\dfrac{dA}{dt} = \dfrac{1}{2}r^2\dot{\theta} = \text{const}$



Kepler's 2nd law

True for any central force

Radial Motion

$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

Lagrange's equation for r: $\dfrac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \dfrac{\partial V(r)}{\partial r} = 0$

Derivative of V is the force: $f(r) = -\dfrac{\partial V(r)}{\partial r}$

EoM: $$m\ddot{r} = mr\dot{\theta}^2 + f(r)$$

Using the angular momentum $l = mr^2\dot{\theta}$

EoM: $$m\ddot{r} = \frac{l^2}{mr^3} + f(r)$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2}\frac{l^2}{mr^2} + V(r)\right)$$

If an ODE does not contain the dependent variable y explicitly, but only its derivatives, then the change of variable $p = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ leads to an equation of one order lower.

$$\ddot{r} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{r}\frac{\mathrm{d}\dot{r}}{\mathrm{d}r}$$

Energy Conservation

$$E = T + V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = \frac{m}{2}\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V(r) = \text{ const}$$

$$\rightarrow \dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}$$

One can solve this (in principle) by

$$t = \int_0^t dt = \int_{r_0}^r \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}} = t(r)$$

Then invert $t(r) \rightarrow r(t)$

Then calculate $\theta(t)$ by integrating $\dot{\theta}(t) = \dfrac{l}{m[r(t)]^2}$

Degrees of Freedom

A particle has 3 degrees of freedom

EoM is 2nd order differential → 6 constants of integration

Each conservation law reduces one constant of integration

We used $\mathbf{L}$ and E → 4 conserved quantities

Left with 2 constants of integration: r_0 and θ_0

only 1 degree of freedom

Of course, we don't have to use conservation laws

However, it's just easier than solving all of Lagrange's equations

3.3 The Equivalent One-dimensional Problem, and Classification of Orbits

Integrating the radial motion isn't always easy, in fact, often impossible

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}$$

We can still get general behavior by looking at

Equivalent potential: $V'(r) \equiv V(r) + \dfrac{l^2}{2mr^2}$

Energy E is conserved and $E - V'$ must be positive

$$E = \frac{m\dot{r}^2}{2} + V'(r) \implies \frac{m\dot{r}^2}{2} = E - V'(r) > 0 \implies E > V'(r)$$

