A basic Introduction to
Lie Groups and Lie Algebras in Physics

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"Matters of elegance should be left to the cobbler."

– Ludwig Boltzmann

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Chapter 1

Lie Groups and Lie Algebras in Physics

1.1 Introduction: Classifying Particles by Fundamental Properties

Particles in physics are classified according to their fundamental properties including mass, electric charge, and **spin**. Spin is a particularly important quantum property that can take integer or half-integer values.

1.1.1 Particle Spin Examples

All known fundamental particles have spins of 0, 1/2, or 1:

- **Spin 0**: Higgs boson - **Spin 1/2**: Matter-type particles (quarks, electrons, neutrinos) - **Spin 1**: Force-carrying particles (photons, gluons, W±, Z bosons)

1.1.2 Mathematical Representation of Spin

The spin value determines the mathematical object used to describe the particle:

Spin	Mathematical Object	Description
0	Scalars	Rank-0 tensors
1/2	Spinors	"Rank-1/2 tensors" (rotate half as much as vectors)
1	Vectors	Rank-1 tensors

Composite particles can have higher spins: - **Spin 3/2**: Delta baryon (represented by spinors spinors spinors) - **Spin 2**: Hypothetical graviton (represented by vectors vectors, related to the metric tensor in general relativity)

1.1.3 Transformation Behavior

Different particle types transform differently under physical transformations (rotations, boosts). For example, in a rotation in the xy-plane by angle :

Scalars (spin 0):

$$ \tag{1}$$

Spinors (spin 1/2):

$$\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{+i\theta/2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Vectors (spin 1):

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

These transformation matrices belong to **Lie groups**, and understanding their structure through **Lie algebras** tells us how particles behave under transformations.

1.1.4 Quantum Operators as Lie Algebra Members

Many quantum mechanical operators belong to Lie algebras:
Hamiltonian: $\hat{H} = i\hbar\frac{\partial}{\partial t}$
Momentum operators:

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}, \quad \hat{p}_y = -i\hbar\frac{\partial}{\partial y}, \quad \hat{p}_z = -i\hbar\frac{\partial}{\partial z}$$

Angular momentum operators:

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Spin operators:

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.2 Group Theory Fundamentals

1.2.1 Definition of a Group

A **group** is a set of elements G that can be combined with an operation satisfying:

1. **Closure**: If $a, b \in G$, then $a \circ b \in G$ 2. **Associativity**: $(a \circ b) \circ c = a \circ (b \circ c)$ 3. **Identity**: There exists $e \in G$ such that $a \circ e = e \circ a = a$ for all a 4. **Inverses**: For every $a \in G$, there exists $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$

1.2.2 Example: Rotation Matrices Form a Group

Identity:

$$\begin{pmatrix} \cos 0 & -\sin 0 & 0 \\ \sin 0 & \cos 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverses:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Combination: Matrix multiplication of rotation matrices produces another rotation matrix.

1.2.3 Lie Groups

A **Lie group** is a **continuous** group (as opposed to discrete).

- **Reflections** are NOT a Lie group (discrete jumps) - **Rotations** ARE a Lie group (smooth, continuous transformations)

Every Lie group has a corresponding **Lie algebra**, which consists of special matrices (generators) that can be exponentiated to produce Lie group elements.

1.3 The Lie Group SO(3) and Its Algebra so(3)

1.3.1 The SO(3) Group

SO(3) = Special Orthogonal 3×3 matrices

- **Orthogonal**: The inverse equals the transpose: $R^{-1} = R^T$ - This means the matrix doesn't change vector length - **Special**: Determinant equals +1: $\det(R) = +1$ - This excludes reflections

Example: Rotation in the xy-plane:

$$R_{xy}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.3.2 The Generator Matrix

The rotation matrix can be written as a matrix exponential:

$$R_{xy}(\theta) = e^{\theta M}$$

where the **generator** matrix M is:

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This generator belongs to the **Lie algebra so(3)** (written in lowercase fraktur font by convention).

1.4 Matrix Exponentials

1.4.1 Taylor Series Definition

For a number x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

For a matrix M:

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} = \frac{M^0}{0!} + \frac{M^1}{1!} + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \frac{M^5}{5!} + \cdots$$

By convention, $M^0 = I$ (identity matrix), just as $x^0 = 1$.

