

# 2\_Ridge\_Regression

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## 1 Ridge Regression

### 1.1 Implementation and Cheat Sheet

This page documents the summary theory and implementation of ridge regression. In particular, it covers:

- Ridge regression
- k-fold cross validation to find best l2-penalty

```
[1]: import pandas as pd
import os
import numpy as np
import re
import csv
from IPython.display import display, Math, Latex
from sklearn import datasets, linear_model
from sklearn.utils import shuffle
import matplotlib.pyplot as plt
%matplotlib inline

def load_csv_data(folder_name, file_name, dtype_dict=None):
    csv_path = os.path.join(folder_name, file_name+".csv")
    return pd.read_csv(csv_path, dtype=dtype_dict)
```

## 2 Loading data

```
[2]: dtype_dict = {'bathrooms':float, 'waterfront':int, 'sqft_above':int, \
                  'sqft_living15':float, 'grade':int, 'yr_renovated':int, \
                  'price':float, 'bedrooms':float, 'zipcode':str, \
                  'long':float, 'sqft_lot15':float, 'sqft_living':float, \
                  'floors':str, 'condition':int, 'lat':float, 'date':str, \
                  'sqft_basement':int, 'yr_built':int, 'id':str, 'sqft_lot':int, \
                  'view':int}
```

```
[3]: df_train_ridge = load_csv_data("data","kc_house_train_data",dtype_dict)
     df_test_ridge = load_csv_data("data","kc_house_test_data",dtype_dict)
```

## 3 Ridge Regression

Regression model:

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \epsilon, \quad (1)$$

where  $\mathbf{y} \in \mathbb{R}^{N \times 1}$ ,  $\mathbf{X} \in \mathbb{R}^{N \times p}$ ,  $\mathbf{w} \in \mathbb{R}^{p \times 1}$ ,  $\epsilon \in \mathbb{R}^{N \times 1}$

Total cost = measure of fit + measure of magnitude of coefficients =  $\text{RSS}(\mathbf{w}) + \lambda \|\mathbf{w}\|_2^2$

where  $\|\mathbf{w}\|_2^2 = w_0^2 + w_1^2 + \dots + w_{p-1}^2$ . Here,  $w_0$  is the intercept, and there are  $p - 1$  actual features, giving a total of  $p$  features.

Here,

$$\text{RSS}(\mathbf{w}) = (\mathbf{y} - \hat{\mathbf{y}})^T(\mathbf{y} - \hat{\mathbf{y}}) = (\mathbf{y} - \mathbf{X}\mathbf{w})^T(\mathbf{y} - \mathbf{X}\mathbf{w}) \quad (2)$$

and

$$\|\mathbf{w}\|_2^2 = \mathbf{w}^T \mathbf{w} \quad (3)$$

Therefore,

$$\text{Total cost} = (\mathbf{y} - \mathbf{X}\mathbf{w})^T(\mathbf{y} - \mathbf{X}\mathbf{w}) + \lambda \mathbf{w}^T \mathbf{w} \quad (4)$$

Taking the gradient of the cost,

$$\nabla(\text{RSS}(\mathbf{w}) + \lambda \|\mathbf{w}\|_2^2) = \nabla((\mathbf{y} - \mathbf{X}\mathbf{w})^T(\mathbf{y} - \mathbf{X}\mathbf{w}) + \lambda \mathbf{w}^T \mathbf{w}) \quad (5)$$

$$= -2\mathbf{X}^T(\mathbf{y} - \mathbf{X}\mathbf{w}) + \lambda(2\mathbf{w}) = -2\mathbf{X}^T(\mathbf{y} - \mathbf{X}\mathbf{w}) + 2\lambda\mathbf{w} \quad (6)$$

The closed form solution can be found by setting the gradient to 0:

$$\nabla(\text{cost}(\hat{\mathbf{w}})) = \nabla(-2\mathbf{X}^T(\mathbf{y} - \mathbf{X}\hat{\mathbf{w}}) + 2\lambda\hat{\mathbf{w}}) = 0 \quad (7)$$

$$\hat{\mathbf{w}}^{\text{ridge}} = (\mathbf{X}^T \mathbf{X} + \lambda \mathbb{I})^{-1} \mathbf{X}^T \mathbf{y} \quad (8)$$

