Theoretical Mechanics
理论力学

Zhou Xiang 周详

Syllabus

- Chapter 0 Preface
- Chapter 1 Survey of the Elementary Principles
- Chapter 2 Variational Principle and Lagrange's Equations
- Chapter 3 The Central Force Problem
- Chapter 4 The Kinematics of Rigid Body Motion

Mid-term exam

- Chapter 5 The Rigid Body Equations of Motion
- Chapter 6 Oscillations
- Chapter 7 The Classical Mechanics of the Special Theory of Relativity
- Chapter 8 The Hamilton Equations of Motion
- Chapter 9 Canonical Transformations

Final term exam

- Chapter 10 Introduction to the Lagrangian and Hamiltonian Formulations for Continuous Systems and Fields

Chapter 9 Canonical Transformations

We firstly show how point transformation, the simplest canonical transformation, works.

Don't panic. We have learn and use point transformation (Chapter 1, Derivation 10, p31).

For a center force problem, one can easily express L in Cartesian coordinates $\mathbf{x} = (x, y, z)$:

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = T - V = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(r(\mathbf{x}))$$

As we have known, L is simplified in spherical coordinates

(r, θ, ϕ) with $\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$

Chapter 9 Canonical Transformations

$\mathbf{x} = (x, y, z) \to q_i$ and $\dot{\mathbf{x}} = (\dot{x}, \dot{y}, \dot{z}) \to \dot{q}_i$

$\boxed{\text{Remember : } q_i \leftrightarrow f_i \text{ and } Q_\mu \leftrightarrow f_\mu^{-1}}$

$(r, \theta, \phi) \to Q_\mu$ and $\qquad (\dot{r}, \dot{\theta}, \dot{\phi}) \to \dot{Q}_\mu$

The point transformation from spherical coordinates to Cartesian coordinates is $q_i = f_i(\mathbf{Q}, t)$ (so that $Q_\mu = f_\mu^{-1}(\mathbf{q}, t)$) and then the transformations of general velocities are

$$\dot{q}_i = \sum_\mu \frac{\partial f_i(\mathbf{Q}, t)}{\partial Q_\mu}\dot{Q}_\mu + \frac{\partial f_i(\mathbf{Q}, t)}{\partial t} \Rightarrow \begin{cases} \dot{x} = \dot{r}\sin\theta\cos\phi + \dot{\theta}r\cos\theta\cos\phi - \dot{\phi}r\sin\theta\sin\phi \\ \dot{y} = \dot{r}\sin\theta\sin\phi + \dot{\theta}r\cos\theta\sin\phi + \dot{\phi}r\sin\theta\cos\phi \\ \dot{z} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta \end{cases}$$

Obviously, we must have

$$L'(\mathbf{Q}, \dot{\mathbf{Q}}, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) = L\left(f_i(\mathbf{Q}, t), \sum_\mu \frac{\partial f_i(\mathbf{Q}, t)}{\partial Q_\mu}\dot{Q}_\mu + \frac{\partial f_i(\mathbf{Q}, t)}{\partial t}, t\right)$$

Action is invariant under point transformations!

Chapter 9 Canonical Transformations

Finally we get NEW Lagrangian from old Lagrangian

$$L'(r,\theta,\phi,\dot{r},\dot{\theta},\dot{\phi},t) = L(\mathbf{x}(r,\theta,\phi),\dot{\mathbf{x}}(r,\theta,\phi,\dot{r},\dot{\theta},\dot{\phi}),t) = \frac{m}{2}(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2) - V(r)$$

The point transformation for Lagrangian is easy, relatively.

For Cartesian coordinates $\mathbf{x} = (x, y, z)$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x},\, p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y},\, p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \text{(p57-58)}$$

$$H(x,y,z,p_x,p_y,p_z) = \dot{\mathbf{x}}\cdot\mathbf{p} - L = \frac{\mathbf{p}^2}{m} - \frac{\mathbf{p}^2}{2m} + V(r) = \frac{\mathbf{p}^2}{2m} + V(r)$$

Our purpose:

Get NEW momenta $\left(P_r, P_\theta, P_\phi\right)$ **directly** from old momenta $\left(p_x, p_y, p_z\right)$

Get NEW Hamiltonian $K(r,\theta,\phi,P_r,P_\theta,P_\phi)$ **directly** from old Hamiltonian $H(x,y,z,p_x,p_y,p_z)$

K is also called H' in other textbooks

The point transformation for Hamiltonian is not hard, definitely or hopefully.

