

Chi-Squared Testing for Conditional Moment Restrictions

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Abstract

Chi-squared (χ^2) tests are proposed for assessing the specification of regression models or general conditional moment restrictions (CMR). The data is partitioned based on explanatory variables into several cells, and the χ^2 tests evaluate whether the difference between the observed average of the dependent variable and its expected value under the model specification, within each cell, arises by chance. In contrast to existing omnibus procedures, χ^2 tests are pivotal and fairly insensitive to the curse of dimensionality. The computation of statistics is straightforward and does not require bootstrapping or smoothing techniques. Importantly, the test asymptotic properties are invariant to sample-dependent partitions, allowing for the preference of certain alternatives with a suitable partition choice. A Monte Carlo study provides evidence of the good performance of χ^2 tests compared to omnibus approaches implemented using bootstrap techniques, particularly in high-dimensional settings. An empirical application regarding returns to education of black people in the US complements our finite sample study.

JEL classification: C12, C31, C52

Keywords: Goodness-of-fit tests; Regression specifications; Distribution-free tests; Regressogram; Neyman-Pearson cells.

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1 Introduction

The identification and estimation of causal relationships often rely on models defined by conditional moment restrictions (CMR), typically in the form of parametric regression models. Ensuring the correct model specification is crucial for drawing valid inferences and providing reliable interpretations. This article introduces chi-squared (χ^2) tests to check the goodness-of-fit of CMR specifications. Leveraging their well-established usefulness in classical goodness-of-fit contexts, these tests offer a valuable complementary approach to existing methodologies, as discussed below.

The need to ensure correct model specification lies at the core of the many model-checking proposals in the literature. These proposals, rooted in the definition of the integrated regression function (IRF), expand upon classical goodness-of-fit tests designed for cumulative density function (CDF) specifications and apply them to the domain of regression analysis. The IRF generalizes the one-to-one relationship between density and the CDF to the regression framework, enabling the extension of all the tests proposed for CDFs to the context of regression. The alternative test statistics are characterized by a functional of the standard empirical process (SEP). The SEP is defined as the difference between the empirical IRF/CDF and its restricted counterpart, based on the assumed model specification, suitably scaled to stabilize the variance.

The omnibus tests are designed to assess whether any deviations of the SEP from zero arise by chance rather than due to model misspecification. Classical test statistics for CDF specifications, which are based on a norm of the SEP, such as the well-known Kolmogorov-Smirnov, Cramér-von Mises, or Anderson-Darling tests, have been extended to the context of regression model specifications by [Bierens \(1982\)](#), [Stute \(1997\)](#), and [Andrews \(1997\)](#). Several tests are based on transformations of the SEP. For instance, test statistics based on Fourier transforms of the SEP, which in fact compare characteristic functions (e.g., [Koutrouvelis and Kellermeier \(1981\)](#)), have been adapted to the regression case by [Bierens \(1982\)](#) and [Bierens and Ploberger \(1997\)](#). Tests based on martingale transforms of the SEP ([Khmaladze \(1982\)](#)) have been extended by [Khmaladze and Koul \(2004\)](#). Additionally, specification tests for Lebesgue densities, which rely on the convolution of the SEP and a kernel function (e.g., [Bickel and Rosenblatt \(1973\)](#)), have been

expanded to regression models by [Hardle and Mammen \(1993\)](#). All of these tests are known as minimum-distance tests.

When the model parameters are estimated from the sample, the SEP converges in distribution to a case-dependent process, and the tests are implemented with the help of bootstrap techniques. Two exceptions exist: (i) tests based on the SEP martingale transform, which converge in distribution to a Brownian motion with tabulated critical values; and (ii) tests based on the convolution of the SEP and a kernel function, suitably centered and standardized, which exhibit a standard normal limit null distribution when the bandwidth parameter vanishes at a suitable rate as the sample size grows. However, the size properties of convolution tests depend on the bandwidth choice, and in practice, the tests are implemented using bootstrap.

Tests based on the SEP exhibit poor power properties in the direction of high-frequency alternatives, as pointed out by [Durbin and Knott \(1972\)](#) based on the study of the SEP's principal components. See also [Stute \(1997\)](#) for the regression case. Tests that rely on convolutions of the SEP and a kernel, with a vanishing bandwidth, are capable of detecting high-frequency alternatives but fail to detect the standard alternatives converging to the model in the null at the parametric rate. All of these tests are affected by the curse of dimensionality, having limited power when the number of explanatory variables is large.

In practice, chi-squared (χ^2) tests, such as the [Pearson \(1900\)](#) test and its subsequent modifications, are among the most popular goodness-of-fit tests for CDF model specification. These tests, after partitioning the data into a number of cells, say L , evaluates whether the difference between the observed and the expected frequencies within each cell occur just by chance. The Pearson test statistics, for instance, can be represented as a quadratic form in the vector of differences between observed and expected frequencies. Note that χ^2 tests are not omnibus; rather, their goal is to detect alternatives where the true probability of the variable falling into a particular cell differs from the expected value specified by the model. However, they have important advantages with respect to tests based on SEP functionals. They are distribution-free and do not require the use of bootstrapping or smoothing techniques. Additionally, these tests are invariant to sample-dependent partitions, which can then be chosen to favor certain alternatives of interest.

Surprisingly χ^2 tests have not been extended to check the specification of CMR. This paper aims to fill this gap by grouping the data according to a partition of the explanatory variables into L classes, and considering χ^2 tests given by quadratic forms in the vector of differences between observed and expected (according to the model) sample averages within each class. The Pearson test analog is a weighted sum of these differences squared, which is in fact a J test for the orthogonality conditions between regression errors and the explanatory variables in each of the L classes. We also consider a generalized Wald test statistic, namely, quadratic form in the vector of differences between observed and expected sample averages using any estimator of the parameters in the model.

Much like the classical χ^2 test statistic, our tests have a limiting null distribution which is invariant to sample-dependent partitions. We propose algorithms for computing balanced cells, each containing approximately the same number of observations, as well as partitions designed to prioritize specific alternatives. For instance, we consider tests based on Neyman-Pearson (NP) cells ([Balakrishnan, Voinov, and Nikulin \(2013\)](#)) built upon deviations of the model under the null hypothesis from a predefined alternative model. Under correct specification of the alternative model, NP cells maximize the distance between the null and alternative models. When there are no specific alternatives to prioritize, we suggest using NP cells that compare the model under the null hypothesis with an auxiliary flexible specification of the regression model.

Monte Carlo simulations shows the good performances of the χ^2 tests compared to tests based on SEP functionals, especially in scenarios with a high number of covariates. The finite-sample study is complemented by an empirical illustration in which we apply the tests to analyze the returns of attending historically black college and universities (HBCU), relative to non-HCBU, for black students in the United States.

The structure of the paper is as follows: In the next section, we introduce the tests for regression specifications. The regression framework allows a natural interpretation of the tests and drastically reduces the notational burden required for the more general case; in Section 3, we discuss the asymptotic properties of the tests under general dependence of the partition cells boundaries from the data; in Section 4, we introduce a set of partitioning algorithms designed to enhance the power of the tests; in Section 5, we analyze the behavior of the tests against Pitman's local alternatives. The main focus of the discussion

is how the partition choice (and the number of cells) influence the sensitivity of the tests in detecting deviations from the null hypothesis; in Section 6, we extend the discussion to general CMR; in Section 7, we present a Monte Carlo study that shows the finite sample performances of the proposed χ^2 tests; in the last section, we provide a real-data application.

2 Chi-Squared Tests for CMR

Let $\{Z_i\}_{i=1}^n = \{(Y_i, X_i')'\}_{i=1}^n$ be an i.i.d sample from the \mathbb{R}^{1+d_x} -valued random vector $Z = (Y, X)'$ with distribution P , where Y is the response variable and X is the d_x -dimensional vector of explanatory variables with support $\mathcal{X} \subset \mathbb{R}^{d_x}$. If $\mathbb{E}[Y^2] < \infty$, the associated regression function, $m(x) := \mathbb{E}[Y|X = x]$, is well-defined and characterizes the (a.s.) optimal predictor of Y , in a mean square error sense. The correct specification hypothesis is stated as,

$$H_0 : m \in \mathcal{M} \tag{1}$$

with $\mathcal{M} = \{m_\theta(\cdot) : \theta \in \Theta\}$ and $\Theta \subset \mathbb{R}^{d_\theta}$ being a family of parametric regression functions and a suitable parameter space, respectively. Thus, under H_0 , there exists a $\theta_0 \in \Theta$ such that the IRF, $M(x) := \mathbb{E}[Y\mathbb{I}_{(-\infty, x]}(X)]$, and its version imposing the null hypothesis restrictions, $M_\theta(x) := \mathbb{E}[m_\theta(X)\mathbb{I}_{(-\infty, x]}(X)]$, are equal in all the borel sets of \mathcal{X} , i.e.

$$M\{A\} = \int_A Y dP = \int_A m_{\theta_0}(X) dP = M_{\theta_0}\{A\} \text{ for all } A \in \mathcal{B}^{d_x}, \tag{2}$$

where \mathcal{B}^{d_x} represent the Borel sigma-field of \mathbb{R}^{d_x} , $M\{A\} = \int_A M(dx)$ and $M_\theta\{A\} = \int_A M_\theta(dx)$ denote the Lebesgue-Stieltjes measure of M and M_θ over A , respectively. Recall that (2) is the definition of the regression function for the specification in H_0 (e.g., see definition 34.1 in Billingsley (2013)), which is equivalent to the following orthogonality conditions,

$$H_0 : \int_A \varepsilon_{\theta_0}(Z) dP = 0 \text{ for all } A \in \mathcal{B}^{d_x},$$

with $\varepsilon_\theta(z) = y - m_\theta(x)$ denoting the regression error.

Intuitively, when X takes values in a finite set and θ_0 is known (i.e., under the simple

hypothesis), a test for H_0 , following the proposal by [Pearson \(1900\)](#), consists of comparing the mean of Y given $x \in \mathcal{X}$ with $m_{\theta_0}(x)$ for all $x \in \mathcal{X}$. In this case, such a test is indeed omnibus (i.e. the alternatives detected from the test coincide with the negation of H_0). For any type of covariates, once the data is partitioned into L cells, say, χ^2 tests assess whether the difference between the expected and observed averages in each cell arose by chance.

Consider the partition $\gamma = (\gamma_1, \dots, \gamma_L)$ of the covariates support \mathcal{X} into L cells and let $\mathbf{I}_\gamma(x) = (\mathbb{I}_{\gamma_1}(x), \dots, \mathbb{I}_{\gamma_L}(x))'$ denote the vector of indicator functions over the sets within γ . The building block of the χ^2 test statistics is the standardized vector of differences between the observed averages of Y and the corresponding averages under the specification of H_0 in each cell,

$$\hat{\Phi}_\gamma(\theta) := \sqrt{n} (\hat{\boldsymbol{\mu}}_\gamma^0 - \hat{\boldsymbol{\mu}}_\gamma(\theta)) := \sqrt{n} \begin{bmatrix} \hat{\mu}_1^0 - \hat{\mu}_1(\theta) \\ \vdots \\ \hat{\mu}_L^0 - \hat{\mu}_L(\theta) \end{bmatrix}, \quad (3)$$

where $\hat{\mu}_l^0 := \hat{M}\{\gamma_l\} = n^{-1} \sum_{i=1}^n Y_i \mathbb{I}_{\gamma_l}(X_i)$ represents the empirical IRF evaluated at γ_l , with $\hat{M}(x) := n^{-1} \sum_{i=1}^n Y_i \mathbb{I}_{(-\infty, x]}(X_i)$, and $\hat{\mu}_l(\theta) := \hat{M}_\theta\{\gamma_l\} = n^{-1} \sum_{i=1}^n m_\theta(X_i) \mathbb{I}_{\gamma_l}(X_i)$ represents its restricted counterpart under H_0 , with $\hat{M}_\theta(x) := n^{-1} \sum_{i=1}^n m_\theta(X_i) \mathbb{I}_{(-\infty, x]}(X_i)$.

Under simple hypothesis, i.e. when θ_0 is known, by the central limit theorem, under the null,

$$\hat{\Sigma}_\gamma(\theta_0)^{-1/2} \hat{\Phi}_\gamma(\theta_0) \xrightarrow{d} N(0, I_L), \quad (4)$$

where $\hat{\Sigma}_\gamma(\theta) := n^{-1} \sum_{i=1}^n \varepsilon_\theta^2(Z_i) \mathbf{I}_\gamma(X_i) \mathbf{I}_\gamma(X_i)'$ estimates

$$\Sigma_{\gamma,0} := \text{Avar} \left(\hat{\Phi}_\gamma(\theta_0) \right) = \mathbb{E} \left[\varepsilon_{\theta_0}^2(Z) \mathbf{I}_\gamma(X) \mathbf{I}_\gamma(X)' \right] = \text{diag}\{\sigma_{0,1}^2, \dots, \sigma_{0,L}^2\}, \quad (5)$$

under H_0 , with $\sigma_{0,l}^2 = \sigma_l^2(\theta_0)$, and $\sigma_l^2(\theta) = \mathbb{E} [\varepsilon_\theta^2(Z) \mathbb{I}_{\gamma_l}(X)]$. Thus, taking for granted that $\rho_l = \mathbb{E} [\mathbb{I}_{\gamma_l}(X)] > 0$ for all l , under H_0 ,

$$\hat{\chi}_{\gamma,0}^2(\theta_0) \xrightarrow{d} \chi_L^2, \quad (6)$$

where

$$\hat{\chi}_{\gamma,0}^2(\theta) := \hat{\Phi}_{\gamma}(\theta)' \hat{\Sigma}_{\gamma}(\theta_0)^{-1} \hat{\Phi}_{\gamma}(\theta) = n \sum_{l=1}^L \frac{(\hat{\mu}_l^0 - \hat{\mu}_l(\theta))^2}{\sigma_l^2}. \quad (7)$$

The test statistic $\hat{\chi}_{\gamma,0}^2(\theta)$ extends the conventional Pearson's chi-squared test to check regression model specifications. It does so by replacing the comparison of observed frequencies with expected frequencies under the null hypothesis with a comparison of averages.

Of course, tests based on (3) are not omnibus, but designed for detecting deviations from H_0 of the type,

$$H_1(\gamma) : \boldsymbol{\mu}_{\gamma}^0 \neq \boldsymbol{\mu}_{\gamma}(\theta) \text{ for all } \theta \in \Theta, \quad (8)$$

where $\boldsymbol{\mu}_{\gamma}^0 = (\mu_{\gamma_1}^0, \dots, \mu_{\gamma_L}^0)' = \mathbb{E}[Y \mathbb{I}_{\gamma}(X)]$ is the vector of expected sample averages of Y in each cell and $\boldsymbol{\mu}_{\gamma}(\theta) = (\mu_{\gamma_1}(\theta), \dots, \mu_{\gamma_L}(\theta))' = \mathbb{E}[m_{\theta}(X) \mathbb{I}_{\gamma}(X)]$ is the vector of expected averages under H_0 . The partitions, therefore, cover a fundamental role in implementing the test and provide a flexible tool to exploit out-of-sample information on the possible alternatives. Neyman-Pearson (NP) cells (Balakrishnan, Voinov, and Nikulin (2013)), for instance, consist of the points in \mathcal{X} where the model under the null and a pre-specified alternative parametrization of $m(\cdot)$ meet (see Example 1 below). Under the pre-specified alternative, NP classes maximize the L2 norm of $\hat{\Phi}_{\gamma}$.

Example 1 (NP Classes, Balakrishnan, Voinov, and Nikulin (2013))

Consider testing the linear model $m_{\theta_0}(X_i) = \theta_{00} + X_i \theta_{01}$ against the alternative specification,

$$H_1 : m_{1,\theta_0^*}(X_i) = \theta_{00} + \theta_{01}X_i + \theta_{02} \sin\left(\frac{50X_i}{2\pi}\right)$$

where θ_0^ (and, thus, θ_0) is a known vector. NP classes split \mathcal{X} over the points where $\theta_{02} \sin\left(\frac{50X_i}{2\pi}\right) = 0$. As a result, under H_1 , $m_{\theta_0}(\cdot)$ is strictly bigger or strictly smaller than $m_{1,\theta_0^*}(\cdot)$ within each cell, and most cell-specific errors have the same sign,*

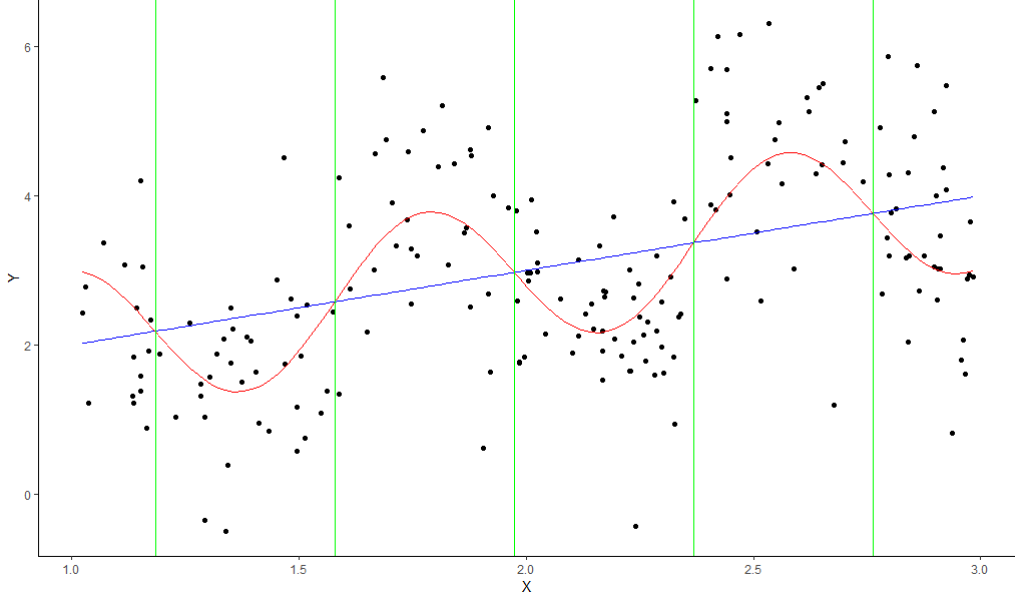


Figure 1: The graph depicts a random draw from the model under H_1 with $\theta_0^* = (1, 1, 1)$. The green lines depict the points where the model under the null (blue line) and under the alternative (red line) meet.

implying that the average error of a single cell is larger than the average error over the union of two contiguous cells (in absolute terms). As a matter of fact, in this example, the average error of (any) two contiguous cells is close to zero.

