1 Linear Regression Basics

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By Raiyan Abdul Baten

University of Rochester, NY, USA

Mail: raiyanabdulbaten@gmail.com

1 Linear Regression Basics

1.1 Implementations and Cheat Sheets

This page documents summary theories and implementations of basic linear regression algorithms. In particular, it covers:

- Simple linear regression
- Multivariate linear regression
- Polynomial linear regression with validation-set based model complexity selection

[1]:
```
import pandas as pd
import os
import numpy as np
import re
import csv
from IPython.display import display, Math, Latex
from sklearn import datasets, linear_model
import matplotlib.pyplot as plt
%matplotlib inline

def load_csv_data(folder_name, file_name, dtype_dict=None):
    csv_path = os.path.join(folder_name, file_name+".csv")
    return pd.read_csv(csv_path, dtype=dtype_dict)
```

2 Loading data

[2]:
```
dtype_dict = {'bathrooms':float, 'waterfront':int, 'sqft_above':int, \
              'sqft_living15':float, 'grade':int, 'yr_renovated':int, \
              'price':float, 'bedrooms':float, 'zipcode':str, \
              'long':float, 'sqft_lot15':float, 'sqft_living':float, \
              'floors':str, 'condition':int, 'lat':float, 'date':str, \
              'sqft_basement':int, 'yr_built':int, 'id':str, 'sqft_lot':int,␣
 ↪'view':int}
```

[3]:
```
df = load_csv_data("data","kc_house_data",dtype_dict)
```

[4]:
```
df.head()
```

[4]:
```
           id             date     price  bedrooms  bathrooms  sqft_living  \
0  7129300520  20141013T000000  221900.0       3.0       1.00       1180.0
1  6414100192  20141209T000000  538000.0       3.0       2.25       2570.0
2  5631500400  20150225T000000  180000.0       2.0       1.00        770.0
3  2487200875  20141209T000000  604000.0       4.0       3.00       1960.0
4  1954400510  20150218T000000  510000.0       3.0       2.00       1680.0

  sqft_lot floors  waterfront  view     …      grade  sqft_above  \
0     5650      1           0     0     …          7        1180
1     7242      2           0     0     …          7        2170
2    10000      1           0     0     …          6         770
3     5000      1           0     0     …          7        1050
4     8080      1           0     0     …          8        1680

   sqft_basement  yr_built  yr_renovated  zipcode      lat     long  \
0              0      1955             0    98178  47.5112 -122.257
1            400      1951          1991    98125  47.7210 -122.319
2              0      1933             0    98028  47.7379 -122.233
3            910      1965             0    98136  47.5208 -122.393
4              0      1987             0    98074  47.6168 -122.045

   sqft_living15  sqft_lot15
0         1340.0      5650.0
1         1690.0      7639.0
2         2720.0      8062.0
3         1360.0      5000.0
4         1800.0      7503.0

[5 rows x 21 columns]
```

3 Simple Linear Regression

Regression model:

$$y_i = w_0 + w_1 x_i + \epsilon_i \tag{1}$$

The residual sum of squares, RSS, is given by:

$$\text{RSS}(w_0, w_1) = \sum_{i=1}^{N} (y_i - (w_0 + w_1 x_i))^2 \tag{2}$$

The parameters w_0 and w_1 can be found by minimizing the RSS:

$$\underset{w_0, w_1}{\text{argmin}} \sum_{i=1}^{N} (y_i - (w_0 + w_1 x_i))^2 \tag{3}$$

Taking the derivatives with respect to w_0 and w_1:

$$\nabla \text{RSS}(w_0, w_1) = \begin{bmatrix} -2\sum_{i=1}^{N}(y_i - (w_0 + w_1 x_i)) \\ -2\sum_{i=1}^{N}(y_i - (w_0 + w_1 x_i))x_i \end{bmatrix} \tag{4}$$

To find the closed form solutions, set the gradients to 0:

$$\begin{bmatrix} -2\sum_{i=1}^{N}(y_i - (\hat{w}_0 + \hat{w}_1 x_i)) \\ -2\sum_{i=1}^{N}(y_i - (\hat{w}_0 + \hat{w}_1 x_i))x_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

$$\hat{w}_0 = \frac{(\sum_{i=1}^{N} y_i)(\sum_{i=1}^{N} x_i^2) - (\sum_{i=1}^{N} x_i)(\sum_{i=1}^{N} x_i y_i)}{N(\sum_{i=1}^{N} x_i^2) - (\sum_{i=1}^{N} x_i)^2} \tag{6}$$

$$\hat{w}_1 = \frac{N(\sum_{i=1}^{N} x_i y_i) - (\sum_{i=1}^{N} x_i)(\sum_{i=1}^{N} y_i)}{N(\sum_{i=1}^{N} x_i^2) - (\sum_{i=1}^{N} x_i)^2} \tag{7}$$

Using Gradient descent:

while not converged:

$$\begin{bmatrix} w_0^{(t+1)} \\ w_1^{(t+1)} \end{bmatrix} <= \begin{bmatrix} w_0^{(t)} \\ w_1^{(t)} \end{bmatrix} + 2\eta \begin{bmatrix} \sum_{i=1}^{N}(y_i - (w_0^{(t)} + w_1^{(t)} x_i)) \\ \sum_{i=1}^{N}(y_i - (w_0^{(t)} + w_1^{(t)} x_i))x_i \end{bmatrix} \tag{8}$$

[5]:
```
def simple_linear_regression(feature, target):
    N=feature.size
    sum_x = np.sum(feature)
    sum_y = np.sum(target)
    sum_x2 = np.dot(feature,feature)
    sum_xy = np.dot(feature,target)
```

```
    intercept = float(sum_y*sum_x2-sum_x*sum_xy)/float(N*sum_x2-(sum_x)**2)
    slope = float(N*sum_xy-sum_x*sum_y)/float(N*sum_x2-(sum_x)**2)
    return intercept, slope
```

[6]:
```
def get_regression_predictions(input_value, intercept, slope):
    predicted_output = intercept+slope*input_value
    return(predicted_output)
```

[7]:
```
w0,w1 = simple_linear_regression(df["sqft_living"],df["price"])
print("Intercept: "+str(w0))
print("Slope: "+str(w1))
y_hat = get_regression_predictions(2650,w0,w1)
print("Prediction for 2650 sqft input: "+str(y_hat))
```

```
Intercept: -43580.743094474085
Slope: 280.6235678974483
Prediction for 2650 sqft input: 700071.711833764
```

Verify with Scikit Learn

[9]:
```
X= df["sqft_living"].values.reshape(-1, 1)
Y= df["price"].values.reshape(-1, 1)
regr = linear_model.LinearRegression()
regr.fit(X,Y)
```

[9]:
```
LinearRegression(copy_X=True, fit_intercept=True, n_jobs=1, normalize=False)
```

