Mathematical derivation:

$\boldsymbol{L}$	Lagrange-Operator
$\boldsymbol{x}$	Position of the Cart
$\dot{\boldsymbol{x}}$	Speed of the Cart
$\ddot{\boldsymbol{x}}$	Acceleration of the Cart
$\boldsymbol{\alpha}$	Angle of the pendulum
$\dot{\boldsymbol{\alpha}}$	Angular velocity of the pendulum
$\ddot{\boldsymbol{\alpha}}$	Angular acceleration of the pendulum
$\boldsymbol{r}$	Length of pendulum
$\boldsymbol{\tau}$	input Torque to pendulum
$\boldsymbol{f}$	Applied force to cart
$\boldsymbol{E}_{K1}$	Kinectic Energy of cart mass
$\boldsymbol{E}_{P1}$	Potential Energy of cart mass
$\boldsymbol{E}_{K2}$	Kinectic Energy of the pendulum mass
$\boldsymbol{E}_{P2}$	Potential Energy of pendulum mass
$\boldsymbol{E}_{kin}$	Kinectic Energy of System
$\boldsymbol{E}_{pot}$	Potential Energy of System
$\boldsymbol{g}$	Gravity constant

Non-Linear Equations of Cart Pendulum System

As per our project, a non-linear system is because of the sine and cosine relationship of the pendulum due to angle. The dynamics of the system can be described through the Lagrange system by finding the kinetic and potential energy of the system. Hence, the Cart Pendulum system can be divided into two sub-systems.

1. **Cart (Gantry) System**
2. **Pendulum system**

Kinetic and Potential of Cart (Gantry) mass System

For the cart system, A mass of $\boldsymbol{m_1}$ with $(\boldsymbol{x}, \boldsymbol{y})$ coordinated is capable of horizontal motion along the x-axis back and forth as shown in fig (a). A horizontal force is applied through the belt attached with the Dc motor caused the push and pull of $\boldsymbol{m_1}$mass. The energy of cart mass $\boldsymbol{m_1}$mass can be defined as,

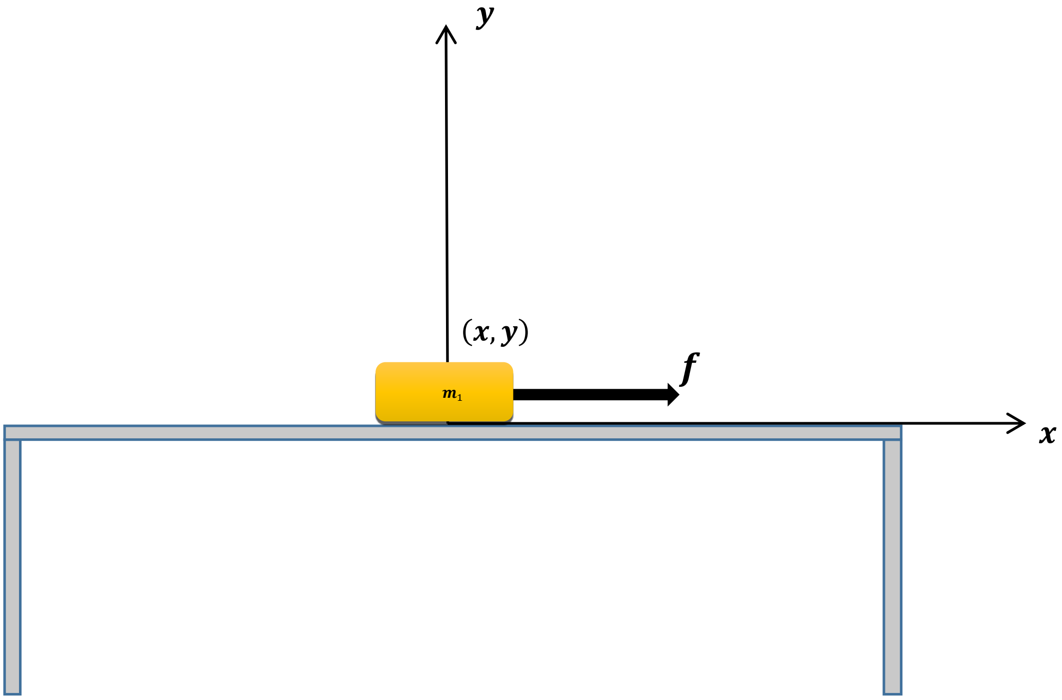


fig (a)