Plot $V'(r)$ and see how it intersects with E

Inverse-Square Force

Consider an attractive $1/r^2$ force

$$f(r) = -\frac{k}{r^2} \Longrightarrow V(r) = -\frac{k}{r}$$

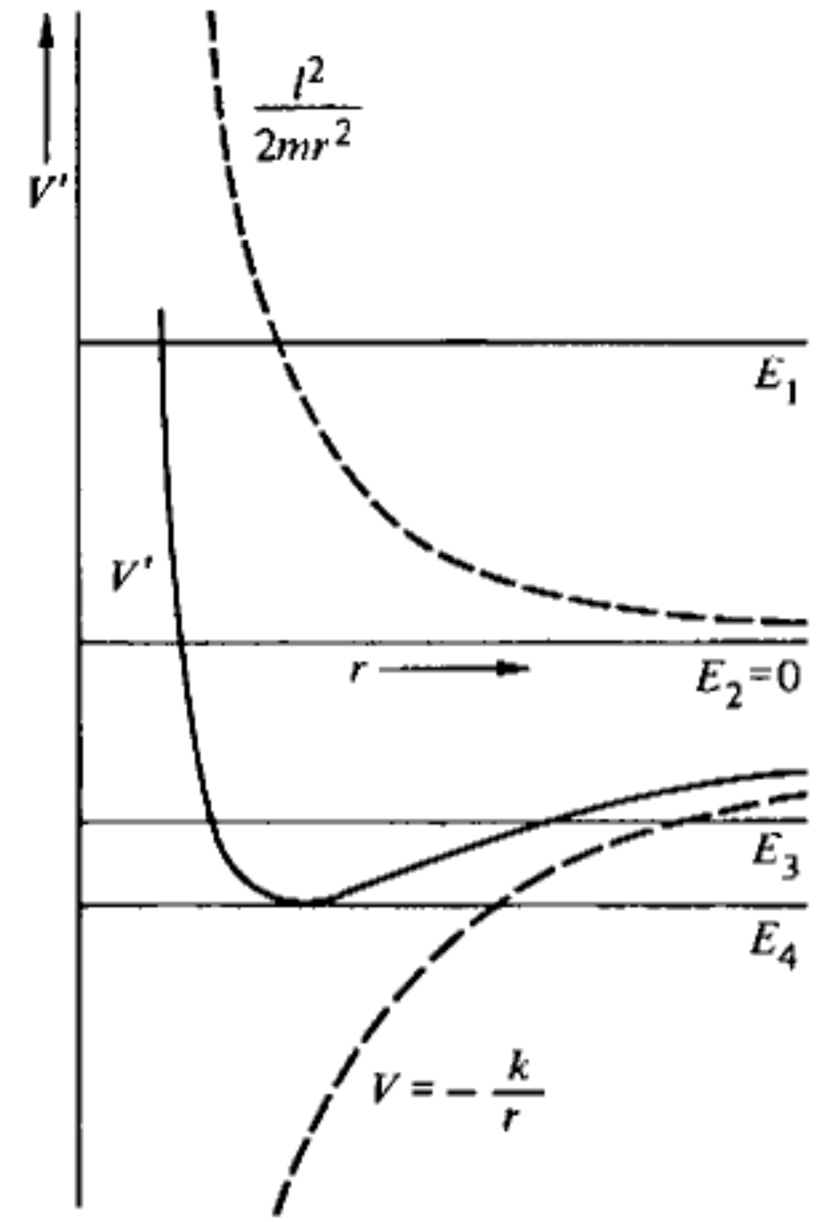
Gravity or electrostatic force

$$V'(r) = -\frac{k}{r} + \frac{l^2}{2mr^2}$$

$1/r^2$ force dominates at large r