Unconscious approach

Get NEW momenta $\left(P_r, P_\theta, P_\phi\right)$ from NEW Lagrangian

$$L'(r,\theta,\phi,\dot{r},\dot{\theta},\dot{\phi},t) = \frac{m}{2}(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2) - V(r)$$

$$P_r = \frac{\partial L'}{\partial \dot{r}} = m\dot{r}, \quad P_\theta = \frac{\partial L'}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad P_\phi = \frac{\partial L'}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi} \quad \Longrightarrow \quad \dot{r} = \frac{P_r}{m},$$
$$\dot{\theta} = \frac{P_\theta}{mr^2}, \quad \dot{\phi} = \frac{P_\phi}{mr^2\sin^2\theta}$$

Get the NEW Hamiltonian from NEW Lagrangian

K is also called H' in other textbooks

$$K(Q_i, P_i, t,) = \sum_i P_i\dot{Q}_i(\mathbf{Q},\mathbf{P},t) - L'(\mathbf{Q},\dot{\mathbf{Q}}(\mathbf{Q},\mathbf{P},t),t)$$
$$= \left(P_r\dot{r} + P_\theta\dot{\theta} + P_\phi\dot{\phi}\right) - \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2\sin^2\theta}\right) + V(r)$$
$$= \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2\sin^2\theta}\right) + V(r)$$

Verify: $K = H' = H$ in this example!

Chapter 9 Canonical Transformations

The **STANDARD** approach

1. Get the transformation between old and NEW coordinates: $q_i = f_i(\mathbf{Q}, t)$

2. Get the transformation of velocities: $\dot{q}_i = \sum_\mu \frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\dot{Q}_\mu + \frac{\partial f_i(\mathbf{Q},t)}{\partial t}$

3. Get old momenta from old Lagrangian $p_i \equiv \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_i}$

4. Get NEW momenta **directly** from old momenta

$$P_\mu \equiv \frac{\partial L'(\mathbf{Q}, \dot{\mathbf{Q}}, t)}{\partial \dot{Q}_\mu} = \sum_i \frac{\partial}{\partial \dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t)\frac{\partial \dot{q}_i}{\partial \dot{Q}_\mu} = \sum_i p_i \left\{\frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\right\}_{i\mu} \Longleftrightarrow p_i = \sum_\mu P_\mu \left\{\frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\right\}^{-1}_{\mu i}$$

5. Get the mapping between old and new phase space

$$(t, Q_\mu, P_\mu) \Longleftrightarrow (t, q_i, p_i) = \left(t, f_i(\mathbf{Q}, t), \sum_\mu P_\mu \left\{\frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\right\}^{-1}_{\mu i}\right)$$

6. Get the NEW Hamiltonian **directly** from old Hamiltonian

$$\begin{aligned}
K(\mathbf{Q}, \mathbf{P}, t) &= \sum_\mu P_\mu \dot{Q}_\mu - L'(\mathbf{Q}, \dot{\mathbf{Q}}, t) \\
&= \sum_\mu \left(\sum_i p_i \left\{\frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\right\}_{i\mu}\right)\left(\sum_j \left\{\frac{\partial f_j(\mathbf{Q},t)}{\partial q_k}\right\}^{-1}_{\mu j}\left(\dot{q}_j - \frac{\partial f_j(\mathbf{Q},t)}{\partial t}\right)_j\right) - L(t, \mathbf{q}, \dot{\mathbf{q}}) \\
&= \sum_i \sum_j p_i \delta_{ij}\left(\dot{q}_j - \frac{\partial f_j(\mathbf{q},t)}{\partial t}\right)_j - L(t, \mathbf{q}, \dot{\mathbf{q}}) \\
&= \sum_i p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t) - \sum_i p_i \frac{\partial f_i(\mathbf{Q},t)}{\partial t} \\
&= H(\mathbf{q}, \mathbf{p}, t) - \sum_i p_i \frac{\partial f_i(\mathbf{Q},t)}{\partial t}
\end{aligned}$$

Attention: Generally K is not equal to H!