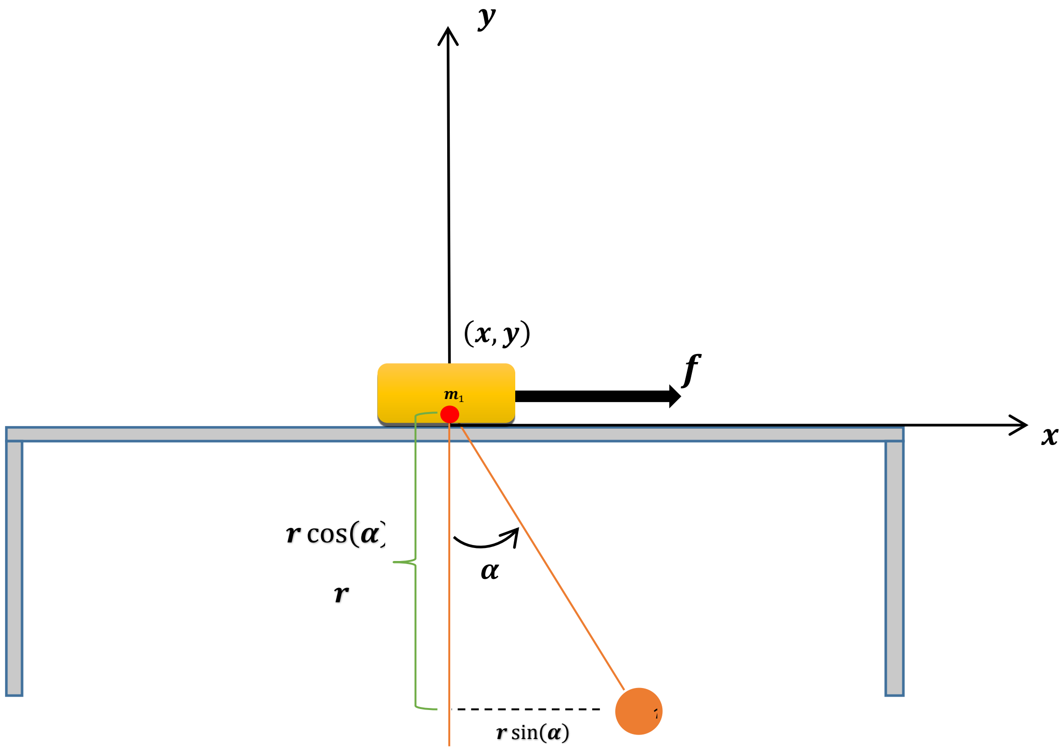
The distance coved by the pendulum mass $\boldsymbol{m_1}$ is $\boldsymbol{x}$ and the derivative of distance will give us the velocity of $\boldsymbol{m_1}$. The potential energy of pendulum mass $\boldsymbol{m_1}$ will be zero because of no vertical movement. Hence, The kinetic and potential energy of the system is given through the given equation below,

$$\boldsymbol{E_{K1}} = \frac{1}{2}\boldsymbol{mv^2} = \frac{1}{2}\boldsymbol{m_1\dot{x}^2} \qquad \text{-----------(1)}$$

$$\boldsymbol{E_{P1}} = \boldsymbol{mgh} = \boldsymbol{m_1 gy} = 0 \qquad \text{----------(2)}$$

Kinetic and Potential of Pendulum mass system

For the pendulum system, A mass of $\boldsymbol{m_2}$ with $(\boldsymbol{x_1}, \boldsymbol{y_1})$ coordinated connected to $\boldsymbol{m_1}$ with revolute joint with 1 degree of freedom capable to produce angular motion freely concerning cart motion.



By tracking the motion of pendulum mass, it has tangential velocity and potion can be calculated by its XY components with global coordinates. Hence coordinate and velocity components of $\boldsymbol{m_2}$ concerning global coordinate can be described by the formulas mentioned below,

$$\boldsymbol{x_1} = \boldsymbol{x} + \boldsymbol{r}\sin(\boldsymbol{\alpha}) \qquad ==> \qquad \dot{\boldsymbol{x}}_1 = \dot{\boldsymbol{x}} + \boldsymbol{r}\dot{\boldsymbol{\alpha}}\cos(\boldsymbol{\alpha})$$
$$\boldsymbol{y_1} = -\boldsymbol{r}\cos(\boldsymbol{\alpha}) \qquad ==> \qquad \dot{\boldsymbol{y}}_1 = \boldsymbol{r}\dot{\boldsymbol{\alpha}}\sin(\boldsymbol{\alpha})$$

