Theoretical Mechanics

理论力学

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Syllabus

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- Chapter 10 Introduction to the Lagrangian and Hamiltonian Formulations for Continuous Systems and Fields

Rotational Motion

We concentrate on the rotational part

Translational part same as a single particle→Easy

Consider total angular momentum $\mathbf{L} = m_i \mathbf{r}_i \times \mathbf{v}_i$

$\mathbf{V}_i$ is given by the rotation $\boldsymbol{\omega}$ as $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$

BAC-CAB rule
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

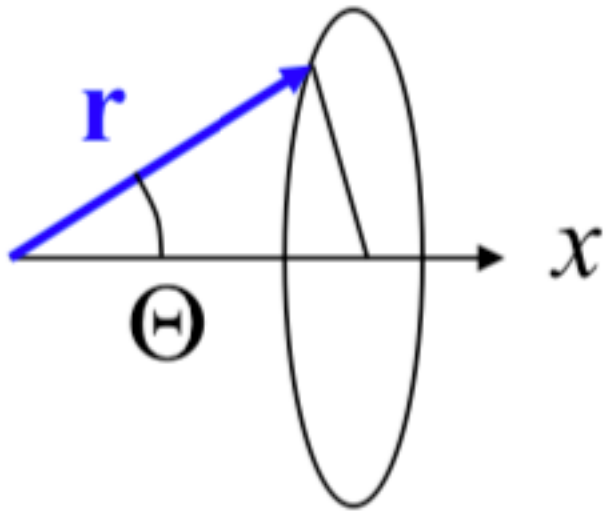
$$\mathbf{L} = m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = m_i \left[\boldsymbol{\omega} r_i^2 - \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega}) \right] =$$

$$\boxed{\begin{bmatrix} m_i \left(r_i^2 - x_i^2\right) & -m_i x_i y_i & -m_i x_i z_i \\ -m_i y_i x_i & m_i \left(r_i^2 - y_i^2\right) & -m_i y_i z_i \\ -m_i z_i x_i & -m_i z_i y_i & m_i \left(r_i^2 - z_i^2\right) \end{bmatrix}} \boldsymbol{\omega}$$

Inertia tensor I
only dependent on the fix point for a rigid body

Inertia Tensor

Diagonal components are familiar moment of inertia

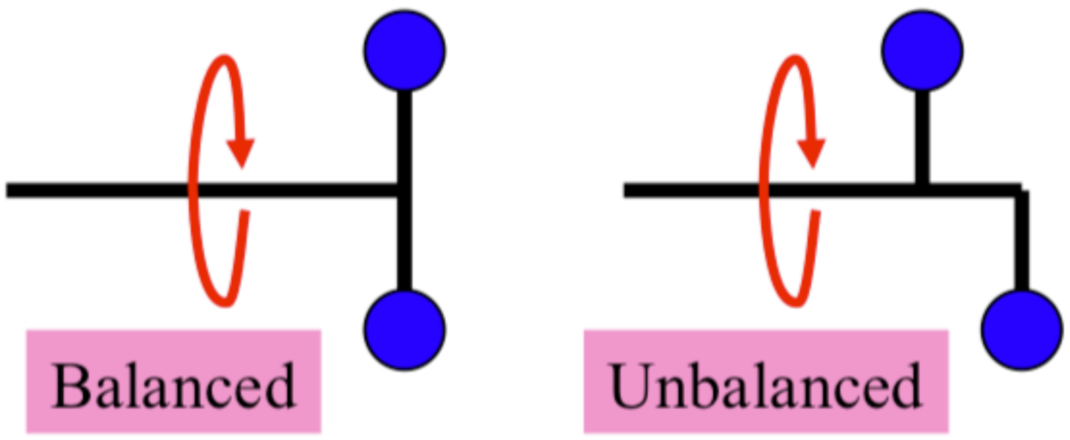


$$I_{xx} = m_i \left(r_i^2 - x_i^2 \right) = m_i r_i^2 \sin^2 \Theta$$

What are the off-diagonal components?