When θ_0 is unknown, the criterion in (7) suggests the following minimum distance estimator, hereafter referred to as the grouped GMM estimator:

$$\hat{\theta}_\gamma := \arg \min_{\theta \in \Theta} \chi_{\gamma, \tilde{\theta}}^2(\theta) \quad (9)$$

where

$$\hat{\chi}_\gamma^2(\theta) := \hat{\Phi}_\gamma(\theta)' \hat{\Sigma}_\gamma(\tilde{\theta})^{-1} \hat{\Phi}_\gamma(\theta) = n \sum_{l=1}^L \frac{(\hat{\mu}_l^0 - \hat{\mu}_l(\theta))^2}{\hat{\sigma}_l^2(\tilde{\theta})},$$

$\hat{\sigma}_l^2(\theta) = n^{-1} \sum_{i=1}^n \varepsilon_{\tilde{\theta}}^2(Z_i) \mathbb{I}_{\gamma_l}(X_i)$, and $\tilde{\theta}$ is some initial \sqrt{n} -consistent estimator of θ_0 . The estimator $\hat{\theta}_\gamma$, analogous to the limited information estimator (or multinomial maximum-likelihood estimator) of the χ^2 test developed by Pearson (1900) and Fisher (1925) for goodness-of-fit distribution model checking (see Cramér (1946)), is a GLS on the aggre-

gated data. In the linear case, i.e. $m_\theta(x) = x'\theta$,

$$\hat{\theta}_\gamma = \left[\sum_{l=1}^L \frac{\hat{\mu}_l^*(\tilde{\theta}) \hat{\mu}_l^*(\tilde{\theta})'}{\hat{\sigma}_l^2(\tilde{\theta})} \right]^{-1} \sum_{l=1}^L \frac{\hat{\mu}_l^*(\tilde{\theta}) \hat{\mu}_l^0}{\hat{\sigma}_l^2(\tilde{\theta})}$$

with $\hat{\mu}_l^*(\theta) = n^{-1} \sum_{i=1}^n \nabla m_\theta(X_i) \mathbb{I}_{\gamma_l}(X_i)$ and $\nabla m_{\tilde{\theta}} = d/d\theta' m_\theta|_{\theta=\tilde{\theta}}$. In the non-linear case, $\hat{\theta}_\gamma$ is obtained with numerical methods. The estimator belongs to the class of minimum-distance estimators considered by [Cristobal, Roca, and Manteiga \(1987\)](#) and [Koul and Ni \(2004\)](#), with the main difference that the regressogram ([Tukey \(1961\)](#)) is used instead of kernels, and the weighting introduced to improve efficiency. Under suitable regularity conditions and the null H_0 ,

$$\sqrt{n}(\hat{\theta}_\gamma - \theta_0) \xrightarrow{d} N \left(0, \left[\boldsymbol{\mu}_{\gamma,0}^{*'} (\Sigma_{\gamma,0})^{-1} \boldsymbol{\mu}_{\gamma,0}^* \right]^{-1} \right),$$

where $\boldsymbol{\mu}_{\gamma,0}^* = \boldsymbol{\mu}_\gamma^*(\theta_0)$, with $\boldsymbol{\mu}_\gamma^*(\theta) = (\mu_1^*(\theta), \dots, \mu_L^*(\theta))'$ and $\mu_l^*(\theta) = \mathbb{E}[\hat{\mu}_l^*(\theta)]$ denoting the matrix of partial derivatives of $\boldsymbol{\mu}_\gamma(\theta)$.

The $\hat{\chi}^2$ test statistics is, for $L > d_\theta$,

$$\hat{\chi}_\gamma^2 := \min_{\theta \in \Theta} \hat{\chi}_\gamma^2(\theta) = \hat{\chi}_\gamma^2(\hat{\theta}_\gamma), \quad (10)$$

which is in fact a J-test on the set of the L , out of the many, orthogonality conditions implied by the null,

$$\mathbb{E}[Y \mathbb{I}_{\gamma_l}(X)] = \mathbb{E}[m_{\theta_0}(X) \mathbb{I}_{\gamma_l}(X)] \quad \text{for all } l \in \{1, 2, \dots, L\}.$$

Thus, under H_0 , and for $L > d_\theta$,

$$\hat{\chi}_\gamma^2 \xrightarrow{d} \chi_{L-d_\theta}^2. \quad (11)$$

Is also well motivated, as suggested in classical goodness-of-fit χ^2 tests (e.g. [Nikulin \(1973\)](#) and [Rao and Robson \(1974\)](#)), to verify the null using the Wald testing principle

based on $\hat{\Phi}_\gamma(\tilde{\theta})$ for some other \sqrt{n} -consistent estimator $\tilde{\theta}$,

$$\hat{\mathcal{W}}_\gamma(\tilde{\theta}) := \hat{\Phi}_\gamma(\tilde{\theta}) \widehat{\text{Avar}}^- \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right) \hat{\Phi}_\gamma(\tilde{\theta}), \quad (12)$$

where $\widehat{\text{Avar}}^- \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$ is a consistent estimator of some generalized inverse of $\text{Avar} \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$, $\text{Avar}^- \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$ say. Under H_0 , regularity conditions, and $\widehat{\text{Avar}}^- \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right) \xrightarrow{p} \text{Avar}^- \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$,

$$\hat{\mathcal{W}}_\gamma(\tilde{\theta}) \xrightarrow{d} \chi_{r(\text{Avar}(\hat{\Phi}_\gamma(\tilde{\theta})))}^2,$$

where for a given square matrix A , $r(A)$ denotes its rank.

Taking for granted the asymptotic linearity of the estimator (see Assumption 3 in the next section), the covariance matrix of $\hat{\Phi}_\gamma(\tilde{\theta})$ is characterized as,

$$\text{Avar} \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right) = \Sigma_{\gamma,0} - \boldsymbol{\mu}_{\gamma,0}^* C'_{\gamma,0} - C_{\gamma,0} \boldsymbol{\mu}_{\gamma,0}^{*'} + \boldsymbol{\mu}_{\gamma,0}^* L_0 \boldsymbol{\mu}_{\gamma,0}^{*'}, \quad (13)$$

where $C_{\gamma,0} = \mathbb{E}[\varepsilon_{\theta_0}(Z) \mathbf{I}_\gamma(X) l_{\theta_0}(Z)']$, $L_0 = \mathbb{E}[l_{\theta_0}(Z) l_{\theta_0}(Z)']$, and $l_{\theta_0}(\cdot)$ is the influence function of $\tilde{\theta}$. When $\text{Avar} \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$ is full rank (e.g., if $\varepsilon_{\theta_0}(\cdot) \mathbf{I}_\gamma(\cdot)$ and $l_{\theta_0}(\cdot)$ have linearly independent components), the Wald test can be performed on any finite splitting of the data. In this case, a valid choice of $\widehat{\text{Avar}}^- \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$ is given by the inverse of the plug-in estimator,

$$\hat{W}_\gamma(\tilde{\theta}) := \hat{\Sigma}_\gamma(\tilde{\theta}) - \hat{\boldsymbol{\mu}}_\gamma^*(\tilde{\theta}) \hat{C}_\gamma(\tilde{\theta})' - \hat{C}_\gamma(\tilde{\theta}) \hat{\boldsymbol{\mu}}_\gamma^{*'}(\tilde{\theta}) + \hat{\boldsymbol{\mu}}_\gamma^*(\tilde{\theta}) \hat{L}(\tilde{\theta}) \hat{\boldsymbol{\mu}}_\gamma^{*'}(\tilde{\theta}), \quad (14)$$

where $\hat{\boldsymbol{\mu}}_\gamma^*(\theta) = (\hat{\mu}_1^*(\theta), \dots, \hat{\mu}_L^*(\theta))'$, $\hat{C}_\gamma(\theta) := n^{-1} \sum_{i=1}^n \varepsilon_\theta(Z_i) \mathbf{I}_\gamma(X_i) l_\theta(Z_i)'$, and $\hat{L}(\theta) := n^{-1} \sum_{i=1}^n l_\theta(Z_i) l_\theta(Z_i)'$. If the covariance matrix is rank deficient and the Moore-Penrose inverse of $\hat{W}_\gamma(\tilde{\theta})$, denoted as $\hat{W}_\gamma^+(\tilde{\theta})$, has rank converging in probability to the one of $\text{Avar} \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$, then $\hat{W}_\gamma^+(\tilde{\theta}) \xrightarrow{p} \text{Avar}^+ \left(\hat{\Phi}_\gamma(\tilde{\theta}) \right)$ (Theorem 2 of [Andrews \(1987\)](#)). However, this need not be the case ([Schott \(2016\)](#), p. 222-224) and more complex methods might be required (see, e.g., [Lütkepohl and Burda \(1997\)](#)). Consider, for instance, the nonlinear least squares (NLLS),

$$\tilde{\theta} = \arg \min_{\theta \in \Theta} \sum_{i=1}^n \varepsilon_\theta^2.$$

Under homoskedasticity, i.e. $\text{Var}(\varepsilon_{\theta_0}|X = x) = \sigma_0^2 = \sum_{l=1}^L \sigma_{0,l}^2$, it holds that, $l_{\theta_0}(z) = \Psi_0^{-1} \nabla m_{\theta_0}(x) \varepsilon_{\theta_0}(z)$, where $\Psi_0 = \mathbb{E} [\nabla m_{\theta_0}(X) \nabla m_{\theta_0}(X)']$. Under H_0 , when $\tilde{\theta}$ is asymptotically more efficient than $\hat{\theta}_\gamma$ and $\Psi_0 - \boldsymbol{\mu}_{\gamma,0}^{*'} \boldsymbol{\mu}_{\gamma,0}^*$ is p.d., $\text{Avar} \left(\hat{\boldsymbol{\Phi}}_\gamma(\tilde{\theta}) \right) = \Sigma_{\gamma,0} - \boldsymbol{\mu}_{\gamma,0}^{*'} \Psi_0^{-1} \boldsymbol{\mu}_{\gamma,0}^*$ is also p.d., and $\hat{\mathcal{W}}(\tilde{\theta}) \xrightarrow{d} \chi_L$. A similar reasoning holds for the probit/logit maximum likelihood estimator (MLE), as discussed in Section 7. Under heteroskedasticity, however, $\text{Avar} \left(\hat{\boldsymbol{\Phi}}_\gamma(\tilde{\theta}) \right)$ is not necessarily an invertible matrix.

Notice that the estimation of the covariance matrix, when $\text{Avar} \left(\hat{\boldsymbol{\Phi}}_\gamma(\tilde{\theta}) \right)$ is full rank, can be avoided by using a random normalizing weighting matrix, as in [Kuan and Lee \(2006\)](#) (see also [Kiefer, Vogelsang, and Bunzel \(2000\)](#) for an early reference). Similarly, the over-identification test of [Lee, Kuan, and Hsu \(2014\)](#) provides a robust version of the $\hat{\chi}^2$ test which does not require estimating $\hat{\Sigma}_\gamma(\tilde{\theta})$. In these cases, the limit null distribution is non-standard but pivotal.

In practice, however, the partition depends randomly on the data. This is the case, for instance, when the partition imposes approximately the same number of observations in each cell or when the cell boundaries depend on the unknown model parameters. Extending the convergence of the statistics above to these cases requires restricting the complexity of the partitioning algorithm, as we discuss in the next section.

3 Data-dependent Cells

When studying the large sample behavior of the statistics, it is crucial to address the inherent influence of the data on the selection of cells ([Watson \(1959\)](#)). [Moore and Spruill \(1975\)](#) were among the first to address this concern in the distribution model check literature providing a rigorous derivation of the null distribution of χ^2 tests with rectangular data-dependent cells. In a more general setting, [Pollard \(1979\)](#) established the result for cells of arbitrary form by utilizing a central limit theorem for empirical measures, while [Andrews \(1988a\)](#) and [Andrews \(1988b\)](#) extended the methodology to conditional distribution testing. In this section, we provide a similar result for the more general CMR testing framework. Specifically, we show that the grouped GMM estimator and the tests keep standard limit distribution when the partition is built with data-dependent cells.

To state the convergence results in a self-contained fashion, we list a minimal set

of assumptions consisting of smoothness conditions and restrictions on the partitioning algorithm complexity.

Assumption 1 (a) $\{Z_i = (Y_i, X_i')'\}_{i=1}^n$ is a sequence of i.i.d. random vectors with $\mathbb{E}|Y_i| < \infty$; (b) $\mathbb{E}[\varepsilon_{\theta_0}^2] < C$, with $C < \infty$; (c) Θ is a compact subset of \mathbb{R}^{d_θ} and θ_0 is an interior point of Θ .

Assumption 2 $m_\theta(\cdot)$ is twice continuously differentiable in a neighborhood Θ_0 of θ_0 , with $\Theta_0 \subset \Theta$. The gradient, $\nabla m_\theta(\cdot) = d/d\theta m_\theta(\cdot)$, is bounded by a square-integrable function $R(\cdot)$ such that $\sup_{\theta \in \Theta_0} |\nabla^{(j)} m_\theta(\cdot)| \leq R(\cdot)$ for all $j \in \{1, \dots, d_\theta\}$, where $\nabla^{(j)}$ denotes the j -th partial derivative, and $\mathbb{E}[R(X)^2] < \infty$.

Assumption 3

(a) The estimator $\tilde{\theta}$ satisfies the following asymptotic expansion under the null,

$$\sqrt{n}(\tilde{\theta} - \theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n l_{\theta_0}(Z_i) + o_p(1)$$

where $\mathbb{E}[l_{\theta_0}(Z)] = 0$ and $L_0 = \mathbb{E}[l_{\theta_0}(Z)l_{\theta_0}(Z)']$ is a finite and non-singular matrix.

(b) The vector-valued function $l_\theta(\cdot)$ is twice continuously differentiable in a neighborhood Θ_0 of θ_0 with first partial derivatives bounded by a square-integrable function $R_2(\cdot)$ such that $\sup_{\theta \in \Theta_0} |\nabla^{(j)} l_\theta(\cdot)| \leq R_2(\cdot)$ for all $j \in \{1, \dots, d_\theta\}$ and $\mathbb{E}[R_2(Z)^2] < \infty$.

Assumptions 1 and 2 are common in the model check literature with omnibus tests (see [Stute and Zhu \(2002\)](#), for instance). Compared to papers developing χ^2 tests based on probability model (e.g., [Tauchen \(1985\)](#)), we require slightly higher smoothness condition of the regression function but leave completely unrestricted the data distribution. Assumption 3(a) holds for most of the estimators used in practice, such as least square or GMM estimators, as well as for identification-robust minimum-distance estimators. While Assumption 3(b) is a technical requirement for the consistency of the plug-in estimator $\hat{W}_\gamma(\tilde{\theta})$.

We also state the necessary global identification and finite-variance conditions for the consistency and asymptotic normality of the grouped GMM estimator.

Assumption 2' (a) $\Sigma_{\gamma,0}$ is positive definite (p.d.); (b) $\mathbb{E}[Y\mathbf{I}_\gamma] = \mathbb{E}[m_\theta(X)\mathbf{I}_\gamma]$ if and only if $\theta = \theta_0$; (c) $\boldsymbol{\mu}_{\gamma,0}^*$ is full rank.

Following Pollard (1979) and Andrews (1988a), the data-dependent partitions are modeled as random functions over a class of properly restricted measurable sets (other approaches, such as the one proposed by Tauchen (1985), postulate cell dependence through a finite-dimensional parameter). Specifically, let \mathbb{C} be a class of measurable sets in \mathcal{X} from which the cells of each partition are drawn, and denote as \mathbb{D} the class of partitions of \mathcal{X} comprised of L sets from \mathbb{C} (L is fixed for all n); that is,

$$\mathbb{D} = \left\{ \gamma = (\gamma_1, \dots, \gamma_L) \in \mathbb{C}^L : \bigcup_{l=1}^L \gamma_l = \mathcal{X}, \gamma_l \cap \gamma_f = \emptyset, \forall l \neq f \right\}, \quad (15)$$

where γ_l and γ_f denote sets of the partition γ . We equip \mathbb{C} with the topology generated by the $L^2(F_x)$ semi-norm, F_x being the distribution of X under P , and give \mathbb{D} the corresponding product topology. This means that two set C_1, C_2 in \mathcal{X} are close if $F_x\{C_1 \Delta C_2\}$ is small, where Δ denotes the symmetric difference operator, $C_1 \Delta C_2 = C_1 \cup C_2 \setminus C_1 \cap C_2$.

Following the literature, we assume that for each sample size n , the corresponding partition is determined by a measurable mapping, denoted as $\hat{\gamma}$, from the underlying probability space to \mathbb{D} , which converges in probability to some fixed partition of cells in \mathbb{D} . In other words, for all $\epsilon > 0$,

$$P(F_x\{\hat{\gamma}_l \Delta \gamma_l\} > \epsilon) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for all } l = 1, 2, \dots, L.$$

Assumption 4 $\hat{\gamma} \xrightarrow{P} \gamma$ for some fixed set of cells $\gamma \in \mathbb{D}$

It is possible to relax Assumption 4 to cases where $\hat{\gamma}$ converges to a random element $\gamma_0 \in \mathbb{D}$. This is done by replacing the convergence in probability with a uniform tightness condition and asymptotic independence between $\hat{\gamma}$ and $\hat{\Phi}_\gamma(\theta_0)$. Refer to Section 6 of Pollard (1979) for a discussion.

Crucially, deriving the limit null distribution requires bounding the complexity of the partitions employed for test construction. To achieve this, we make the assumption that the cells are drawn from a class with a finite Vapnik-Cervonenkis (VC) dimension.

Assumption 5 \mathbb{C} is a VC class of sets.

This assumption is convenient because it is independent of the data distribution P , yet it is general enough for our purposes. For example, algorithms that generate a finite number of straight edges and the class of hyper ellipsoids are VC classes. Furthermore, unions, intersections, differences, and complements of VC classes are also VC classes (Andrews (1988a) and Pollard (1984) provide a thorough discussion, see also Section 2.6 in Van Der Vaart (1996)). A less stringent condition assumes that \mathbb{C} is a Donsker class for the underlying probability measure (Pollard (1979)). While this allows for a wider range of admissible partitions, verifying Donsker assumptions may prove more challenging in practice.

The following theorems show that the asymptotic distribution of the grouped GMM estimator and of the test statistics are unaffected by data-dependent cells. All the proofs are relegated to the appendix.

Theorem 1 *Let Assumptions 1, 2, 2', 4, 5 hold. Then, under the null hypothesis H_0 ,*

$$\sqrt{n}(\hat{\theta}_{\hat{\gamma}} - \theta_0) \xrightarrow{d} N\left(0, \left[\boldsymbol{\mu}_{\gamma,0}^{*\prime}(\Sigma_{\gamma,0})^{-1}\boldsymbol{\mu}_{\gamma,0}^*\right]^{-1}\right),$$

Theorem 2 *Let Assumptions 1, 2, 4, 5, and the null hypothesis H_0 hold. Then,*

(a) *Under Assumption 2' and $L > d_\theta$,*

$$\hat{\chi}_{\hat{\gamma}}^2 \xrightarrow{d} \chi_{L-d_\theta}^2.$$

(b) *Under Assumption 3, and $\widehat{\text{Avar}}^-\left(\hat{\boldsymbol{\Phi}}_{\hat{\gamma}}(\tilde{\theta})\right) \xrightarrow{p} \text{Avar}^-\left(\hat{\boldsymbol{\Phi}}_{\hat{\gamma}}(\tilde{\theta})\right)$,*

$$\hat{\mathcal{W}}_{\hat{\gamma}}(\tilde{\theta}) \xrightarrow{d} \chi_{r(\text{Avar}(\hat{\boldsymbol{\Phi}}_{\hat{\gamma}}(\tilde{\theta})))}^2.$$

Thus, Theorems 1 and 2 offer formal justification for the use of data-dependent partitioning algorithms in both estimating the grouped GMM and constructing the $\hat{\chi}^2$ test and the Wald test. In the next section, we propose various partitioning procedures, discussing

methods based on unsupervised clustering and model-based techniques that leverage information on the null hypothesis and pre-specified alternatives.