[10]:
```
regr.intercept_
```

[10]:
```
array([-43580.74309447])
```

[11]:
```
regr.coef_
```

[11]:
```
array([[280.6235679]])
```

4 Multi-Regression: Multiple features

Regression model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \epsilon, \tag{9}$$

where $\mathbf{y} \in \mathbb{R}^{N\times 1}, \mathbf{X} \in \mathbb{R}^{N\times p}, \mathbf{w} \in \mathbb{R}^{p\times 1}, \epsilon \in \mathbb{R}^{N\times 1}$

Residual sum of squares

$$\mathrm{RSS}(\mathbf{w}) = (\mathbf{y} - \widehat{\mathbf{y}})^{\mathrm{T}}(\mathbf{y} - \widehat{\mathbf{y}}) = (\mathbf{y} - \mathbf{X}\mathbf{w})^{\mathrm{T}}(\mathbf{y} - \mathbf{X}\mathbf{w}) \tag{10}$$

Taking the gradient

$$\nabla \mathrm{RSS}(\mathbf{w}) = \nabla((\mathbf{y}-\mathbf{X}\mathbf{w})^{\mathrm{T}}(\mathbf{y}-\mathbf{X}\mathbf{w})) = -2\mathbf{X}^{\mathrm{T}}(\mathbf{y}-\mathbf{X}\mathbf{w}) \tag{11}$$

Finding closed solutions

$$\nabla \mathrm{RSS}(\mathbf{w}) = 0\hat{\mathbf{w}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y} \tag{12}$$

Using Gradient Descent init $t = 1, \mathbf{w}^{(1)} = 0$ or randomly or smartly

while $||\nabla \mathrm{RSS}(\mathbf{w}^{(t)})|| >$threshold:

$$\mathbf{w}^{(t+1)} <= \mathbf{w}^{(t)} + 2\eta\mathbf{X}^{\mathrm{T}}(\mathbf{y}-\mathbf{X}\mathbf{w}^{(t)}) \tag{13}$$

[12]:
```
def predict_outcome(feature_matrix, weights):
    predictions = np.matmul(feature_matrix,weights)
    return(predictions)
```

[13]:
```
def compute_RSS(X,y,w):
    RSS = np.matmul(np.transpose(y-predict_outcome(X,w)),y-predict_outcome(X,w))
    return RSS
```

[14]:
```
def regression_gradient_descent(feature_matrix, target, initial_weights,␣
↪step_size, tolerance):
    converged = False
    weights = np.array(initial_weights)
    while not converged:
        gradient_RSS = -2*np.matmul(np.transpose(feature_matrix),target-np.
↪matmul(feature_matrix,weights))
        weights = weights - step_size*gradient_RSS
        # print(np.linalg.norm(gradient_RSS))
        if np.linalg.norm(gradient_RSS) < tolerance:
            converged = True
    return(weights)
```

[15]:
```
def regression_closed_form(feature_matrix, target):
    weights = np.matmul(np.linalg.inv(np.matmul(np.
↪transpose(feature_matrix),feature_matrix)),\
                        np.matmul(np.transpose(feature_matrix),target))
    return(weights)
```

[16]:
```
def dataframe_prepare(dataframe,features,target):
    dataframe["constant"]=1
    one_padded_features=["constant"]
    one_padded_features.extend(features)
    X = dataframe[one_padded_features].values
    Y = dataframe[target].values
```

```
    return X,Y
```

[17]:
```
X,Y=dataframe_prepare(df,['sqft_living', 'sqft_living15'],['price'])
initial_weights = np.array([[-100000.], [1.], [1.]])
step_size = 1e-12
tolerance = 1e7
weights = regression_gradient_descent(X, Y, initial_weights, step_size,␣
↪tolerance)
print("The weights from gradient descent are: ")
print(weights)
weights_closed = regression_closed_form(X, Y)
print("The weights from closed form solution are: ")
print(weights_closed)
```

```
The weights from gradient descent are:
[[-9.99999581e+04]
 [ 2.42236995e+02]
 [ 6.85301426e+01]]
The weights from closed form solution are:
[[-9.88630845e+04]
 [ 2.42215593e+02]
 [ 6.80410303e+01]]
```

Verify with Scikit Learn

[18]:
```
regr = linear_model.LinearRegression()
regr.fit(X,Y)
```

[18]:
```
LinearRegression(copy_X=True, fit_intercept=True, n_jobs=1, normalize=False)
```

[19]:
```
regr.intercept_
```

[19]:
```
array([-98863.08452929])
```

[20]:
```
regr.coef_
```

[20]:
```
array([[  0.        , 242.21559297,  68.04103034]])
```

5 Polynomial Regression: Selection of Model Complexity

Plot RSS vs model complexity (max degree of polynomial) and pick the complexity that gives the minimum RSS.

[21]:
```
def polynomial_dataframe(feature, degree):
    poly_dataframe = pd.DataFrame()
    poly_dataframe["power_1"] = feature
```

```
    if degree > 1:
        for power in range(2, degree+1):
            name = 'power_' + str(power)
            poly_dataframe[name] = feature.apply(lambda x: x**power)
    return poly_dataframe
```

[22]:

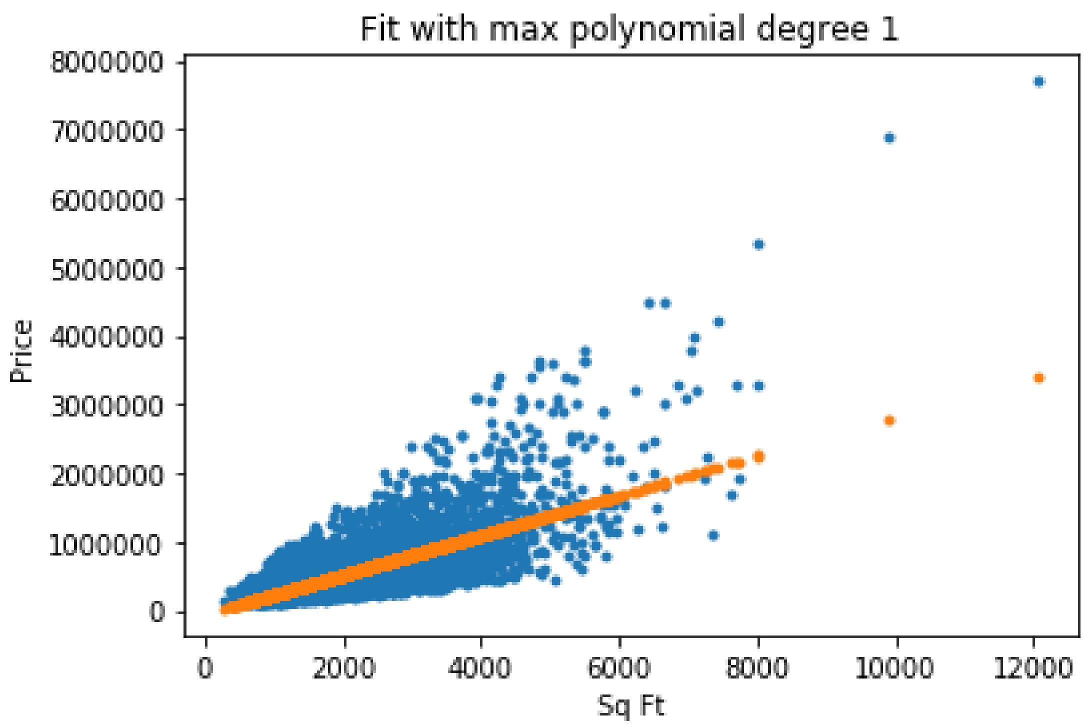
```
df_wk3_train = load_csv_data("data","wk3_kc_house_train_data",dtype_dict)
df_wk3_valid = load_csv_data("data","wk3_kc_house_valid_data",dtype_dict)
df_wk3_test = load_csv_data("data","wk3_kc_house_test_data",dtype_dict)

#for saving the learned weights and RSS values
weights_list=[]
RSS=[]

for degree in range(1,16): #degrees from 1 to 15
    #generate polynomial features with upto 'degree' degrees
    poly_train = polynomial_dataframe(df_wk3_train["sqft_living"], degree)
    poly_train["price"]=df_wk3_train["price"]
    features = ["power_"+str(i) for i in range(1,degree+1)]
    X_train,Y_train=dataframe_prepare(dataframe=poly_train,\
                                      features=features,target=['price'])

    #learn weights on training set
    weights = regression_closed_form(X_train, Y_train)
    weights_list.append(weights)

    plt.figure(degree)
    plt.plot(poly_train['power_1'],poly_train["price"],'.',
    poly_train['power_1'], predict_outcome(X_train,weights),'.')
    plt.title("Fit with max polynomial degree "+str(degree))
    plt.xlabel("Sq Ft")
    plt.ylabel("Price")

    #find RSS on validation set
    poly_valid = polynomial_dataframe(df_wk3_valid["sqft_living"], degree)
    poly_valid["price"]=df_wk3_valid["price"]
    X_valid,Y_valid=dataframe_prepare(dataframe=poly_valid,\
                                      features=features,target=['price'])
    RSS_temp = compute_RSS(X_valid,Y_valid,weights)
    RSS.append(RSS_temp)

#find best RSS on validation set
val, idx = min((val, idx) for (idx, val) in enumerate(RSS))
print("Best RSS on validation set is "+str(val)+", corresponding to degree␣
 ↪"+str(idx+1))

#final run on test set
```