I_{yx} produces L_y when the object is turned around x axis

Imagine turning something like:



Unbalanced one has non-zero off-diagonal components, which represents "wobbliness" of rotation

Inertia Tensor

Using $\left(x_i, y_i, z_i\right) \rightarrow \left(x_{i1}, x_{i2}, x_{i3}\right)$

$$\mathbf{I} = \begin{bmatrix} m_i\left(r_i^2 - x_i^2\right) & -m_i x_i y_i & -m_i x_i z_i \\ -m_i y_i x_i & m_i\left(r_i^2 - y_i^2\right) & -m_i y_i z_i \\ -m_i z_i x_i & -m_i z_i y_i & m_i\left(r_i^2 - z_i^2\right) \end{bmatrix} \rightarrow$$

$$I_{jk} = m_i\left(r_i^2\delta_{jk} - x_{ij}x_{ik}\right)$$

We can also deal with continuous mass distribution $\rho(\mathbf{r})$

$$I_{jk} = \int \rho(\mathbf{r})\left(r^2\delta_{jk} - x_j x_k\right) d\mathbf{r}$$

5.2 Tensors

$\mathbf{I}$ can be considered as the quotient of $\mathbf{L}$ and $\boldsymbol{\omega}$, since $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$.

The quotient of two quantities is often not a member of the same class as the dividing factors, but may belong to a more complicated class.

$\mathbf{I}$ is a new type of quantity, a tensor of the second rank.

A tensor of the first rank is completely equivalent to a vector since $T_i' = a_{ij}T_j$ where $\{a_{ij}\} = \mathbf{A}$.

The 9 components of a tensor of the 2nd-rank transforms as $T_{ij}' = a_{ik}a_{jl}T_{kl}$.

5.3 The inertia tensor and the moment of inertia

The kinetic energy of motion about a point is

$$T = \frac{1}{2} \sum m_i v_i^2,$$

Where $\mathbf{v}_i$ is the velocity of the ith particle relative to the fixed point as measured in the space axes.

$$T = \frac{1}{2} \sum m_i \mathbf{v}_i \cdot (\boldsymbol{\omega} \times \mathbf{r}_i) = \frac{\boldsymbol{\omega}}{2} \cdot \sum m_i (\mathbf{r}_i \times \mathbf{v}_i) = \frac{\boldsymbol{\omega} \cdot \mathbf{L}}{2} = \frac{\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}}{2}$$

If $\boldsymbol{\omega} = \omega \mathbf{n}$, then $T = \dfrac{\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}}{2} = \dfrac{\omega^2}{2} \mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n} = \dfrac{1}{2} I \omega^2$, where
$I \equiv \mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n} = \sum m_i \left[r_i^2 - (\mathbf{r}_i \cdot \mathbf{n})^2 \right]$.

I is called the moment of inertia about the axis of rotation.

$$I = \sum m_i (\mathbf{r}_i \times \mathbf{n}) \cdot (\mathbf{r}_i \times \mathbf{n}) = \frac{2T}{\omega^2}$$

Shifting Origin

Origin of body axes does not have to be at the CoM

It's convenient – Separates translational/rotational motion

If it isn't, *I* can be easily translated

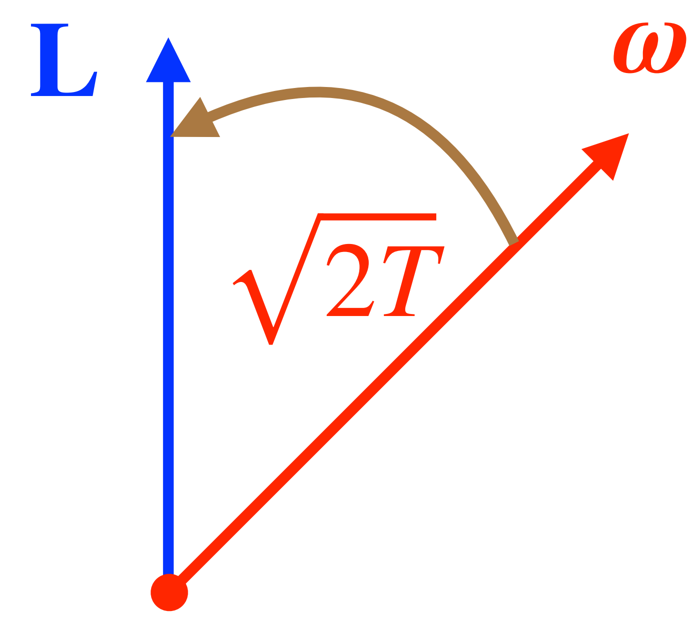
$$\text{from origin} \rightarrow \quad \mathbf{r}_i = \mathbf{R} + \mathbf{r}_i' \quad \leftarrow \text{from CoM}$$

$$I = m_i \left(\mathbf{r}_i \times \mathbf{n}\right)^2 = m_i \left[\left(\mathbf{R} + \mathbf{r}_i'\right) \times \mathbf{n}\right]^2$$

$$= \underbrace{M(\mathbf{R} \times \mathbf{n})^2}_{\textbf{\textit{I} of CoM}} + \underbrace{m_i \left(\mathbf{r}_i' \times \mathbf{n}\right)^2}_{\textbf{\textit{I} from CoM}} + \underbrace{\cancel{2m_i(\mathbf{R} \times \mathbf{n})\left(\mathbf{r}_i' \times \mathbf{n}\right)}}_{\sum_i \mathbf{r}_i' = 0}$$