4 Partitioning Procedures

The partitioning algorithms can be classified into two categories: covariates-based methods and model-based methods, depending on the information used to split the sample. Covariates-based methods partition the data based exclusively on the characteristics of the population. This includes simple methods such as dividing the data into cubic cells, as well as more complex methodologies like k-means clustering, hierarchical clustering, and nearest-neighbor clustering, to mention only a few (see chapter 14 of [Hastie, Tibshirani, and Friedman \(2009\)](#) for a review of unsupervised clustering methods). In contrast, model-based methods exploit information about the model under the null hypothesis and about pre-specified alternatives to increase the power of the tests. The objective of these procedures is to generate cells where the difference between the average prediction under the null hypothesis and the alternative model is maximized, facilitating the detection of deviations from the null hypothesis towards the alternative.

We will first discuss the construction of partitions based on the union of statistically equivalent blocks (SEB) ([Gessaman \(1970\)](#)), which is a simple and intuitive method to partition the data into balanced cells, i.e., cells with (approximately) equal numbers of observations. This method can be applied directly to the matrix of covariates or to a lower-dimensional projection of the data.

Next, we will discuss the construction of Neyman-Pearson (NP) classes ([Balakrishnan, Voinov, and Nikulin \(2013\)](#)). These partitioning algorithms aim to maximize the probability of rejecting the null hypothesis by comparing the parametric fit under the null with the parametric fit under a pre-specified alternative. This method is particularly useful when the researcher's primary concern is rejecting a subset of deviations critical for the application at hand.

Lastly, when the alternative is left unrestricted, we propose the construction of NP classes using an auxiliary flexible specification of the regression model (see [Davidson, MacKinnon, et al. \(2004\)](#) Ch. 15.2 for a review of tests based on artificial regressions). The

procedure assumes a structured alternative model, which allows us to capture the most critical deviations from the null hypothesis by exploiting the conditional mean dependence of the parametric residuals from the fitted values under the alternative. Tests built with these partitions can be seen as a formalization of the statistical practice of looking at the residual-fitted values scatter plot to capture model misspecification and heteroskedasticity (e.g., [Cook \(1994\)](#)).

4.1 Union of Statistically Equivalent Blocks (SEB)

This algorithm offers a straightforward and intuitive approach to partition the data while ensuring that each cell contains approximately an equal number of observations. The SEB algorithm is easy to apply and aligns with statistical practice, which involves dividing the data into equiprobable cells when testing against unknown probability distributions (see, for instance, [Greenwood and Nikulin \(1996\)](#)).

Let \mathbb{F} be an $n \times d_F$ design matrix. The procedure involves sorting observations by column values and grouping them into S blocks at each iteration, resulting in a partition with S^{d_F} cells. Since the number of cells increases exponentially with d_F , applying the procedure directly to the raw design matrix, \mathbb{X} , can be problematic for datasets with large or moderate dimensions of covariates. Even for small choices of S , the final partition may become too fine for the validity of the tests' asymptotic approximation.

To avoid creating excessively fine partitions, one can divide the data by applying the SEB algorithm to a lower-dimensional projection of the data or to some known function of the covariates. In the Monte Carlo simulations of section 7, for example, we partitioned the data using the SEB algorithm on the first q principal components of \mathbb{X} (where $q < d_x$) and on the fitted values of the parametric model under the null hypothesis. We refer to these methods as **PSEB** and **FSEB**, respectively. It's worth noting that when the design matrix is one-dimensional (i.e., $d_F = 1$), the algorithm splits the sorted vector into S blocks.

Algorithm 1 (SEB Algorithm)

- (1) *Sort the observations based on the first column of \mathbb{F} and split them into S blocks, $S > 1$, containing $\lfloor n/S \rfloor$ observations each.*

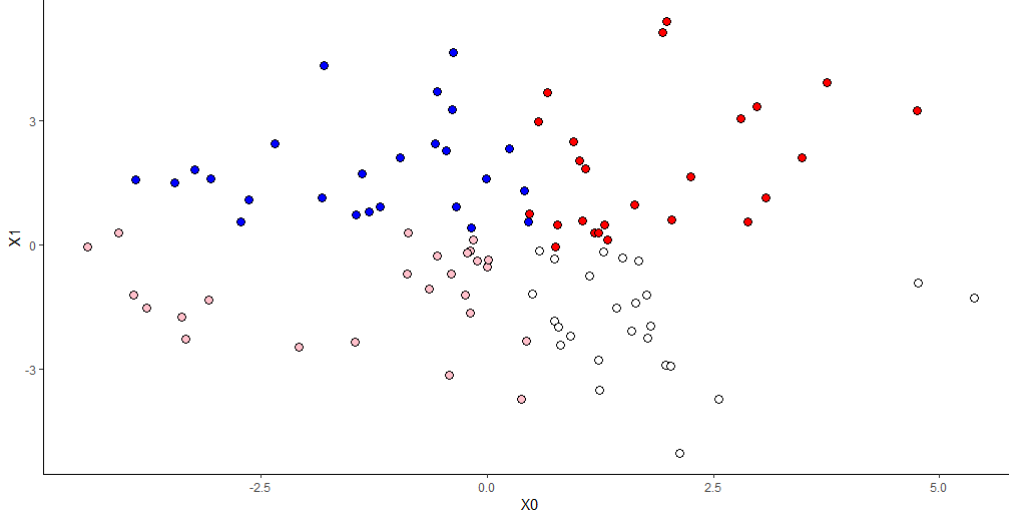


Figure 2: SEB partitioning on raw data with $d_F = 2$ and $S = 2$. The sample is extracted from a bivariate standard normal.

- (2) For each block, sort the observations based on the next column of \mathbb{F} and split them into S blocks containing $\lfloor n/S^2 \rfloor$ observations each.
- (3) Iterate until the last column of \mathbb{F} is reached.

4.2 Neyman-Pearson (NP) Classes

Frequently, the researcher's primary concern is rejecting a subset of deviations critical for the application at hand. For instance, when estimating Mincer's earning regression, it is common to compare it with alternative specifications of the log-income profile (e.g. [Polachek et al. \(2008\)](#)). Similarly, in the [Cox \(1972\)](#) model, a parsimonious parametric specification of the baseline hazard is often compared to more flexible models (e.g. [Seetharaman and Chintagunta \(2003\)](#)).

In this case, a procedure that splits the data where the deviations from the null toward the alternative hypothesis are the largest is preferable. To this end, we consider the construction of Neyman-Pearson (NP) classes built over the intersection points between the null and the alternative specification. These classes, in the probability distribution model checking literature, increase (or even maximize) Pearson's measure of discrepancy between the null and the alternative hypothesis (see Remark 3.3 in [Balakrishnan, Voinov, and Nikulin \(2013\)](#)).

Let H_1 be a given alternative parametric specification of the regression function,

$$H_1 : m \in \mathcal{M}_1$$

with $\mathcal{M}_1 = \{m_{1,\theta^*}(\cdot) : \theta^* \in \Theta^*\}$ being a family of parametric regression functions and $\Theta^* \subset \mathbb{R}^{d_{\theta^*}}$ a suitable parameter space. We partition the data using Neyman-Pearson classes, splitting over the points where the models under H_0 and under H_1 meet, i.e. where $m_{1,\tilde{\theta}^*}(\cdot) = m_{\tilde{\theta}}(\cdot)$, $\tilde{\theta}^*$ being a \sqrt{n} -consistent estimator of θ_0^* under H_1 . By doing so, the difference between the average prediction under the null and the alternative model in each class is the largest possible, making it easier to detect deviations from the null toward H_1 .

If X is a univariate variable, finding the intersection set, $\{x \in \mathcal{X} : m_{1,\tilde{\theta}^*}(x) - m_{\tilde{\theta}}(x) = 0\}$, and dividing the data accordingly is a straightforward process. However, when X is multivariate, the intersection set includes surfaces, and splitting \mathcal{X} across different dimensions becomes significantly more complex. Following the original proposal of [Greenwood and Nikulin \(1996\)](#), rather than splitting across the intersections, we partition \mathcal{X} into two classes where the difference between the two models is bigger or smaller than zero,

$$\begin{aligned}\hat{\gamma}_1 &= \{x \in \mathcal{X} : m_{1,\tilde{\theta}^*}(x) - m_{\tilde{\theta}}(x) > 0\}, \\ \hat{\gamma}_2 &= \{x \in \mathcal{X} : m_{1,\tilde{\theta}^*}(x) - m_{\tilde{\theta}}(x) \leq 0\},\end{aligned}$$

Notice that, if the alternative model is correctly specified, i.e. $\mathbb{E}[Y|X] = m_{1,\theta^*}(X)$ a.s. for some $\theta^* \in \Theta^*$, then $\hat{\gamma}_1$ is the set of points where the null model underestimates the regression function, and $\hat{\gamma}_2$ is the set of points where the null model overestimates the regression function. In this case, χ^2 tests with NP classes detect deviations of the type,

$$H_1(\gamma) : \mathbb{E}[(m_{1,\theta^*}(X) - m_{\theta}(X))\mathbb{I}_{\gamma_l}(X)] \neq 0 \text{ for some } l \in \{1, 2\} \text{ and for all } \theta \in \Theta.$$

where $\gamma_1 = \{x \in \mathcal{X} : m_{1,\tilde{\theta}^*}(x) - m_{\theta}(x) > 0\}$ and $\gamma_2 = \{x \in \mathcal{X} : m_{1,\tilde{\theta}^*}(x) - m_{\theta}(x) \leq 0\}$. That is, if the alternative model is correct, χ^2 tests with Neyman-Pearson classes are capable of detecting any deviations from it, at least asymptotically.

It is convenient to further divide the data into additional cells to ensure $L > d_{\theta}$

and perform the J-test. This is done by employing the SEB procedure, using a fixed value of $S = L/2$, on the respective vector of fitted values $\{m_{\tilde{\theta}}(X_i)\}_{i \in \gamma_l}$ within each cell. The advantages of using SEB on the fitted values are duals: on the one hand, we reduce the splitting to a uni-dimensional problem; on the other hand, we exploit the dependence between the residuals and the fitted values under the alternative to generate large aggregate squared residuals.

Algorithm 2 (Parametric Neyman-Pearson (PNP) Algorithm)

- (0) Fix an even number of cells, L , and a minimum number of observations in each cell, n_{min} .
- (1) Compute the fitted values under the null $\{m_{\tilde{\theta}}(X_i)\}_{i=1}^n$.
- (2) Compute the fitted values under the alternative $\{m_{1,\tilde{\theta}^*}(X_i)\}_{i=1}^n$.
- (3) Split the data into two cells, $\hat{\gamma}_1$ and $\hat{\gamma}_2$, where $\hat{\gamma}_1 = \{x \in \mathcal{X} : m_{1,\tilde{\theta}^*} - m_{\tilde{\theta}}(x) > 0\}$ and $\hat{\gamma}_2 = \{x \in \mathcal{X} : m_{1,\tilde{\theta}^*} - m_{\tilde{\theta}}(x) \leq 0\}$.
- (4) If the number of observations in either cell is less than n_{min} , merge them.
- (5) If $L > 2$, split each cell $\hat{\gamma}_l$, $l = 1, 2$, into $L/2$ cells by applying the SEB algorithm on the fitted values $\{m_{\tilde{\theta}}(X_i)\}_{i \in \hat{\gamma}_l}$ of each cell.

This is not the only solution to determine additional splits. The key point for the NP procedure is that, in each cell, the differences between the two models have the same sign. Once this is granted, we can split the two-cell NP partition with any other procedure.

Example 2 (Parametric Neyman-Pearson Classes)

Consider the same scenario as in Example 1, but now under a composite hypothesis, where the model parameters are unknown both under the null hypothesis and under the alternative. After obtaining the OLS estimates for both the null and alternative models, the Neyman-Pearson critical regions are defined based on the points $x \in \mathcal{X}$ where the equation $(\tilde{\theta}_{00} - \tilde{\theta}_{00}^*) + (\tilde{\theta}_{01} - \tilde{\theta}_{01}^*)x - \tilde{\theta}_{02}^* \sin(50x/2\pi) = 0$ holds. The resulting partition is depicted in Figure 3.

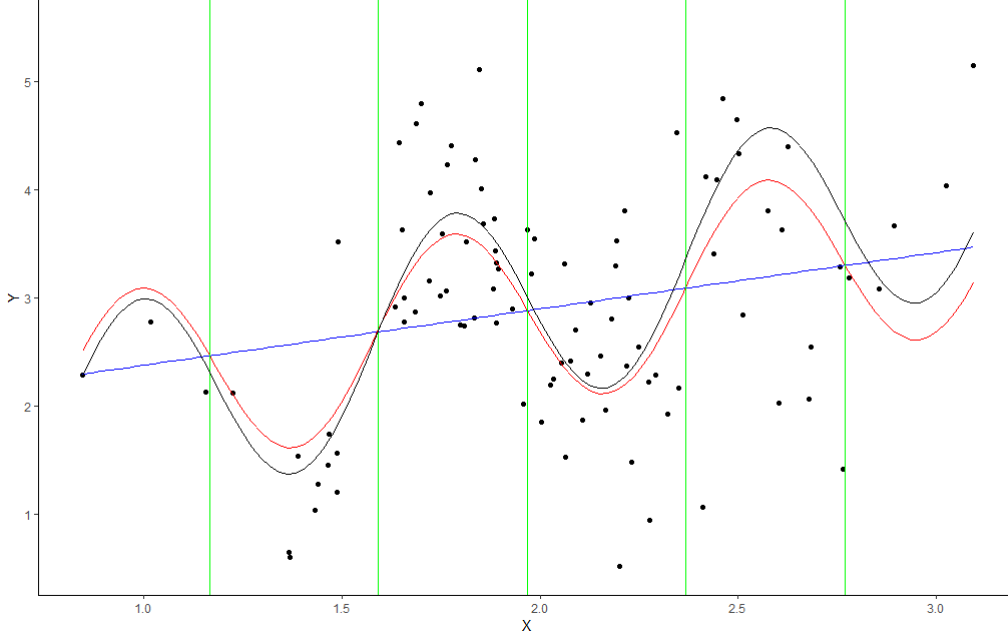


Figure 3: Neyman-Pearson classes with parametric estimate of the regression function. The data is generated under H_1 (black line) with $\theta^* = (1, 1, 1)$. The green line depicts the points where the estimates under the null (blue line) and the estimates under the alternative (red line) meet.

4.3 Flexible Neyman-Pearson (FNP) Classes

When considering the unrestricted alternative, Neyman-Pearson classes correspond to subsets of points where the model under the null hypothesis consistently overestimates and underestimates the (unknown) true regression model. In other words, these classes help us distinguish situations where $m(x)$ is bigger or smaller than $m_{\theta_0}(x)$, which characterize the alternative scenario.

Additionally, differences between the parametric model and the regression function become evident when we examine how the parametric residual relates to various function of the covariates vector, such as the model fitted values. To harness this connection, we adopt a structured alternative model of the form $\mathbb{E}[\varepsilon_{\theta_0}(Z)|X] = f(m_{\theta_0}(X))$ a.s., for some $\theta_0 \in \Theta$, where $f(\cdot)$ represents an unknown smooth function (see [Han, Ma, Ren, and Wang \(2023\)](#) for a similar approach). Consequently, under the null hypothesis, the regression error is independent in mean from the assumed regression function, i.e. $\mathbb{E}[\varepsilon_{\theta_0}(Z)|m_{\theta_0}(X)] = 0$. In contrast, under the alternative hypothesis, a systematic dependence in the means of the errors emerges.

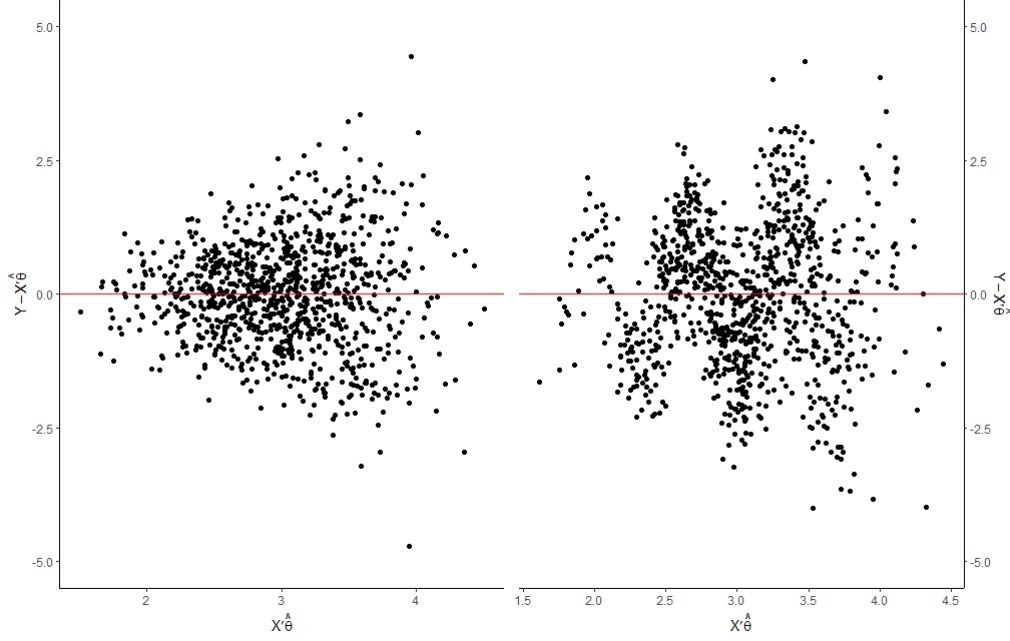


Figure 4: Residual-Fitted values scatter plot under the null (left panel) and under the alternative (right panel). Data generated using the model in (26) (pag.27) under homoskedasticity, $a = 0$, with $c = 0$ (left panel) and $c = 50$ (right panel).