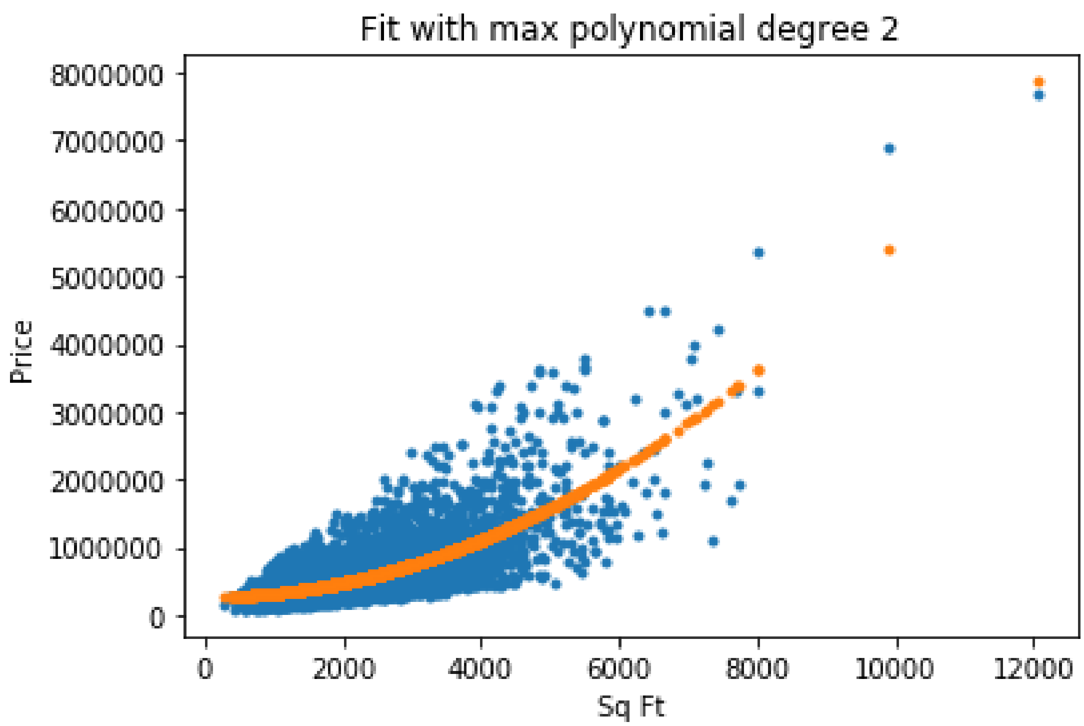
```
poly_test = polynomial_dataframe(df_wk3_test["sqft_living"], idx+1) #using the␣
↪degree with best RSS
poly_test["price"]=df_wk3_test["price"]
features = ["power_"+str(i) for i in range(1,idx+2)]
X_test,Y_test=dataframe_prepare(dataframe=poly_test,\
                                features=features,target=['price'])
RSS_test =  compute_RSS(X_test,Y_test,weights_list[idx])
print("RSS on test set with degree "+str(idx+1)+": "+str(RSS_test))

plt.figure(16)
plt.plot(range(1,16),[RSS[i][0][0] for i in range(0,15)],'.')
plt.title("Model Complexity")
plt.xlabel("Complexity")
plt.ylabel("RSS")
```

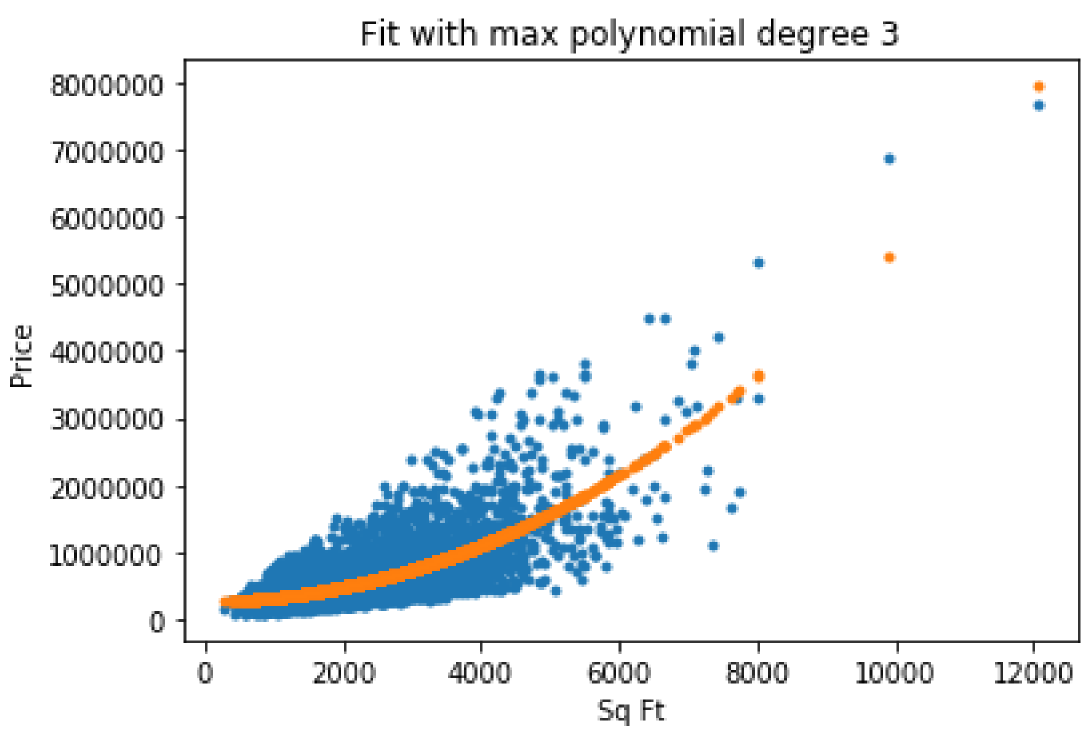
```
Best RSS on validation set is [[6.20045619e+14]], corresponding to degree 5
RSS on test set with degree 5: [[1.35567153e+14]]
```

