理论力学第 16 次作业

8.1

(a)

$$L(q,\dot{q},t) = p_i\dot{q}_i - H(q,p,t)$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$dL = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial H}{\partial q_i}dq_i - \frac{\partial H}{\partial p_i}dp_i - \frac{\partial H}{\partial t}dt$$
$$= \dot{p}_i dq_i + p_i d\dot{q}_i - \frac{\partial H}{\partial t}dt$$

对比 L 的全微分，可得

$$\frac{\partial L}{\partial q_i} = \dot{p}_i$$
$$\frac{\partial L}{\partial \dot{q}_i} = p_i$$
$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

所以

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

(b)

$$L^{'}(p,\dot{p},t) = -q_i\dot{p}_i - H(q,p,t)$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$dL^{'} = -\dot{p}_i dq_i - q_i d\dot{p}_i - \frac{\partial H}{\partial q_i}dq_i - \frac{\partial H}{\partial p_i}dp_i - \frac{\partial H}{\partial t}dt$$

$$=-\dot{q}_i dp_i - q_i d\dot{p}_i - \frac{\partial H}{\partial t}dt$$

对比$L^{'}$的全微分，可得

$$\frac{\partial L^{'}}{\partial p_i}=-\dot{q}_i$$
$$\frac{\partial L^{'}}{\partial \dot{p}_i}=-q_i$$
$$\frac{\partial L^{'}}{\partial t}=-\frac{\partial H}{\partial t}$$

所以

$$\frac{d}{dt}\left(\frac{\partial L^{'}}{\partial \dot{p}_i}\right)-\frac{\partial L^{'}}{\partial p_i}=0$$

8.2

令$L^{'}=L+\dot{F}(q,t)$

$$S=\int_{t_1}^{t_2} L dt$$
$$S^{'}=\int_{t_1}^{t_2}\left(L+\frac{dF}{dt}\right)dt=S+F(t_2)-F(t_1)$$

所以

$$\delta S=\delta S^{'}$$

$$\frac{d}{dt}\left(\frac{\partial L^{'}}{\partial \dot{q}}\right)-\frac{\partial L^{'}}{\partial q}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}+\frac{d}{dt}\left(\frac{\partial \dot{F}}{\partial \dot{q}}\right)-\frac{\partial \dot{F}}{\partial q}=0$$
$$\frac{d}{dt}\left(\frac{\partial \dot{F}}{\partial \dot{q}}\right)-\frac{\partial \dot{F}}{\partial q}=0$$
$$\frac{\partial \dot{F}}{\partial \dot{q}}=\frac{\partial F}{\partial q}$$

$$H=p_i\dot{q}_i-L$$
$$H^{'}\left(q_i^{'},p_i^{'},t\right)=p_i^{'}\dot{q}_i^{'}-L^{'}=H+\frac{\partial \dot{F}}{\partial \dot{q}}\dot{q}-\dot{F}=H-\frac{\partial F}{\partial t}$$

其中

$$\dot{q}_i^{'}=\dot{q}_i,\ \dot{p}_i^{'}=\dot{p}_i+\frac{\partial \dot{F}}{\partial \dot{q}_i}=\dot{p}_i+\frac{\partial F}{\partial q_i}$$

$$\because \dot{q}_i^{'}=\dot{q}_i=\frac{\partial H^{'}}{\partial p_i^{'}}=\frac{\partial H^{'}}{\partial p_i}\frac{\partial p_i}{\partial p_i^{'}}+\frac{\partial H^{'}}{\partial q_i}\frac{\partial q_i}{\partial p_i^{'}}=\frac{\partial H}{\partial p_i}$$
$$\therefore \dot{q}_i=\frac{\partial H}{\partial p_i}$$

$$\because \dot{p}_i^{'} = \dot{p}_i + \frac{\partial \dot{F}}{\partial q_i} =- \frac{\partial H^{'}}{\partial q_i^{'}} =- \frac{\partial H^{'}}{\partial q_i}\frac{\partial q_i}{\partial q_i^{'}} - \frac{\partial H^{'}}{\partial p_i}\frac{\partial p_i}{\partial q_i^{'}} =- \frac{\partial H}{\partial q_i} + \frac{\partial^2 F}{\partial q_i \partial t} + \frac{\partial^2 F}{\partial q_i^2}$$

$$\therefore \dot{p}_i =- \frac{\partial H}{\partial q_i}$$

8.3

$$G(\dot{q}_i,\dot{p}_i,t) = q_i\dot{p}_i - L$$

$$dG = \dot{p}_i dq_i + q_i d\dot{p}_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial t}dt$$
$$=- p_i d\dot{q}_i + q_i d\dot{p}_i - \frac{\partial L}{\partial t}dt$$

对比 G 的全微分，可得

$$p_i =- \frac{\partial G}{\partial \dot{q}_i}$$
$$q_i = \frac{\partial G}{\partial \dot{p}_i}$$
$$\frac{\partial L}{\partial t} =- \frac{\partial G}{\partial t}$$