1.4.2 Computing Powers of the Generator

For $M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$:

$$M^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M^1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M^3 = M \cdot M^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -M$$

$$M^4 = M^3 \cdot M = (-M) \cdot M = -M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M^5 = M^3 \cdot M^2 = (-M) \cdot M^2 = -M^3 = -(-M) = M$$

The pattern repeats: $M^5 = M$, $M^6 = M^2$, $M^7 = M^3$, etc.

1.4.3 Evaluating the Matrix Exponential

$$e^{\theta M} = \sum_{n=0}^{\infty} \frac{(\theta M)^n}{n!} = M^0 \frac{\theta^0}{0!} + M^1 \frac{\theta^1}{1!} + M^2 \frac{\theta^2}{2!} + M^3 \frac{\theta^3}{3!} + M^4 \frac{\theta^4}{4!} + \cdots$$

Substituting the powers:

$$e^{\theta M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\theta^0}{0!} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\theta^1}{1!} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\theta^2}{2!} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\theta^3}{3!} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\theta^4}{4!} + \cdots$$

Looking at the top-left entry:

$$\frac{\theta^0}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = \cos\theta$$

Looking at the entry in position (1,2):

$$-\frac{\theta^1}{1!} + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots = -\sin\theta$$

Looking at the entry in position (2,1):

$$\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots = \sin\theta$$

Looking at the entry in position (2,2):

$$\frac{\theta^0}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = \cos\theta$$

Looking at the bottom-right entry:

$$1 + 0 + 0 + \cdots = 1$$

Therefore:

$$e^{\theta M} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is indeed the rotation matrix in the xy-plane!

1.5 Recipe for Finding Generators

1.5.1 Derivative Method

For a scalar: If we have $e^{\theta s}$ where s is a number:

$$\frac{d}{d\theta}e^{\theta s} = se^{\theta s}$$

Setting $\theta = 0$:

$$\left.\frac{d}{d\theta}e^{\theta s}\right|_{\theta=0} = se^0 = s \cdot 1 = s$$

For a matrix: If we have $e^{\theta M}$ where M is a matrix:

$$\frac{d}{d\theta}e^{\theta M} = Me^{\theta M}$$

Proof using Taylor series:

$$\frac{d}{d\theta}e^{\theta M} = \frac{d}{d\theta}\sum_{n=0}^{\infty}\frac{(\theta M)^n}{n!} = \sum_{n=0}^{\infty}\frac{n\theta^{n-1}M^n}{n!} = \sum_{n=0}^{\infty}\frac{M\theta^{n-1}M^{n-1}}{(n-1)!} = Me^{\theta M}$$

Setting $\theta = 0$:

$$\left.\frac{d}{d\theta}e^{\theta M}\right|_{\theta=0} = Me^{0M} = MI = M$$

1.5.2 General Recipe

To find the generator M of a Lie group matrix R:

$$M = \left.\frac{dR}{d\theta}\right|_{\theta=0}$$

Two steps: 1. Take the derivative with respect to the parameter 2. Set the parameter to zero

1.5.3 Example: xy-plane Rotation

$$R_{xy}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 1: Take derivative:

$$\frac{d}{d\theta}R_{xy}(\theta) = \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2: Set $\theta = 0$:

$$M = \begin{pmatrix} -\sin 0 & -\cos 0 & 0 \\ \cos 0 & -\sin 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matches the generator we used earlier!

1.5.4 All Three Rotation Generators

Rotation in xy-plane:

$$R_{xy}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad g_{xy} = \left.\frac{dR_{xy}}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rotation in yz-plane:

$$R_{yz}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \quad \Rightarrow \quad g_{yz} = \left.\frac{dR_{yz}}{d\phi}\right|_{\phi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Rotation in zx-plane:

$$R_{zx}(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \quad \Rightarrow \quad g_{zx} = \left.\frac{dR_{zx}}{d\psi}\right|_{\psi=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

These three generators form a basis for the Lie algebra so(3).

1.6 Exponential of Derivative Operators

This is a brief aside showing that derivatives can also be exponentiated using the Taylor series definition.

1.6.1 Translation Operator

Using the exponential of a derivative:

$$e^{a\frac{d}{dx}} f(x) = f(x+a)$$

This operator translates (shifts) a function by amount a.

