A Complete Course in Cosmology

Lecture Notes

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Contents

Chapter 1

Foundations of Cosmological Geometry

1.1 The Metric: Describing Space

The fundamental mathematical object that represents the geometry of space is the **metric**. The metric encodes all information about distances, angles, and curvature in spacetime.

Definition 1.1. *A metric tensor $g_{\mu\nu}$ allows us to calculate the invariant interval ds^2 between nearby events in spacetime.*

1.2 Homogeneity and Isotropy

The observed universe exhibits two crucial symmetries on large scales:

- **Homogeneity**: The universe appears the same at all locations (no preferred position)
- **Isotropy**: The universe appears the same in all directions (no preferred direction)

Remark 1.2. *These are statistical properties valid on scales larger than ∼100 Mpc. On smaller scales, the universe is clearly inhomogeneous (galaxies, clusters, voids).*

1.3 The Friedmann-Lemaître-Robertson-Walker (FLRW) Metric

Given the assumptions of homogeneity and isotropy, the most general metric for the universe takes the form:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\, d\Omega_2^2\right]$$

where:

- t is cosmic time
- $a(t)$ is the **scale factor** (describes how distances change with time)
- $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the angular part
- k is the **curvature parameter** taking values $+1$, 0, or -1

The spatial part depends on the curvature:

$$ds^2 = -dt^2 + a^2(t)\begin{cases} dr^2 + \sin^2 r\, d\Omega_2^2 & k = +1 \text{ (closed, positive curvature)} \\ dr^2 + r^2\, d\Omega_2^2 & k = 0 \text{ (flat)} \\ dr^2 + \sinh^2 r\, d\Omega_2^2 & k = -1 \text{ (open, negative curvature)} \end{cases}$$

Important

Physical Interpretation:

- $k = +1$: Closed universe (like the surface of a 3-sphere)
- $k = 0$: Flat universe (Euclidean geometry)
- $k = -1$: Open universe (hyperbolic geometry)

1.4 The Scale Factor and Expansion

The scale factor $a(t)$ determines how proper distances evolve with time. If we define $a(t_0) = 1$ today, then:

Proper distance between two points separated by coordinate distance Δr:

$$D(t) = a(t)\Delta r$$

Velocity of recession:

$$v = \dot{D} = \dot{a}(t)\Delta r$$

1.5 Hubble's Law

From the relation $v = \dot{a}(t)\Delta r$ and $D = a(t)\Delta r$, we can write:

$$v = \frac{\dot{a}}{a}D$$

This is **Hubble's Law**. The quantity:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

is the **Hubble parameter**. It depends on time but not on position (homogeneity).

Important

Today's value: $H_0 \approx 70$ km s^{-1} Mpc^{-1} is called the **Hubble constant**.

Key Insight: The universe is expanding everywhere. This is not expansion *into* something, but expansion *of* space itself.

Chapter 2

Einstein's Field Equations and Friedmann Cosmology

2.1 Einstein's Field Equations

The fundamental equation relating spacetime geometry to matter-energy content is:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}T^{\mu\nu}$$

where:

- $R^{\mu\nu}$ is the Ricci curvature tensor
- R is the Ricci scalar
- $g^{\mu\nu}$ is the metric tensor
- $T^{\mu\nu}$ is the stress-energy tensor
- G is Newton's gravitational constant

Physical meaning: The left side describes the curvature of spacetime; the right side describes the distribution of matter and energy.

2.2 The Friedmann Equation

For the FLRW metric, the time-time component ($\mu = \nu = 0$) of Einstein's equations gives:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

This can be rearranged as:

$$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}}$$

or equivalently:

$$\boxed{H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}}$$

This is the **Friedmann equation**, the central equation of cosmology.

2.3 Critical Density and the Density Parameter

Define the **critical density**:

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$$

This is the density required for a flat universe ($k = 0$).