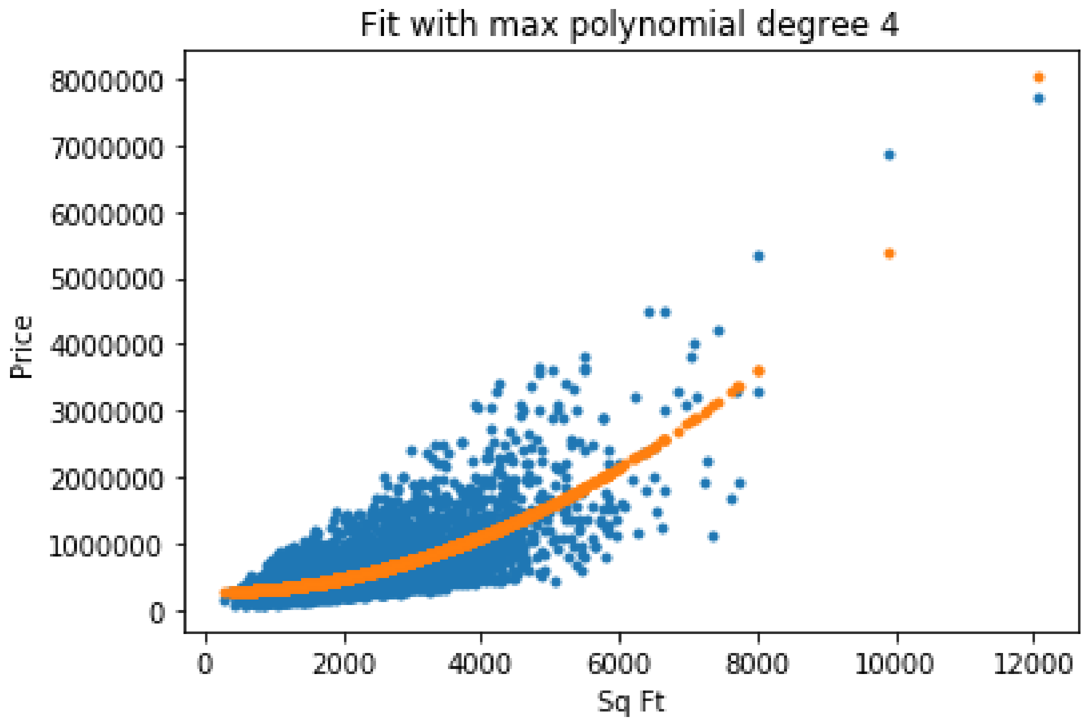
[22]:

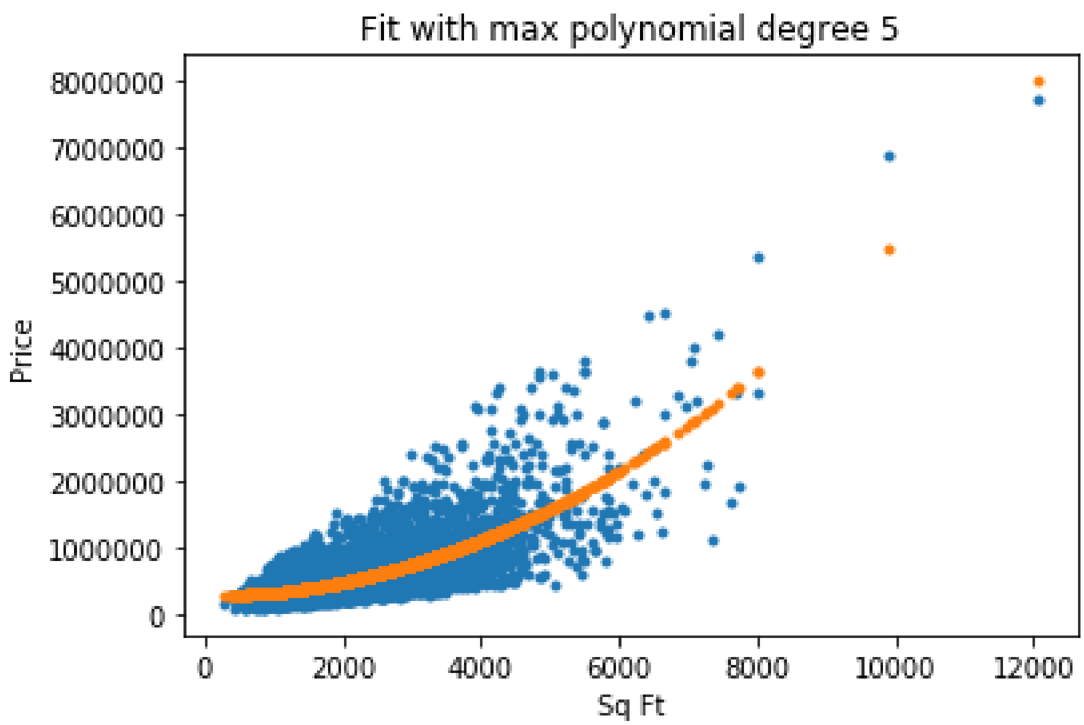
```
Text(0,0.5,'RSS')
```

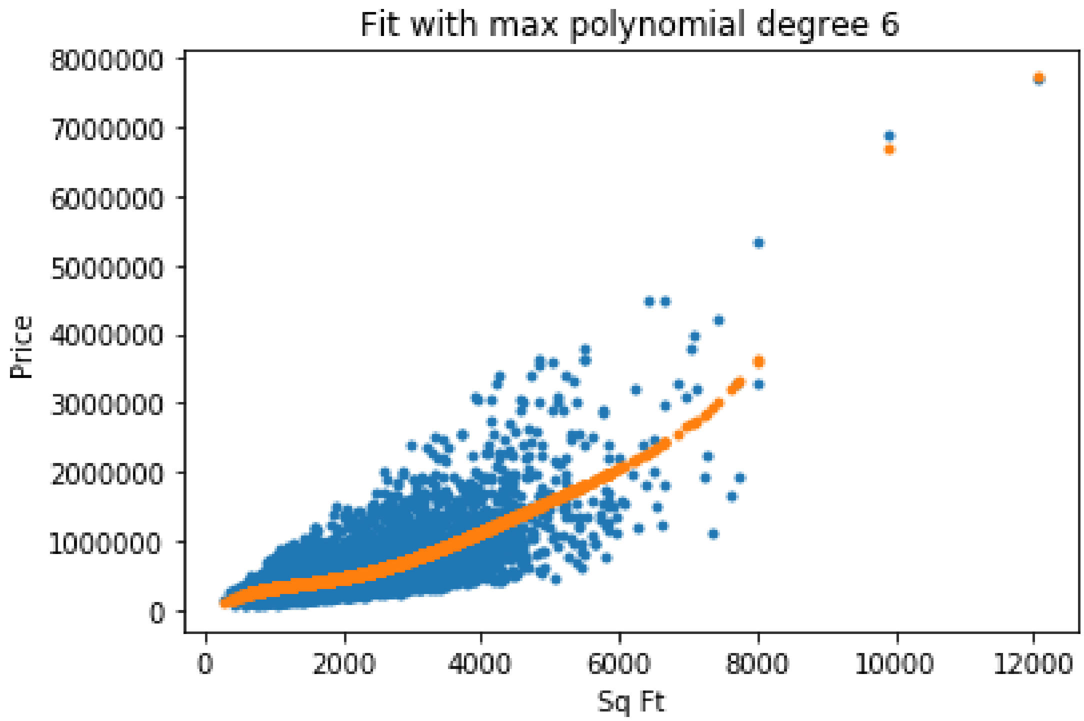



Fit with max polynomial degree 2
Price
Sq Ft
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
2000
4000
6000
8000
10000
12000

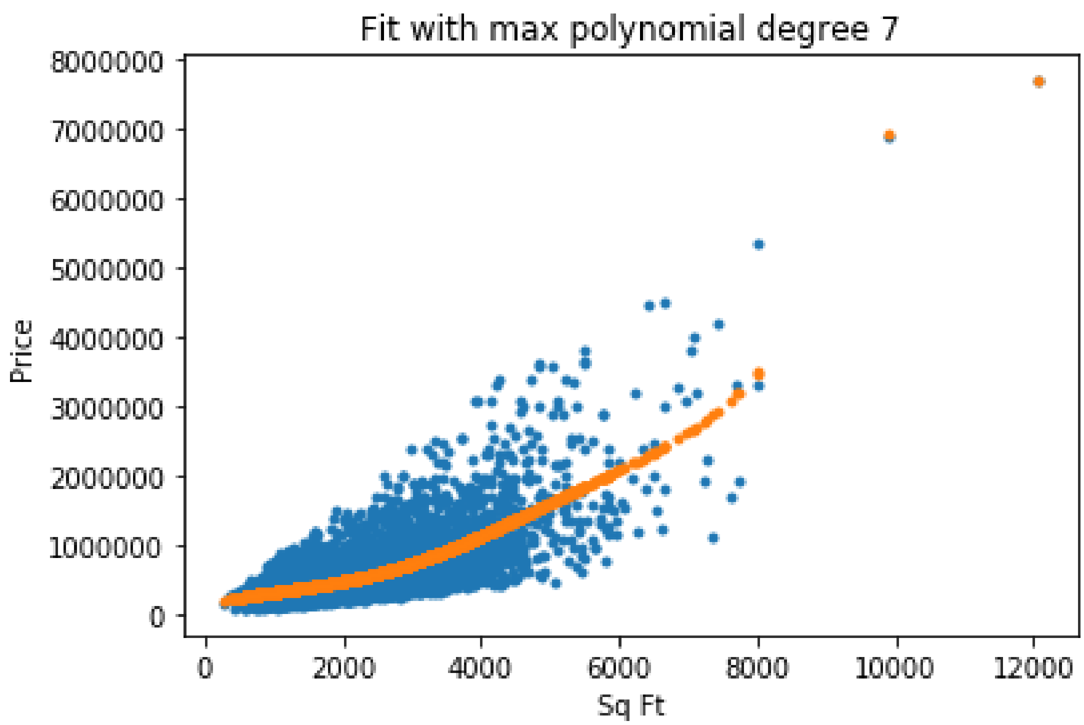

Fit with max polynomial degree 3
Price
Sq Ft
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
2000
4000
6000
8000
10000
12000


Fit with max polynomial degree 4
Price
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000
Sq Ft

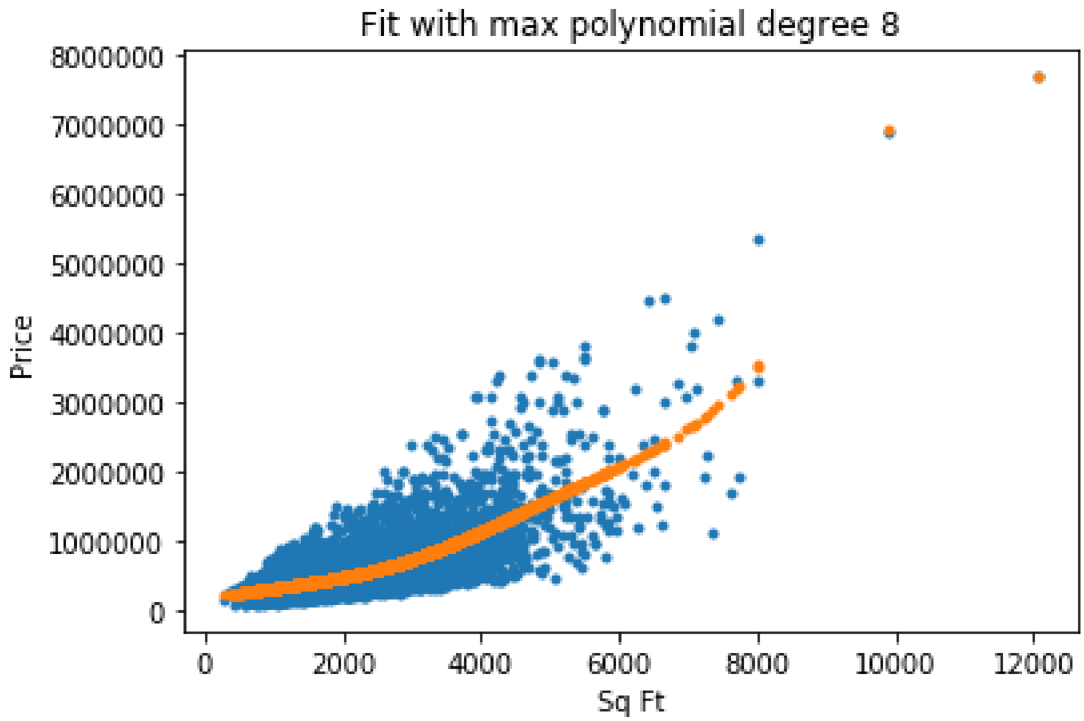

Fit with max polynomial degree 5
Price
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000
Sq Ft