Proof using Taylor series:

The Taylor series of $f(x)$ around x_0 is:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x_0}$$

Setting $x = x_0 + \Delta x$:

$$f(x_0 + \Delta x) = \sum_{n=0}^{\infty} \frac{(\Delta x)^n}{n!} \left.\frac{d^n f}{dx_0^n}\right|_{x_0}$$

Using the chain rule, $\frac{d}{dx_0} = \frac{d(x_0+\Delta x)}{dx_0}\frac{d}{d(x_0+\Delta x)} = 1 \cdot \frac{d}{d(x_0+\Delta x)}$:

$$f(x_0 + \Delta x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\Delta x \frac{d}{dx_0}\right)^n f(x_0) = e^{\Delta x \frac{d}{dx_0}} f(x_0)$$

1.6.2 Connection to Quantum Mechanics

Just as rotation matrices generate rotations when exponentiated:

$$e^{\theta g_{xy}} = R_{xy}(\theta)$$

The derivative operator generates translations when exponentiated:

$$e^{a\frac{d}{dx}} = T_x(a)$$

where $T_x(a)\psi(x) = \psi(x+a)$.

In quantum mechanics, the **momentum operator** contains a derivative:

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

because momentum is the generator of spatial translations:

$$T_x(a) = e^{a\frac{i}{\hbar}\hat{p}_x} = e^{a\frac{d}{dx}}$$

The factor of i makes the operator Hermitian, which is required for observables in quantum mechanics.

1.7 Properties of SO(3) Generators

1.7.1 Traceless Property

Theorem: For a matrix A,

$$\det(e^A) = e^{\mathrm{tr}(A)}$$

Proof for diagonal matrices:

If $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, then:

$$e^A = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix}$$

Therefore:

$$\det(e^A) = e^{\lambda_1} \cdot e^{\lambda_2} \cdot e^{\lambda_3}$$

Also:

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$

So:

$$e^{\mathrm{tr}(A)} = e^{\lambda_1+\lambda_2+\lambda_3} = e^{\lambda_1} \cdot e^{\lambda_2} \cdot e^{\lambda_3}$$

Thus: $\det(e^A) = e^{\mathrm{tr}(A)}$

Since eigenvalues, determinant, and trace are unchanged by coordinate transformations, this theorem holds for any diagonalizable matrix, including our SO(3) generators (provided we allow complex entries).

Applying to SO(3):

For a rotation matrix $R \in SO(3)$, we know $\det(R) = +1$.

If $R(\theta) = e^{\theta M}$:

$$\det(R(\theta)) = 1$$
$$\det(e^{\theta M}) = 1$$
$$e^{\theta \cdot \mathrm{tr}(M)} = 1$$

For this to be true for all values of θ:

$$\mathrm{tr}(M) = 0$$

Conclusion: All SO(3) generators are **traceless**.

1.7.2 Antisymmetric Property

For an SO(3) matrix R, we know $R^{-1} = R^T$ (orthogonality).

Therefore:

$$R \cdot R^T = I$$

Taking the derivative of both sides:

$$\frac{d}{d\theta}(R(\theta) \cdot R(\theta)^T) = \frac{dI}{d\theta}$$

Using the product rule on the left:

$$\frac{dR(\theta)}{d\theta} \cdot R(\theta)^T + R(\theta) \cdot \frac{dR(\theta)^T}{d\theta} = 0$$

Now express R as an exponential: $R(\theta) = e^{\theta M}$

The transpose of an exponential is:

$$e^{\theta M^T} = \left(\sum_{n=0}^{\infty} \frac{(\theta M)^n}{n!}\right)^T = \sum_{n=0}^{\infty} \frac{(\theta M^T)^n}{n!} = e^{\theta M^T}$$

So: $(e^{\theta M})^T = e^{\theta M^T}$

The derivative of the exponential brings down a factor of M (by chain rule):

$$\frac{d}{d\theta} e^{\theta M} = M e^{\theta M}$$

Substituting into our equation:

$$M e^{\theta M} \cdot e^{\theta M^T} + e^{\theta M} \cdot M^T e^{\theta M^T} = 0$$

Setting $\theta = 0$ (all exponentials become the identity):

$$M \cdot I \cdot I + I \cdot M^T \cdot I = 0$$

$$M + M^T = 0$$

$$M^T = -M$$

Conclusion: All SO(3) generators are **antisymmetric**.