This motivates the adoption of an auxiliary regression of $\varepsilon_{\theta_0}(Z)$ on a fixed-order polynomial expansion of the fitted values,

$$\varepsilon_{\tilde{\theta}}(Z_i) = \beta_0 + \sum_{j=1}^q \beta_j m_{\tilde{\theta}}(X_i)^j + \epsilon_i,$$

and use it to split where the predicted residuals are greater or smaller than zero. The procedure consists of using the auxiliary model

$$\mathcal{M}_{BP} = \left\{ \beta_0 + \beta_1 m_{\tilde{\theta}}(x) + \beta_2 m_{\tilde{\theta}}(x)^2 + \cdots + \beta_q m_{\tilde{\theta}}(x)^q : \beta \in \mathbb{B} \subset \mathbb{R}^q \right\},$$

to split the sample. Denote as $\tilde{\beta} = (\tilde{\beta}_1, \dots, \tilde{\beta}_q)$ the OLS estimates of β . Under the null hypothesis, $\tilde{\theta} = \theta_0 + o_p(1)$, and $\tilde{\beta}_0 = \tilde{\beta}_1 = \cdots = \tilde{\beta}_q = o_p(1)$, implying that the fitted values do not have predictive power over the residuals. Under the alternative, some of the estimated coefficients converge to a non-zero limit, and the fitted values exhibit predictive power on the residuals. The dependence, then, is used to split cases based on the predicted residuals to generate cells with residuals of the same sign and, thus, larger aggregate residuals.

Algorithm 3 (Flexible Neyman-Pearson (FNP) Algorithm)

- (0) Fix an even number of cells, L , and a minimum number of observations in each cell, n_{min} .
- (1) Compute the fitted values under the null $\{m_{\bar{\theta}}(X_i)\}_{i=1}^n$ and obtain the residuals, $\{\varepsilon_{\bar{\theta}}(Z_i)\}_{i=1}^n$.
- (2) Perform an auxiliary regression of $\varepsilon_{\bar{\theta}}(Z_i)$ on a fixed-order polynomial expansion of the fitted values and obtain the vector of predicted residuals, $\{\hat{\varepsilon}_{\bar{\theta}}(Z_i)\}_{i=1}^n$.
- (3) Split the data into two cells, $\hat{\gamma}_1$ and $\hat{\gamma}_2$, where $\hat{\gamma}_1 = \{x \in \mathcal{X} : \hat{\varepsilon}_{\bar{\theta}}(x) > 0\}$ and $\hat{\gamma}_2 = \{x \in \mathcal{X} : \hat{\varepsilon}_{\bar{\theta}}(x) \leq 0\}$.
- (4) If the number of observations in either cell is less than n_{min} , merge them.
- (5) If $L > 2$, split each cell $\hat{\gamma}_l$, $l = 1, 2$, into $L/2$ cells by applying the SEB algorithm on the fitted values $\{m_{\bar{\theta}}(X_i)\}_{i \in \hat{\gamma}_l}$ of each cell.

5 Local Power

We study the local power of the χ^2 test statistics under a functional Pittman drift. Specifically, we consider the following sequence of local alternatives,

$$H_{1,n} : m(x) = m_{\theta_0}(x) + \frac{1}{\sqrt{n}}h(x) \text{ a.s.}, \quad (16)$$

where $h(X)$ is a random variable representing departures from the null hypothesis with $\mathbb{E}[h(X)^2] < \infty$ and $0 \leq P(h(X) = 0) < 1$.

Let $\delta_{\gamma} = (\delta_1, \dots, \delta_L)$ represents the vector of expected distances between the null and alternative specifications in each cell of γ , with $\delta_l := \mathbb{E}[h(X)\mathbb{I}_{\gamma_l}(X)]$ for $l = 1, \dots, L$. Under $H_{1,n}$ and θ_0 known, the test statistics converge to a noncentral chi-squared distribution with L degrees of freedom,

$$\begin{aligned} \hat{\chi}_{\hat{\gamma},0}^2 &\xrightarrow{d} \chi_L^2(\lambda), \\ \hat{\mathcal{W}}_{\hat{\gamma}}(\theta_0) &\xrightarrow{d} \chi_L^2(\lambda), \end{aligned} \quad (17)$$

where,

$$\lambda = \delta'_\gamma (\Sigma_{\gamma,0})^{-1} \delta_\gamma = \sum_{l=1}^L \frac{\delta_l^2}{\sigma_{0,l}^2}. \quad (18)$$

is the non-centrality parameter.

Equations (17) and (18) highlight an important trade-off when choosing L . On one hand, the inequality,

$$\frac{\delta_l^2}{\sigma_{0,l}^2} + \frac{\delta_f^2}{\sigma_{0,f}^2} \geq \frac{(\delta_l + \delta_f)^2}{\sigma_{0,l}^2 + \sigma_{0,f}^2},$$

shows that λ is non-decreasing for nested partitions. That is, as the partition becomes finer with more cells, the test becomes more capable of detecting smaller deviations from the null hypothesis, resulting in increased power. On the other hand, the (global) power of χ^2 tests under the alternative hypothesis $H_{1,n}$ decreases as the number of cells increases. This decline in power is due to the higher variability of the limit distribution in (17) associated with a larger number of cells. As the number of cells grows, the distribution becomes more spread out, leading to a higher chance of observing test statistics falling in less extreme regions and, thus, not rejecting. Similar trade-offs have already been noted in the classical goodness-of-fit testing literature (see, e.g., [Kallenberg, Oosterhoff, and Schriever \(1985\)](#)).

When θ_0 is unknown, under $H_{1,n}$, the estimators of θ_0 typically follow the asymptotic expansion in Assumption 3' below (see, e.g., [Newey \(1985\)](#)).

Assumption 3' *Under $H_{1,n}$, the estimator $\tilde{\theta}$ satisfies,*

$$\sqrt{n}(\tilde{\theta} - \theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n l_{\theta_0}(Z_i) + \delta + o_p(1)$$

where the function $l_\theta(\cdot)$ is as defined in Assumption 3 and $\delta \in \mathbb{R}^{d_\theta}$ is a non-zero vector.

It's easy to show, for instance, that Assumption 3' is satisfied by the grouped GMM estimator, with

$$\delta = - \left[\boldsymbol{\mu}_{\gamma,0}^* (\Sigma_{\gamma,0})^{-1} \boldsymbol{\mu}_{\gamma,0}^{*'} \right]^{-1} \boldsymbol{\mu}_{\gamma,0}^* (\Sigma_{\gamma,0})^{-1} \delta_\gamma.$$

It follows that under $H_{1,n}$ and the same assumption of Theorem 2,

$$\begin{aligned}\hat{\chi}_{\hat{\gamma}}^2 &\xrightarrow{d} \chi_{L-d_\theta}^2(\lambda_1), \\ \hat{\mathcal{W}}_{\hat{\gamma}}(\tilde{\theta}) &\xrightarrow{d} \chi_{r(\text{Avar}(\hat{\Phi}_{\gamma}(\tilde{\theta})))}^2(\lambda_2),\end{aligned}\tag{19}$$

where

$$\lambda_1 = \delta_{\gamma}' \left[(\Sigma_{\gamma,0})^{-1} - (\Sigma_{\gamma,0})^{-1} \mu_{\gamma,0}^* (\mu_{\gamma,0}^{*\prime} (\Sigma_{\gamma,0})^{-1} \mu_{\gamma,0}^*)^{-1} \mu_{\gamma,0}^{*\prime} (\Sigma_{\gamma,0})^{-1} \right] \delta_{\gamma} \tag{20}$$

$$\lambda_2 = (\delta_{\gamma} - \mu_{\gamma,0}^* \delta)' \text{Avar}^{-}(\hat{\Phi}_{\gamma}(\tilde{\theta})) (\delta_{\gamma} - \mu_{\gamma,0}^* \delta) \tag{21}$$

are the respective non-centrality parameters. Comparing the power of these two tests is not a straightforward task, given the differences in both degrees of freedom and non-centrality parameters in their respective distributions. In fact, scenarios can be constructed where the $\hat{\chi}^2$ test exhibits greater power than the Wald test, and vice versa. Similar considerations apply to traditional goodness-of-fit χ^2 tests, as discussed in [Moore and Spruill \(1975\)](#) and [Moore \(1977\)](#), among others.

If instead of the non-centrality parameters, we consider the squared L2 norm of the drift,

$$\|\delta_{\gamma}\|_2^2 = \delta_{\gamma}' \delta_{\gamma} = \sum_{l=1}^L \delta_l^2 = \sum_{l=1}^L \mathbb{E} [h(X) \mathbb{I}_{\gamma_l}(X)]^2,$$

then, for any pair of δ_l and δ_f such that $\text{sgn}(\delta_l) = \text{sgn}(\delta_f)$, the inequality $(\delta_l + \delta_f)^2 \geq \delta_l^2 + \delta_f^2$ shows that the optimal partitioning for $\|\delta_{\gamma}\|_2^2$ consists of two cells: one containing the points where $h(x) = \sqrt{n}(m(x) - m_{\theta_0}(x))$ takes only positive values, and the other where it takes only negative values. In essence, the optimal partitioning of $\|\delta_{\gamma}\|_2^2$ is achieved with two Neyman-Pearson classes.

Proposition 1 *The euclidean norm of the drifts, $\|\delta_{\gamma}\|_2$, is maximized by two Neyman-Pearson classes, $\gamma^* = \{\gamma_i^*\}_{i=1}^2$,*

$$\gamma_1^* = \{x \in \mathcal{X} : h(x) \geq 0\} \quad \gamma_2^* = \{x \in \mathcal{X} : h(x) < 0\}$$

This suggests that, in some sense, Neyman-Pearson classes correspond to optimization

criteria for the non-centrality parameter. Of course, a rigorous optimization result should account for the dependence of the denominators on the cell boundaries. Doing so, however, has the inevitable consequence of determining splitting conditions that depend on the heteroskedasticity function.

6 General CMR

To extend the analysis to general moment restrictions, we introduce a vector of response variables, Y , taking values in $\mathcal{Y} \subset \mathbb{R}^{d_y}$, with $d_y \geq 1$, and a generalized residual vector (Wooldridge (1990)), $\boldsymbol{\varepsilon}_\theta(\cdot) : \mathbb{R}^{d_y} \times \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_\varepsilon}$ with $\boldsymbol{\varepsilon}_\theta(\cdot) = (\varepsilon_{1,\theta}(\cdot), \dots, \varepsilon_{d_\varepsilon,\theta}(\cdot))'$, defining parametric relationships between Y and X . The null hypothesis is defined as before,

$$\int_A \boldsymbol{\varepsilon}_{\theta_0}(Z) dP = 0 \text{ for all } A \in \mathcal{B}^{d_x}.$$

The generality of this framework allows testing for a wide range of econometric models such as simultaneous equation model identified by instrumental variables (Newey (1990)).

When the dimension of the generalized residual is bigger than one, it might be optimal to consider a partition for each component of $\boldsymbol{\varepsilon}_\theta(\cdot)$. In particular, for each $j \in \{1, \dots, d_\varepsilon\}$, let \mathbb{D}_j be a class of partitions of \mathcal{X} comprised of L_j sets from \mathbb{C} (L_j is fixed for all n); that is,

$$\mathbb{D}_j = \left\{ \boldsymbol{\gamma}_j = (\gamma_{j,1}, \dots, \gamma_{j,L_j})' \in \mathbb{C}^{L_j} : \cup_{l=1}^{L_j} \gamma_{j,l} = \mathcal{X}, \gamma_{j,l} \cap \gamma_{j,f} = \emptyset, \forall l \neq f \right\}, \quad (22)$$

We let $\mathcal{E}_\theta(\cdot)$ be the $\bar{L} \times \bar{L}$ block diagonal matrix of generalized residuals with main diagonal elements given by $\{\varepsilon_{j,\theta}(\cdot) I_{L_j}\}_{j=1}^{d_\varepsilon}$, where $\bar{L} = \sum_j L_j$. If $L_1 = L_2 = \dots = L_{d_\varepsilon} = L$, then $\mathcal{E}_\theta(\cdot) = \text{diag}[\boldsymbol{\varepsilon}_\theta(\cdot)] \otimes I_L$, where $\text{diag}[\boldsymbol{\varepsilon}_\theta(\cdot)] = \text{diag}\{\varepsilon_{1,\theta}(\cdot), \dots, \varepsilon_{d_\varepsilon,\theta}(\cdot)\}$ is the $d_\varepsilon \times d_\varepsilon$ diagonal matrix with the components of $\boldsymbol{\varepsilon}_\theta(\cdot)$ on the main diagonal and \otimes denotes the Kronecker product.

The χ^2 test statistics can be expressed as quadratic forms of

$$\hat{\boldsymbol{\Phi}}_\gamma(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathcal{E}_\theta(Z_i) \mathbf{I}_\gamma(X_i), \quad (23)$$

where $\mathbf{I}_\gamma(\cdot) = (\mathbf{I}'_{\gamma_1}, \dots, \mathbf{I}'_{\gamma_{d_\varepsilon}})'$ is the vector of indicator functions over all the partitions. While the covariance matrices of $\hat{\Phi}_\gamma(\theta_0)$ and $\hat{\Phi}_\gamma(\tilde{\theta})$ under the null are given by,

$$\Sigma_{\gamma,0} = \mathbb{E}[\mathcal{E}_{\theta_0}(Z)\mathbf{I}_\gamma(X)\mathbf{I}_\gamma(X)'\mathcal{E}_{\theta_0}(Z)'], \quad (24)$$

and,

$$\text{Avar}\left(\hat{\Phi}_\gamma(\tilde{\theta})\right) = \Sigma_{\gamma,0} - \boldsymbol{\mu}_{\gamma,0}^* C'_{\gamma,0} - C_{\gamma,0} \boldsymbol{\mu}_{\gamma,0}^{*'} + \boldsymbol{\mu}_{\gamma,0}^* L_0 \boldsymbol{\mu}_{\gamma,0}^{*'}, \quad (25)$$

where $\boldsymbol{\mu}_{\gamma,0}^* = \mathbb{E}[\nabla \mathcal{E}_{\theta_0}(Z)\mathbf{I}_\gamma(X)]$ is the Jacobian matrix, $C'_{\gamma,0} = \mathbb{E}[\mathcal{E}_{\theta_0}(Z)\mathbf{I}_\gamma(X)l_{\theta_0}(Z)']$, and L_0 is defined as before.

It's important to note that $\Sigma_{\gamma,0}$ is not necessarily diagonal, unless $\varepsilon_\theta(\cdot)$ consists of orthogonal components or $d_\varepsilon = 1$. As a result, in the general framework, the advantages of the $\hat{\chi}^2$ test in terms of implementability are no longer applicable. Nevertheless, the $\hat{\chi}^2$ test remains a viable choice, especially when the rank of $\text{Avar}\left(\hat{\Phi}_\gamma(\tilde{\theta})\right)$ is unknown

7 Monte Carlo Study

We simulate the rejection rates of the χ^2 tests. The experiment is divided into two parts: first, we verify the null hypothesis of linearity in a regression model, and then we check the correct specification of a probit model for a dichotomous response variable. To provide a basis for comparison with existing tests for H_0 , we report the rejection rates of the tests proposed by [Stute \(1997\)](#) and [Stute and Zhu \(2002\)](#) for regression specifications based on marked residuals processes, as well as the specification test for propensity score specifications proposed by [Sant'Anna and Song \(2019\)](#) (hereafter, SS).

Linear Model

The data generating process (DGP) is given by,

$$Y_i = \sum_{j=1}^{d_x} X_{j,i} + b \sin\left(\frac{c \sum_{j=1}^{d_x} X_{j,i}}{2\pi}\right) + \sigma(X_i)\epsilon_i \quad (26)$$

where $X_i = (X_{1,i}, \dots, X_{d_x,i})$ is a vector of mutually independent covariates distributed uniformly over $[0,1]$, $X_{i,j} \sim U[0,1]$, the error distributes normally and independently from X , $\epsilon|X \sim N(0,1)$,

$$\sigma^2(X) = \frac{g(X)}{\mathbb{E}[g(X,a)]},$$

and $g(X,a) = e^{aX_1}$, with $\mathbb{E}[\sigma^2(X)] = 1$. The model under the null corresponds to $b = 0$, while for the alternative models we set $b = 0.5$. The parameter a controls the heteroskedasticity severity, with $a = 0$ corresponding to homoskedasticity, while c governs the extent of departures from linearity. In the first part of the simulations, we verify the size of the test for $a \in \{0, 3\}$, we then fix $a = 3$ and proceed to examine the tests power for $c \in \{10, 50\}$. Notably, with $c = 50$, the deviations manifest at higher frequencies making them more challenging to discern from sampling error. We generate samples of size $n \in \{100, 200, 500, 1000\}$ with a dimension of the covariate vector $d_x \in \{5, 10\}$. The number of Monte Carlo repetition, R , is set at $R = 5000$ for $n = 100$, $R = 2500$ for $n = 200$, $R = 1250$ for $n = 500$, $R = 625$ for $n = 1000$. We report rejection rates at the 5% nominal level, results at 1% and 10% are similar and available upon request.

The $\hat{\chi}^2$ test is built as described in Section 2 using the grouped GMM estimator,

$$\hat{\theta}_\gamma = \left[\sum_{l=1}^L \frac{\bar{X}_l \bar{X}_l'}{\hat{\sigma}_l^2(\tilde{\theta})} \right]^{-1} \sum_{l=1}^L \frac{\bar{X}_l \hat{\mu}_l^0}{\hat{\sigma}_l^2(\tilde{\theta})}.$$

where $\bar{X}_l = n^{-1} \sum_{i=1}^n X_i \mathbb{I}_{\gamma_l}(X_i)$, and $\tilde{\theta}$ is the OLS estimator. Given our data-generating process, one can easily show that the influence function of $\tilde{\theta}$, $l_{\theta_0}(z) = \mathbb{E}[X X']^{-1} z \varepsilon_{\theta_0}(z)$, and $\varepsilon_{\theta_0}(\cdot) \mathbf{I}_\gamma(z)$ have linearly independent components (however, the independence does not hold when X includes a constant term). Thus, $\text{Avar}(\hat{\Phi}_\gamma(\tilde{\theta}))$, is full rank and the Wald test can be implemented using the plug-in estimator $\hat{W}_\gamma(\tilde{\theta})$,

$$\begin{aligned} \hat{W}_\gamma(\tilde{\theta}) &= \mathbb{E}_n [\varepsilon_{\tilde{\theta}}(Z_i)^2 \mathbf{I}_\gamma(X_i) \mathbf{I}_\gamma(X_i)'] - \mathbb{E}_n [\mathbf{I}_\gamma(X_i) X_i'] \mathbb{E}_n [X_i X_i']^{-1} \mathbb{E}_n [\varepsilon_{\tilde{\theta}}(Z_i)^2 X_i \mathbf{I}_\gamma(X_i)'] - \\ &\quad - \mathbb{E}_n [\varepsilon_{\tilde{\theta}}(Z_i)^2 \mathbf{I}_\gamma(X_i) X_i'] \mathbb{E}_n [X_i X_i']^{-1} \mathbb{E}_n [X_i \mathbf{I}_\gamma(X_i)'] + \\ &\quad + \mathbb{E}_n [\mathbf{I}_\gamma(X_i) X_i'] \mathbb{E}_n [X_i X_i']^{-1} \mathbb{E}_n [\varepsilon_{\tilde{\theta}}(Z_i)^2 X_i X_i'] \mathbb{E}_n [X_i X_i']^{-1} \mathbb{E}_n [X_i \mathbf{I}_\gamma(X_i)']. \end{aligned}$$

We examine the performance of the test under the following partitioning methods:

SEB on the first principal component (**PSEB**), SEB on the vector of fitted values (**FSEB**), parametric NP (**PNP**), and Flexible NP splitting (**FNP**). When $d_x = 5$, we consider partitions with $L \in \{4, 6, 8\}$ cells, and when $d_x = 10$, we consider partitions with $L \in \{8, 10, 12\}$ cells. The minimum number of observations in each cell of the initial two-cell partition of the FNP and PNP algorithms is set to $n_{min} = n/5$. To construct the parametric NP cells, we employ the parametric fit of the model under the correctly specified alternative (where c is known). Tests constructed in this manner utilize additional information about the alternative specifications of the regression function and, therefore, are anticipated to possess higher power compared to other tests. Including them allows for a comprehensive comparison with the other feasible partitioning methods. In the FNP method, instead, we run a regression of the parametric residuals on a polynomial expansion of the fitted values of order $q = 3$ and use the predicted residuals to produce the initial two-cell split.