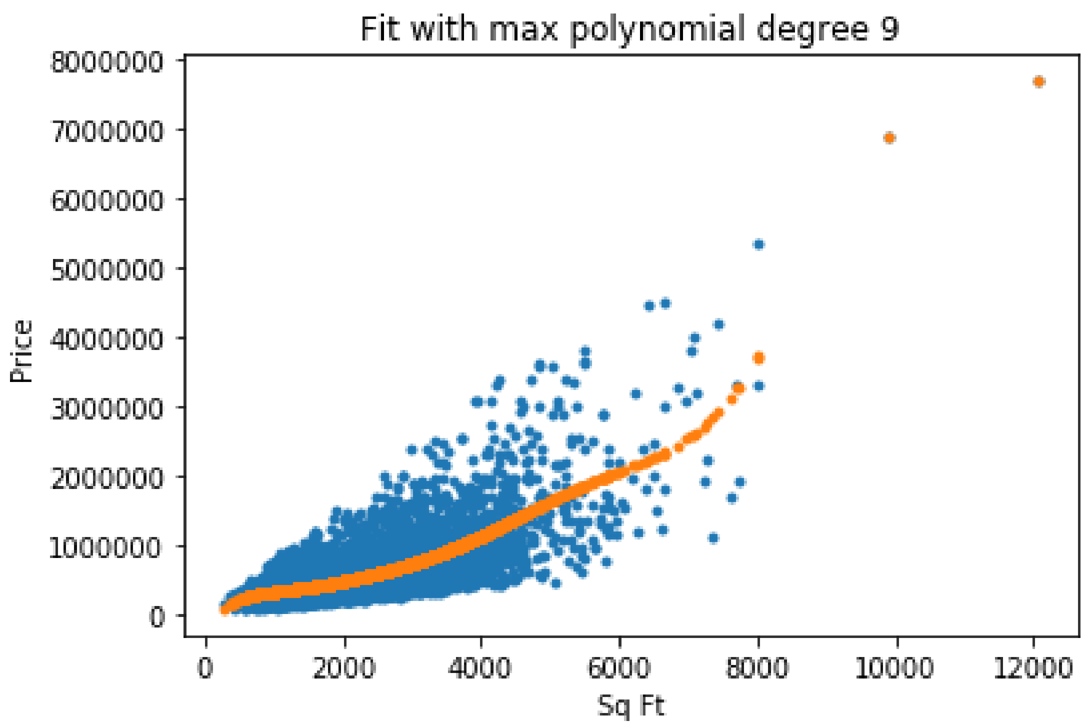


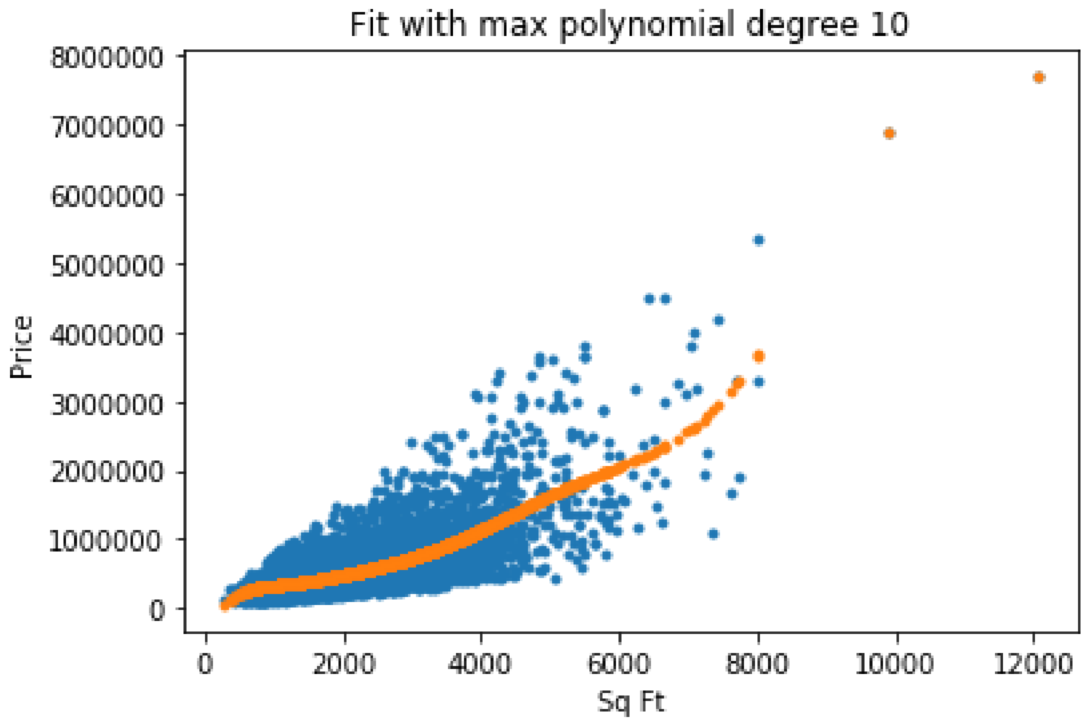
Fit with max polynomial degree 6
Price
Sq Ft
0
1000000
2000000
3000000
4000000
5000000
6000000
7000000
8000000
0
2000
4000
6000
8000
10000
12000