1.7.3 Summary for SO(3)

The Lie algebra so(3) consists of all 3×3 matrices that are: - **Traceless**: $\text{tr}(M) = 0$ - **Antisymmetric**: $M^T = -M$

In set notation:

$$\mathfrak{so}(3) = \{M \in \mathbb{R}^{3\times 3} : M^T = -M, \text{tr}(M) = 0\}$$

1.8 Warning: Matrix Exponent Rules Don't Always Work

For ordinary numbers, we have:

$$e^a \cdot e^b = e^{a+b}$$

This is NOT always true for matrices!

For matrices A and B:

$$e^A \cdot e^B \neq e^{A+B} \quad \text{(in general)}$$

1.8.1 Why Matrix Exponent Rules Fail

Expanding as Taylor series:

$$e^A \cdot e^B = \left(I + A + \frac{A^2}{2} + \cdots\right)\left(I + B + \frac{B^2}{2} + \cdots\right)$$

$$= I + A + B + \frac{1}{2}(A^2 + 2AB + B^2) + \cdots$$

$$e^{A+B} = I + (A + B) + \frac{(A + B)^2}{2} + \cdots$$

$$= I + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \cdots$$

Comparing the second-order terms: - From e^Ae^B: $\frac{1}{2}(A^2+2AB+B^2)$ - From e^{A+B}: $\frac{1}{2}(A^2+AB+BA+B^2)$ These are only equal if $AB = BA$ (i.e., if A and B **commute**).

Conclusion:

$$e^Ae^B = e^{A+B} \quad \text{ONLY if } AB = BA$$

or equivalently, only if the **commutator** $[A, B] = AB - BA = 0$.

1.8.2 Special Case: Antisymmetric Matrices

For an antisymmetric matrix M (where $M^T = -M$):

$$M \cdot M^T = M \cdot (-M) = -M \cdot M = -M^2 = M^T \cdot M$$

So M does commute with M^T, and we can write:

$$e^M \cdot e^{M^T} = e^{M+M^T}$$

for antisymmetric matrices.

1.8.3 Baker-Campbell-Hausdorff Formula

The general relationship between e^Xe^Y and exponentials is given by:

$$e^Xe^Y = e^Z$$

where:

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots$$

This is an infinite series involving nested commutators.

1.9 What is a Lie Algebra?

An **algebra** is a vector space where addition, subtraction, and multiplication are defined (division is not required).

Generator matrices can be thought of as vectors: - We can **add** generators: $g_{xy} + g_{yz}$ gives another generator - We can **subtract** generators: $g_{xy} - g_{yz}$ gives another generator - We can **scale** generators: $c \cdot g_{xy}$ gives another generator

But what about **multiplication**?

1.9.1 Why Matrix Multiplication Doesn't Work

Adding two traceless, antisymmetric matrices gives another traceless, antisymmetric matrix.

Subtracting two traceless, antisymmetric matrices gives another traceless, antisymmetric matrix.

But multiplying two antisymmetric matrices does NOT always give an antisymmetric matrix.

Example:

$$g_{xy} \cdot g_{yz} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix squares to zero: $(g_{xy}g_{yz})^2 = 0$

Exponentiating:

$$e^{\theta(g_{xy}g_{yz})} = I + \theta(g_{xy}g_{yz}) + 0 + 0 + \cdots = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is NOT a rotation matrix! It's a shear transformation.

1.9.2 The Commutator (Lie Bracket)

Instead of using matrix multiplication as our algebra operation, we use the **commutator**:

$$[A, B] = AB - BA$$

In the context of Lie algebras, this is called the **Lie bracket**.

Example:

$$[g_{xy}, g_{yz}] = g_{xy}g_{yz} - g_{yz}g_{xy}$$

Computing $g_{yz} \cdot g_{xy}$:

$$g_{yz} \cdot g_{xy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Therefore:

$$[g_{xy}, g_{yz}] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = g_{zx}$$

The commutator of two generators gives another generator!

1.9.3 The so(3) Commutation Relations

Working out all the commutators:

$$[g_{xy}, g_{yz}] = g_{zx}$$
$$[g_{zx}, g_{xy}] = g_{yz}$$
$$[g_{yz}, g_{zx}] = g_{xy}$$

These are the **commutation relations** for the Lie algebra so(3).