The **density parameter** is:

$$\Omega = \frac{\rho}{\rho_{\text{crit}}}$$

The Friedmann equation can be rewritten as:

$$\Omega - 1 = \frac{k}{a^2 H^2}$$

Thus:

- $\Omega > 1 \Rightarrow k = +1$ (closed)
- $\Omega = 1 \Rightarrow k = 0$ (flat)
- $\Omega < 1 \Rightarrow k = -1$ (open)

Chapter 3

Matter, Radiation, and the Equation of State

3.1 The Stress-Energy Tensor for Perfect Fluids

For a homogeneous, isotropic fluid, the stress-energy tensor takes the form:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}$$

where:

- ρ is energy density
- P is pressure
- u^μ is the 4-velocity of the fluid

In the rest frame: $T^{00} = \rho$ and $T^{ii} = P$.

3.2 Equation of State

The relationship between pressure and energy density is called the **equation of state**:

$$P = w\rho$$

where w is a constant characterizing the fluid.
Examples:

- **Matter (non-relativistic):** $w = 0$ (negligible pressure)
- **Radiation (relativistic):** $w = 1/3$
- **Vacuum energy:** $w = -1$

3.3 Energy Conservation and $\rho(a)$

From energy conservation (or equivalently, the continuity equation $\nabla_\mu T^{\mu\nu} = 0$), we can derive how energy density changes with the scale factor.
Derivation:
The total energy in a comoving volume $V \propto a^3$ is:

$$E = \rho V$$

The first law of thermodynamics states:

$$dE = -P\,dV$$

Therefore:

$$dE = d(\rho V) = \rho\,dV + V\,d\rho = -P\,dV$$

This gives:

$$V\,d\rho = -(P+\rho)\,dV$$

Substituting $P = w\rho$:

$$V\,d\rho = -(1+w)\rho\,dV$$

Separating variables:

$$\frac{d\rho}{\rho} = -(1+w)\frac{dV}{V}$$

Integrating:

$$\ln\rho = -(1+w)\ln V + \text{const}$$

$$\rho = \frac{C}{V^{1+w}}$$

Since $V \propto a^3$:

$$\boxed{\rho(a) = \frac{C}{a^{3(1+w)}}}$$

Specific cases:
For matter ($w = 0$):

$$\boxed{\rho_m = \frac{\rho_{m,0}}{a^3}}$$

where $\rho_{m,0}$ is the matter density when $a = 1$.

Physical interpretation: Matter density decreases as volume increases—particles get diluted.

For radiation ($w = 1/3$):

$$\boxed{\rho_r = \frac{\rho_{r,0}}{a^4}}$$

Physical interpretation: Radiation density decreases faster than matter because not only do photons get diluted ($\propto 1/a^3$), but each photon also loses energy due to cosmological redshift ($\propto 1/a$).

For vacuum energy ($w = -1$):

$$\boxed{\rho_\Lambda = \text{constant}}$$

Physical interpretation: Vacuum energy density does not dilute—it is a property of space itself.

3.4 Physical Origin of $w = 0$ for Matter

Question 3.1. *Why is pressure negligible for non-relativistic matter?*

Answer 3.2. *Pressure arises from the kinetic motion of particles. For a particle of mass m moving with velocity v:*

- *Rest mass energy: $E_{rest} = mc^2$*
- *Kinetic energy: $E_{kin} = \frac{1}{2}mv^2$ (non-relativistic)*

The ratio is:

$$\frac{E_{kin}}{E_{rest}} = \frac{v^2}{2c^2} \ll 1$$

when $v \ll c$.