However, we do not want to penalize the intercept  $w_0$  for being large. Therefore, use a modified

$$\mathbb{I}^{\text{mod}} = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \text{ with } \mathbb{I} \in \mathbb{R}^{p \times p}$$

Using Gradient Descent:

init  $t = 1, \mathbf{w}^{(1)} = 0$  or randomly or smartly

while  $\|\nabla \text{RSS}(\mathbf{w}^{(t)})\| > \text{threshold}$ :

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + 2\eta(\mathbf{X}^T(\mathbf{y} - \mathbf{X}\mathbf{w}^{(t)}) - \lambda \mathbb{I}^{\text{mod}} \mathbf{w}) \quad (9)$$

```
[4]: def ridge_regression_gradient_descent(feature_matrix, target, initial_weights,\
                                           step_size, l2_penalty,\
                                           max_iterations=100):

    weights = np.array(initial_weights)
    I = np.identity(weights.size)
    I[0][0]=0
    for iteration in range(max_iterations):
        gradient_RSS = -2*np.matmul(np.transpose(feature_matrix),target-np.
        matmul(feature_matrix,weights))
        gradient_regularization = 2*l2_penalty*np.matmul(I,weights)
        total_cost=gradient_RSS+gradient_regularization
        weights = weights - step_size*total_cost
    return(weights)
```

```
[5]: def ridge_regression_closed_form(feature_matrix,target,l2_penalty):
    I = np.identity(feature_matrix.shape[1])
    I[0][0]=0
    weights = np.matmul(np.linalg.inv(np.matmul(np.
    transpose(feature_matrix),feature_matrix)+l2_penalty*I),\
    np.matmul(np.transpose(feature_matrix),target))
    return(weights)
```

```
[6]: def dataframe_prepare(dataframe,features,target):
    dataframe["constant"]=1
    one_padded_features=["constant"]
    one_padded_features.extend(features)
    X = dataframe[one_padded_features].values
    Y = dataframe[target].values
    return X,Y
```

```
[7]: def predict_outcome(feature_matrix, weights):
      predictions = np.matmul(feature_matrix, weights)
      return(predictions)
```

```
[8]: simple_features = ['sqft_living', 'sqft_living15']
X_train, Y_train = dataframe_prepare(df_train_ridge, simple_features, ['price'])
X_test, Y_test = dataframe_prepare(df_test_ridge, simple_features, ['price'])
initial_weights = np.array([[ -100000.], [0.], [0.]])
step_size = 1e-12
max_iterations = 10000

multiple_weights_small_penalty = ridge_regression_gradient_descent(X_train,
    ↪ Y_train, initial_weights=initial_weights,
    ↪ step_size=step_size, l2_penalty=1.
    ↪ 5e-5, max_iterations=max_iterations)
print("The small-penalty weights from ridge gradient descent are: ")
print(multiple_weights_small_penalty)

weights_closed = ridge_regression_closed_form(X_train, Y_train, l2_penalty=1.
    ↪ 5e-5)
print("The small-penalty weights from ridge closed form solution are: ")
print(weights_closed)
```

The small-penalty weights from ridge gradient descent are:

```
[[ -9.99999771e+04]
 [ 2.45183819e+02]
 [ 6.51587638e+01]]
```

The small-penalty weights from ridge closed form solution are:

```
[[ -1.00262175e+05]
 [ 2.45188714e+02]
 [ 6.52715852e+01]]
```

### Verify with Scikit Learn

```
[9]: l2_small_penalty = 1.5e-5
model = linear_model.Ridge(alpha=l2_small_penalty, max_iter=10000, tol=None,
    ↪ normalize=True)
model.fit(X_train, Y_train)
```

```
[9]: Ridge(alpha=1.5e-05, copy_X=True, fit_intercept=True, max_iter=10000,
      normalize=True, random_state=None, solver='auto', tol=None)
```

```
[10]: model.intercept_
```

```
[10]: array([ -100259.76546574])
```

```
[11]: model.coef_
```

```
[11]: array([[ 0.          , 245.18141126,  65.27802084]])
```