The **unconscious** approach

1. Get the transformation between old and NEW coordinates: $q_i = f_i(\mathbf{Q}, t)$

2. Get the transformation of velocities:

$$\dot{q}_i = \sum_\mu \frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\dot{Q}_\mu + \frac{\partial f_i(\mathbf{Q},t)}{\partial t} \quad \text{or}$$

$$\dot{Q}_\mu = \sum_i \left\{\frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\right\}^{-1}_{\mu i}\left(\dot{q}_i - \frac{\partial f_i(\mathbf{Q},t)}{\partial t}\right)_i$$

$$\sum_\mu \left\{\frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu}\right\}_{i\mu}\left\{\frac{\partial f_j(\mathbf{Q},t)}{\partial Q_\mu}\right\}^{-1}_{\mu j} = \delta_{ij}$$

3. Get old momenta from old Lagrangian $p_i \equiv \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_i}$

4.Get NEW Lagrangian $L'(\mathbf{Q}, \dot{\mathbf{Q}}, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t)$

5. Get NEW momenta from NEW Lagrangian $P_\mu \equiv \frac{\partial L'(\mathbf{Q}, \dot{\mathbf{Q}}, t)}{\partial \dot{Q}_\mu}$

and then $\dot{Q}_\mu = Q_\mu(\mathbf{Q}, \mathbf{P}, t)$

6. Get the NEW Hamiltonian from NEW Lagrangian

$$K(Q_\mu, P_\mu, t,) = \sum_\mu P_\mu \dot{Q}_\mu(\mathbf{Q}, \mathbf{P}, t) - L'(\mathbf{Q}, \dot{\mathbf{Q}}(\mathbf{Q}, \mathbf{P}, t), t)$$

Chapter 9 Canonical Transformations

STANDARD approach

Get NEW momenta $\left(P_r, P_\theta, P_\phi\right)$ directly from old momenta $\left(p_x, p_y, p_z\right)$

$$P_\mu = \sum_i p_i \left\{ \frac{\partial f_i(\mathbf{Q},t)}{\partial Q_\mu} \right\}_{i\mu} \Longleftrightarrow p_i = \sum_\mu P_\mu \left\{ \frac{\partial f_i(\mathbf{q},t)}{\partial Q_\mu} \right\}^{-1}_{\mu i}$$

$$\begin{aligned} P_r &= p_x \frac{\partial x}{\partial r} + p_y \frac{\partial y}{\partial r} + p_z \frac{\partial z}{\partial r} = m\dot{x}\frac{\partial x}{\partial r} + m\dot{y}\frac{\partial y}{\partial r} + m\dot{z}\frac{\partial z}{\partial r} \\ &= m(\dot{r}\sin\theta\cos\phi + \dot{\theta}r\cos\theta\cos\phi - \dot{\phi}r\sin\theta\sin\phi)(\sin\theta\cos\phi) \\ &\quad + m(\dot{r}\sin\theta\sin\phi + \dot{\theta}r\cos\theta\sin\phi + \dot{\phi}r\sin\theta\cos\phi)(\sin\theta\sin\phi) \\ &\quad + m(\dot{r}\cos\theta - \dot{\theta}r\sin\theta)(\cos\theta) \\ &= m\dot{r} \end{aligned}$$

$$P_r = \frac{\partial L'}{\partial \dot{r}} = m\dot{r}$$

$$\begin{aligned} \dot{x} &= \dot{r}\sin\theta\cos\phi + \dot{\theta}r\cos\theta\cos\phi - \dot{\phi}r\sin\theta\sin\phi \\ \dot{y} &= \dot{r}\sin\theta\sin\phi + \dot{\theta}r\cos\theta\sin\phi + \dot{\phi}r\sin\theta\cos\phi \\ \dot{z} &= \dot{r}\cos\theta - \dot{\theta}r\sin\theta \end{aligned}$$

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

$$\begin{pmatrix} P_r \\ P_\theta \\ P_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \frac{1}{r}\cos\theta\cos\phi & -\frac{1}{r}\csc\theta\sin\phi \\ \sin\theta\sin\phi & \frac{1}{r}\cos\theta\sin\phi & \frac{1}{r}\csc\theta\cos\phi \\ \cos\theta & -\frac{1}{r}\sin\theta & 0 \end{pmatrix} \begin{pmatrix} P_r \\ P_\theta \\ P_\phi \end{pmatrix}$$

Get NEW Hamiltonian $K(r, \theta, \phi, P_r, P_\theta, P_\phi)$ directly from old Hamiltonian $H(x, y, z, p_x, p_y, p_z)$

$$\begin{aligned} K(\mathbf{Q}, \mathbf{P}, t,) &= H(\mathbf{q}, \mathbf{p}, t) - \sum_i p_i \frac{\partial f_i(\mathbf{Q},t)}{\partial t} = H(\mathbf{q}, \mathbf{p}, t) \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(\sqrt{x^2 + y^2 + z^2}) \\ &= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r) \end{aligned}$$

Chapter 9 Canonical Transformations

For point transformations, the **unconscious** approach only manually seems much easier than the **STANDARD** approach.