Together with the rules for addition, subtraction, and the Lie bracket, the three generators $\{g_{xy}, g_{yz}, g_{zx}\}$ form the Lie algebra so(3).

1.9.4 Abstract Definition of Lie Bracket

In more abstract treatments, the Lie bracket is defined by these properties:

1. **Alternating**: $[x, x] = 0$ 2. **Jacobi Identity**: $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$

The matrix commutator $[A, B] = AB - BA$ satisfies both these properties, so it is a valid Lie bracket.

1.10 Lie Algebras as Tangent Spaces

A Lie group is a continuous space (a manifold). Each point in the space is a different group element.

1.10.1 Visualizing the Lie Group

Consider the curve of xy-plane rotations parameterized by :

$$R_{xy}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- At $\theta = 0$: We get the identity matrix (no rotation) - As increases: We trace out a curve through the Lie group SO(3)

When we take the derivative $\frac{d}{d\theta}R_{xy}(\theta)$, we're finding the **tangent vectors** along this curve.

When we evaluate at $\theta = 0$, we get the **tangent vector at the identity**:

$$\frac{d}{d\theta}R_{xy}(\theta)\bigg|_{\theta=0} = g_{xy}$$

The generator g_{xy} is a tangent vector at the identity!

The set of all generators forms a **tangent space** at the identity element of the Lie group. This tangent space IS the Lie algebra.

1.10.2 Proofs of Lie Algebra Properties

The following proofs show that the tangent space at the identity satisfies the requirements of a Lie algebra. These proofs work for any Lie group G with Lie algebra $\mathfrak{g}$.

Proof 1: Sum of Tangent Vectors is a Tangent Vector

Consider two paths in the Lie group:

$$A(s) = e^{sa}, \quad B(s) = e^{sb}$$

At $s = 0$, both give the identity: $A(0) = B(0) = I$

Their derivatives at $s = 0$ give generators:

$$\frac{dA}{ds}\bigg|_{s=0} = a, \quad \frac{dB}{ds}\bigg|_{s=0} = b$$

Now consider the product path:

$$C(s) = A(s) \cdot B(s)$$

Taking the derivative using the product rule:

$$\frac{d}{ds}C(s) = \frac{d}{ds}A(s) \cdot B(s) + A(s) \cdot \frac{d}{ds}B(s)$$

At $s = 0$:

$$\frac{dC}{ds}\bigg|_{s=0} = \frac{dA}{ds}\bigg|_{s=0} \cdot B(0) + A(0) \cdot \frac{dB}{ds}\bigg|_{s=0} = a \cdot I + I \cdot b = a + b$$

Therefore, $a + b$ is a tangent vector (tangent to the curve $C(s)$ at the identity).

Important note: Although the product of exponentials is not generally equal to the exponential of the sum:

$$e^{sA}e^{sB} \neq e^{s(A+B)}$$

both formulas have the **same first-order term** when expanded as Taylor series. This means they have the same tangent vector at the identity:

$$\frac{d}{ds}(e^{sA}e^{sB})\Big|_{s=0} = A + B = \frac{d}{ds}e^{s(A+B)}\Big|_{s=0}$$

There are infinitely many curves through the identity with the same tangent vector. However, when we treat the tangent vector as a **generator** and exponentiate it, it generates a unique curve: e^{sa}.

Proof 2: Scalar Multiple of a Tangent Vector is a Tangent Vector

Consider the path $A(s)$ and form a new path $A(rs)$ where r is a constant:

Taking the derivative using the chain rule:

$$\frac{d}{ds}A(rs) = \frac{d(rs)}{ds} \cdot \frac{dA(rs)}{d(rs)} = r \cdot \frac{dA}{ds}$$

At $s = 0$:

$$\frac{d}{ds}A(rs)\Big|_{s=0} = r \cdot \frac{dA}{ds}\Big|_{s=0} = ra$$

Therefore, ra is a tangent vector (tangent to the curve $A(rs)$ at the identity).

Proof 3: Commutator of Tangent Vectors is a Tangent Vector

This is the key proof that establishes the Lie bracket structure.

Consider a path with three segments:

$$D(s,t) = A(s) \cdot B(t) \cdot A^{-1}(s)$$

This is called a **conjugation**: ABA^{-1}.

