Division of Applied Mathematics

Brown University

Subject : APMA 2070-Deep Learning for Scientists and Engineers Year : Spring 2026

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Homework : 3 Due Date : 03/14/2026

Instructions

— This homework emphasizes **numerical reasoning, diagnostics, and failure analysis**, not just correct code.

1. You may use AI tools for syntax, debugging, or plotting.
2. All explanations, interpretations, and conclusions must be your own.
3. You are responsible for defending every result you submit.

Task 1 : Function Approximation and Representation Limits

Consider the function

$$f(x) = \begin{cases} 5 + \sum_{k=1}^{6} \sin(kx), & x < 0, \\ \cos(10x), & x \geq 0. \end{cases}$$

(a) Train shallow neural networks with `tanh` and `ReLU` activations using **40 and 80 training points**, with both equi-spaced and randomly sampled data.

(b) Plot :

— the learned function,

— its first derivative.

Clearly mark regions where derivative errors are largest.

(c) Explain why derivative errors concentrate near specific locations in the domain.

(d) Identify **one failure mode for each activation function** that cannot be resolved by increasing width. Support your answer using either :

— derivative plots, or

— Fourier spectra of the learned function.

Task 2 : Learning Differential Operators

Consider the following functions :

$$(a)\ f(x,y,z) = \sqrt{xy} + z,$$
$$(b)\ f(x,y,z) = \sqrt{x^2 + y^2 + 5z^2},$$
$$(c)\ f(x,y,z) = x^7 y^5,$$
$$(d)\ f(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

(a) Compute the Laplacian using automatic differentiation. Identify where the operator is ill-defined or singular and explicitly mask those regions.

(b) Where possible, compare the automatic differentiation result with the analytical Laplacian.

(c) Perturb the inputs by 10^{-6} and recompute the Laplacian.

— Which function is most unstable ?

— Is the instability mathematical or numerical ?

— How does this relate to conditioning ?

Reflection (Mandatory)

— Identify one numerical result in this homework that surprised you. Explain what mathematical or floating-point mechanism caused it.

Task 3 : Width, Depth, and Generalization

Consider the function

$$f(x) = \begin{cases} 5 + \sum_{k=1}^{4} \sin(kx), & x \in [-\pi, 0), \\ \cos(10x), & x \in [0, \pi]. \end{cases}$$

Train ReLU networks with widths

$$\{10,\ 30,\ 100,\ 300,\ 1000\}$$

using **200 training points**.

(a) Compute training and testing errors separately. Estimate approximation error versus optimization error and identify the width beyond which optimization dominates.

(b) Perform a seed sensitivity analysis. Plot the **full distribution** of errors and identify whether failures are rare or systematic. Explain what this implies about the loss landscape.

(c) Repeat using **two hidden layers**. Identify one regime where deeper networks perform worse and explain why. Propose one architectural or training modification to address this issue (no implementation required).

(d) Repeat with only **20 training points**. Discuss the role of estimation error and explain why ReLU networks behave differently from smooth activations.

Mandatory Reflection

Answer concisely :

1. Identify one result that contradicted your intuition.
2. Explain the underlying mechanism (representation, optimization, or sampling).
3. State one follow-up question you would investigate if this were a research project.