### Explore 0-penalty and infinity-penalty effects

```
[12]: multiple_weights_0_penalty = ridge_regression_gradient_descent(X_train,
    ↪Y_train, initial_weights=initial_weights,\
    ↪step_size=step_size, l2_penalty=0,
    ↪max_iterations=max_iterations)
print("The 0-penalty weights from ridge gradient descent are: ")
print(multiple_weights_0_penalty)

weights_closed = ridge_regression_closed_form(X_train, Y_train, l2_penalty=0)
print("The 0-penalty weights from ridge closed form solution are: ")
print(weights_closed)

multiple_weights_high_penalty = ridge_regression_gradient_descent(X_train,
    ↪Y_train, initial_weights=initial_weights,\
    ↪step_size=step_size,
    ↪l2_penalty=1e11, max_iterations=max_iterations)
print("The high-penalty weights from ridge gradient descent are: ")
print(multiple_weights_high_penalty)

weights_closed = ridge_regression_closed_form(X_train, Y_train, l2_penalty=1e11)
print("The high-penalty weights from ridge closed form solution are: ")
print(weights_closed)
```

The 0-penalty weights from ridge gradient descent are:

```
[[-9.9999771e+04]
 [ 2.45183819e+02]
 [ 6.51587638e+01]]
```

The 0-penalty weights from ridge closed form solution are:

```
[[-1.00262175e+05]
 [ 2.45188714e+02]
 [ 6.52715852e+01]]
```

The high-penalty weights from ridge gradient descent are:

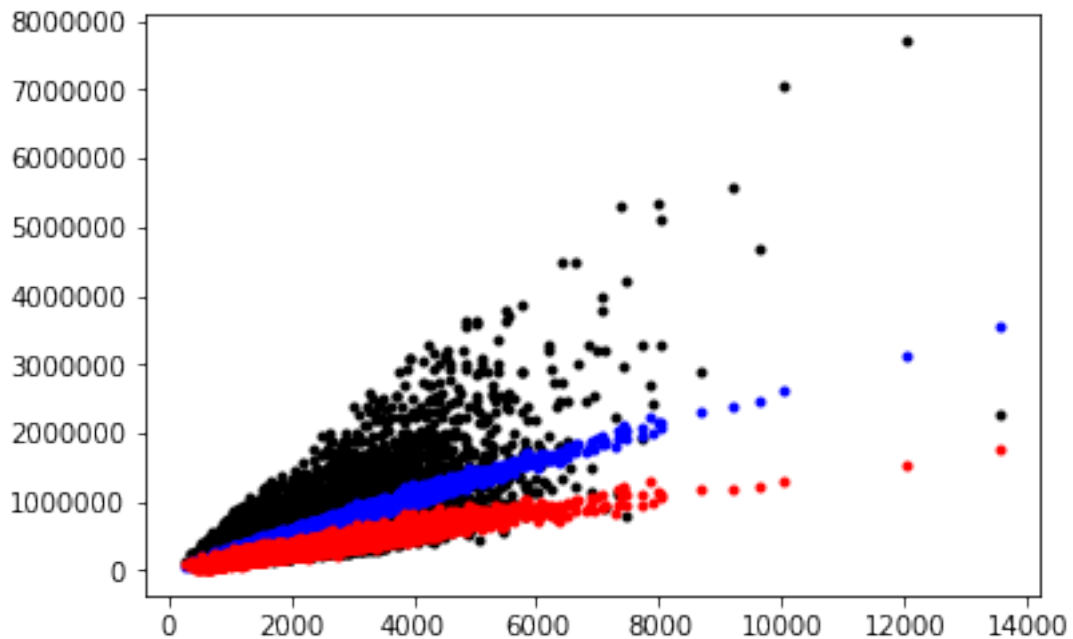
```
[[-9.99169711e+04]
 [ 1.04826258e+02]
 [ 9.19077002e+01]]
```

The high-penalty weights from ridge closed form solution are:

```
[4.25149484e+05]
[3.47447641e+01]
[2.11214072e+01]]
```

```
[13]: plt.plot(X_train[:,1], Y_train, 'k.',
    ↪X_train[:,1], predict_outcome(X_train, multiple_weights_0_penalty), 'b.',
    ↪X_train[:,1], predict_outcome(X_train, multiple_weights_high_penalty), 'r.
    ↪')
```

```
[13]: [<matplotlib.lines.Line2D at 0x10b3d6898>,
      <matplotlib.lines.Line2D at 0x10b3d6b38>,
      <matplotlib.lines.Line2D at 0x10b3d6f98>]
```