Taking the partial derivative with respect to t (holding s constant):

$$\frac{\partial}{\partial t}D(s,t) = A(s) \cdot \frac{\partial B(t)}{\partial t} \cdot A^{-1}(s)$$

At $t = 0$:

$$\frac{\partial}{\partial t}D(s,0) = A(s) \cdot b \cdot A^{-1}(s)$$

This is a tangent vector at the identity (for each fixed value of s).

Now take the partial derivative with respect to s, using the product rule:

$$\frac{\partial}{\partial s}\left(\frac{\partial D}{\partial t}\Big|_{t=0}\right) = \frac{\partial}{\partial s}(A(s) \cdot b \cdot A^{-1}(s))$$

$$= \frac{\partial A}{\partial s} \cdot b \cdot A^{-1}(s) + A(s) \cdot b \cdot \frac{\partial A^{-1}}{\partial s}$$

At $s = 0$:

$$\frac{\partial^2 D}{\partial s \partial t}\Big|_{s=0,t=0} = \frac{\partial A}{\partial s}\Big|_{s=0} \cdot b \cdot I + I \cdot b \cdot \frac{\partial A^{-1}}{\partial s}\Big|_{s=0}$$

Now, since $A(s) \cdot A^{-1}(s) = I$, taking the derivative:

$$\frac{\partial A}{\partial s} \cdot A^{-1} + A \cdot \frac{\partial A^{-1}}{\partial s} = 0$$

At $s = 0$:

$$a \cdot I + I \cdot \frac{\partial A^{-1}}{\partial s}\Big|_{s=0} = 0$$

$$\left.\frac{\partial A^{-1}}{\partial s}\right|_{s=0} = -a$$

Substituting back:

$$\left.\frac{\partial^2 D}{\partial s \partial t}\right|_{s=0,t=0} = a \cdot b \cdot I + I \cdot b \cdot (-a) = ab - ba = [a, b]$$

Interpretation: - For fixed s, $\frac{\partial D}{\partial t}|_{t=0}$ is a tangent vector - As s varies, we get a path of tangent vectors - Taking $\frac{\partial}{\partial s}$ gives tangent vectors to this path-of-tangent-vectors - At $s = 0, t = 0$, this equals $[a, b]$

Since tangent vectors in a vector space also belong to that same vector space, the commutator $[a, b]$ belongs to the tangent space (the Lie algebra).

1.10.3 Summary: Vector Space and Lie Algebra

We've proven that the tangent space at the identity has: - **Sum**: $a + b$ is a tangent vector - **Scalar multiplication**: ra is a tangent vector - **Lie bracket**: $[a, b]$ is a tangent vector

These properties make it a **Lie algebra**.

1.11 Structure Coefficients

For a general Lie algebra with basis generators $\{g_1, g_2, \ldots, g_n\}$:

The Lie bracket of two basis generators doesn't always give back a single basis generator, but it always gives a **linear combination** of basis generators:

$$[g_i, g_j] = \sum_k f_{ij}^k g_k$$

The coefficients f_{ij}^k are called the **structure coefficients** or **structure constants** of the Lie algebra.

1.11.1 Structure Coefficients for so(3)

Relabeling our generators:

$$g_1 = g_{yz}, \quad g_2 = g_{zx}, \quad g_3 = g_{xy}$$

The commutation relations become:

$$[g_1, g_2] = g_3$$
$$[g_3, g_1] = g_2$$
$$[g_2, g_3] = g_1$$

Reading off the structure coefficients:

$$f_{12}^3 = 1, \quad f_{31}^2 = 1, \quad f_{23}^1 = 1$$

Since the commutator is antisymmetric ($[g_i, g_j] = -[g_j, g_i]$):

$$f_{ji}^k = -f_{ij}^k$$

So:

$$f_{21}^3 = -1, \quad f_{13}^2 = -1, \quad f_{32}^1 = -1$$

All other structure coefficients are zero.