The energy of Pendulum mass $\boldsymbol{m_2}$ can be defined as,

$$\boldsymbol{E_{K2}} = \frac{1}{2}\boldsymbol{m_2}(\dot{\boldsymbol{x}}_1{}^2 + \dot{\boldsymbol{y}}_1{}^2)$$
$$\boldsymbol{E_{P2}} = \boldsymbol{m_2 g y_1}$$

By placing the value of $\boldsymbol{y_1}, \dot{\boldsymbol{x}}_1$ and $\dot{\boldsymbol{y}}_1$ in the above equation we get the kinetic and potential energy of Pendulum mass $\boldsymbol{m_2}$

$$\boldsymbol{E_{K2}} = \frac{1}{2}\boldsymbol{m_2}(\dot{\boldsymbol{x}}^2 + \boldsymbol{r}^2\dot{\boldsymbol{\alpha}}^2 + 2\dot{\boldsymbol{x}}\dot{\boldsymbol{\alpha}}\boldsymbol{r}\cos\boldsymbol{\alpha})$$
$$\boldsymbol{E_{P2}} = -\boldsymbol{m_2 g r}\cos\boldsymbol{\alpha}$$

Now applying the Lagrange operator,

$$\boldsymbol{L} = \boldsymbol{E_{kin}} - \boldsymbol{E_{pot}} = (\boldsymbol{E_{K1}} - \boldsymbol{E_{K2}}) - (\boldsymbol{E_{P1}} - \boldsymbol{E_{P2}})$$

$$\boldsymbol{L} = \frac{(\boldsymbol{m_1}+\boldsymbol{m_2})}{2}\dot{\boldsymbol{x}}^2 + \frac{\boldsymbol{m_2}}{2}\boldsymbol{r}^2\dot{\boldsymbol{\alpha}}^2 + \boldsymbol{m_2}\dot{\boldsymbol{x}}\dot{\boldsymbol{\alpha}}\boldsymbol{r}\cos\boldsymbol{\alpha} + \boldsymbol{m_2 g r}\cos\boldsymbol{\alpha} \qquad -----\textbf{(3)}$$

To determine the force and torque applying the descriptive differential equations, for force

$$\boldsymbol{f} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\frac{\boldsymbol{\partial L}}{\boldsymbol{\partial \dot{x}}}\right) - \left(\frac{\boldsymbol{\partial L}}{\boldsymbol{\partial x}}\right)$$

$$\boldsymbol{f} = (\boldsymbol{m_1} + \boldsymbol{m_2})\ddot{\boldsymbol{x}} + \boldsymbol{m_2 r \ddot{\alpha} cos\alpha} - \boldsymbol{m_2 \dot{\alpha}^2 r sin\alpha}$$

$$\ddot{\boldsymbol{x}} = \left(\boldsymbol{f} - \boldsymbol{m_2 r \ddot{\alpha} cos\alpha} + \boldsymbol{m_2 \dot{\alpha}^2 r sin\alpha}\right)\frac{1}{(\boldsymbol{m_1}+\boldsymbol{m_2})} \qquad -----(\mathrm{A})$$

For torque,

$$\boldsymbol{\tau} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\frac{\boldsymbol{\partial L}}{\boldsymbol{\partial \dot{\alpha}}}\right) - \left(\frac{\boldsymbol{\partial L}}{\boldsymbol{\partial \alpha}}\right)$$

$$\boldsymbol{\tau} = \boldsymbol{m_2 r^2 \ddot{\alpha}} + \boldsymbol{m_2 \ddot{x} r cos\alpha} + \boldsymbol{m_2 g r sin\alpha}$$

$$\ddot{\boldsymbol{\alpha}} = -\left(\frac{\boldsymbol{cos\alpha}}{\boldsymbol{r}}\ddot{x} + \frac{\boldsymbol{g sin\alpha}}{\boldsymbol{r}}\right) \qquad ------(B)$$