Since pressure is related to kinetic energy density, while total energy density is dominated by rest mass, we have:

$$P \ll \rho \quad \Rightarrow \quad w \approx 0$$

3.5 Physical Origin of $w = 1/3$ for Radiation

Question 3.3. *Why does radiation have $w = 1/3$?*

Answer 3.4. ***Answer through kinetic theory:***
Consider a box filled with photons, each with energy ϵ and momentum $|\vec{p}| = \epsilon/c$ (in units where $c = 1$, we have $|\vec{p}| = \epsilon$).
Setup:

- *Number density of photons: $n = N/V$*
- *Photons move isotropically in all directions*

Calculating pressure on a wall:
Consider photons hitting a wall perpendicular to the x-axis at angle θ to the normal.
In time Δt, photons within distance $c\Delta t \cos\theta$ from the wall will hit it.
Momentum transfer *per collision: $\Delta p = 2\epsilon\cos\theta$ (elastic reflection)*
Force per photon:

$$F = \frac{\Delta p}{\Delta t} = \frac{2\epsilon\cos\theta}{\Delta t}$$

Number of photons hitting *the wall of area A in time Δt:*

$$N_{hit} = (c\Delta t\cos\theta \cdot A) \cdot n \cdot \frac{1}{2}$$

(The factor $1/2$ accounts for only half the photons moving toward the wall)
Pressure *(force per unit area):*

$$P = \frac{F \cdot N_{hit}}{A} = \frac{2\epsilon\cos\theta}{\Delta t} \cdot \frac{c\Delta t\cos\theta \cdot A \cdot n}{2A}$$

$$P = \epsilon nc\cos^2\theta$$

Angular averaging:
Photons move in all directions isotropically. The average of $\cos^2\theta$ over all solid angles:
In 3D, direction cosines satisfy: $n_x^2 + n_y^2 + n_z^2 = 1$
By symmetry: $\langle n_x^2\rangle = \langle n_y^2\rangle = \langle n_z^2\rangle = 1/3$
Therefore: $\langle\cos^2\theta\rangle = 1/3$
Final result:
Since energy density $\rho = n\epsilon c$:

$$P = \rho\langle\cos^2\theta\rangle = \frac{1}{3}\rho$$

$$\boxed{w = \frac{1}{3}}$$

Chapter 4

Vacuum Energy and the Cosmological Constant

4.1 What is Vacuum Energy?

Vacuum energy is the energy density of "empty" space. Even in the absence of particles, quantum field theory predicts that the vacuum has a non-zero energy density due to:

- Zero-point fluctuations of quantum fields
- Virtual particle-antiparticle pairs
- Quantum vacuum polarization

Key property: Vacuum energy density is a universal constant—it does not change with time or location. We denote it as ρ_0 or ρ_Λ.

4.2 The Cosmological Constant Λ

Vacuum energy can be incorporated into Einstein's equations through the **cosmological constant** Λ:

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

The Friedmann equation becomes:

$$H^2 = \frac{8\pi G}{3}\rho_{\text{matter+radiation}} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

4.3 Negative Pressure: The Physics of Tension

Question 4.1. *Can energy density be negative?*

Answer 4.2. *No. Energy density must be non-negative.*

Question 4.3. *Can pressure be negative?*

Answer 4.4. *Yes! Negative pressure is called **tension**.*
Analogy: *Consider a spring stretched between two walls:*

- *As you pull the walls apart (increase volume), the spring exerts greater tension*
- *The potential energy stored in the spring increases (unlike ordinary matter/radiation where energy decreases when volume increases)*

4.4 Deriving $w = -1$ for Vacuum Energy

Given: Vacuum energy density ρ_0 is constant.

From $E = \rho V$:

$$dE = \rho_0\, dV$$

From thermodynamics:

$$dE = -P\, dV$$

Therefore:

$$\rho_0\, dV = -P\, dV$$

$$\boxed{P = -\rho_0}$$

Thus: $w = P/\rho = -1$

Physical interpretation: Vacuum energy does not dilute as the universe expands. It is an intrinsic property of space itself. As space expands, more vacuum energy is created, which is possible because vacuum energy has negative pressure—it does work *on* the universe as it expands.

4.5 Vacuum-Dominated Universe Solutions

With only vacuum energy, the Friedmann equation is:

$$H^2 = \frac{\dot{a}^2}{a^2} = \Lambda - \frac{k}{a^2}$$

Let's examine all six possible cases.