1.11.2 Levi-Civita Symbol

For so(3), the structure coefficients can be written compactly using the **antisymmetric symbol** (Levi-Civita symbol):

$$[g_i, g_j] = \sum_k \epsilon_{ijk} g_k$$

where:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if any indices are repeated} \end{cases}$$

1.12 Summary: SO(3) Lie Group and Algebra

1.12.1 The Lie Group SO(3)

Definition:

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} : R^{-1} = R^T, \det(R) = +1\}$$

Rotation matrices:

$$R_{xy}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{yz}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$R_{zx}(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}$$

1.12.2 The Lie Algebra so(3)

Definition:

$$\mathfrak{so}(3) = \{M \in \mathbb{R}^{3\times 3} : M^T = -M, \mathrm{tr}(M) = 0\}$$

Generator matrices:

$$g_{xy} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_{zx} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Exponential map:

$$e^{\theta g_{xy}} = R_{xy}(\theta), \quad e^{\phi g_{yz}} = R_{yz}(\phi), \quad e^{\psi g_{zx}} = R_{zx}(\psi)$$

Recipe for generators:

$$g = \left.\frac{dR}{d\theta}\right|_{\theta=0}$$

Commutation relations:

$$[g_{xy}, g_{yz}] = g_{zx}$$

$$[g_{zx}, g_{xy}] = g_{yz}$$

$$[g_{yz}, g_{zx}] = g_{xy}$$

1.13 The Lorentz Group SO(1,3)

The **Lorentz group** SO(1,3) consists of spacetime transformations that preserve the spacetime interval:

$$s^2 = +ct^2 - x^2 - y^2 - z^2$$

This is the metric signature with one plus sign and three minus signs (hence the notation 1,3).
- **Special (S)**: $\det(\Lambda) = +1$ (excludes spatial reflections) - **Orthochronous (+)**: $\Lambda^0_0 > 0$ (excludes time reflections)

The group includes: - **3 rotations** (in xy, yz, zx planes) - **3 boosts** (in tx, ty, tz planes - relative motion along each axis)

1.13.1 Lorentz Transformation Matrices

Rotations (same as SO(3)):

$$\Lambda_{xy}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\Lambda_{zx}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Boosts:

$$\Lambda_{tx}(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_{ty}(\phi) = \begin{pmatrix} \cosh\phi & 0 & \sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_{tz}(\phi) = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix}$$

1.13.2 Generators of SO(1,3)

Using the recipe $g = \frac{d\Lambda}{d\theta}|_{\theta=0}$:

Rotation generators (denoted with J):

$$J_{xy} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_{yz} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_{zx} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

These are **traceless and antisymmetric**.

Boost generators (denoted with K):

$$K_{tx} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_{ty} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_{tz} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

These are **traceless and symmetric**.

1.13.3 The Lie Algebra so(1,3)

The six generators (3 rotations + 3 boosts) form a basis for the Lie algebra so(1,3).

Commutation relations:

Rotation-Rotation commutators (give rotations):

$$[J_{yz}, J_{zx}] = J_{xy}$$

$$[J_{xy}, J_{yz}] = J_{zx}$$

$$[J_{zx}, J_{xy}] = J_{yz}$$

Boost-Boost commutators (give rotations):

$$[K_{tx}, K_{ty}] = -J_{xy}$$

$$[K_{tz}, K_{tx}] = -J_{zx}$$

$$[K_{ty}, K_{tz}] = -J_{yz}$$

Rotation-Boost commutators (give boosts):

$$[J_{yz}, K_{ty}] = +K_{tz}$$

$$[J_{yz}, K_{tz}] = -K_{ty}$$

$$[J_{zx}, K_{tz}] = +K_{tx}$$

$$[J_{zx}, K_{tx}] = -K_{tz}$$

$$[J_{xy}, K_{tx}] = +K_{ty}$$

$$[J_{xy}, K_{ty}] = -K_{tx}$$

All other commutation relations give zero.

1.13.4 SO(3) as a Sub-algebra

Note that the commutation relations for the J generators alone form a closed set:

$$[J_{yz}, J_{zx}] = J_{xy}$$

$$[J_{xy}, J_{yz}] = J_{zx}$$

$$[J_{zx}, J_{xy}] = J_{yz}$$

This shows that **so(3) lives inside so(1,3)** as a sub-algebra.

1.14 Representations: Spin-1 vs Spin-1/2

The SO(3) and SO(1,3) matrices we've been discussing operate on vectors: - SO(3): operates on 3D spatial vectors - SO(1,3): operates on 4D spacetime vectors

Since vectors are rank-1 tensors, these are called the **spin-1 representation**.

However, there are OTHER sets of matrices that satisfy the SAME commutation relations but have different dimensions. These are different **representations** of the same Lie algebra.