To the implementation of the above non-linear equation of cart pendulum system in Simulink put the value of $\ddot{\boldsymbol{x}}$ in equation (A) and the value of $\ddot{\boldsymbol{\alpha}}$ in equation (B) will give us the new equations mentioned below,

$$\ddot{\boldsymbol{x}} = \frac{\boldsymbol{f}}{\boldsymbol{M} - \boldsymbol{m_2 cos\alpha^2}} + \frac{\boldsymbol{m_2 g}}{\boldsymbol{M} - \boldsymbol{m_2 cos\alpha^2}} \boldsymbol{cos\alpha sin\alpha} + \frac{\boldsymbol{m_2 r}}{\boldsymbol{M} - \boldsymbol{m_2 cos\alpha^2}} \boldsymbol{sin\alpha \dot{\alpha}^2} \qquad ------(\boldsymbol{C})$$

$$\ddot{\boldsymbol{\alpha}} = \frac{\boldsymbol{cos\alpha}}{\boldsymbol{r}(\boldsymbol{m_2 cos\alpha^2} - \boldsymbol{M})}\boldsymbol{f} + \frac{\boldsymbol{m_2 cos\alpha sin\alpha}}{(\boldsymbol{m_2 cos\alpha^2} - \boldsymbol{M})}\dot{\boldsymbol{\alpha}}^2 + \frac{\boldsymbol{M sin\alpha}}{\boldsymbol{r}(\boldsymbol{m_2 cos\alpha^2} - \boldsymbol{M})}\boldsymbol{g} \qquad ------(\boldsymbol{D})$$

$$\ddot{\boldsymbol{\alpha}} = -\frac{\boldsymbol{cos\alpha}}{\boldsymbol{r}(M - \boldsymbol{m_2 cos\alpha^2})}\boldsymbol{f} - \frac{\boldsymbol{m_2 cos\alpha sin\alpha}}{(\boldsymbol{M} - \boldsymbol{m_2 cos\alpha^2})}\dot{\boldsymbol{\alpha}}^2 - \frac{\boldsymbol{M sin\alpha}}{\boldsymbol{r}(\boldsymbol{M} - \boldsymbol{m_2 cos\alpha^2})}\boldsymbol{g} \qquad ------(\boldsymbol{E})$$

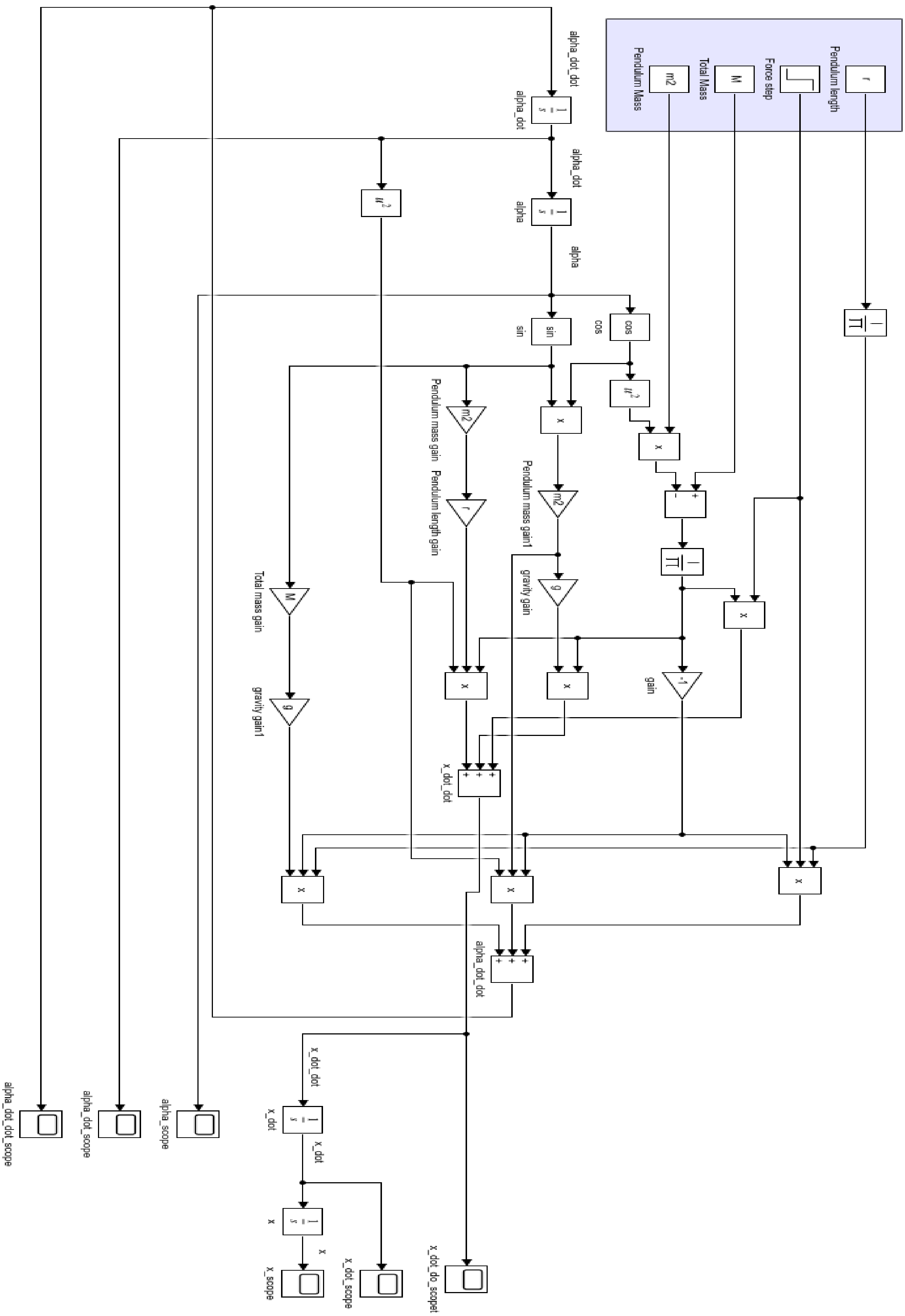
Implementation of Non-linear modal in Matlab Simulink