## 4 k-fold cross validation to find l2-penalty

```
[14]: def k_fold_cross_validation(k, l2_penalty, data,
    ↪output, features=False, degree=False):
    N = len(data)
    RSS_list=[]
    for i in range(k):
        #get the fold splits for the current iteration
        start = int(np.ceil((N*i)/k))
        end = int(np.ceil((N*(i+1))/k-1))
        training_data = data[0:start].append(data[end+1:N])
        validation_data = data[start:end+1]
        training_target = output[0:start].append(output[end+1:N])
        validation_target= output[start:end+1]

        #training on training split
        model = linear_model.Ridge(alpha=l2_penalty, normalize=True)
        if (features):
            model.fit(training_data[features], training_target)
```

```

else:
    model.fit(training_data, training_target)

    #RSS on validation split
    validation_data.loc[:, 'price'] = validation_target
    if (degree):
        features = ["power_" + str(i) for i in range(1, degree + 1)]
    ↵
    ↪ X, Y = dataframe_prepare(dataframe=validation_data, features=features, target=['price'])
        weights = np.concatenate((np.array([model.intercept_]), model.
    ↪ coef_), axis=0).reshape(-1, 1) #merging intercept and coeffs
        RSS = compute_RSS(X, Y, weights)
        RSS_list.append(RSS)
    return RSS_list, np.mean(RSS_list)

```

```

[15]: def compute_RSS(X, y, w):
        RSS = np.matmul(np.transpose(y - predict_outcome(X, w)), y - predict_outcome(X, w))
        return RSS

```

```

[16]: df_train_ridge_shuffled = shuffle(df_train_ridge)
        features = ['sqft_living', 'sqft_living15']
        mean_RSS_list = []

        #get best l2-penalty using 10-fold cross validation
        l2 = np.logspace(-9, 9, num=26)
        for l2_penalty in l2:
            _, mean_RSS = k_fold_cross_validation(k=10, l2_penalty=l2_penalty, ↵
            ↪ data=df_train_ridge_shuffled, ↵
            ↪ output=df_train_ridge_shuffled["price"], features=['sqft_living', ↵
            ↪ 'sqft_living15'], degree=False)
                #print(mean_RSS)
                mean_RSS_list.append(mean_RSS)

        val, idx = min((val, idx) for (idx, val) in enumerate(mean_RSS_list))
        print("Best l2-penalty based on lowest RSS:" + str(l2[idx]))

        #training new model using best l2-penalty
        model = linear_model.Ridge(alpha=l2[idx], normalize=True)
        model.fit(df_train_ridge_shuffled[features], df_train_ridge_shuffled["price"])
        print("Weights of final model: ")
        print(model.intercept_)
        print(model.coef_)

        #testing on test set

        X_test, Y_test = dataframe_prepare(dataframe=df_test_ridge, features=features, target=['price'])

```

```
weights = np.concatenate((np.array([model.intercept_]), model.coef_),axis=0).
    ↳reshape(-1,1)
test_RSS = compute_RSS(X_test,Y_test,weights)
print("Test set RSS with the best l2-penalty: "+str(test_RSS))
```

/Users/raihanabdulbaten/Dropbox/ROC/hands-on-ml/my\_codes/env/lib/python3.6/site-packages/pandas/core/indexing.py:537: SettingWithCopyWarning:  
A value is trying to be set on a copy of a slice from a DataFrame.  
Try using .loc[row\_indexer,col\_indexer] = value instead

See the caveats in the documentation: <http://pandas.pydata.org/pandas-docs/stable/indexing.html#indexing-view-versus-copy>

```
self.obj[item] = s
/Users/raihanabdulbaten/Dropbox/ROC/hands-on-ml/my_codes/env/lib/python3.6/site-packages/ipykernel_launcher.py:2: SettingWithCopyWarning:  
A value is trying to be set on a copy of a slice from a DataFrame.  
Try using .loc[row_indexer,col_indexer] = value instead
```