1.14.1 Multiple Representations Example

Just as the complex number i can be represented by different matrices that all square to -1:

$$i \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad i \to \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Similarly, there are multiple matrix representations of the SO(3) and SO(1,3) generators.

1.14.2 Spin-1/2 Representation

There exist **2×2 matrices** that follow the same commutation relations as the SO(3) generators:

$$g_{yz} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad g_{zx} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad g_{xy} = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

For SO(1,3):

$$J_{yz} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad J_{zx} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_{xy} = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$K_{tx} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K_{ty} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad K_{tz} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are the **spin-1/2 representation** because they transform **spinors** (2-component objects).

The transformation matrices for spinors: - **Pauli spinors**: Transform under SO(3) (rotations only) - **Weyl spinors**: Transform under SO(1,3) (rotations and boosts)

1.15 Math Convention vs Physics Convention

Depending on the textbook, the generators may look different due to conventions.

1.15.1 Math Convention

Generators are: - **Traceless**: $\mathrm{tr}(g) = 0$ - **Antisymmetric**: $g^T = -g$

Exponential formula:

$$e^{\theta g} = R(\theta)$$

Example:

$$g_{xy} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1.15.2 Physics Convention

Generators are multiplied by $-i$:

$$\tilde{g} = -ig$$

This makes them: - **Traceless**: $\mathrm{tr}(\tilde{g}) = 0$ - **Hermitian**: $\tilde{g}^{T*} = \tilde{g}$ (complex conjugate transpose equals itself)

Exponential formula (with compensating factor of i):

$$e^{i\theta\tilde{g}} = R(\theta)$$

Example:

$$\tilde{g}_{xy} = -ig_{xy} = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{g}_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \tilde{g}_{zx} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

1.15.3 Why Physicists Use This Convention

In quantum mechanics, **observables must be represented by Hermitian operators** because: 1. Hermitian operators have **real eigenvalues** 2. Measurements must give real numbers, not complex numbers

Since Lie algebra generators correspond to observables (momentum, angular momentum, spin), physicists multiply by $-i$ to make them Hermitian.

Examples:

Momentum operator eigenvalue equation:

$$\hat{p}\psi = p\psi$$

where p is a real number (the measured momentum).

The operator is:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

Spin operator eigenvalue equation:

$$\hat{S}_y\psi = +\frac{\hbar}{2}\psi$$

where $\frac{\hbar}{2}$ is real.

The operator is:

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

This is Hermitian: $\hat{S}_y^{T*} = \hat{S}_y$.

1.15.4 Summary of Conventions

Convention	Generator Properties	Exponential Formula
Math	Traceless, Antisymmetric: $g^T = -g$	$e^{\theta g} = R(\theta)$
Physics	Traceless, Hermitian: $\tilde{g}^{T*} = \tilde{g}$	$e^{i\theta\tilde{g}} = R(\theta)$

The physics convention requires an extra factor of i in the exponential to compensate for the $-i$ in the generator definition.

1.16 Key Concepts Summary

1. **Particle spin** determines the mathematical representation: - Spin 0 $\rightarrow$ Scalars - Spin 1/2 $\rightarrow$ Spinors - Spin 1 $\rightarrow$ Vectors
2. **Groups** are sets with closure, associativity, identity, and inverses.
3. **Lie groups** are continuous groups (e.g., rotations).
4. **Lie algebras** are the tangent spaces at the identity of Lie groups, consisting of generators.
5. **Generators** can be found by: $M = \frac{dR}{d\theta}\Big|_{\theta=0}$
6. **Matrix exponentials** connect Lie algebras to Lie groups: $R = e^{\theta M}$

7. **SO(3) generators** are traceless and antisymmetric.
8. **The Lie bracket** (commutator) $[A, B] = AB - BA$ is the multiplication operation in Lie algebras.
9. **so(3) commutation relations**: $[g_{xy}, g_{yz}] = g_{zx}$ (and cyclic permutations)
10. **The Lorentz group SO(1,3)** includes 3 rotations and 3 boosts, with so(3) as a sub-algebra.
11. **Different representations** of the same Lie algebra exist (spin-1, spin-1/2, etc.)
12. **Observables in quantum mechanics** = Hermitian operators = Lie algebra generators (in physics convention)