In order to make a comparison with pre-existing tests, we report the performance of two minimum-distance tests for regression specifications based on marked residuals processes indexed by real vectors and real numbers; refer to [Stute \(1997\)](#) and [Stute and Zhu \(2002\)](#), respectively. Let $R_{1,n}(x_1)$ represent the marked empirical process of the residuals indexed by $x_1 \in \mathbb{R}^{d_x}$, and let $R_{2,n}(x_2)$ represent the marked empirical process of the residuals indexed by $x_2 \in \mathbb{R}$,

$$R_{1,n}(x_1) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(Y_i - X_i' \tilde{\theta} \right) \mathbb{I}\{X_i \leq x_1\},$$

$$R_{2,n}(x_2) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(Y_i - X_i' \tilde{\theta} \right) \mathbb{I}\{X_i' \tilde{\theta} \leq x_2\}.$$

Incidentally, the marked residual process indexed by partitions generated with the **FSEB** and the **SP** methods,

$$\hat{\Phi}_{\hat{\gamma}_l}(\tilde{\theta}) = n^{-1/2} \sum_{i=1}^n \varepsilon_{\tilde{\theta}}(Z_i) \mathbb{I}\{X_i' \tilde{\theta} \in C_l\},$$

where $\bigcup_{l=1}^L C_l = \mathbb{R}$, is a finite cells version of the process used in the [Stute and Zhu \(2002\)](#) test. The test statistics consist of a functional of $R_{1,n}(\cdot)$ and $R_{2,n}(\cdot)$. In these simulations,

we only consider the Kolmogorov-Smirnov functional,

$$KS1 = \sup_{x_1 \in \mathbb{R}^{d_x}} |R_{1,n}(x_1)|,$$

$$KS2 = \sup_{x_2 \in \mathbb{R}} |R_{2,n}(x_2)|.$$

Since these tests have non-pivotal limiting distribution, we approximate it with a Wild bootstrap procedure (see [Stute, Manteiga, and Quindimil \(1998\)](#)) using $B = 500$ bootstrap repetitions for each replication.

The simulation results for the linear model presented in Tables 1 (size control) and 2 (power) reveal several key findings. In Table 1, both the $\hat{\mathcal{W}}$ and $\hat{\chi}^2$ tests generally exhibit effective size control. Nevertheless, some size distortion persists, particularly depending on the partitioning methods and the number of cells, even for large sample sizes. This distortion arises from irregularities in the replication of the partitioning algorithm, which generates cells with too few observations, particularly when the number of observations in each cell of the initial two-cell split is close to the minimum limit set in the algorithm. Such irregularities disappear when n is large enough or when n_{min} is set to a higher value. Notably, the $KS1$ test experiences pronounced size distortions when $d_x = 10$, while the $KS2$ test demonstrates greater robustness in the face of higher covariate dimensions.

Turning to Table 2, our focus shifts to assessing the tests' ability to detect deviations from linearity when the null hypothesis is false ($b = 0.5$). The $\hat{\mathcal{W}}$ test generally exhibits higher power compared to the $\hat{\chi}^2$ test, especially in cases with larger values of c . However, power varies depending on the choice of partitioning method and other parameters. For example, the $\hat{\mathcal{W}}$ test, when employing model-based partitioning, consistently outperforms omnibus tests in most settings, while the J-test performs better than the $KS2$ test primarily in scenarios involving high-frequency deviations ($c = 50$). As expected, both χ^2 tests using **PNP** partitions consistently exhibit the highest power across all scenarios, given their ability to leverage out-of-sample information. However, they exhibit low power when the partitioning is done with the **PSEB** method.

Table 1: Linear Model: Size ($b = 0$)

				PSEB		FSEB		FNP		PNP			
n	d_x	a	L	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	KS1	KS2
100	5	0	4	5.18	NA	4.86	NA	6.46	NA	4.86	NA	5.14	6.40
			6	4.88	5.70	4.32	5.84	4.88	6.06	4.32	5.84		
			8	3.83	5.44	4.48	5.86	4.24	5.92	4.48	5.86		
100	5	3	4	4.88	NA	5.09	NA	6.32	NA	5.09	NA	4.86	7.12
			6	4.42	5.72	3.98	5.08	4.38	5.72	3.98	5.08		
			8	3.62	4.90	3.42	4.62	3.92	5.26	3.42	4.62		
100	10	0	8	4.96	NA	4.74	NA	5.66	NA	4.74	NA	1.46	8.14
			10	4.92	NA	4.68	NA	5.28	NA	4.68	NA		
			12	4.72	6.24	4.54	6.24	4.60	6.38	4.54	6.24		
100	10	3	8	4.68	NA	4.94	NA	4.80	NA	4.94	NA	0.78	9.16
			10	4.06	NA	3.66	NA	4.40	NA	3.66	NA		
			12	3.48	5.88	3.56	6.75	3.86	6.26	3.56	6.75		
200	5	0	4	4.72	NA	4.56	NA	6.12	NA	4.56	NA	5.56	6.75
			6	4.16	5.08	4.51	3.64	5.04	5.56	4.51	3.64		
			8	4.72	4.44	4.92	5.68	5.16	6.44	4.92	5.68		
200	5	3	4	4.44	NA	4.56	NA	5.92	NA	4.56	NA	5.56	6.56
			6	4.60	5.40	4.00	5.20	5.40	6.64	4.00	5.20		
			8	4.48	5.52	4.08	5.48	4.44	5.60	4.08	5.48		
200	10	0	8	5.28	NA	4.08	NA	6.00	NA	4.08	NA	2.80	6.60
			10	4.08	NA	5.20	NA	5.20	NA	5.20	NA		
			12	4.68	5.64	4.32	6.24	4.76	6.12	4.32	6.24		
200	10	3	8	4.63	NA	4.96	NA	5.20	NA	4.96	NA	1.40	6.04
			10	5.20	NA	4.36	NA	4.56	NA	4.36	NA		
			12	4.68	6.28	4.00	5.76	4.36	5.96	4.00	5.76		
500	5	0	4	5.28	NA	5.28	NA	6.00	NA	5.28	NA	6.40	7.28
			6	5.04	5.20	5.52	6.56	5.28	5.68	5.52	6.56		
			8	5.04	4.72	5.12	6.24	6.08	7.20	5.12	6.24		
500	5	3	4	4.16	NA	4.96	NA	6.48	NA	4.96	NA	5.76	5.84
			6	5.04	4.32	4.32	3.92	5.52	5.44	4.32	3.92		
			8	4.24	4.08	5.20	6.32	5.68	6.00	5.20	6.32		
500	10	0	8	5.60	NA	4.72	NA	6.16	NA	4.72	NA	4.32	6.32
			10	4.72	NA	4.80	NA	5.12	NA	4.80	NA		
			12	5.04	5.20	5.44	5.52	5.76	5.68	5.44	5.52		
500	10	3	8	5.04	NA	4.32	NA	5.12	NA	4.32	NA	3.20	5.60
			10	3.92	NA	5.28	NA	4.08	NA	5.28	NA		
			12	4.72	5.12	4.16	4.08	4.56	5.76	4.16	4.08		
1000	5	0	4	4.74	NA	5.62	NA	5.62	NA	5.62	NA	3.40	5.92
			6	3.85	4.88	5.03	4.59	5.77	5.92	5.03	4.59		
			8	4.88	5.77	4.74	5.92	6.37	6.22	4.74	5.92		
1000	5	3	4	5.77	NA	5.77	NA	6.37	NA	5.77	NA	3.55	7.11
			6	4.59	5.77	6.51	6.51	6.07	5.92	6.51	6.51		
			8	6.51	6.07	3.70	4.00	4.44	4.59	3.70	4.00		
1000	10	0	8	5.03	NA	5.33	NA	6.07	NA	5.33	NA	6.37	4.88
			10	5.48	NA	5.77	NA	5.48	NA	5.77	NA		
			12	5.62	6.22	5.48	5.18	5.48	6.07	5.48	5.18		
1000	10	3	8	4.74	NA	5.48	NA	4.14	NA	5.48	NA	6.22	5.62
			10	5.03	NA	4.74	NA	3.40	NA	4.74	NA		
			12	4.44	5.92	4.44	5.03	7.11	6.07	4.44	5.03		

Rejection rates at the nominal level $\alpha = 0.05$ of the $\hat{\chi}^2$ and the Wald test under different partitioning methods. n , d_x , a , L , respectively denote the sample size, the number of covariates, the degree of heteroskedasticity, and the number of cells. The last two columns report the rejection rates of the Kolmogorov-Smirnov tests.

Table 2: Linear Model: Power ($b = 0.5$)

				PSEB		FSEB		FNP		PNP			
n	d_x	c	L	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	KS1	KS2
100	5	10	4	22.02	NA	39.12	NA	46.76	NA	44.60	NA	31.54	35.75
			6	18.32	6.74	37.01	15.34	40.34	16.36	40.08	30.12		
			8	15.16	6.02	30.71	19.60	33.18	19.54	33.28	35.64		
100	5	50	4	4.80	NA	6.36	NA	8.34	NA	72.14	NA	5.20	7.98
			6	4.48	5.38	7.04	7.32	7.66	8.16	63.72	42.62		
			8	3.70	4.51	7.60	8.18	7.43	8.10	55.18	53.12		
100	10	10	8	5.98	NA	27.80	NA	28.48	NA	40.64	NA	1.58	28.16
			10	4.97	NA	24.18	NA	24.32	NA	34.69	NA		
			12	4.76	6.34	20.08	16.08	19.44	15.16	28.04	32.80		
100	10	50	8	4.78	NA	4.97	NA	5.34	NA	53.24	NA	0.66	8.59
			10	4.24	NA	4.54	NA	4.84	NA	45.84	NA		
			12	3.48	5.70	4.26	6.48	4.26	6.30	39.58	35.42		
200	5	10	4	40.64	NA	73.36	NA	81.12	NA	80.80	NA	59.84	65.08
			6	35.44	7.12	75.44	25.92	79.04	24.76	79.96	49.64		
			8	31.92	6.20	72.44	36.12	75.28	34.12	75.12	62.92		
200	5	50	4	5.12	NA	12.28	NA	18.28	NA	96.52	NA	5.04	10.24
			6	4.40	5.44	19.91	10.96	20.92	12.04	94.96	59.28		
			8	4.56	4.68	22.88	17.96	23.40	16.44	92.96	82.44		
200	10	10	8	8.00	NA	72.40	NA	75.40	NA	82.60	NA	4.04	50.44
			10	7.24	NA	69.16	NA	71.04	NA	79.16	NA		
			12	6.68	6.68	66.04	30.24	65.68	29.48	74.96	53.08		
200	10	50	8	4.92	NA	7.88	NA	8.00	NA	91.08	NA	1.44	7.40
			10	4.56	NA	7.52	NA	7.12	NA	88.36	NA		
			12	4.16	5.76	7.16	6.84	6.80	7.04	86.04	56.72		
500	5	10	4	81.04	NA	99.04	NA	99.92	NA	99.84	NA	94.72	98.16
			6	75.36	8.80	99.20	40.24	99.44	40.48	99.60	68.72		
			8	71.28	9.27	99.60	69.19	99.68	64.72	99.52	92.40		
500	5	50	4	4.40	NA	35.04	NA	50.72	NA	100.00	NA	5.36	19.60
			6	5.68	5.44	73.44	33.43	70.88	30.24	100.00	75.68		
			8	5.28	6.88	82.64	57.92	81.28	55.92	100.00	96.64		
500	10	10	8	14.16	NA	99.84	NA	99.92	NA	100.00	NA	17.52	94.40
			10	11.76	NA	99.84	NA	100.00	NA	99.92	NA		
			12	10.00	5.76	99.84	59.84	99.84	56.72	99.84	81.44		
500	10	50	8	4.56	NA	42.64	NA	38.40	NA	100.00	NA	4.40	10.00
			10	5.28	NA	45.52	NA	45.20	NA	100.00	NA		
			12	5.28	5.12	48.96	23.12	49.28	21.92	100.00	80.32		
1000	5	10	4	97.48	NA	100.00	NA	100.00	NA	100.00	NA	99.85	100.00
			6	96.29	10.96	100.00	59.11	100.00	51.70	100.00	80.44		
			8	95.70	14.66	100.00	84.59	100.00	82.51	100.00	98.07		
1000	5	50	4	5.48	NA	68.29	NA	77.77	NA	100.00	NA	4.59	45.03
			6	5.62	5.77	99.55	57.33	96.44	49.48	100.00	85.33		
			8	6.96	6.07	99.40	87.11	99.85	84.29	100.00	99.11		
1000	10	10	8	22.07	NA	100.00	NA	100.00	NA	100.00	NA	37.92	100.00
			10	21.92	NA	100.00	NA	100.00	NA	100.00	NA		
			12	16.44	8.00	100.00	79.85	100.00	78.96	100.00	91.55		
1000	10	50	8	4.59	NA	93.48	NA	84.29	NA	100.00	NA	4.29	20.00
			10	5.03	NA	94.37	NA	95.55	NA	100.00	NA		
			12	4.29	5.62	96.88	50.07	94.66	48.74	100.00	91.11		

Rejection rates at the nominal level $\alpha = 0.05$ of the $\hat{\chi}^2$ and the Wald test under different partitioning methods. n , d_x , c , L , respectively denote the sample size, the number of covariates, the departure from the null, and the number of cells. The last two columns report the rejection rates of the Kolmogorov-Smirnov tests.

Probit Model

We consider the data generating processes denoted as DGP.6, DGP.7, DGP.8, DGP.9, and DGP.10 in SS. The data consists of random draws $\{(D_i, X_i')'\}_{i=1}^n$, where $D = \mathbb{I}\{D^* > 0\}$ and D^* is generated as,

$$D^* = -\frac{\sum_{j=1}^{10} X_j}{6} - \epsilon \quad (DGP.6),$$

under the null hypothesis. Here, $X_1 = Z_1$, $X_2 = (Z_1 + Z_2)/\sqrt{2}$, $X_j = Z_j$, for $j = 3, \dots, 10$, with $Z_j \sim N(0, 1)$, for all j , and $\epsilon \sim N(0, 1)$ independently of X . The alternative models for the latent variable are given by,

$$\begin{aligned} D^* &= -1 - \frac{\sum_{j=1}^{10} X_j}{10} + \frac{X_1 X_2}{2} - \epsilon \quad (DGP.7), \\ D^* &= -1 - \frac{\sum_{j=1}^{10} X_j}{10} + \frac{X_1 \sum_{j=2}^5 X_j}{4} - \epsilon \quad (DGP.8), \\ D^* &= -1.5 - \frac{\sum_{j=1}^{10} X_j}{6} + \frac{\sum_{j=1}^{10} X_j^2}{10} - \epsilon \quad (DGP.9), \\ D^* &= \frac{-0.1 + 0.1 \sum_{j=1}^5 X_j}{\exp(-0.2 \sum_{j=1}^{10} X_j)} - \epsilon \quad (DGP.10). \end{aligned}$$

We generate $R = 1000$ samples of sizes $n \in \{200, 600, 1000\}$. The value of R is the same of [Sant'Anna and Song \(2019\)](#), allowing fair comparisons with their results.

The null hypothesis to test is given by,

$$H_0 : p(X) = \Phi(X'\theta_0) \text{ a.s., for some } \theta_0 \in \Theta,$$

where, $p(X) = P(D = 1|X)$, $X = (1, X_1, \dots, X_{10})$, and $\Phi(x)$ denotes the CDF of the standard normal at x .

Since the model is nonlinear, the grouped GMM estimator does not have a closed-form solution, necessitating the use of numerical methods for estimation. In these simulations and in the empirical application discussed in the last section, we employ the "gmm" package in R ([Chaussé \(2010\)](#)) to obtain the estimator. This function requires the specification of the moment conditions and the weighting matrix, which we set equal to $n^{-1/2}\hat{\Phi}_\gamma(\theta)$ and $\Sigma_\gamma(\tilde{\theta})$, respectively. Here, $\tilde{\theta}$ represents the (linear) probit maximum likelihood estimator

(MLE):

$$\tilde{\theta} = \arg \max_{\theta \in \Theta} \sum_{i=1}^n \{D_i \log \Phi(X_i' \theta) + (1 - D_i) \log(1 - \Phi(X_i' \theta))\}.$$

We employ the Nelder-Mead algorithm to minimize the χ^2 criterion, setting a maximum of 3500 iterations and a tolerance of e^{-25} . To mitigate the risk of converging to local minima, we iteratively run the algorithm. Initially, we use the MLE estimates as starting points, and subsequently, we apply the algorithm sequentially to perturbations of the final estimates obtained at each step. This iterative process continues until the difference between the minimized criteria falls below $\tilde{\epsilon} = 0.01$. It's worth noting that this procedure does not guarantee the attainment of the global minimum, and we may encounter slight size distortions as a result.

The probit MLE estimator is also employed in the construction of the Wald test statistics. In this case, the covariance matrix is given by,

$$\text{Avar}(\Phi_{\gamma}(\theta)) = \Sigma_{\gamma}(\theta) - \mu_{\gamma,0}^* \mathcal{I}^{-1} \mu_{\gamma,0}^*,$$

where $\mu_{\gamma,0}^* = \mathbb{E}[\phi(X'\theta_0)X\mathbf{I}_{\gamma}(X)']$, $\phi(x)$ denoting the density of the standard normal at x , and,

$$\mathcal{I} = \mathbb{E} \left[\frac{\phi(X'\theta_0)^2 XX'}{\Phi(X'\theta_0)(1 - \Phi(X'\theta_0))} \right],$$

is the Fisher information matrix. Under the null, the MLE estimator is more efficient than the grouped GMM estimator and $\text{Avar}(\Phi_{\gamma}(\theta))$ is positive definite. Therefore, it can be estimated using the plug-in estimator,

$$\hat{W}_{\gamma}(\tilde{\theta}) = \hat{\Sigma}_{\gamma}(\tilde{\theta}) - \hat{\mu}_{\gamma,0}^* \left(\frac{1}{n} \sum_{i=1}^n \frac{\phi(X_i' \tilde{\theta})^2 XX'}{\Phi(X_i' \tilde{\theta})(1 - \Phi(X_i' \tilde{\theta}))} \right)^{-1} \hat{\mu}_{\gamma,0}^{*'},$$

where $\hat{\sigma}_{\gamma_l}^2(\theta) = n^{-1} \sum_{i=1}^n \Phi(X_i' \tilde{\theta})(1 - \Phi(X_i' \tilde{\theta})) \mathbb{I}_{\gamma_l}(X_i)$, and $\hat{\mu}_{\gamma,0}^* = n^{-1} \sum_{i=1}^n \phi(X_i' \tilde{\theta}) \mathbf{I}_{\gamma}(X_i) X_i'$.