Case 1: Flat Vacuum Universe ($\Lambda > 0$, $k = 0$)

$$\frac{\dot{a}}{a} = \sqrt{\Lambda}$$

This integrates to:

$$\boxed{a(t) = a_0 e^{\sqrt{\Lambda}\, t} = a_0 e^{Ht}}$$

where $H = \sqrt{\Lambda} =$ constant.

This is called **de Sitter space**. The universe expands exponentially, with constant Hubble parameter.

Case 2: Closed Vacuum Universe ($\Lambda > 0$, $k = +1$)

$$\dot{a}^2 = \Lambda a^2 - 1$$

This can be written as:

$$\dot{a}^2 - \Lambda a^2 = -1$$

This is analogous to a classical energy equation: kinetic energy minus potential energy equals total energy (-1).

Solution:

$$\boxed{a(t) = \frac{1}{\sqrt{\Lambda}} \cosh\left(\sqrt{\Lambda}\, t\right)}$$

Expanding the hyperbolic function:

$$a(t) = \frac{1}{2\sqrt{\Lambda}}(e^{\sqrt{\Lambda}\, t} + e^{-\sqrt{\Lambda}\, t})$$

Properties:

- Symmetric in time around $t = 0$ (minimum at the "throat")
- Exponentially expanding for $|t| \gg 1/\sqrt{\Lambda}$
- Geometrically equivalent to de Sitter space (different coordinate system)

Visualization: The spatial geometry is a 3-sphere, and the spacetime forms a hyperboloid.

Case 3: Open Universe with Negative Cosmological Constant ($\Lambda < 0$, $k = -1$)

$$\dot{a}^2 = -|\Lambda|a^2 + 1$$

or:

$$\dot{a}^2 + |\Lambda|a^2 = 1$$

This is analogous to a particle in a potential well.

Behavior: The universe oscillates! It expands, reaches a maximum size, then contracts in a periodic cycle.

Remark 4.5. *Cases with $\Lambda < 0$ and $k = +1$ are not physical (left side of Friedmann equation would be negative).*

4.6 The Vacuum Energy Problem

Question 4.6. *What would be the most natural value for vacuum energy density?*

Answer 4.7. *Based on dimensional analysis using fundamental constants c, $\hbar$, and G, we can construct the **Planck energy density**:*

$$\rho_{Planck} \sim \frac{\hbar c}{l_P^4} \sim M_P^4 \sim 10^{76}\ GeV^4$$

where $l_P = \sqrt{\hbar G/c^3} \approx 10^{-35}$ m is the Planck length.

Observation: *The measured vacuum energy density is:*

$$\rho_{observed} \sim 10^{-47}\ GeV^4$$

The problem:

$$\frac{\rho_{observed}}{\rho_{Planck}} \sim 10^{-123}$$

This is the worst prediction in the history of physics! *Why is the vacuum energy density 123 orders of magnitude smaller than our "natural" expectation? This is called the **cosmological constant problem** and remains one of the deepest unsolved problems in physics.*

Chapter 5

Observational Cosmology: Measuring the Universe

5.1 The Omega Parameters

It is convenient to express the Friedmann equation in terms of dimensionless density parameters. Today (at $a = 1$):

$$H_0^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{\Lambda}{3} - k$$

Define:

$$\Omega_m = \frac{8\pi G\rho_m}{3H_0^2} = \frac{\rho_m}{\rho_{\text{crit}}}$$

$$\Omega_r = \frac{8\pi G\rho_r}{3H_0^2} = \frac{\rho_r}{\rho_{\text{crit}}}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$$

$$\Omega_k = -\frac{k}{H_0^2}$$

The Friedmann equation becomes:

$$\boxed{\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1}$$

Current observational values:

- $\Omega_m \approx 0.3$ (includes dark matter)
- $\Omega_r \approx 10^{-5}$ (negligible today)
- $\Omega_\Lambda \approx 0.7$
- $\Omega_k \approx 0 \pm 0.01$ (universe is very flat)

5.2 Hubble's Original Measurement

Historical context (1920s-1930s):

Edwin Hubble discovered the expansion of the universe by:

1. Measuring **distances** to galaxies using standard candles (Cepheid variables)
2. Measuring **velocities** from redshifts
3. Plotting luminosity (distance) versus redshift

He found: $v \propto d$, establishing Hubble's Law.