Geometric view of tensors

We have two fundamental relations of the tensor $\mathbf{I}$ of a rigid body:



(1) $\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$

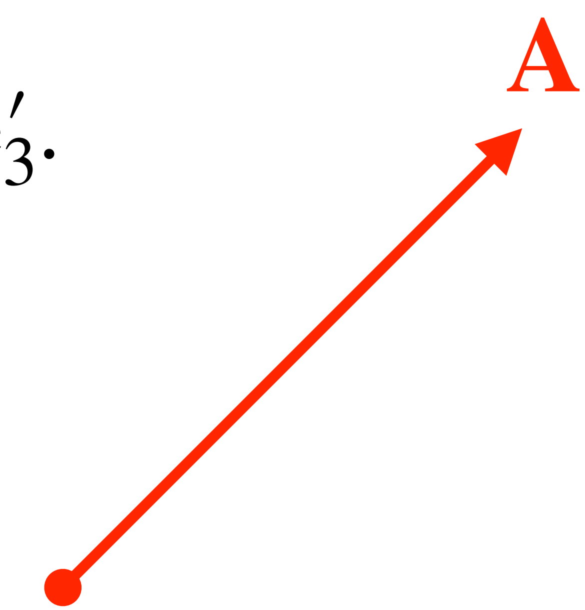
(2) $T = \dfrac{\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}}{2}$

If we ignore the units of $\mathbf{I}$, $\boldsymbol{\omega}$ and T, then in Eq. (1) $\mathbf{I}$ maps a vector $\boldsymbol{\omega}$ to another vector $\mathbf{L}$ in a 3-dimension vector space; in Eq. (2) $\mathbf{I}$ maps a vector $\boldsymbol{\omega}$ to a scalar T.

Therefore, a tensor $\mathbf{I}$ is a linear transform operator in Eq. (1) which transform $\boldsymbol{\omega}$ to $\mathbf{L}$; and a metric tensor in Eq. (2) which defines the "magnitude" of a vector $\boldsymbol{\omega}$ as $\sqrt{2T}$.

Geometric view of tensors

As we all know, a vector **A** is invariant for the choice of the bases in a vector space. Its geometric interpretation is easily visualized.



In the other word, its algebraic interpretation is

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 = A_1'\mathbf{e}_1' + A_2'\mathbf{e}_2' + A_3'\mathbf{e}_3'.$$

In matrix form

$$\mathbf{A} = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} A_1' & A_2' & A_3' \end{bmatrix} \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix},$$ where A_i and A_i' are called as coordinates

Geometric view of tensors

If $\mathbf{M}$, a transform matrix, transfers $\mathbf{e}_i$ to $\mathbf{e}_i'$, then we have

$$\mathbf{A} = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \mathbf{M}^{-1}\mathbf{M} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} A_1' & A_2' & A_3' \end{bmatrix} \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix}.$$

Remember: if $\mathbf{M}$ is a **clockwise** rotation, then $\mathbf{M}^{-1}$ is a **counter clockwise** rotation.

Since there is no absolute bases, two observers usually give different coordinates for a vector $\mathbf{A}$.

Therefore the coordinates of a vector in different bases should transfer as

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = (\mathbf{M}^{-1})^T \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = (\mathbf{M}^T)^{-1} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix},$$

where $(\mathbf{M}^T)^{-1}$ is the transformation matrix of coordinates.

Geometric view of tensors

We, physicists, usually ignore the superscripts of $(\mathbf{M}^T)^{-1}$ and name $\mathbf{M}$ as the transformation matrix of coordinates. Why?

Then $A'_i = M_{ij}A_j$, where M_{ij} is the components of $\mathbf{M}$.

A vector, also a tensor of the first rank, in a Cartesian 3-dimensional space is defined as a quantity having 3 components which transform under an orthogonal transformation of coordinates, $\mathbf{M}$, according to the rule $A'_i = M_{ij}A_j$.

Why do we define a vector in such a complex way?

Because it makes the different coordinates A_i and A'_i refer to the same vector.

Geometric view of tensors

The meaning of the relation between different coordinates is invariant with the formula forms:

$$A_i' = M_{ij}A_j \Leftrightarrow \mathbf{A}' = \mathbf{M}\mathbf{A} \Leftrightarrow \begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Note: From now on, $\mathbf{A}$ is a 3×1 coordinate matrix not $\mathbf{A} = \sum A_i \mathbf{e}_i$, although they refer to the same vector!