NEW Lagrangian L' is non-vanishing (not always true for canonical transformations)

Legendre transformation always true

The point transformation not only changes (x, y, z) to (r, θ, ϕ) but also changes (p_x, p_y, p_z) to (P_r, P_θ, P_ϕ) which is

$$Q_\mu = Q_\mu \left(q_1, \ldots, q_n, \qquad\qquad , t\right)$$
$$P_\mu = P_\mu \left(q_1, \ldots, q_n, p_1, \ldots, p_n, t\right)$$

Canonical transformations general have forms as

$$Q_\mu = Q_\mu \left(q_1, \ldots, q_n, p_1, \ldots, p_n, t\right)$$
$$P_\mu = P_\mu \left(q_1, \ldots, q_n, p_1, \ldots, p_n, t\right)$$

Chapter 9 Canonical Transformations

Goal: To find transformations

$$Q_i = Q_i\left(q_1, \ldots, q_n, p_1, \ldots, p_n, t\right) \quad P_i = P_i\left(q_1, \ldots, q_n, p_1, \ldots, p_n, t\right)$$

that satisfy Hamilton's equation of motion

Old New

$$\dot{q}_i = \frac{\partial H}{dp_i} \quad \dot{p}_i = -\frac{\partial H}{dq_i} \Longrightarrow \dot{Q}_i = \frac{\partial K}{dP_i} \quad \dot{P}_i = -\frac{\partial K}{dQ_i}$$

K is the transformed Hamiltonian $K = K(Q, P, t)$

Hamilton's principle requires

$$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H(q, p, t) \right) dt = 0 \text{ and}$$

$$\delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K(Q, P, t) \right) dt = 0$$

General Transformation

$$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H(q,p,t) \right) dt = 0 \text{ and } \delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K(Q,P,t) \right) dt = 0$$

Two types of transformations are possible

Scale transformation: $\sum_i P_i \dot{Q}_i - K = \lambda \left(\sum_i p_i \dot{q}_i - H \right)$

Canonical transformation: $\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = \sum_i p_i \dot{q}_i - H$

Combined, we find Extended Canonical transformation

$$\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = \lambda \left(\sum_i p_i \dot{q}_i - H \right)$$

Scale Transformation

We can always change the scale of (or unit we use to measure) coordinates and momenta

$$P_i = \nu p_i \quad Q_i = \mu q_i$$

To satisfy Hamilton's principle, we can define

$$K(P, Q, t) = \mu\nu H(p, q, t)$$

$$\Longrightarrow \sum_i P_i \dot{Q}_i - K = \mu\nu\left(\sum_i p_i \dot{q}_i - H\right)$$

Scale transformation is trivial

We now concentrate on – Canonical transformations

Canonical Transformation

$$\sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} = \sum_i p_i \dot{q}_i - H$$

Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K \right) dt = \delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H - \frac{dF}{dt} \right) dt = - \delta [F]_{t_1}^{t_2} = 0$$

Satisfied if $\delta q = \delta Q = 0$ at t_1 and t_2

don't care about δp and δP

F can be any function partly of q, p , partly of Q, P and t

There only 4 types of basic combinations (q, Q, t), (q, P, t), (p, Q, t), (p, P, t)

It defines a canonical transformation

生成函数（产生函数、母函数、产生子……）

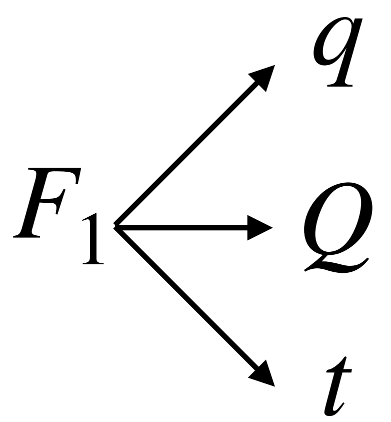
Call it the **generating function** (or **generator**) of the transformation