5.3 Age of the Universe from H_0

For a matter-dominated universe ($\rho \propto a^{-3}$):

Friedmann equation: $\dot{a}^2 \propto a^{-1}$

This gives: $a(t) \propto t^{2/3}$

Therefore:

$$H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{t^{-1/3}}{t^{2/3}} = \frac{2}{3t}$$

Today:

$$\boxed{H_0 = \frac{2}{3T}}$$

where T is the age of the universe.

Or approximately: $H_0 \sim 1/T$ for order-of-magnitude estimates.

5.4 Dark Matter from Galaxy Rotation Curves

Historical expectation: Matter glows and gravitates together.

Galactic rotation curves:

For a galaxy with most luminous mass concentrated near the center, Newtonian gravity predicts:

$$\frac{GMv^2}{r} = \frac{GM}{r^2} \quad \Rightarrow \quad v \propto \frac{1}{\sqrt{r}}$$

Expected: Velocity should decrease with distance from center.

Observed: Velocity remains approximately constant (flat rotation curves) at large radii.

Implication:

If $v =$ constant and $v^2 = GM(r)/r$, then:

$$M(r) \propto r$$

The mass must increase linearly with radius, implying a large halo of invisible matter.

5.5 Redshift

Definition:

Light emitted at wavelength λ_{emit} is observed at wavelength λ_{obs}:

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}}$$

Relation to scale factor:

Consider a photon traveling from a distant galaxy to us. The metric is:

$$ds^2 = -dt^2 + a^2(t)dr^2$$

For a light ray ($ds^2 = 0$) traveling radially toward us:

$$dt = -a(t)dr$$

The wavelength of light scales with the scale factor:

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a_{\text{today}}}{a(t_{\text{emit}})}$$

With $a_{\text{today}} = 1$:

$$\boxed{1 + z = \frac{1}{a(t_{\text{emit}})}}$$

or:

$$\boxed{z = \frac{1}{a} - 1}$$

Physical interpretation: Redshift directly measures how much the universe has expanded since the light was emitted.

Chapter 6

Thermal History of the Universe

6.1 The Cosmic Microwave Background (CMB)

Most important fact: The most abundant particles in the universe today are photons from the **Cosmic Microwave Background** (CMB).

The CMB is thermal radiation with a near-perfect blackbody spectrum at temperature:

$$T_{\text{CMB,today}} \approx 2.7 \text{ K}$$

Peak wavelength: $\lambda_{\text{peak}} \approx 1$ mm (microwave region).

6.2 Blackbody Radiation

Historical context (pre-1900):

Attempts to derive the spectrum of thermal radiation using classical physics led to the **Rayleigh-Jeans Law**:

$$I(\nu, T) = \frac{2k_B T \nu^2}{c^2}$$

where I is energy per unit volume per unit frequency.

Problem: As $\nu \to \infty$, $I \to \infty$ (the **ultraviolet catastrophe**).

Planck's solution (1900):

Introducing the quantum hypothesis ($E = h\nu$), Planck derived:

$$\boxed{I(\nu, T) \propto \frac{\nu^3}{e^{h\nu/k_B T} - 1}}$$

This is the **Planck distribution**.