Geometric view of tensors

Back to $\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} \Leftrightarrow \mathbf{L} = \mathbf{I}\boldsymbol{\omega} \Leftrightarrow \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

For another observer, he/she has

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} \Leftrightarrow \mathbf{L}' = \mathbf{I}'\boldsymbol{\omega}' \Leftrightarrow \begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = \begin{pmatrix} I_{11}' & I_{12}' & I_{13}' \\ I_{21}' & I_{22}' & I_{23}' \\ I_{31}' & I_{32}' & I_{33}' \end{pmatrix} \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix},$$

where $\begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$ and $\begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

Geometric view of tensors

According to the matrix rules

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \mathbf{I}\mathbf{M}^{-1}\mathbf{M}\boldsymbol{\omega} \Rightarrow \mathbf{M}\mathbf{L} = \mathbf{M}\mathbf{I}\mathbf{M}^{-1}\mathbf{M}\boldsymbol{\omega}$$
$$\Rightarrow \mathbf{L}' = \mathbf{M}\mathbf{I}\mathbf{M}^{-1}\boldsymbol{\omega}' = \mathbf{I}'\boldsymbol{\omega}'$$

Here $\boldsymbol{\omega}' = \mathbf{M}\boldsymbol{\omega}$.

We have $\mathbf{I}' = \mathbf{M}\mathbf{I}\mathbf{M}^{-1}$ and then $I'_{ij} = M_{ik}I_{kl}M^{-1}_{lj} = M_{ik}M_{jl}I_{kl}$.

Remember: $\mathbf{M}^{-1} = \mathbf{M}^{T} = \tilde{\mathbf{M}}$ for an orthogonal transformation matrix $\mathbf{M}$.

Although the components of $\mathbf{I}'$ and $\mathbf{I}$ are different, they refer to the same quantity. $\mathbf{I}'$ is naturally similar to $\mathbf{I}$.

Unfortunately we cannot show it geometrically.

Geometric view of tensors

无尽藏是唐朝武周时期的一位比丘尼。

一天，她对六祖慧能说：“我读《涅般经》好多年了，但仍有许多不明白的地方，希望能得到你的指教。”

慧能回答道：“我不识字，请你把经读给我听，也许我能帮你解疑。”

比丘尼忍不住笑着说：“你连字都不认识，怎么谈得上解释经典呢？”

慧能认真地告诉她：“真理与文字不是一回事。真理就像天上的明月，文字只是指月的手指；手指能指出明月的所在，但手指并不就是明月，看明月也并不一定非用手指不可。”

比丘尼感觉这话很有道理，于是就将经文读给慧能听。慧能一句一句为她解释，使尼师大受启迪。

Geometric view of tensors

Back to $T = \dfrac{\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}}{2}$

$$T = \frac{\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}}{2} \Leftrightarrow 2T = \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} = \boldsymbol{\omega}'^T \hat{\mathbf{I}}' \boldsymbol{\omega}' = \boldsymbol{\omega}^T \mathbf{M}^T \hat{\mathbf{I}}' \mathbf{M} \boldsymbol{\omega}$$

We have $\mathbf{I} = \mathbf{M}^T \hat{\mathbf{I}}' \mathbf{M}$. $\hat{\mathbf{I}}'$ is naturally congruent (合同) to $\mathbf{I}$.

We already have $\mathbf{I}' = \mathbf{M}\mathbf{I}\mathbf{M}^{-1} \Leftrightarrow \mathbf{I} = \mathbf{M}^{-1}\mathbf{I}'\mathbf{M}$.

If $\mathbf{M}^{-1} = \mathbf{M}^T$, $\hat{\mathbf{I}}' = \mathbf{I}'$.

M is an orthogonal transformation matrix!

Geometric view of tensors

Last but not least, one can find an orthogonal similarity transformation $\mathbf{M}$ which make $\mathbf{I}'$ and $\mathbf{I}$ refer to the **same** tenor in

(1) $\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$

(2) $T = \dfrac{\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}}{2}$.

The vector space V of $\boldsymbol{\omega}$ can be decomposed into a direct sum of 3 invariant subspaces U_i where $\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} \in U_i$ if $\boldsymbol{\omega} \in U_i$.

The most important task in studying of the motion about a fixed point for a rigid body is to find the 3 **invariant subspaces** so that the tensor $\mathbf{I}$ can be expressed in the simplest ways.

There is no absolute bases but the simplest set of bases. Why? (Broken symmetry!)

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