We compare our method to the test proposed in SS, which is based on the following empirical process,

$$R_{3,n}(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(D_i - \Phi(X_i' \tilde{\theta}) \right) \left(\mathbb{I}\{\Phi(X_i' \tilde{\theta}) \leq u\} - g(X_i, \tilde{\theta})' \Delta^{-1}(\tilde{\theta}) G(u, \tilde{\theta}) \right),$$

where $g(X, \theta) = d/d\theta \Phi(X'\theta)$ is the score function, $\Delta(\theta) = n^{-1} \sum_{i=1}^n g(X, \theta)g(X, \theta)'$, and $G(u, \theta) = n^{-1} \sum_{i=1}^n g(X, \theta) \mathbb{I}\{\Phi(X'\theta) \leq u\}$. The additional term in the right parenthesis is the projection of $\mathbb{I}\{\Phi(X'\tilde{\theta}) \leq u\}$ on $g(X, \theta)$ and has the purpose of making the limit null distribution of the test independent from the estimation method used for $\tilde{\theta}$. The test is given by the Kolmogorov-Smirnov functional,

$$\mathbf{KS3} = \sup_{u \in \mathbb{R}} |R_{3,n}(u)|,$$

and the limit null distribution is approximated using a multiplier bootstrap procedure, described in the paper, with $B = 999$ bootstrap repetitions.

We apply the same partitioning methods as those used in the linear model simulations, considering L values from the set $\{10, 12, 14\}$ for the number of cells. For both the FNP and PNP algorithms, we set the minimum number of observations in each cell of the initial two-cell partition to $n_{\min} = n/5$. The PNP partitions under the null are constructed using the specification of DGP.6 as the alternative model. In the FNP method, we employ a polynomial expansion of the fitted values with an order of $q = 3$.

The simulation results for the probit model are presented in Table 3. Under DGP.6, the Wald test exhibits rejection rates that closely match the nominal significance level of $\alpha = 0.05$, even for moderate sample sizes and across all partitioning methods. This alignment holds true even when dealing with moderate sample sizes and across all partitioning methods. In contrast, the $\hat{\chi}^2$ test consistently displays size distortions, even with large sample sizes. The distortions stem from the approximation involved in the numerical methods used for the minimization of the χ^2 criterion. These approximations can lead the test to either over-reject or under-reject, depending on how far the approximation is from the minimum. Achieving greater accuracy in the approximation can be attained by increasing the number of iterations in the minimization algorithm and reducing the tolerance level. However, this enhanced accuracy comes at the expense of increased computational demands.

Rejection rates of the χ^2 tests under the other DGPs demonstrate the tests' effectiveness in detecting deviations from the null hypothesis. Regardless of the DGP or the partitioning method employed, the Wald test outperforms the $\hat{\chi}^2$ test.

Table 3: Probit Model: Size and Power

			PSEB		FSEB		FNP		PNP		
DGP	n	L	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	KS3
6	200	10	7.60	NA	7.00	NA	5.90	NA	6.50	NA	6.10
		12	6.40	5.92	7.30	8.37	6.80	7.05	6.00	8.56	
		14	7.60	2.51	6.90	1.87	6.50	2.33	7.00	1.98	
6	600	10	5.80	NA	5.20	NA	5.09	NA	4.39	NA	5.20
		12	4.39	6.90	3.80	6.20	4.80	8.20	4.00	7.12	
		14	5.60	2.00	5.30	2.60	5.30	1.70	5.50	2.81	
6	1000	10	4.59	NA	6.20	NA	5.20	NA	5.30	NA	6.00
		12	5.40	7.90	6.20	9.10	6.00	8.30	6.10	8.50	
		14	3.90	2.19	6.10	2.40	5.80	4.10	4.70	3.50	
7	200	10	89.70	NA	11.10	NA	89.30	NA	90.00	NA	14.90
		12	87.80	72.49	11.89	9.14	88.20	77.05	88.90	74.51	
		14	88.30	58.41	15.10	4.50	87.50	60.88	88.00	59.16	
7	600	10	100.00	NA	32.60	NA	100.00	NA	100.00	NA	33.00
		12	100.00	100.00	30.50	22.10	100.00	98.40	100.00	98.20	
		14	100.00	99.90	28.60	10.30	100.00	94.60	100.00	99.50	
7	1000	10	100.00	NA	51.50	NA	100.00	NA	100.00	NA	52.90
		12	100.00	100.00	49.90	37.70	100.00	99.20	100.00	98.70	
		14	100.00	100.00	45.40	20.70	100.00	98.80	100.00	99.00	
8	200	10	44.40	NA	19.40	NA	24.00	NA	82.69	NA	18.70
		12	44.40	23.31	20.90	13.37	25.40	16.07	82.19	67.65	
		14	39.20	11.20	20.10	6.08	21.20	4.59	80.50	47.01	
8	600	10	95.70	NA	70.00	NA	81.10	NA	100.00	NA	62.10
		12	94.10	73.30	66.80	46.70	78.20	60.00	100.00	98.10	
		14	95.00	54.10	67.00	32.30	78.00	43.70	100.00	96.09	
8	1000	10	100.00	NA	90.90	NA	96.40	NA	100.00	NA	82.10
		12	100.00	92.70	88.50	77.20	96.00	87.40	100.00	99.80	
		14	99.90	84.60	88.70	63.10	95.90	78.50	100.00	99.80	
9	200	10	27.30	NA	12.70	NA	25.80	NA	14.39	NA	15.10
		12	24.70	17.36	13.60	9.60	24.40	18.88	14.80	10.99	
		14	24.00	6.67	13.30	3.89	22.20	11.03	13.10	4.41	
9	600	10	71.30	NA	33.60	NA	64.20	NA	31.30	NA	36.10
		12	70.59	48.00	34.40	21.40	62.90	51.45	32.40	19.53	
		14	66.70	25.40	29.20	8.61	57.30	27.74	31.20	8.61	
9	1000	10	93.10	NA	53.20	NA	88.30	NA	53.40	NA	57.50
		12	91.80	72.70	52.80	38.60	86.30	76.00	49.90	32.50	
		14	93.40	53.10	49.10	16.30	84.70	54.55	49.80	16.90	
10	200	10	9.10	NA	8.69	NA	10.60	NA	10.40	NA	10.10
		12	8.60	9.62	9.90	11.13	11.30	11.79	10.50	9.54	
		14	10.50	2.52	7.50	2.03	9.19	2.88	8.10	2.64	
10	600	10	28.49	NA	25.30	NA	29.10	NA	30.10	NA	28.10
		12	27.80	13.60	22.80	21.40	26.40	23.82	28.19	20.20	
		14	23.40	3.80	21.70	9.10	24.20	11.70	25.60	9.60	
10	1000	10	56.39	NA	54.20	NA	54.70	NA	62.20	NA	51.70
		12	52.00	18.70	50.20	40.20	52.70	41.80	58.30	39.20	
		14	47.50	6.00	45.00	20.39	45.10	21.10	55.10	21.80	

Rejection rates at the nominal level $\alpha = 0.05$ of the $\hat{\chi}^2$ and the Wald test under different partitioning methods. *DGP*, *n*, and *L*, respectively denote the data generating process, the sample size, and the number of cells. The last column report the rejection rates of the [Sant'Anna and Song \(2019\)](#) Kolmogorov-Smirnov test.

In contrast with the linear model simulations, χ^2 tests using the **PSEB** partitioning

exhibit significantly superior performance compared to those under **FSEB** partitioning. Interestingly, under DGP.9, the tests utilizing **PSEB** partitioning even surpass those using the **PNP** method. Part of this notable performance improvement can be attributed to the non-zero covariance between the first two regressors, which enhances the informativeness of the principal components reduction. Furthermore, with the exception of DGP.10, where the performance of $\hat{\mathcal{W}}$ is comparable to that of the KS3 test, the Wald test consistently outperforms the test proposed by SS under various partitioning schemes, including **PSEB**, **FNP**, and **PNP**, with slight enhancements observed in the case of **FSEB** partitioning. The comparison between KS3 and the $\hat{\chi}^2$ test exhibits greater nuance, depending on the specific DGPs and partitioning algorithms considered. For example, under DGP.7, the $\hat{\chi}^2$ test outperforms KS3 under **PSEB** partitioning, while the opposite is true under **FSEB** partitioning.

8 Empirical Application

We present an empirical application of our testing procedures by revisiting the analysis conducted by [Fryer Jr and Greenstone \(2010\)](#) on the impact of attending Historically Black Colleges and Universities (HBCU), versus non-HBCU, on the labor market and educational outcomes of African American students. The question of whether HBCUs are comparatively more effective than Traditional White Institutions (TWIs) in advancing academic achievement and economic success for black students holds significant policy implications. HBCUs have historically relied heavily on federal government financial support and have been consistently underfunded in comparison to TWIs, as discussed by [Price and Viceisza \(2023\)](#). This financial disparity can substantially influence the quality of education provided by HBCUs, potentially limiting the opportunities and experiences accessible to their students. Conversely, criticism has been raised against the existence of HBCUs, as they can create competition for limited educational resources, potentially leading to inefficiencies and missed opportunities for collaboration. The suggestion of merging HBCUs with other institutions or redirecting resources toward existing TWI could lead to a more efficient and cost-effective higher education system.

Our analysis is based on the National Longitudinal Survey of the High School Class

of 1972 (NLS) used by [Fryer Jr and Greenstone \(2010\)](#), which is publicly available on OpenICPSR¹. The survey is a nationally representative sample of 23,451 high school seniors in 1972, with follow-up interviews conducted in 1973, 1974, 1976, 1979, and 1986. This dataset provides rich information concerning these students, encompassing details about their demographics, academic performance, and labor market outcomes.

We examine the outcomes of African American students who pursued higher education after high school and were tracked throughout the 1972-1986 period. The sample consists of 624 individuals, of which 260 attended an HBCU and 364 attended a TWI. The outcomes considered are the logarithm of hourly wage in 1986, the probability of receiving a bachelor's degree, and the probability of obtaining a graduate degree.

Table 4: Summary Statistics of Black Students (NLS)

	HSBCU			TWI		
	Mean	St. deviation	NA	Mean	St. deviation	NA
ln(wage)	2.16	0.64	57	2.11	0.51	71
Bachelor's degree	0.67	0.46	0	0.62	0.48	2
Graduate degree	0.14	0.35	0	0.16	0.37	2
SAT	685.55	133.46	143	793.68	183.39	212
ACT	12.86	4.02	201	14.73	5.31	269
Family income	3.45	2.50	59	3.89	2.70	75
Father education	1.71	0.96	4	1.96	1.16	9
Mother education	1.95	1.08	3	2.08	1.11	4
Private high school	0.03	0.17	0	0.05	0.23	0
Female	0.66	0.47	0	0.64	0.47	0
South	0.87	0.33	0	0.48	0.50	0

The table report mean, standard deviation, and number of missing values of the outcomes and characteristics of African American who attended college. The columns HSBCU and TWI refer to the type of college attended. In the first three rows, the outcomes are the logarithm of hourly wage in 1986, the probability of receiving a bachelor's degree, and the probability of enrolling in graduate school. The remaining rows report the covariates. The total sample size is $n = 624$, with $n_1 = 260$ students enrolled in HBCU, and $n_2 = 364$ in TWI.

Following [Fryer Jr and Greenstone \(2010\)](#), the set of covariates consists of home environment variables and pre-college characteristics. The former include family income, mother and father's education, and whether or not a student lives in the south. The latter consists of SAT and ACT scores, high school GPA, and whether or not the student attended a private school. The treatment of missing values is as [Fryer Jr and Greenstone \(2010\)](#), and consists of generating a dummy variable for each covariate denoting whether the value is

¹Ann Arbor, MI: Inter-university Consortium for Political and Social Research, 2019-10-12. <https://doi.org/10.3886/E113736V1>

missing or not. The relative NA is set to 0. Differently from [Fryer Jr and Greenstone \(2010\)](#), however, we do not partition the set of covariates into categories.

We estimate the effect of attending an HBCU institution under the unconfoundedness framework (see, e.g., [Rosenbaum and Rubin \(1983\)](#)). Let $Y_i = Y_i(1)D + Y_i(0)(1 - D_i)$ denote the outcome of the i -th individual, where $Y_i(1)$ and $Y_i(0)$ are the potential outcomes under treatment and control respectively, and D_i is a binary variable indicating whether the i -th individual attended an HBCU. The unconfoundedness setup requires the following independence and overlap assumptions.

Assumption 6 $(Y_i(1), Y_i(0)) \perp\!\!\!\perp D_i | X_i$ and $0 < p(x) < 1 \quad \forall x \in \mathcal{X}$.

Here X_i denotes the vector of home-environment and pre-college characteristics, while $p(X) = P(D = 1|X)$ denotes the propensity score. The independence assumption implies that, once we account for an individual's characteristics, the decision to attend an HBCU university is akin to a random choice. Under Assumption 6, the average treatment effect (ATE) and the average treatment effect on the treated (ATT) are identified ([Rosenbaum \(1987\)](#)) by,

$$\text{ATE} = \mathbb{E} \left[\left(\frac{D}{p(X)} - \frac{1-D}{1-p(X)} \right) Y \right] \quad \text{and} \quad \text{ATT} = \frac{\mathbb{E} \left[\left(D - \frac{p(X)(1-D)}{1-p(X)} \right) Y \right]}{\mathbb{E}[D]}.$$

This motivates a two-step procedures where first we obtain an estimate of the propensity score, and then we estimate the ATE and ATT by using the analog principle. If there are doubts about the validity of the independence assumption, the estimates of the ATE and ATT below should be interpreted as statistical associations rather than causal effects.

The estimation of the propensity score is done by fitting a probit model, i.e., $p(X) = \Phi(X'\theta_0)$, where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution. We consider two different specifications of the propensity score model,

- (a) This specification includes all the covariates listed in Table 4, along with the inclusion of the relative missing values dummy.
- (b) In addition to the elements in specification (a), specification (b) introduces interactions between the variables "south" and "SAT" as well as "south" and "family

income.”

Specification (b) is more comprehensive and accommodates the consideration of varying effects of family income and SAT scores on individuals originating from the southern region. This choice is motivated by the observation that the south is home to a significant number of HBCU, thus implying that attending an HBCU might entail lower mobility costs for individuals hailing from the southern region.

To ensure that the procedure above returns stable and consistent estimates of the causal parameters, it is important to check the correct specification of the propensity score estimates. That is, $H_0 : \mathbb{E}[D|X] = \Phi(X'\theta_0)$ for some $\theta_0 \in \Theta$. We report the p-values of the $\hat{\chi}^2$, the Wald test, and the [Sant’Anna and Song \(2019\)](#) test, together with the relative ATE and ATT estimates, for the two specification of $p(X)$.

The parameter θ_0 is estimated through MLE and the χ^2 tests are built as in Section 7. The J test, $\hat{\chi}^2$, is estimated by numerical approximation using iteratively the Nelder-Mead algorithm with 150000 iteration, a tolerance of e^{-31} , and $\tilde{\epsilon} = 0.0001$, while the SS test is implemented using $B = 100000$ bootstrap replications. The construction of PNP partition is done using the opposite specification as reference, meaning that the test for specification (a) assumes the alternative model to be specification (b), and vice-versa. The FNP partition uses a polynomial expansion of the fitted values of order $q = 3$, and the PSEB partition is built using only the first principal component. Following common practice, the ATE and ATT are estimated from the subsample of observations with estimated propensity score outside of the $[0.05, 0.95]$ interval.

Table 5 presents the results of our specification tests. In summary, the evidence suggests that model (a) is not correctly specified, whereas model (b) appears to offer a better fit to the data, although minor misalignments are still present. In our set of 40 specification tests (20 for each partitioning method), the null hypothesis of correct specification for model (a) is rejected 11 times at a 10% significance level using the Wald test, whereas for model (b), it is rejected only 3 times. Notably, most of these rejections occur when employing **PNP** partitioning, which is based on comparing specifications (a) and (b), and under **PSEB** partitioning. The effectiveness of the latter in detecting deviations is supported by the Monte Carlo study in Section 7. The $\hat{\chi}^2$ test, on the other hand, rejects the null hypothesis 3 times out of 12 trials for model (a) and 3 times out

of 8 for model (b). However, it's important to exercise caution as these rejections are predominantly associated with the largest values of L . The relatively small number of within-cell observations in these cases can influence the test's size control. Regarding the KS test from [Sant'Anna and Song \(2019\)](#), it rejects the null hypothesis for model (a) but not for model (b). However, the Cramer-von Mises (CvM) functional does not detect misspecifications for either model.

Table 5: Tests Results

Spec.	L	PSEB		FSEB		FNP		PNP		KS	CvM
		\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$	\hat{W}	$\hat{\chi}^2$		
(a)	2	0.91	NA	0.13	NA	0.65	NA	0.02	NA	0.02	0.45
	4	0.05	NA	0.18	NA	0.88	NA	0.08	NA		
	6	0.11	NA	0.55	NA	0.02	NA	0.20	NA		
	8	0.06	NA	0.25	NA	0.80	NA	0.13	NA		
	10	0.03	NA	0.53	NA	0.64	NA	0.22	NA		
	12	0.26	NA	0.80	NA	0.08	NA	0.03	NA		
	14	0.15	NA	0.18	NA	0.45	NA	0.03	NA		
	16	0.20	0.09	0.41	0.27	0.85	0.40	0.10	0.00		
	18	0.12	0.48	0.14	0.41	0.15	0.48	0.30	0.04		
	20	0.10	0.48	0.45	0.85	0.14	0.24	0.33	0.32		
(b)	2	0.74	NA	0.41	NA	0.66	NA	0.77	NA	0.53	0.53
	4	0.52	NA	0.27	NA	0.50	NA	0.48	NA		
	6	0.53	NA	0.68	NA	0.60	NA	0.68	NA		
	8	0.60	NA	0.46	NA	0.79	NA	0.34	NA		
	10	0.63	NA	0.76	NA	0.88	NA	0.93	NA		
	12	0.29	NA	0.65	NA	0.80	NA	0.71	NA		
	14	0.01	NA	0.80	NA	0.78	NA	0.91	NA		
	16	0.03	NA	0.67	NA	0.85	NA	0.38	NA		
	18	0.07	0.03	0.73	0.36	0.69	0.04	0.61	0.02		
	20	0.22	0.47	0.93	0.84	0.84	0.27	0.33	0.41		

The table reports the p-values of the $\hat{\chi}^2$ and the Wald test, under different partitioning methods, for the null hypothesis of correct specification of model (a) and (b). L denotes the number of cells. The last two columns report the p-values of the Kolmogorov-Smirnov and the Cramer-von Mises functional of the [Sant'Anna and Song \(2019\)](#) test.