See the caveats in the documentation: <http://pandas.pydata.org/pandas-docs/stable/indexing.html#indexing-view-versus-copy>

```
Best l2-penalty based on lowest RSS:0.003019951720402013
Weights of final model:
-99770.3868120372
[243.73362872  66.54793933]
Test set RSS with the best l2-penalty: [[2.7020516e+14]]
```

## 4.1 Revisiting the polynomial regression problem with k-fold CV

Please refer to the polynomial regression code in the file ‘Linear Regression Basics’. Here a maximum polynomial degree of 15 is used (which performed horribly without regularization). Using l2-penalty, the performance is improved.

```
[17]: train_valid_shuffled =_
    ↳load_csv_data("data","wk3_kc_house_train_valid_shuffled",dtype_dict)
test = load_csv_data("data","wk3_kc_house_test_data",dtype_dict)
```

```
[18]: def polynomial_dataframe(feature, degree):
    poly_dataframe = pd.DataFrame()
    poly_dataframe["power_1"] = feature
    if degree > 1:
        for power in range(2, degree+1):
            name = 'power_' + str(power)
            poly_dataframe[name] = feature.apply(lambda x: x**power)
    return poly_dataframe
```



```
[19]: degree=15
poly15_data = polynomial_dataframe(train_valid_shuffled['sqft_living'], degree)
mean_RSS_list=[]

#get best l2-penalty using 10-fold cross validation
l2 = np.logspace(-9, 9, num=26)
for l2_penalty in l2:
    _ , mean_RSS = k_fold_cross_validation(k=10, l2_penalty=l2_penalty,
    →data=poly15_data,
    →output=train_valid_shuffled["price"],features=False,degree=degree)
    #print(mean_RSS)
    mean_RSS_list.append(mean_RSS)

val, idx = min((val, idx) for (idx, val) in enumerate(mean_RSS_list))
print("Best l2-penalty based on lowest RSS:"+str(l2[idx]))

#training new model using best l2-penalty
model = linear_model.Ridge(alpha=l2[idx], normalize=True)
model.fit(poly15_data, train_valid_shuffled["price"])

#testing on test set
poly15_test = polynomial_dataframe(test['sqft_living'], degree)
poly15_test['price']=test['price']
features = ["power_"+str(i) for i in range(1,degree+1)]
X_test,Y_test=dataframe_prepare(dataframe=poly15_test,features=features,target=['price'])
weights = np.concatenate((np.array([model.intercept_]), model.coef_),axis=0).
    →reshape(-1,1)
test_RSS = compute_RSS(X_test,Y_test,weights)
print("Test set RSS with the best l2-penalty: "+str(test_RSS))
```

/Users/raihanabdulbaten/Dropbox/ROC/hands-on-ml/my\_codes/env/lib/python3.6/site-packages/pandas/core/indexing.py:357: SettingWithCopyWarning:  
A value is trying to be set on a copy of a slice from a DataFrame.  
Try using .loc[row\_indexer,col\_indexer] = value instead

See the caveats in the documentation: <http://pandas.pydata.org/pandas-docs/stable/indexing.html#indexing-view-versus-copy>

```
self.obj[key] = _infer_fill_value(value)
```

/Users/raihanabdulbaten/Dropbox/ROC/hands-on-ml/my\_codes/env/lib/python3.6/site-packages/pandas/core/indexing.py:621: SettingWithCopyWarning:  
A value is trying to be set on a copy of a slice from a DataFrame.  
Try using .loc[row\_indexer,col\_indexer] = value instead

See the caveats in the documentation: <http://pandas.pydata.org/pandas-docs/stable/indexing.html#indexing-view-versus-copy>

```
self.obj[item_labels[indexer[info_axis]]] = value
```

/Users/raihanabdulbaten/Dropbox/ROC/hands-on-ml/my\_codes/env/lib/python3.6/site-

```
packages/ipykernel_launcher.py:2: SettingWithCopyWarning:  
A value is trying to be set on a copy of a slice from a DataFrame.  
Try using .loc[row_indexer,col_indexer] = value instead
```

See the caveats in the documentation: <http://pandas.pydata.org/pandas-docs/stable/indexing.html#indexing-view-versus-copy>

```
Best l2-penalty based on lowest RSS:0.08317637711026708  
Test set RSS with the best l2-penalty: [[1.37392658e+14]]
```