Centrifugal force dominates at small r

A dip forms in the middle



Unbounded Motion

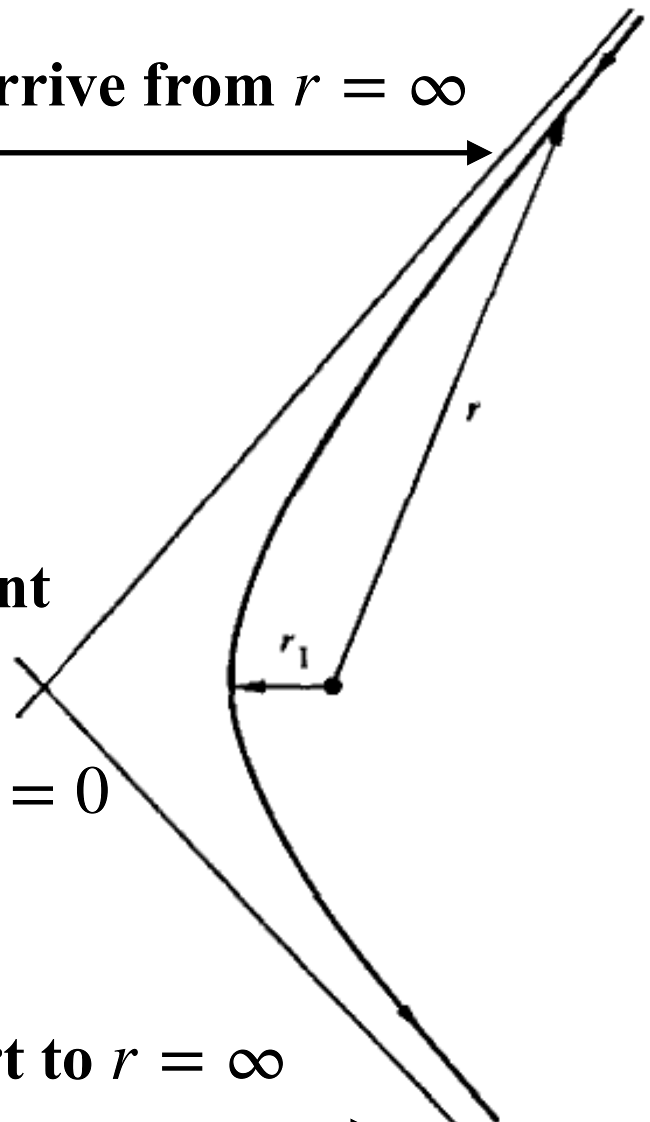
$$E = E_1 \rightarrow r > r_1$$

$$E_1 = V'\left(r_1\right)$$

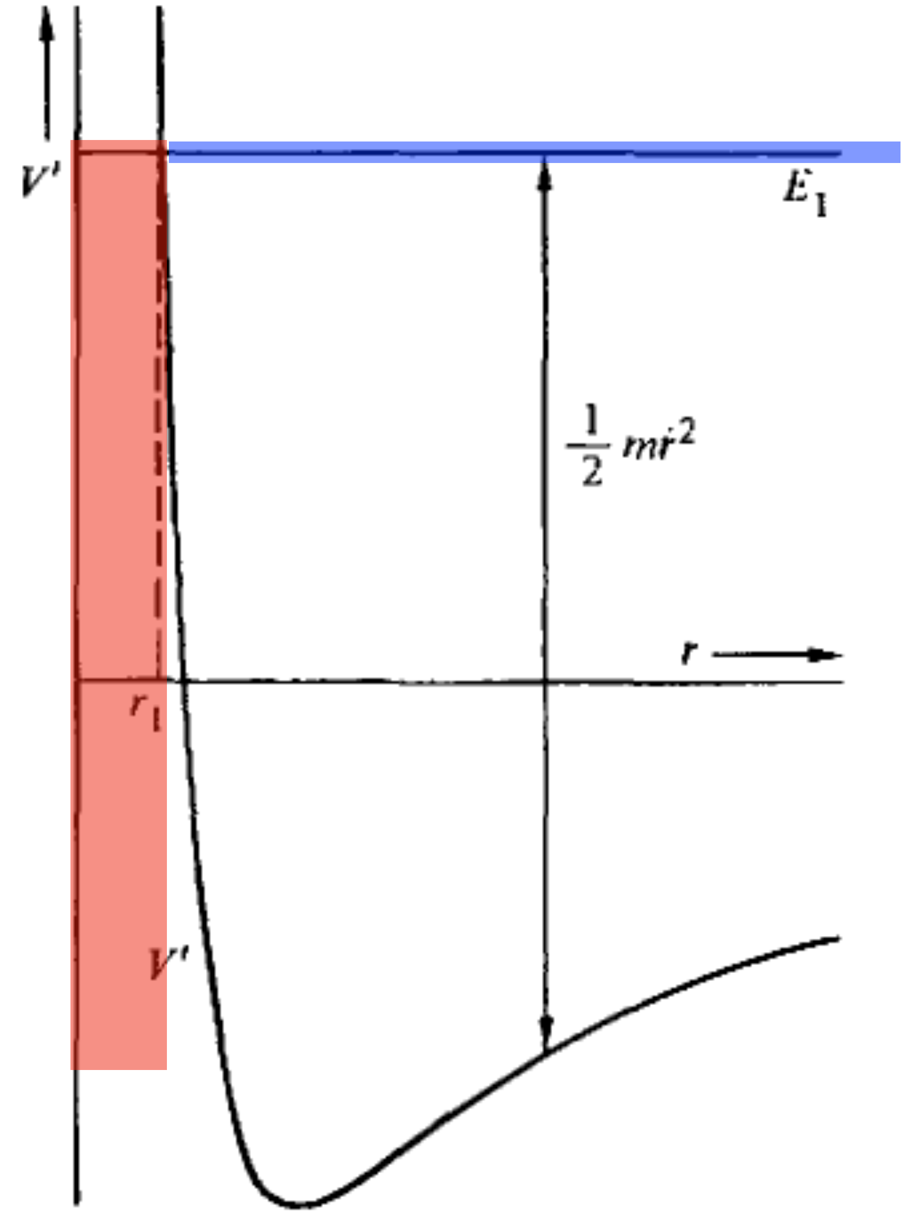
Arrive from $r = \infty$

Turning point

$E = V' \rightarrow \dot{r} = 0$



depart to $r = \infty$