Type-1 Generator

We request

$$\sum_i P_i\dot{Q}_i - K + \frac{dF}{dt} = \sum_i p_i\dot{q}_i - H \Longrightarrow \mathrm{d}F = \sum_i p_i \mathrm{d}q_i - \sum_i P_i \mathrm{d}Q_i + (K-H)\mathrm{d}t$$

Type-1 Generator: $F = F_1(q, Q, t)$: Type-1



Obviously, $\mathrm{d}F = \sum_i \frac{\partial F_1}{\partial q_i}\mathrm{d}q_i + \sum_i \frac{\partial F_1}{\partial Q_i}\mathrm{d}Q_i + \frac{\partial F_1}{\partial t}\mathrm{d}t$

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = P_i\dot{Q}_i - K + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} = p_i\dot{q}_i - H$$

$$\sum_i \left(P_i + \frac{\partial F_1}{\partial Q_i}\right)\mathrm{d}Q_i - \sum_i \left(p_i - \frac{\partial F_1}{\partial q_i}\right)\mathrm{d}q_i - \left(K - H - \frac{\partial F_1}{\partial t}\right)\mathrm{d}t = 0$$

Obviously, $\mathrm{d}Q_i$, $\mathrm{d}q_i$ and $\mathrm{d}t$ are independent to each other

We have $p_i = \dfrac{\partial F_1(q, Q, t)}{\partial q_i}$, $P_i = -\dfrac{\partial F_1(q, Q, t)}{\partial Q_i}$ and $K = H + \dfrac{\partial F_1}{\partial t}$

Finding the Generator

$$\sum_i P_i \dot{Q}_i - K + \frac{d}{dt} F_1(q, Q, t) = \sum_i p_i \dot{q}_i - H$$

Let's look for a generating function

Suppose $K(Q, P, t) = H(q, p, t)$ for simplicity

$$\Longrightarrow \frac{d}{dt} F_1(q, Q, t) = \sum_i \left(p_i \dot{q}_i - P_i \dot{Q}_i \right)$$

Easiest way to satisfy this would be

$$F = F(q, Q) \qquad \frac{\partial F}{\partial q_i} = p_i \qquad \frac{\partial F}{\partial Q_i} = -P_i$$

Trivial example: $F(q, Q) = \sum_i q_i Q_i$

$$p_i = Q_i \quad P_i = -q_i$$

In the Hamiltonian formalism, you can freely swap the coordinates and the momenta

Harmonic Oscillator

Consider a 1-dimensional harmonic oscillator

$$H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \text{ with } \omega^2 \equiv \frac{k}{m}$$

Sum of squares→Can we make them sine and cosine?

Suppose $p = f(P)\cos Q \quad q = \dfrac{f(P)}{m\omega}\sin Q$

$$\Longrightarrow K = H = \frac{\{f(P)\}^2}{2m}$$ ←Q is cyclic→P is constant

Trick is to find $f(P)$ so that the transformation is canonical

How?

Harmonic Oscillator

Let's try a Type-1 generator

$$F_1(q,Q,t) \quad p = \frac{\partial F_1}{\partial q} \quad P = -\frac{\partial F_1}{\partial Q} \Longleftarrow \boxed{\left.\frac{\partial p}{\partial Q}\right|_q = \frac{\partial^2}{\partial Q \partial q}F_1 = \frac{\partial^2}{\partial q \partial Q}F_1 = -\left.\frac{\partial P}{\partial q}\right|_Q}$$

Express p as a function of q and Q

$$p = f(P)\cos Q \quad q = \frac{f(P)}{m\omega}\sin Q \Longrightarrow p = m\omega q \cot Q \Longrightarrow \boxed{\left.\frac{\partial p}{\partial Q}\right|_q = -\frac{m\omega q}{\sin^2 Q}}$$

Integrate with q: $\displaystyle\int p \mathrm{d}q = \int \frac{\partial F_1}{\partial q}\mathrm{d}q \Longrightarrow F_1 = \frac{m\omega q^2}{2}\cot Q + f(Q)$

$$\Longrightarrow P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q} - f'(Q) \Longrightarrow \boxed{-\left.\frac{\partial P}{\partial q}\right|_Q = -\frac{m\omega q}{\sin^2 Q}}$$

We set $f(Q) = 0$ (why?) and then we are getting somewhere

Harmonic Oscillator

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \quad P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

We need to turn $H(q, p)$ into $K(Q, P)$

Solve the above equations for q and p

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \quad p = \sqrt{2Pm\omega} \cos Q$$

Now work out the Hamiltonian

$$K = H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \omega P$$

Things don't get much simpler than this...