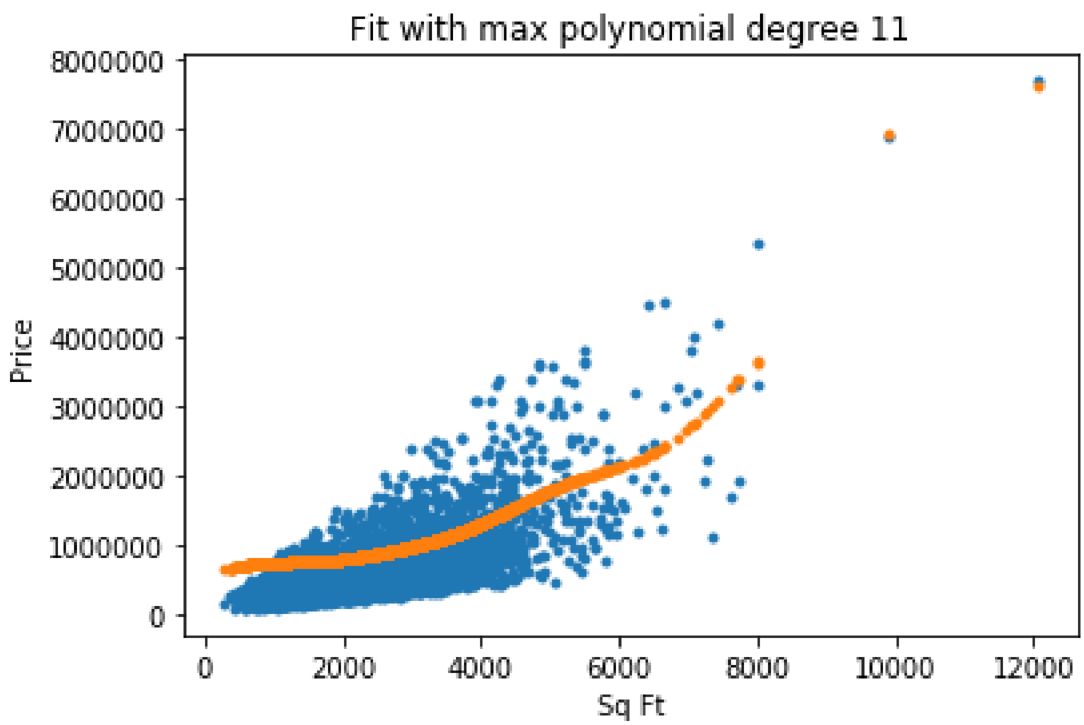


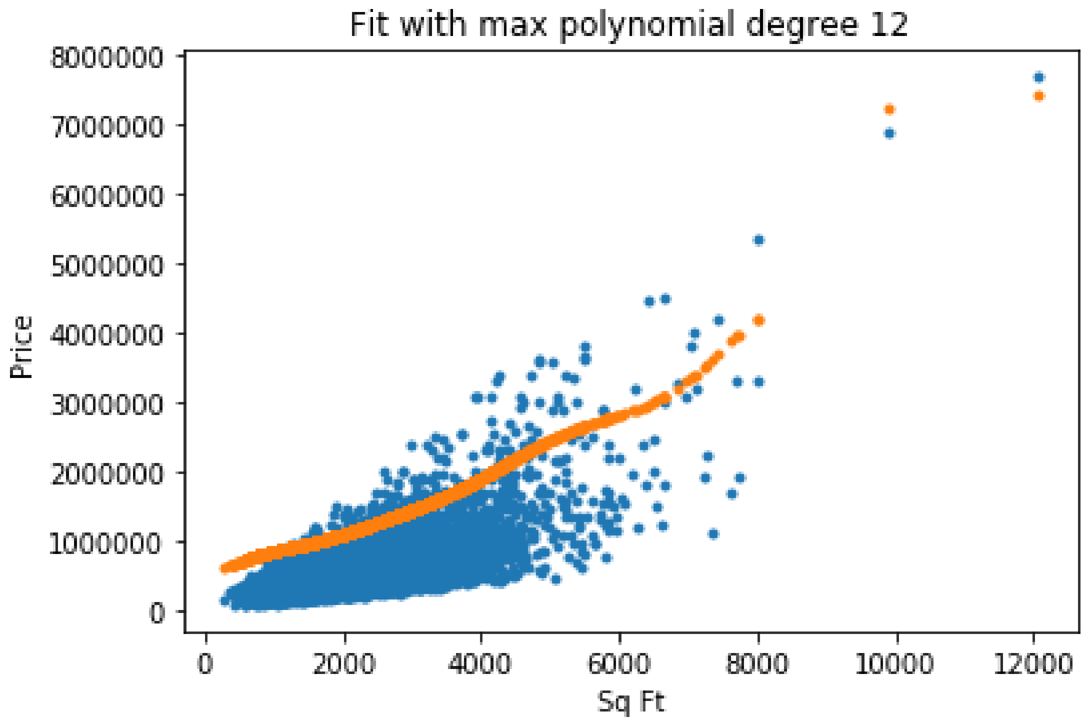
Fit with max polynomial degree 7
Price
Sq Ft
0
1000000
2000000
3000000
4000000
5000000
6000000
7000000
8000000
0
2000
4000
6000
8000
10000
12000

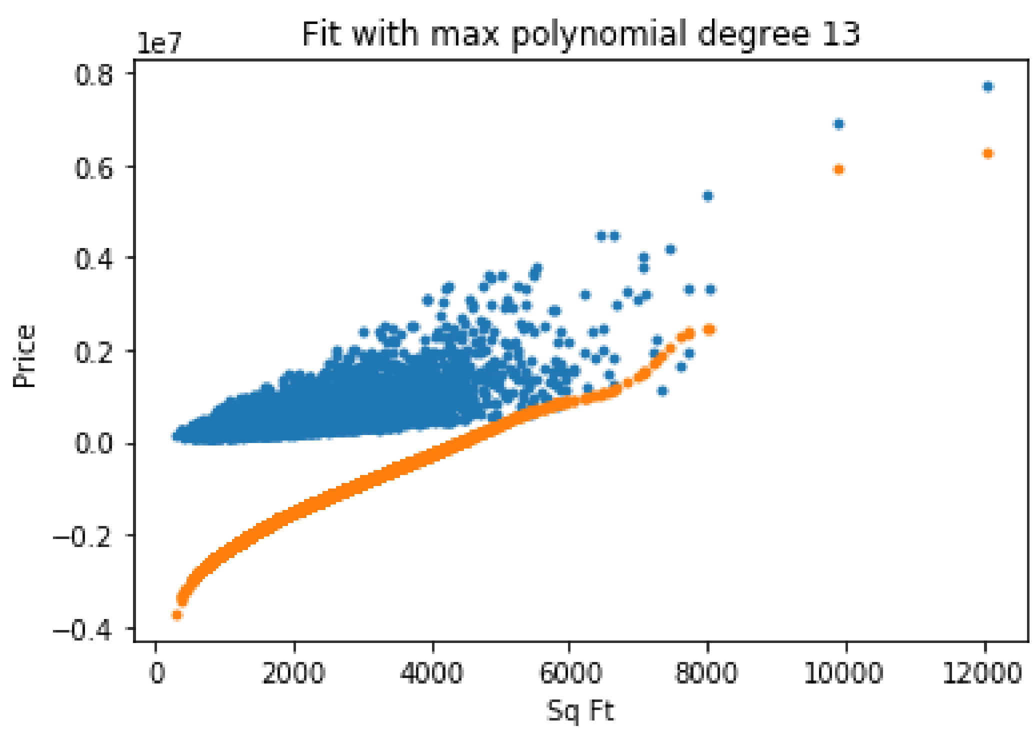

Fit with max polynomial degree 8
Price
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000
Sq Ft