6.3 Why is the CMB a Blackbody?

Question 6.1. *The universe is not in thermal equilibrium today, so why does the CMB have a perfect blackbody spectrum?*

Answer 6.2. *The CMB was in thermal equilibrium in the early universe. After photons decoupled from matter, the blackbody shape was "frozen in"—it just redshifts.*

Scaling of blackbody spectrum:

The Planck distribution depends on the ratio ν/T:

$$I(\nu, T) \propto \frac{\nu^3}{e^{h\nu/kT} - 1}$$

In an expanding universe:

- *Frequency redshifts: $\nu \propto 1/a$*
- *Temperature decreases: $T \propto 1/a$*

The ratio ν/T remains constant! Therefore, the blackbody shape is preserved.

$$\boxed{T(a) = \frac{T_0}{a}}$$

As the universe expands, the blackbody spectrum simply shifts to longer wavelengths (lower temperatures), but maintains its characteristic shape.

6.4 Decoupling and Recombination

Question 6.3. *At what epoch did photons "decouple" from matter?*

Answer 6.4. *When the universe cooled enough for electrons and protons to combine into neutral hydrogen atoms.*

Ionization energy of hydrogen: $E_{ion} = 13.6\ eV$

Naïve estimate:

$$T_{decouple} \sim \frac{E_{ion}}{k_B} \sim 10^5\ K$$

Problem: *Photons have a distribution of energies. Even at lower temperatures, some photons in the high-energy tail can ionize hydrogen.*

Refined calculation:

The probability of a photon having energy E is:

$$P(E) \propto e^{-E/k_B T}$$

The photon-to-baryon ratio:

$$\frac{N_\gamma}{N_b} \approx 10^8$$

For significant recombination, the number of photons with $E > E_{ion}$ must drop below the number of baryons:

$$e^{-E_{ion}/k_B T} \times 10^8 \sim 1$$

$$\frac{E_{ion}}{k_B T} \approx \ln\left(10^8\right) \approx 20$$

$$\boxed{T_{decouple} \approx \frac{E_{ion}}{20 k_B} \approx 4000\ K}$$

Redshift at decoupling:

$$z_{decouple} = \frac{T_{decouple}}{T_{today}} \approx \frac{4000}{2.7} \approx 1100$$

Physical picture:

*Before decoupling ($z > 1100$): Universe is ionized plasma. Photons scatter frequently off free electrons (Thomson scattering). The universe is **opaque**.*

*After decoupling ($z < 1100$): Electrons combine with protons to form neutral atoms. Photons travel freely. The universe becomes **transparent**.*

What we observe: *The CMB is a snapshot of the universe at $z \approx 1100$. We cannot see electromagnetic radiation from earlier times—the universe was opaque.*

Can we observe earlier epochs?

Yes, using:

1. ***Neutrinos*** *(if we could detect them)—they decoupled even earlier*

2. ***Gravitational waves*** *from the early universe (primordial GWs)*

Both can propagate through the opaque ionized plasma.

Chapter 7

The Matter-Antimatter Asymmetry

7.1 The Puzzle

Observation: The universe today contains:

- Lots of matter (protons, neutrons, electrons)
- Almost no antimatter (antiprotons, antineutrons, positrons)

Photon-to-baryon ratio:

$$\frac{N_\gamma}{N_b} \approx 10^8$$

Question 7.1. *Why is there more matter than antimatter?*

7.2 Baryon Number

Definition: The **baryon number** B of a particle:

$$B = \frac{(\text{number of quarks}) - (\text{number of antiquarks})}{3}$$

Examples:

- Proton (uud): $B = +1$
- Neutron (udd): $B = +1$
- Antiproton: $B = -1$
- Electron: $B = 0$
- Photon: $B = 0$

Baryon number conservation is observed in all known processes (in the Standard Model).

7.3 The Sakharov Conditions

Question 7.2. *What conditions are necessary to generate a matter-antimatter asymmetry?*

Answer 7.3. ***Answer (Andrei Sakharov, 1967):*** *Three conditions are necessary and (probably) sufficient:*

1. ***Baryon number violation:*** *There must be processes that change the net baryon number.*
2. ***C and CP violation:*** *Charge conjugation and combined CP symmetry must be violated, so matter and antimatter behave differently.*
3. ***Departure from thermal equilibrium:*** *The universe must be out of equilibrium to avoid C symmetry being restored.*

All three conditions are satisfied in the real universe:

1. *Baryon number violation: Predicted in GUTs (though rare)*
2. *C and CP violation: Observed in weak interactions*
3. *Out of equilibrium: The expanding universe is inherently out of equilibrium*

Chapter 8

Inflation: Solving the Puzzles of the Early Universe

8.1 The Homogeneity Problem

Naïve expectation:

A box of gas is homogeneous because particles collide and exchange energy, reaching statistical equilibrium.