Harmonic Oscillator

$$K = \omega P = E$$

Solving the problem is trivial

$$P = \text{const} = \frac{E}{\omega} \quad \dot{Q} = \frac{\partial K}{\partial P} = \omega \quad Q = \omega t + \alpha$$

$$p = \sqrt{2Pm\omega}\cos Q = \sqrt{2mE}\cos(\omega t + \alpha)$$

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha)$$

What's the new Lagrangian L'?

$$H(q,p) = \frac{p^2}{2m} + \frac{m^2\omega^2 q^2}{2} \qquad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \Longrightarrow p = m\dot{q}$$

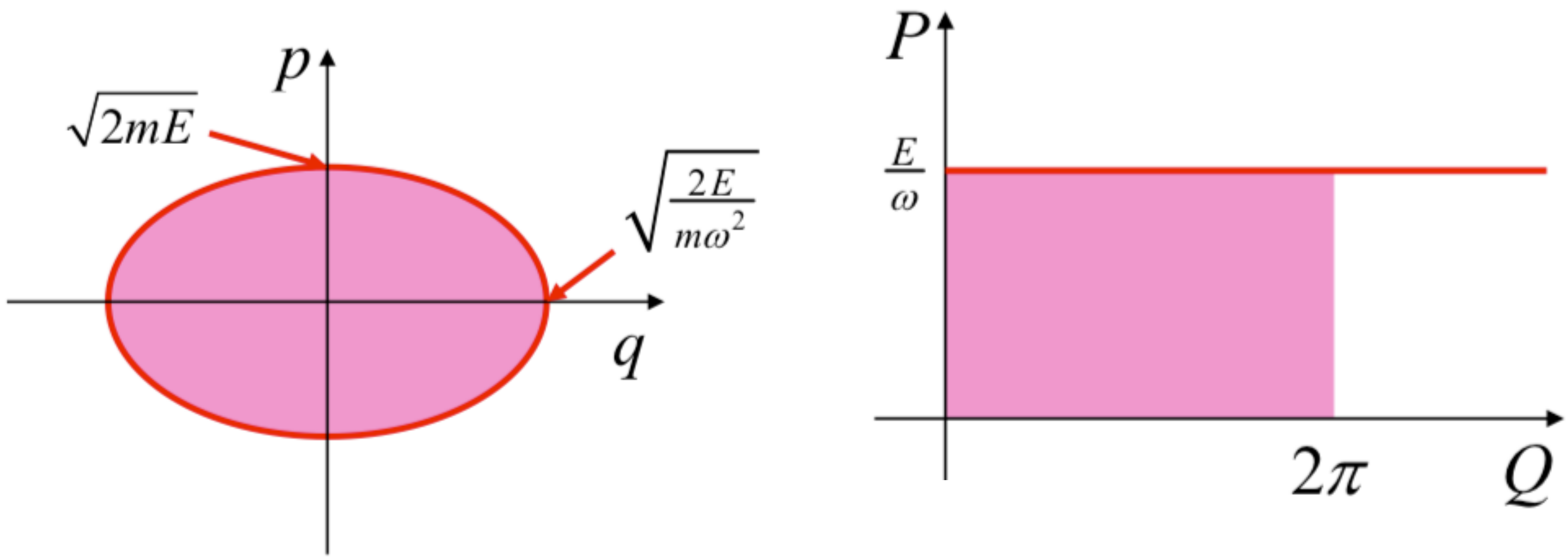
$$L(q,\dot{q},t) = p\dot{q} - H(q,p,t) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$$

$$K = \omega P \qquad \dot{Q} = \frac{\partial K}{\partial P} = \omega$$

$$L'(Q,\dot{Q},t) = P\dot{Q} - K(Q,P,t) = P\omega - \omega P = 0$$

Phase Space

Oscillator moves in the p-q and P-Q phase spaces



One cycle draws the same area $\dfrac{2\pi E}{\omega}$ in both spaces

The area swept by a cyclic system in the phase space is invariant