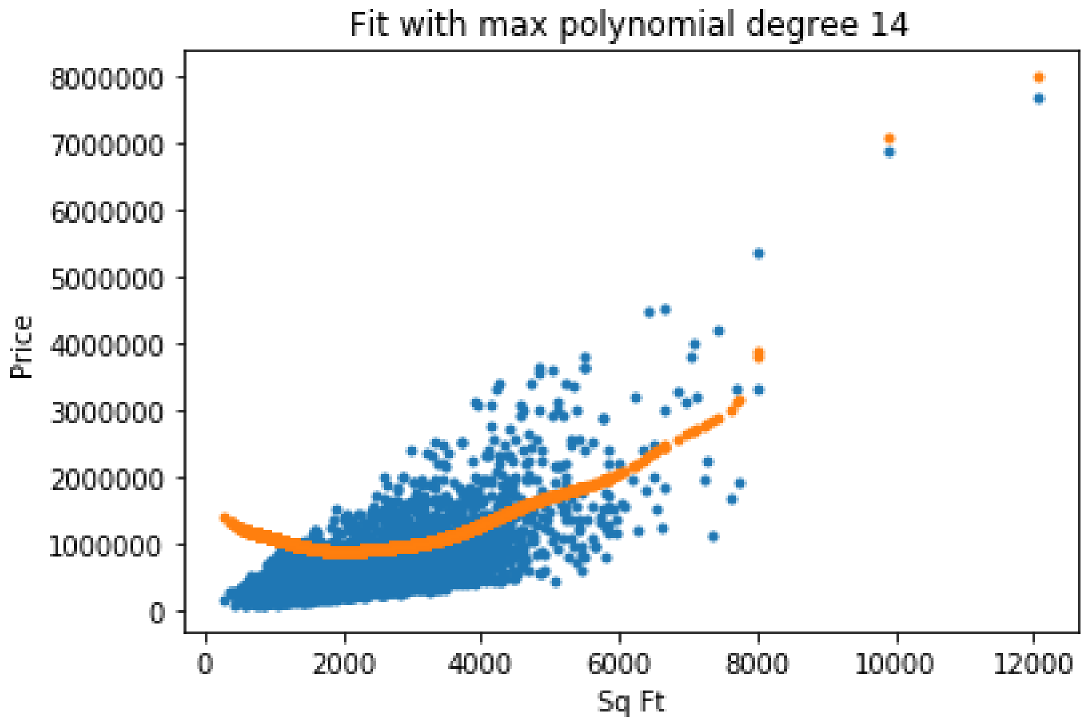

Fit with max polynomial degree 9
Price
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000
Sq Ft

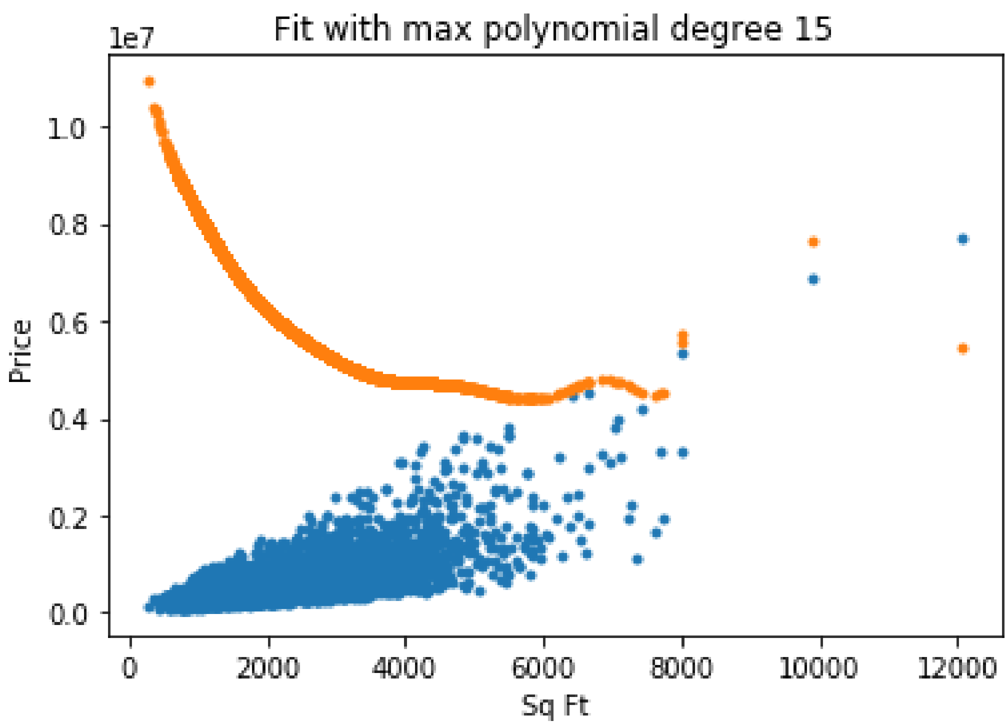

Fit with max polynomial degree 10
Price
Sq Ft
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000

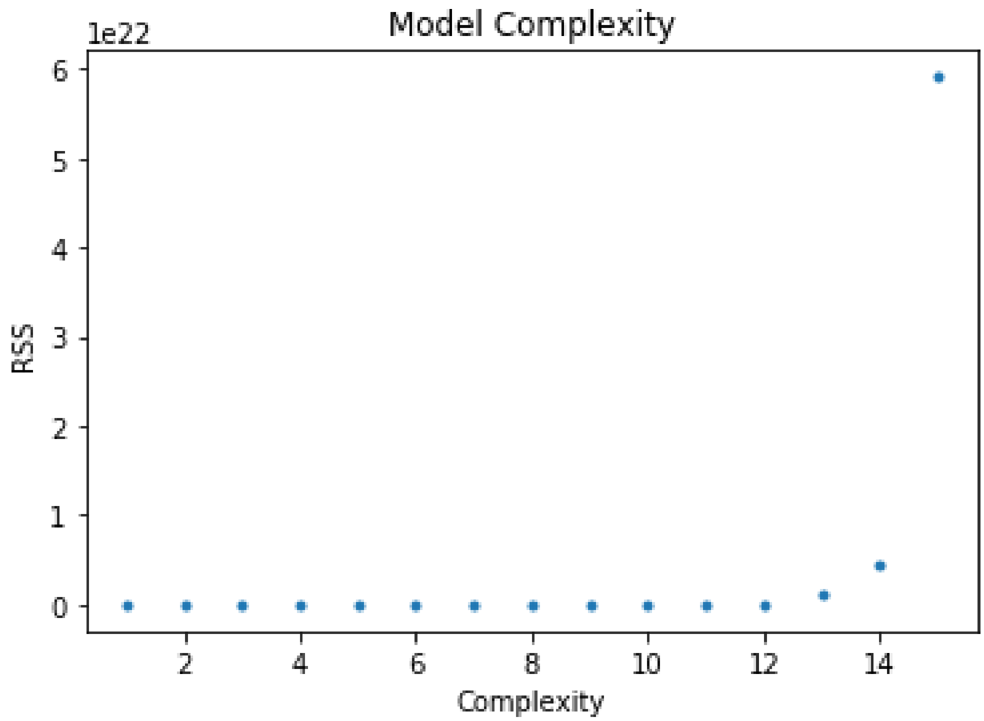

Fit with max polynomial degree 11
Price
Sq Ft
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000


Fit with max polynomial degree 12
Price
Sq Ft
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000


Fit with max polynomial degree 13
1e7
Price
Sq Ft
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
2000
4000
6000
8000
10000
12000


Fit with max polynomial degree 14
Price
8000000
7000000
6000000
5000000
4000000
3000000
2000000
1000000
0
0
2000
4000
6000
8000
10000
12000
Sq Ft


Fit with max polynomial degree 15
1e7
Price
1.0
0.8
0.6
0.4
0.2
0.0
0
2000
4000
6000
8000
10000
12000
Sq Ft


Model Complexity
1e22
RSS
Complexity
0
1
2
3
4
5
6
2
4
6
8
10
12
14