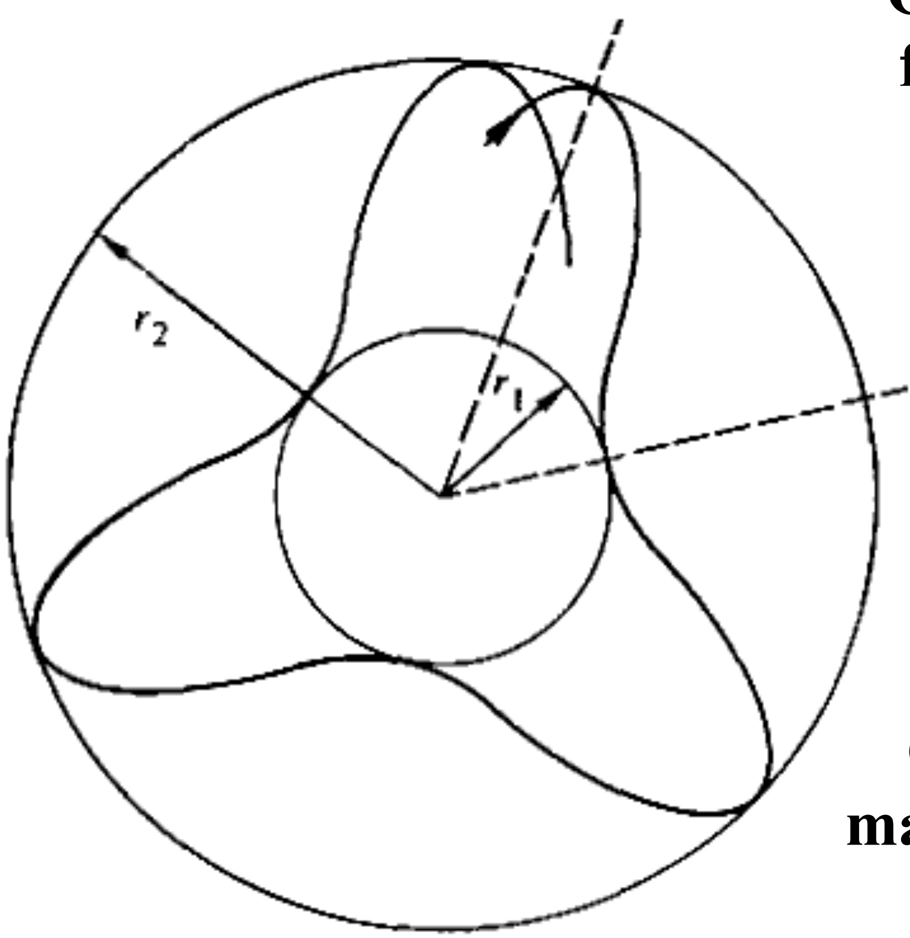
A $1/r^2$ force would make a hyperbola

Bounded Motion

$E = E_2 \rightarrow r_1 < r < r_2$

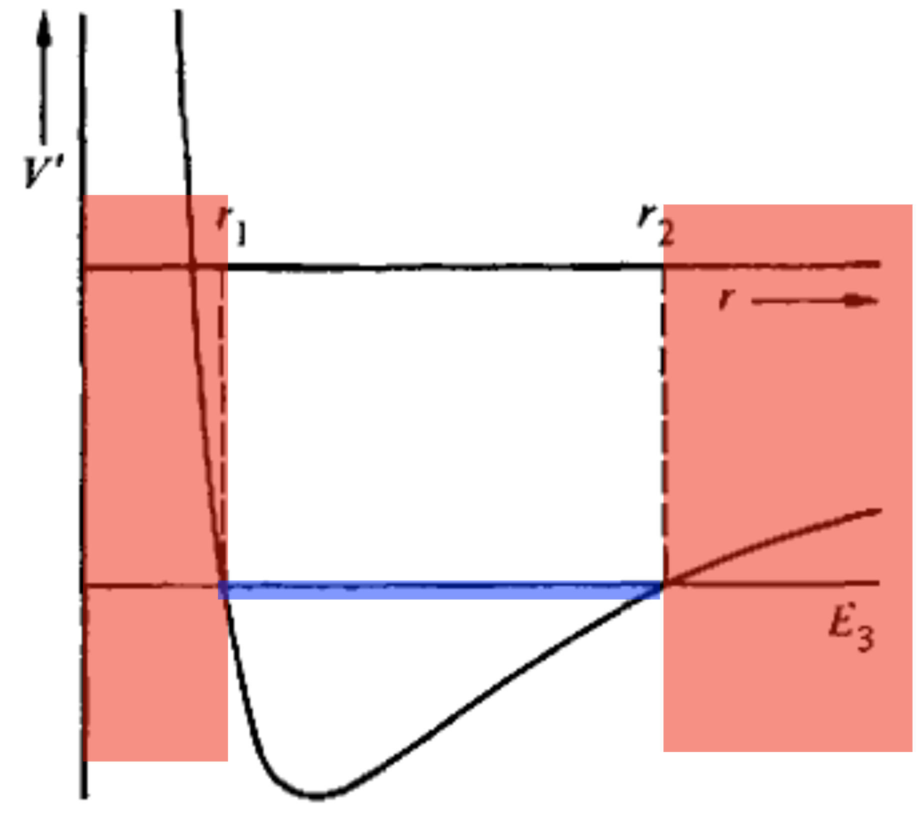
Particle is confined

between tow circles



Goes back and forth between two radii

Orbit may or may not be closed



A $1/r^2$ force would make an ellipse