The treatment effects exhibit significant variation between the two distinct propensity score specifications. In specification (a), the ATE estimates are positive and statistically significant at the 10% nominal level for all three outcomes. However, the ATT estimates for these three outcomes are too noisy and the null of zero effect cannot be rejected. When we account for the distinct influence of family income and SAT scores on students hailing from the South, we find that neither the ATE nor the ATT estimates are stastically different from zero. Thus, under our preferred model specification, attending HBCU institutions seems to have only a minimal (albeit positive) impact on the labor market and

educational outcomes of African American students. In terms of these long-run outcomes, the evidence suggests that HBCUs are not more effective than TWIs in advancing the economic success and academic achievement of black students. These findings contrast with the results reported by [Fryer Jr and Greenstone \(2010\)](#), who found evidence of a positive effect of HBCU attendance on the wages of black students in 1986 and their likelihood of attaining a bachelor’s degree. Nonetheless, the results of [Fryer Jr and Greenstone \(2010\)](#) are consistent with our findings when considering the probability of obtaining a graduate degree.

Table 6: ATE and ATT Estimates

	Specification (a)			Specification (b)		
	ln(wage)	Col. Degree	Grad. Degree	ln(wage)	Col. Degree	Grad. Degree
ATE	0.14 (0.06) [0.02]	0.09 (0.05) [0.10]	0.08 (0.04) [0.06]	0.06 (0.04) [0.14]	0.02 (0.03) [0.46]	0.02 (0.03) [0.54]
ATT	0.02 (0.06) [0.65]	0.01 (0.06) [0.86]	0.03 (0.04) [0.46]	0.03 (0.05) [0.45]	0.00 (0.03) [0.83]	0.00 (0.03) [0.80]

The table reports estimates of the ATE and ATT using two different specifications of the propensity score for the following outcomes: the logarithm of hourly wage in 1986, the probability of obtaining a bachelor’s degree, and the probability of obtaining a graduate degree. Standard errors are reported in parentheses, and p-values are enclosed in brackets. The estimates are obtained by excluding observations with a propensity score outside the interval [0.05, 0.95].

It’s important to note that our results may be influenced by selection bias, primarily because of the absence of wage outcomes. To mitigate this concern, one possible approach could involve employing the correction method proposed by [Heckman \(1979\)](#) and assessing the overall model fit through our testing procedures. However, addressing this issue lies beyond the scope of the current section.

9 Conclusion

In conclusion, this article introduces a novel approach for validating the correct specification of conditional moment restriction (CMR) models, which are fundamental for identifying and estimating causal relationships. The proposed method harnesses the well-established chi-squared (χ^2) tests, commonly used in classical goodness-of-fit contexts, to assess the goodness-of-fit of CMR specifications. Traditionally, CMR model checks have

been adapted from tests for the cumulative density function (CDF) based on functionals of the standard empirical process (SEP), but these often exhibit limitations, especially when dealing with high-dimensional data and high-frequency alternatives. The introduced χ^2 tests, on the other hand, are distribution-free, do not necessitate complex bootstrapping or smoothing techniques, and offer flexibility in partitioning the data to favor specific alternative hypotheses. Thus, providing a valid complementary instruments to the existing model checking procedures. Monte Carlo simulations and empirical evidence highlight the effectiveness of these χ^2 tests, especially in scenarios involving a large number of covariates. The empirical analysis showcases the practical application of these tests in assessing the returns of attending historically black colleges and universities (HBCUs) for black students in the United States. The results suggest that once we account for the effect of family income and SAT scores on students hailing from the South, HBCUs are not more effective than traditional white institutions (TWIs) in advancing the economic success and academic achievement of black students.

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A Appendix A

A.1 Lemmas

We first state auxiliary lemmas for the propositions and theorems in the main text. We let \rightsquigarrow denote weak convergence on $l^\infty(\mathbb{D})$ (see definition 13.3 in [Van Der Vaart \(1996\)](#), hereafter VW), where $l^\infty(\mathbb{D})$ is the space of all real-valued functions that are uniformly bounded on \mathbb{D} , and \xrightarrow{d} denote convergence of real-valued random variables. Throughout, to highlight the dependency on the partition, we denote as $\hat{\Phi}_\theta(\gamma) := \hat{\Phi}_\gamma(\theta)$ and $\hat{\Phi}_0(\gamma) := \hat{\Phi}_\gamma(\theta_0)$.

Lemma 1 *Under the null H_0 ,*

(a) if Assumption 1, 2, and 2' hold, then $\Sigma_{\gamma}(\tilde{\theta}) = \Sigma_{\gamma,0} + o_p(1)$.

(b) if Assumption 1, 2, 3 hold, and $\text{Avar}(\hat{\Phi}_{\gamma}(\tilde{\theta}))$ is p.d., then $\hat{W}_{\gamma}(\tilde{\theta}) = \text{Avar}(\hat{\Phi}_{\gamma}(\tilde{\theta})) + o_p(1)$

Lemma 2 Under the null hypothesis H_0 , Assumptions 1, and 5,

$$\hat{\Phi}_0(\cdot) \rightsquigarrow \Phi_0(\cdot) \text{ as a process on } l^\infty(\mathbb{D}),$$

where $\Phi_0(\cdot)$ is an \mathbb{R}^L -valued Gaussian process with zero mean vector and covariance structure given by,

$$\mathbb{E}[\Phi_0(\gamma)\Phi_0(\tilde{\gamma})] = \mathbb{E}[\varepsilon_{\theta_0}(Z)^2 \mathbf{I}_{\gamma}(X)\mathbf{I}_{\tilde{\gamma}}(X)'] \quad \forall \gamma, \tilde{\gamma} \in \mathbb{D}.$$

Lemma 3 Under the null hypothesis H_0 , and Assumptions 1-5, it holds that:

(a) $\sup_{\gamma \in \mathbb{D}} \left| \hat{\Phi}_{\hat{\theta}}(\gamma) - (\hat{\Phi}_0(\gamma) - \hat{\mu}_{\gamma}^*(\theta_0)\sqrt{n}(\hat{\theta} - \theta_0)) \right| = o_p(1)$.

(b) $\hat{\mu}_{\gamma}^*(\theta_0) = \mu_{\gamma,0}^* + o_p(1)$.

Lemma 4 Under the null H_0 , and Assumptions 4,5,

(a) if Assumption 1, 2, and 2' hold, then $\Sigma_{\tilde{\gamma}}(\tilde{\theta}) = \Sigma_{\gamma,0} + o_p(1)$.

(b) if Assumption 1, 2, 3 hold, and $\text{Avar}(\hat{\Phi}_{\gamma}(\tilde{\theta}))$ is p.d., then $\hat{W}_{\tilde{\gamma}}(\tilde{\theta}) = \text{Avar}(\hat{\Phi}_{\gamma}(\tilde{\theta})) + o_p(1)$

A.2 Proofs

For any class of functions \mathcal{F} , we denote as $\{P_n f : f \in \mathcal{F}\}$ the empirical measure indexed by \mathcal{F} , such that $P_n f = n^{-1} \sum f(Z_i)$; alike, we use Pf for the population measure, $Pf = \int f(Z)dP$. We say that a class of functions is: i) Glivenko-Cantelli for P (hereafter, P -GC) whenever $\sup_{f \in \mathcal{F}} |P_n - P|f = o_p(1)$; ii) P -Donsker if $\{\sqrt{n}(P_n - P)f : f \in \mathcal{F}\}$ converge in distribution to a tight random element in the space $l^\infty(\mathcal{F})$. Throughout, we refer to both classes of sets with finite VC dimension and classes of functions with finite VC subgraph dimension as VC classes. These classes, having uniformly bounded covering

numbers (Theorem 2.6.7 in VW), are Glivenko-Cantelli and Donsker (see Theorem 2.4.3 and 2.5.2 in VW) for any probability measure on the sample space, provided that they have integrable and square-integrable envelope function, respectively.

Proof of Lemma 1. For the first part of the Lemma, by the weak law of large numbers (WLLN) and a mean value theorem argument (MVT), suffices to show that

$$\frac{1}{n} \sum_{i=1}^n (\varepsilon_{\tilde{\theta}}^2(Z_i) - \varepsilon_{\theta_0}^2(Z_i)) \mathbb{I}_{\gamma_l}(X_i) = I + II + III = o_p(1)$$

for each $l \in 1, 2, \dots, L$, where,

$$\begin{aligned} I &= (\tilde{\theta} - \theta_0)' \frac{1}{n} \sum_{i=1}^n \nabla m_{\tilde{\theta}}(X_i) \nabla m_{\tilde{\theta}}(X_i)' \mathbb{I}_{\gamma_l}(X_i) (\tilde{\theta} - \theta_0), \\ II &= \frac{2}{n} \sum_{i=1}^n m_{\theta_0}(X_i) \nabla m_{\tilde{\theta}}(X_i)' \mathbb{I}_{\gamma_l}(X_i) (\tilde{\theta} - \theta_0), \\ III &= \frac{2}{n} \sum_{i=1}^n Y_i (m_{\tilde{\theta}}(X_i) - m_{\theta_0}(X_i)) \mathbb{I}_{\gamma_l}(X_i), \end{aligned}$$

and $|\tilde{\theta} - \theta_0| \leq |\tilde{\theta} - \theta_0|$. The triangle inequality, Assumption 2, and the consistency of $\tilde{\theta}$ show that, $|I| \leq d_{\theta}^2 \left\| \tilde{\theta} - \theta_0 \right\|^2 n^{-1} \sum_{i=1}^n R(X_i)^2 = o_p(1)$, where $\|\cdot\|$ denotes the euclidean norm. By a similar reasoning,

$$\begin{aligned} |II| &\leq d_{\theta} \left\| \tilde{\theta} - \theta_0 \right\| \frac{2}{n} \sum_{i=1}^n m_{\theta_0}(X_i) R(X_i) \\ &\leq d_{\theta} \left\| \tilde{\theta} - \theta_0 \right\| (\mathbb{E}[Y^2])^{1/2} (\mathbb{E}[R(X)^2])^{1/2} + o_p(1) = o_p(1) \end{aligned}$$

where the last inequality follows from the WLLN, the law of iterated expectation, and Cauchy-Schwarz inequality. Finally, after expanding again around θ_0 it is easy to see that $|III| \leq d_{\theta} \left\| \tilde{\theta} - \theta_0 \right\| 2n^{-1} \sum_{i=1}^n Y_i R(X_i) \mathbb{I}_{\gamma_l}(X_i) = o_p(1)$.

For the second part of the lemma, we need to show that $\hat{L}(\tilde{\theta}) = L_0 + o_p(1)$, $\hat{C}_{\gamma}(\tilde{\theta}) = C_{\gamma,0} + o_p(1)$, and $\hat{\mu}_{\gamma}^*(\tilde{\theta}) = \mu_{\gamma,0}^* + o_p(1)$. By the usual MVT argument and the law of large numbers,

$$L_n = L_0 + I + II + III + o_p(1)$$

with,

$$\|I\| = \left\| \frac{1}{n} \sum_{i=1}^n \nabla l_{\bar{\theta}}(Z_i) (\hat{\theta} - \theta_0) (\hat{\theta} - \theta_0)' \nabla l_{\bar{\theta}}(Z_i)' \right\| \leq d_{\theta}^4 \|\hat{\theta} - \theta_0\|^2 \frac{1}{n} \sum_{i=1}^n R_2^2(Z_i) = o_p(1),$$

$$\begin{aligned} \|II\| &= \left\| \frac{1}{n} \sum_{i=1}^n \nabla l_{\bar{\theta}}(Z_i) (\hat{\theta} - \theta_0) l'_{\theta_0}(Z_i) \right\| \leq d_{\theta}^2 \|\hat{\theta} - \theta_0\| \frac{1}{n} \sum_{i=1}^n \|l_{\theta_0}(Z_i)\| R_2(Z_i) \\ &\leq d_{\theta}^2 \|\hat{\theta} - \theta_0\| \left(\frac{1}{n} \sum_{i=1}^n \|l_{\theta_0}(Z_i)\|^2 \right)^{1/2} \left(\frac{1}{n} \sum_{i=1}^n R_2^2(Z_i) \right)^{1/2} = o_p(1). \end{aligned}$$

Alike, we write $\hat{C}_{\gamma}(\tilde{\theta})$ as,

$$\hat{C}_{\gamma}(\tilde{\theta}) = I - II - III + C_{\gamma,0} + o_p(1),$$

where,

$$\begin{aligned} I &= \frac{1}{n} \sum_{i=1}^n l_{\theta_0}(Z_i) \nabla m_{\bar{\theta}}(X_i)' (\hat{\theta} - \theta_0) \mathbf{L}_{\gamma}(X_i)' \\ II &= \frac{1}{n} \sum_{i=1}^n \nabla l_{\bar{\theta}}(Z_i)' (\hat{\theta} - \theta_0) \varepsilon_{\theta_0}(Z_i) \mathbf{L}_{\gamma}(X_i)' \\ III &= \frac{1}{n} \sum_{i=1}^n \nabla l_{\bar{\theta}}(Z_i) (\hat{\theta} - \theta_0) (\hat{\theta} - \theta_0)' \nabla m_{\bar{\theta}}(X_i) \mathbf{L}_{\gamma}(X_i)' \end{aligned}$$

By Assumptions 2, and 3,

$$\begin{aligned} \|I\| &\leq \|\hat{\theta} - \theta_0\| \frac{1}{n} \sum_{i=1}^n \|l_{\theta_0}(Z_i)\| \|\nabla m_{\bar{\theta}}(X_i)\| \|\mathbf{L}_{\gamma}(X_i)\| \\ &\leq \sqrt{L} \|\hat{\theta} - \theta_0\| \left(\frac{1}{n} \sum_{i=1}^n \|l_{\theta_0}(Z_i)\|^2 \right)^{1/2} \left(\frac{1}{n} \sum_{i=1}^n R^2(X_i) \right)^{1/2} = o_p(1), \end{aligned}$$

An analogous reasoning shows that $\|II\| = o_p(1)$, and,

$$\begin{aligned}\|III\| &\leq \left\| \hat{\theta} - \theta_0 \right\|^2 \frac{1}{n} \sum_{i=1}^n \|\nabla l_{\bar{\theta}}(Z_i)\| \|\nabla m_{\bar{\theta}}(X_i)\| \|\mathbf{I}_{\gamma}(X_i)\| \\ &\leq \sqrt{L} d_{\bar{\theta}}^3 \left\| \hat{\theta} - \theta_0 \right\|^2 \left(\frac{1}{n} \sum_{i=1}^n R^2(X_i) \right)^{1/2} \left(\frac{1}{n} \sum_{i=1}^n R_2^2(Z_i) \right)^{1/2} = o_p(1).\end{aligned}$$

Finally, $\hat{\boldsymbol{\mu}}_{\gamma}^*(\tilde{\theta}) = \boldsymbol{\mu}_{\gamma,0}^* + o_p(1)$ follows from the proof of Lemma 3 below. ■

Proof of Lemma 2. By Lemma 2.6.17 in VW and Assumption 5, both \mathbb{D} and $\{\mathbf{I}_{\gamma}(x) : \gamma \in \mathbb{D}\}$ are VC classes. Therefore, $\mathcal{F} = \{\varepsilon_{\theta_0}(z)\mathbf{I}_{\gamma}(x) : \gamma \in \mathbb{D}\}$ is a VC class (Lemma 2.6.18 in VW), with square integrable envelope function $F = |\varepsilon_{\theta_0}|$, and, hence, is P -Donsker. The convergence of the finite-dimensional distributions (fidis) of $\hat{\Phi}_0(\cdot)$ to those of $\Phi_0(\cdot)$, by the multivariate central limit theorem, characterize the limit process. ■

Proof of Lemma 3. By an MVT argument,

$$\hat{\Phi}_{\bar{\theta}}(\gamma) = \hat{\Phi}_0(\gamma) - I' \sqrt{n}(\hat{\theta} - \theta_0) - \hat{\boldsymbol{\mu}}_{\gamma}^{*'}(\theta_0) \sqrt{n}(\hat{\theta} - \theta_0)$$

where,

$$I = \hat{\boldsymbol{\mu}}_{\gamma}^*(\bar{\theta}) - \hat{\boldsymbol{\mu}}_{\gamma}^*(\theta_0) = \frac{1}{n} \sum_{i=1}^n (\nabla m_{\bar{\theta}}(X_i) - \nabla m_{\theta_0}(X_i)) \mathbf{I}_{\gamma}(X_i)'$$

and $|\bar{\theta} - \theta_0| \leq |\tilde{\theta} - \theta_0|$. The class $\{\nabla m_{\theta}(x) : \theta \in \Theta\}$ is a collection of continuous mapping, $\theta \rightarrow \nabla m_{\theta}$, over the compact metric space Θ with integrable envelope function $R(\cdot)$ and, therefore, is P -GC (e.g., Example 19.8 in [Van der Vaart \(2000\)](#)). Thus,

$$\begin{aligned}\sup_{\gamma \in \mathbb{D}} \|I\| &\leq \sup_{\gamma \in \mathbb{D}} \frac{1}{n} \sum_{i=1}^n \|\nabla m_{\bar{\theta}}(X_i) - \nabla m_{\theta_0}(X_i)\| \|\mathbf{I}_{\gamma}(X_i)\| \\ &\leq \sqrt{L} \frac{1}{n} \sum_{i=1}^n \|\nabla m_{\bar{\theta}}(X_i) - \nabla m_{\theta_0}(X_i)\| = o_p(1).\end{aligned}$$

where the last equality follows from an application of the uniform law of large numbers