But the universe is different!

- Gravitational interactions dominate on large scales
- Gravity amplifies inhomogeneities (overdense regions attract more matter)

Paradox: Given that gravity enhances density contrasts, why is the universe so homogeneous?

Answer: The universe must have started *extremely* homogeneous.

Quantitative constraint:

The observed density fluctuations in the CMB:

$$\frac{\delta\rho}{\rho} \sim 10^{-5}$$

This requires explanation! Why were initial conditions so special?

8.2 The Flatness Problem

Observation: The universe today is extraordinarily flat:

$$\Omega_k = 0 \pm 0.01$$

Evolution of curvature:

From the Friedmann equation:

$$\Omega - 1 = \frac{k}{a^2 H^2}$$

In a matter-dominated universe ($H^2 \propto a^{-3}$):

$$|\Omega - 1| \propto a$$

In a radiation-dominated universe ($H^2 \propto a^{-4}$):

$$|\Omega - 1| \propto a^2$$

Implication: $\Omega = 1$ is an unstable fixed point. Any small deviation grows with time.

Observational constraint:

To have $|\Omega - 1| < 0.01$ today requires:

$$|\Omega - 1|_{\text{Planck}} < 10^{-60}$$

at the Planck time ($t \sim 10^{-43}$ s).

Question 8.1. *Why was the universe tuned to flatness to 60 decimal places?*

8.3 The Inflationary Solution

Idea (Alan Guth, 1981): The early universe underwent a period of exponential expansion.

Mechanism: Inflation is driven by a scalar field ϕ (the "inflaton") with:

- Large potential energy $V(\phi)$
- Slow evolution

Duration: At least ~ 60 **e-foldings**:

$$a_{\text{final}}/a_{\text{initial}} \gtrsim e^{60} \sim 10^{26}$$

8.4 The Inflaton Field

Energy density of a scalar field:

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi)$$

- Kinetic energy: $\frac{\dot{\phi}^2}{2}$
- Potential energy: $V(\phi)$
- No spatial gradients: $\nabla\phi = 0$ (field is uniform)

Pressure:
For a scalar field:

$$P = \frac{\dot{\phi}^2}{2} - V(\phi)$$

Equation of state parameter:

$$w = \frac{P}{\rho} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}$$

Slow-roll regime: If $V(\phi) \gg \dot{\phi}^2$:

$$w \approx \frac{-V(\phi)}{V(\phi)} = -1$$

The scalar field behaves like vacuum energy!

8.5 Equation of Motion for the Inflaton

Lagrangian:

$$\mathcal{L} = a^3(t)\left[\frac{\dot{\phi}^2}{2} - V(\phi)\right]$$

The factor a^3 accounts for the changing volume.

Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt}(a^3\dot{\phi}) = -a^3\frac{dV}{d\phi}$$

Expanding:

$$3a^2\dot{a}\dot{\phi} + a^3\ddot{\phi} = -a^3\frac{dV}{d\phi}$$

Dividing by a^3:

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0}$$

Physical interpretation:

This looks like the equation of motion for a particle moving in potential $V(\phi)$ with **friction** proportional to $3H$.

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}$$

The Hubble expansion acts like a viscous medium, damping the motion of ϕ.

