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Comparison of Two Methods of Incorporating an Integral Action in Linear Quadratic Regulator

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Abstract: Comparison of two Methods of incorporating an integral action in the linear quadratic controller is presented. First Method of incorporating an integral action in linear quadratic regulator (LQR) is augmenting system matrix with error in sate variable. Second Method of incorporating integral action is introducing $\dot{u}$ in performance index instead of u to get multivariable proportional plus integral controller. For comparison of both the Methods state space model of two CSTR's in series is used for simulation studies. A MATLAB code is developed for solving nonlinear matrix equations.

Key Words: linear quadratic regulator (LQR), incorporating integral action

1. INTRODUCTION

The classical (conventional) control theory, concerned with single input and single output (SISO), is mainly based on Laplace transforms theory. The modern control theory, concerned with multiple inputs and multiple outputs (MIMO), is based on the state variable representation in terms of a set of first order differential (or difference) equations. The modern control theory dictates that all the state variables should be fed back after suitable weighting. Modern control theory depends on well-established matrix theory, which is amenable for the large scale computer simulation (Naidu, 2000).

The fact that the state variable representation uniquely specifies the transfer function while there are a number of state variable representations for a given transfer function, reveals the fact that state variable is a more complete description of a system (Naidu, 2000). Classical control design techniques have been successfully applied to linear, time-invariant, single-input, single output (SISO) systems. Typical performance criteria are system time response to step response or ramp input characterize by rise time, settling time, peak overshoot, and steady state accuracy; and the frequency response of the system is characterized by gain and phase margins, and bandwidth.

In modern control theory, the optimal control problem is to find a control which causes the dynamical system to reach a target or follow a state variable (or trajectory) and at the same time extremize a performance index. In optimal control theory, the design is usually with respect to a performance index. In the so called linear quadratic control, the term linear refers to the plant being linear and the term quadratic refers to the performance index that involves the square or quadratic of an error, and/or control.

Basically the classical control theory using frequency domain deals with single input and single output (SISO) systems, whereas modern control theory works with time domain for SISO and multi-input and multi-output (MIMO) systems. Although modern control and optimal control appeared to be very attractive, it lacks a very important feature of robustness. That is, controllers designed based on LQR theory failed to be robust to measurement noise, external disturbances and un-modelled dynamics.

An excellent review is presented by Johnson (1993) on LQG applications in the process industries. Many applications of the LQG theory have been reported in the control of ships, space vehicles etc. In the process industries, very few applications of the LQG have been reported such as in paper industry control of paper machine basis weight and moisture control, in mineral industry control of drying, grinding, raw material mixing and cement rotary kiln and in steel industry control of closed refining, a continuous annealing line and hot strip finishing mill.

Recent applications of LQR reported in the literature are control of wind turbine generator system (Endusa et al., 2008), variable speed wind turbine (Yao et al., 2009), ball and beam system (Pang et al., 2011), wind energy conversion system based on doubly fed induction generator (Bachir & Kamal, 2011), three-phase three-wire shunt active filter (Bachir & Kamal, 2009), etc.

For uncertainties in the system parameters linear quadratic controller gives an offset. To eliminate the offset, an integral action is to be incorporated in the linear quadratic controller. In this work comparison of two methods of incorporating an integral action in the linear quadratic controller is presented. First method, of incorporating an integral action in the linear quadratic, is by augmenting the system matrix with error in the state variable. The second method is by introducing $\dot{u}$ (rather than u) in the performance index to get the multivariable proportional plus integral controller (Douglas, 1972). There is no work reported in the literature on comparison of these two Methods. For the present simulation studies, the control problem of CSTR's in series is considered (Li & Christofides, 2007).

The paper is organized as follows: in section 2, linear quadratic regulator theory is reviewed. In section 3, an integral action has been incorporated in the LQR by augmenting system matrix with error in the state variable. In section 4, LQRI control strategy via state feedback is given. In section 5, incorporating an integral action in the linear quadratic (LQ) controller by introducing $\dot{u}$ in performance index instead of u is presented. State space model of two CSTR's in series taken for simulation studies is presented in section 6. Simulation results and discussion are presented in the section 7. Section 8 gives the conclusions of the present work. In APPENDIX-A, comparison of both Methods of incorporating an integral action in the LQ controller is presented. In APPENDIX-B, MATLAB code, to solve 4-non- linear equations, is given.