(e.g., Davidson (1994), Theorem 21.6). For the second part of the Lemma is sufficient to prove that,

$$|II| = \left| \frac{1}{n} \sum_{i=1}^n \nabla^{(j)} m_{\theta_0}(X_i) (\mathbf{I}_{\tilde{\gamma}}(X_i) - \mathbf{I}_{\gamma}(X_i)) \right| = o_p(1),$$

for each $j \in \{1, \dots, d_\theta\}$. To see this is true, notice that by Assumption 5, $\mathbb{D}\Delta\mathbb{D} = \{\gamma_1 \Delta \gamma_2 : \gamma_1, \gamma_2 \in \mathbb{D}\}$ is a class of subsets of unions of VC classes, and hence is VC. Therefore, $\{|\nabla^{(j)} m_{\theta_0}(x)| \mathbf{I}_{\tilde{\gamma}}(x) : \tilde{\gamma} \in \mathbb{D}\Delta\mathbb{D}\}$ is also VC with integrable envelope $R(\cdot)$ and, hence, P -GC. Thus, for each $j \in \{1, \dots, d_\theta\}$,

$$\begin{aligned} |II| &\leq \frac{1}{n} \sum_{i=1}^n |\nabla^{(j)} m_{\theta_0}(X_i)| \mathbf{I}_{\hat{\gamma} \Delta \gamma}(X_i) \\ &\leq \sup_{\tilde{\gamma} \in \mathbb{D}\Delta\mathbb{D}} (P_n - P) |\nabla^{(j)} m_{\theta_0}| \mathbf{I}_{\tilde{\gamma} \Delta \gamma} + \mathbb{E} [|\nabla^{(j)} m_{\theta_0}(X)| \mathbf{I}_{\tilde{\gamma} \Delta \gamma}(X)] \\ &\leq o_p(1) + \mu_R(\hat{\gamma} \Delta \gamma) = o_p(1) \end{aligned}$$

where $\mu_R(\hat{\gamma} \Delta \gamma) = (\mu_R(\hat{\gamma}_1 \Delta \gamma_1), \dots, \mu_R(\hat{\gamma}_L \Delta \gamma_L))'$, and $\mu_R(A) = \int_A R(X) dP$ is a (signed) measure absolutely continuous with respect to P . The last equality follows from Assumption 4. ■

Proof of Lemma 4. For each element on the main diagonal of $\hat{\Sigma}_{\tilde{\gamma}}(\tilde{\theta}) - \hat{\Sigma}_{\gamma}(\tilde{\theta})$ write,

$$\frac{1}{n} \sum_{i=1}^n \varepsilon_{\hat{\theta}}(Z_i)^2 (\mathbb{I}_{\hat{\gamma}_l}(X_i) - \mathbb{I}_{\gamma_l}(X_i)) = I + II + III$$

where

$$\begin{aligned} I &= \frac{1}{n} \sum_{i=1}^n \varepsilon_{\theta_0}(Z_i)^2 (\mathbb{I}_{\hat{\gamma}_l}(X_i) - \mathbb{I}_{\gamma_l}(X_i)) \\ II &= (\hat{\theta} - \theta_0)' \frac{1}{n} \sum_{i=1}^n \nabla m_{\tilde{\theta}}(X_i) \nabla m_{\tilde{\theta}}(X_i)' (\mathbb{I}_{\hat{\gamma}_l}(X_i) - \mathbb{I}_{\gamma_l}(X_i)) (\hat{\theta} - \theta_0) \\ III &= -(\hat{\theta} - \theta_0)' \frac{2}{n} \sum_{i=1}^n \varepsilon_{\theta_0}(Z_i) \nabla m_{\tilde{\theta}}(X_i) (\mathbb{I}_{\hat{\gamma}_l}(X_i) - \mathbb{I}_{\gamma_l}(X_i)) \end{aligned}$$

The class $\{\varepsilon_{\theta_0}(z)^2 \mathbb{I}_{\tilde{\gamma}_l} : \tilde{\gamma} \in \mathbb{C}\Delta\mathbb{C}\}$ is VC with integrable envelope function $\varepsilon_{\theta_0}^2$ and, hence, is P -GC. Therefore, $|I| \leq \mu_\sigma(\hat{\gamma}_l \Delta \gamma_l) + o_p(1) = o_p(1)$, by Assumption 4. Also, $|II| \leq$

$\sqrt{L}d_\theta^2 \left\| \hat{\theta} - \theta_0 \right\|^2 n^{-1} \sum_{i=1}^n R(X_i)^2 = o_p(1)$, by Assumptions 1-2 and the consistency of $\hat{\theta}$, and $|III| \leq \sqrt{L}d_\theta \left\| \hat{\theta} - \theta_0 \right\| n^{-1} \sum_{i=1}^n \varepsilon_{\theta_0} R(X_i) = o_p(1)$ by Cauchy-Schwarz inequality. Thus, $\hat{\Sigma}_{\hat{\gamma}}(\tilde{\theta}) = \hat{\Sigma}_{\hat{\gamma}}(\hat{\theta}) + o_p(1)$, and the first part of the lemma follows from Lemma 1(a).

For the second part of the lemma, notice that by Lemma 3(b) (and the proof of the first part of Lemma 3), $\hat{\mu}_{\hat{\gamma}}^*(\hat{\theta}) = \mu_{\gamma,0}^* + o_p(1)$, and for each element of $C_n(\hat{\gamma}) - C_n(\gamma_0)$ it holds that,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \varepsilon_{\hat{\theta}}(Z_i) l_{\hat{\theta},j} (\mathbb{I}_{\hat{\gamma}_l}(X_i) - \mathbb{I}_{\gamma_l}(X_i)) &\leq \left(\frac{1}{n} \sum_{i=1}^n \varepsilon_{\hat{\theta}}(Z_i)^2 \mathbb{I}_{\hat{\gamma}_l \Delta \gamma_l}(X_i) \right)^{1/2} \left(\frac{1}{n} \sum_{i=1}^n l_{\hat{\theta},j}^2 \right)^{1/2} \\ &= o_p(1) O_p(1), \end{aligned}$$

where $l_{\theta,j}$ denotes the j -th component of l_θ and the last equality follows from the first part of this proof and Lemma 1. ■

Proof of Theorem 1. The consistency of the grouped GMM estimator follows from $\sup_{\theta \in \Theta} |Q_n(\theta) - Q_0(\theta)| = o_p(1)$, where $Q_0(\theta) = \mathbb{E} [\varepsilon_\theta(Z) \mathbf{I}_\gamma(X)]' (\Sigma_{\gamma,0})^{-1} \mathbb{E} [\varepsilon_\theta(Z) \mathbf{I}_\gamma(X)]$ and $Q_n(\theta) = n^{-1} \hat{\chi}_{\hat{\gamma}, \hat{\theta}}^2(\theta)$ (see Theorem 2.1 in [Newey and McFadden \(1994\)](#), for instance). To see that the condition holds, notice that,

$$\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_\theta(Z_i) \mathbf{I}_{\hat{\gamma}}(X_i) - \mathbb{E} [\varepsilon_\theta(Z) \mathbf{I}_\gamma(X)] \right| \leq \sup_{\theta \in \Theta} |I| + \sup_{\theta \in \Theta} |II| + \sup_{\theta \in \Theta} |III|$$

where

$$\begin{aligned} I &= (P_n - P) \varepsilon_\theta \mathbf{I}_\gamma, \\ II &= \frac{1}{n} \sum_{i=1}^n \varepsilon_{\theta_0}(Z_i) (\mathbf{I}_{\hat{\gamma}}(X_i) - \mathbf{I}_\gamma(X_i)), \\ III &= \frac{1}{n} \sum_{i=1}^n [\nabla m_{\hat{\theta}}(Z_i) (\mathbf{I}_{\hat{\gamma}}(X_i) - \mathbf{I}_\gamma(X_i))]' (\theta - \theta_0). \end{aligned}$$

The mapping $\theta \rightarrow \varepsilon_\theta$ is continuous over the compact Θ , with

$$\mathbb{E} \left[\sup_{\theta \in \Theta} \varepsilon_\theta(Z) \right] \leq \mathbb{E} \left[\sup_{\bar{\theta}, \theta \in \Theta} \varepsilon_{\theta_0}(Z) - \nabla m_{\bar{\theta}}(X)' (\theta - \theta_0) \right] \leq d_\theta \mathbb{E} [R(Z)] D < \infty,$$

by Assumption 2 and the compactness of Θ , where D denotes the diameter of Θ . Therefore, both $\{\varepsilon_\theta(z) : \theta \in \Theta\}$ and $\{\varepsilon_\theta(z)\mathbf{I}_\gamma(X) : \theta \in \Theta\}$ are P -GC classes (see Corollary 8.6 in [Giné and Zinn \(1984\)](#), for instance) and, hence, $\sup_{\theta \in \Theta} |I| = o_p(1)$. Then, by Cauchy-Schwarz inequality and Assumptions 1 and 4,

$$\sup_{\theta \in \Theta} |II| \leq \left(\frac{1}{n} \sum_{i=1}^n \varepsilon_{\theta_0}^2(Z_i) \right)^{1/2} \left(\frac{1}{n} \sum_{i=1}^n \mathbf{I}_{\hat{\gamma} \Delta \gamma}(X_i) \right)^{1/2} = o_p(1).$$

Finally,

$$\begin{aligned} \sup_{\theta \in \Theta} ||II| &\leq \sup_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \|\mathbf{I}_{\hat{\gamma} \Delta \gamma}(X_i)\| \|\nabla m_{\bar{\theta}}(X_i)\| \|\theta - \theta_0\| \\ &\leq d_\theta D \frac{1}{n} \sum_{i=1}^n \|\mathbf{I}_{\hat{\gamma} \Delta \gamma}(X_i)\| R(X_i) = o_p(1), \end{aligned}$$

where the second inequality follows from Assumption 2 and the compactness of Θ . This result, together with the consistency of $\hat{\Sigma}_{\hat{\gamma}}(\tilde{\theta})$ implies that $\sup_{\theta \in \Theta} |Q_n(\theta) - Q_0(\theta)| = o_p(1)$ and, therefore, $\hat{\theta}_{\hat{\gamma}} = \theta_0 + o_p(1)$. For the asymptotic normality: by Assumptions 2 and 1(c), the first-order conditions of the minimization problem are satisfied with probability approaching one, $\hat{\boldsymbol{\mu}}_{\hat{\gamma}}^*(\hat{\theta}_{\hat{\gamma}}) \hat{\Sigma}_{\hat{\gamma}}(\tilde{\theta})^{-1} n^{-1/2} \hat{\boldsymbol{\Phi}}_{\hat{\gamma}}(\hat{\theta}_{\hat{\gamma}}) = 0$. Expanding $\hat{\boldsymbol{\Phi}}_{\hat{\gamma}}(\hat{\theta}_{\hat{\gamma}})$ around θ_0 and solving gives the Bahadur representation,

$$\sqrt{n}(\hat{\theta}_{\hat{\gamma}} - \theta_0) = - \left[\hat{\boldsymbol{\mu}}_{\hat{\gamma}}^*(\hat{\theta}_{\hat{\gamma}}) \hat{\Sigma}_{\hat{\gamma}}(\tilde{\theta})^{-1} \hat{\boldsymbol{\mu}}_{\hat{\gamma}}^{*'}(\bar{\theta}) \right]^{-1} \hat{\boldsymbol{\mu}}_{\hat{\gamma}}^*(\hat{\theta}_{\hat{\gamma}}) \hat{\Sigma}_{\hat{\gamma}}(\tilde{\theta})^{-1} \hat{\boldsymbol{\Phi}}_0(\hat{\gamma})$$

where $|\bar{\theta} - \theta_0| \leq |\hat{\theta}_{\hat{\gamma}} - \theta_0|$.

From the proof of Lemma 3, it follows that $\hat{\boldsymbol{\mu}}_{\hat{\gamma}}^*(\theta) - \hat{\boldsymbol{\mu}}_{\hat{\gamma}}^*(\theta_0) = o_p(1)$ for any $\theta \xrightarrow{p} \theta_0$. Thus by consistency of $\hat{\theta}_{\hat{\gamma}}$ and Lemma 1(b), it follows that both $\hat{\boldsymbol{\mu}}_{\hat{\gamma}}^*(\hat{\theta}_{\hat{\gamma}})$ and $\hat{\boldsymbol{\mu}}_{\hat{\gamma}}^*(\bar{\theta})$ converge in probability to $\boldsymbol{\mu}_{\gamma,0}^*$. Finally, by Assumptions 4-5, and the uniform continuity of the sample paths of $\Phi_0(\cdot)$,

$$\Phi_0(\hat{\gamma}) \xrightarrow{d} N(0, \Sigma_{\gamma,0}).$$

The result follows by Slutsky's lemma. ■

Proof of Theorem 2. By Lemma 3, 4, and Assumption 3 (or Theorem 1 and Assump-

tion 2' for the $\hat{\chi}^2$ tests) both test statistics are asymptotically equivalent to the following quadratic form,

$$q(\hat{\theta}, W, \hat{\gamma}) = \left(\Phi_0(\hat{\gamma}) - \sqrt{n} \boldsymbol{\mu}_{\gamma,0}^* \bar{l}_{\theta_0} \right)' W^{-1} \left(\Phi_0(\hat{\gamma}) - \sqrt{n} \boldsymbol{\mu}_{\gamma,0}^* \bar{l}_{\theta_0} \right), \quad (\text{A1})$$

where $\bar{l}_{\theta_0} = n^{-1} \sum_{i=1}^n l_{\theta_0}(Z_i)$ and W^{-1} is the probability limit of the weighting matrix W_n^{-1} . In particular, the couple $(\hat{\theta}, W_n^{-1})$ is equal to $(\hat{\theta}_{\hat{\gamma}}, \hat{\Sigma}_{\hat{\gamma}}(\tilde{\theta}))$ in the $\hat{\chi}^2$ test and $(\tilde{\theta}, \widehat{\text{Avar}}^-(\hat{\Phi}_{\gamma}(\tilde{\theta})))$ in the Wald test. The functional

$$\phi(z, w, \gamma) = (z(\gamma) - \boldsymbol{\mu}_{\gamma,0}^* w)' W^{-1} (z(\gamma) - \boldsymbol{\mu}_{\gamma,0}^* w),$$

mapping $(\hat{\Phi}_0(\cdot), \sqrt{n} \bar{l}_{\theta_0}, \hat{\gamma})$ into $q(\hat{\theta}, W, \hat{\gamma})$ is continuous with respect to the product topology on $l^\infty(\mathbb{D}) \times \mathbb{R}^L \times \mathbb{D}$ (see Lemma 4 in Andrews (1988a)). Thus, Theorem 2 follows by establishing the limit null distribution of $(\Phi_0(\hat{\gamma}) - \boldsymbol{\mu}_{\gamma,0}^* \sqrt{n} \bar{l}_{\theta_0})$ and an application of the continuous mapping theorem (e.g., Theorem 1.3.6 in VW). Lemma 2, Assumptions 1, 4, 5, and the central limit theorem imply that $(\Phi_0(\cdot), \sqrt{n} \bar{l}_{\theta_0}, \hat{\gamma})$ is a uniformly tight process on \mathbb{D} with fidis converging weakly to those of $(\Phi_0(\cdot), l_0, \gamma_0)$, where $l_0 \stackrel{d}{=} N(0, L_0)$ and $\mathbb{E}[l_0 \Phi_0(\gamma)] = \mathbb{E}[l_{\theta_0}(Z) \varepsilon_{\theta_0}(Z) \mathbf{I}_{\gamma}(X)']$. Thus,

$$(\hat{\Phi}_0(\cdot), \sqrt{n} \bar{l}, \hat{\gamma}) \rightsquigarrow (\Phi_0(\cdot), l_0, \gamma_0) \text{ on } l^\infty(\mathbb{D}),$$

and by the continuous mapping theorem,

$$q(\hat{\theta}, W, \hat{\gamma}) \xrightarrow{d} \tilde{Y}' W^{-1} \tilde{Y},$$

where $\tilde{Y} \stackrel{d}{=} N(0, \Sigma_{\tilde{Y}})$ and $\Sigma_{\tilde{Y}} = \text{Avar}(\hat{\Phi}_{\gamma}(\hat{\theta}))$. In the Wald test, where W^{-1} is a generalized inverse of $\Sigma_{\tilde{Y}}$, $\tilde{Y}' W^{-1} \tilde{Y} \stackrel{d}{=} \chi_{r(\text{Avar}(\hat{\Phi}_{\gamma}(\hat{\theta})))}^2$ by Theorem 7.3(i) in Rao and Mitra (1972). In the $\hat{\chi}^2$ test, $W = \Sigma_{\gamma,0}$, and $\Sigma_{\tilde{Y}} = \text{Avar}(\hat{\Phi}_{\gamma}(\hat{\theta}_{\hat{\gamma}})) = \Sigma_{\gamma,0} - \boldsymbol{\mu}_{\gamma}^* (\boldsymbol{\mu}_{\gamma}^* \Sigma_{\gamma,0}^{-1} \boldsymbol{\mu}_{\gamma}^*) \boldsymbol{\mu}_{\gamma}^*$. Since $\text{Var}(\Sigma_{\gamma,0}^{-1/2} \tilde{Y}) = \Sigma_{\gamma,0}^{-1/2} (\Sigma_{\tilde{Y}}) \Sigma_{\gamma,0}^{-1/2}$ is idempotent with rank equal to $L - d_{\theta}$, it follows that $\tilde{Y}' W^{-1} \tilde{Y} = (\Sigma_{\gamma,0}^{-1/2} \tilde{Y})' (\Sigma_{\gamma,0}^{-1/2} \tilde{Y}) \stackrel{d}{=} \chi_{L-d_{\theta}}^2$, proving the theorem. ■

Proof of Proposition 1. Let

$$d_{\gamma}(I_L) = \delta(\gamma)' \delta(\gamma) = \sum_{l=1}^L \delta_l^2 = \sum_{l=1}^L \mathbb{E} [h(X) \mathbb{I}_{\gamma_l}(X)]^2.$$

If $h(x) > 0$ ($h(x) \leq 0$) for all $x \in \mathcal{X}$, then $\delta_l \geq 0$ ($\delta_l \leq 0$) for all $l = 1, 2, \dots, L$, and, since $(\delta_l + \delta_f)^2 \geq (\delta_l^2 + \delta_f^2)$ for each δ_l, δ_f such that $\text{sign}(\delta_l) = \text{sign}(\delta_f)$, it follows that,

$$\sum_{l=1}^L \delta_l^2 \leq \left(\sum_{l=1}^L \delta_l \right)^2.$$

Thus, a unique cells maximize the drifts norm.

Consider, instead, the case $h(x) = 0$ for some $x \in \mathcal{X}$, then for any finite split of \mathcal{X} with $L > 1$, we have that

$$\delta_l^2 + \delta_f^2 = (\delta_l^+ + \delta_l^-)^2 + (\delta_f^+ + \delta_f^-)^2 \leq (\delta_l^+ + \delta_f^+)^2 + (\delta_l^- + \delta_f^-)^2 = \delta_l^{*2} + \delta_f^{*2},$$

where δ_l^+ and δ_l^- denote the positive and negative part of δ_l , respectively. Therefore, any partition is dominated by a partition with the same number of cells obtained by merging the positive and negative part of each δ_l into a new cell. Since the δ_l^* are either non-negative or non-positive, by merging them in two cells containing only the positive and negative δ_l^* we obtain a partition with two cells that dominates any partition with $L > 1$ cells. Of course, since

$$(\delta_l^+ + \delta_l^-)^2 \leq (\delta_l^+)^2 + (\delta_l^-)^2,$$

the two-cells partition dominates the one-cell partition as well. Thus, the two-cells partition into positive and negative values maximizes the drifts norm. ■