8.6 Friedmann Equation During Inflation

$$H^2 = \frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right]$$

Slow-roll condition: $V(\phi) \gg \dot{\phi}^2$

$$H^2 \approx \frac{8\pi G}{3}V(\phi)$$

If $V(\phi)$ is approximately constant (slowly varying):

$$H \approx \text{constant}$$

Exponential expansion:

$$\frac{\dot{a}}{a} = H = \text{constant}$$

$$\boxed{a(t) = a_0 e^{Ht}}$$

Doubling time:

$$t_{\text{double}} = \frac{\ln 2}{H} \sim \frac{1}{H}$$

For early universe with large $V(\phi)$:

$$t_{\text{double}} \sim 10^{-32}\text{ s}$$

The universe doubles in size every 10^{-32} seconds!

8.7 How Inflation Solves the Problems

Flatness Problem

Recall:

$$\Omega_k = -\frac{k}{a^2H^2}$$

During inflation, $H \approx$ constant while a grows exponentially:

$$|\Omega_k| \propto \frac{1}{a^2} \to 0$$

Inflation drives the universe toward flatness regardless of initial conditions.

Horizon Problem

Before inflation: Widely separated regions were never in causal contact.

During inflation: A tiny causally-connected patch is stretched to enormous size.

After inflation: The entire observable universe originated from a single causally-connected region.

All parts of the CMB had the same temperature because they originated from the same small patch.

Monopole Problem

Dilution by expansion:

Number density: $n \propto a^{-3}$

After 60 e-foldings:

$$n_{\text{final}} = n_{\text{initial}} \times e^{-180} \sim 10^{-78}$$

Any monopoles produced before inflation are diluted to undetectable levels.

Homogeneity Problem

Quantum fluctuations:

During inflation, quantum fluctuations in ϕ are stretched to macroscopic scales.

These generate density perturbations:

$$\frac{\delta\rho}{\rho} \sim \frac{H^2}{2\pi\dot{\phi}}$$

Inflation predicts:

- Nearly scale-invariant spectrum (observed!)
- Gaussian fluctuations (observed!)
- $\delta\rho/\rho \sim 10^{-5}$ (observed!)

The observed density fluctuations are *predictions* of inflation, not fine-tuning!

Appendix A

Appendix A: Summary of Key Equations

Metric and Expansion

FLRW metric:

$$ds^2 = -dt^2 + a^2(t)[dr^2 + S_k^2(r)d\Omega_2^2]$$

Hubble parameter:

$$H(t) = \frac{\dot{a}}{a}$$

Hubble's law:

$$v = HD$$

Friedmann Equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

With components:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) - \frac{k}{a^2}$$

In terms of Ω parameters:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$$

Equation of State and Scaling

$$P = w\rho$$

$$\rho(a) = \frac{\rho_0}{a^{3(1+w)}}$$

Component	w	$\rho(a)$
Matter	0	ρ_0/a^3
Radiation	1/3	ρ_0/a^4
Vacuum	-1	ρ_0

Redshift

$$1 + z = \frac{1}{a(t_{\text{emit}})}$$

Temperature:

$$T(a) = \frac{T_0}{a}$$

Scale Factor Evolution

Matter-dominated:

$$a(t) \propto t^{2/3}$$

Radiation-dominated:

$$a(t) \propto t^{1/2}$$

Vacuum-dominated (de Sitter):

$$a(t) = e^{Ht}$$

Appendix B

Appendix B: Important Epochs

Epoch	Redshift z	Temperature	Description
Today	0	2.7 K	Dark energy dominated
Matter-Λ equality	$\sim$0.3	$\sim$4 K	Transition to acceleration
Matter-radiation equality	$\sim$3000	$\sim$8000 K	Radiation $\rightarrow$ matter dominance
Recombination/Decoupling	$\sim$1100	$\sim$4000 K	Atoms form, CMB released
Electron-positron annihilation	$\sim 10^{10}$	$\sim 10^{10}$ K	Pairs annihilate
Quark-hadron transition	$\sim 10^{13}$	$\sim 10^{13}$ K	Quark-gluon plasma $\rightarrow$ hadrons
Inflation	$\sim 10^{26+}$	?	Exponential expansion