Matlab Code:

```
% non-linear modal of cart pendulum system

m1=3       % cart mass
m2=.5      % pendulum mass
r=0.5      % length
g=9.81     % gravitational
M=m1+m2    % addition of both masses
```

Simulink Model:



Output Graphs:

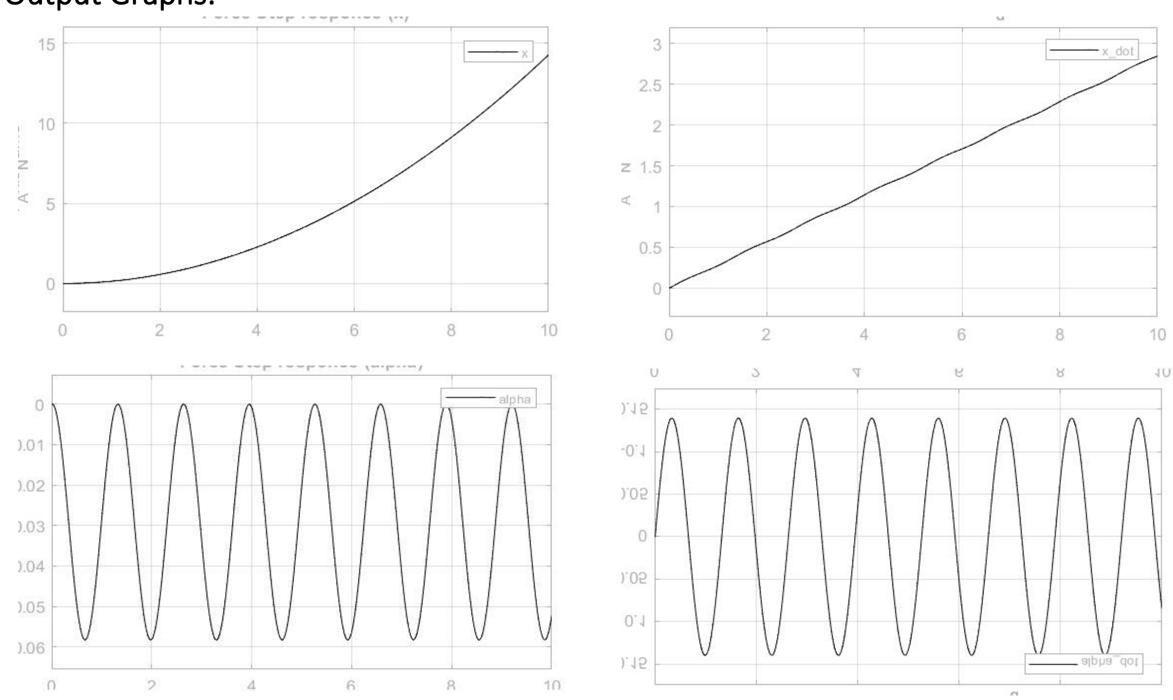


Figure 1: Non-Linear Cart Pendulum Model Response