2. LINEAR QUADRATIC REGULATOR (LQR)

For a linear time invariant (LTI), multi-input multi-output (MIMO) system is described as (Douglas, 1972;Brian, & Moore, 1989):

$$\dot{x}(t) = A\cdot x(t) + B\cdot u(t) \quad (1)$$
$$y(t) = C\cdot x(t) \quad (2)$$

Where initially the system is at optimum steady state conditions, $x(0)=0$. A is the process plant matrix $(n\times n)$, B is the input matrix $(n\times m)$ and C is $(p \times n)$. Equation (1) is called as state equation. Equation (2) is called as an output equation. The vector $x(t)$ is the state of the system at time t. The vector $\dot{x}(t)$ is the time derivative of the state of the system at time t.

Vector $u(t)$ is an independent input to the system. The vector $y(t)$ is the output of the system. The matrices A, B and C are the constant matrices containing parameters of the overall system.

The Linear Quadratic Regulator (LQR) consists of determining the control law,

$$u = -K\cdot x \quad (3)$$

By minimizing the performance index (cost function),

$$J = \int_0^\infty (x^TQx + u^TRu)dt \quad (4)$$

Where, Q and R are the state weighting matrix and control weighting matrix respectively. Q should be positive semi definite matrix and R should be positive definite matrix. Both Q & R matrices are square and symmetric matrices. Here J is always a scalar.The optimal control law is given by Douglas (Douglas, 1972) as:

$$K = R^{-1}B^TP \quad (5)$$

where P is the solution of algebraic Riccati equation (ARE),

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \quad (6)$$

Algebraic Riccati equation (6) can be solved by MATLAB's built in functions *care* & *lqr* commands.

3. INCORPORATING AN INTEGRAL ACTION IN LQR BY AUGMENTING SYSTEM MATRIX WITH ERROR IN THE STATE VARIABLE:

The LQR control law essentially gives a multivariable proportional regulator. An integral action has been incorporated in the LQR controller in order to eliminate the offset. An integral action has been incorporated in the LQR controller by augmenting system with error in the state variable. One new state is added to the space model of the system. The new state variable is the integral of the state variable concentration in the exit of the second CSTR.
Defining integral state q as follows:

$$\dot{q} = r - y = r - Cx \quad (7)$$

Here r, $\dot{q}$ & Cx are scalars for single variable system and vectors for multivariable system.
Therefore the system (1) and (2) after augmenting with integral state (7) becomes,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \quad (8)$$
$$y = \tilde{C}\tilde{x} \quad (9)$$

Where,

$$\dot{\tilde{x}} = \begin{bmatrix}\dot{x}\\ \dot{q}\end{bmatrix},\quad \tilde{A} = \begin{bmatrix}A & 0\\ -C & 0\end{bmatrix},\quad \tilde{x} = \begin{bmatrix}x\\ q\end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix}B\\ 0\end{bmatrix},\quad \tilde{C} = \begin{bmatrix}C & 0\end{bmatrix}$$

The new augmented matrices are used to calculate the state feedback optimal controller gain.

4. LQRI CONTROL STRATEGY VIA STATE FEEDBACK

Let the controller has the linear state feedback form,

$$\tilde{u} = -\tilde{K}(\tilde{x} - \tilde{r}) \quad (10)$$

Where $\tilde{r}$ new vector of reference input and $\tilde{K}$ is state feedback optimal control gain. State feedback optimal controller gain is the unknown controller matrix with dimensions (1×n) which we will get after solving algebraic Riccati equation (ARE) for the augmented system matrices.
Substituting (10) in (8) yields the closed loop system (11),

$$\dot{x} = (\tilde{A} - \tilde{B}\tilde{K})\tilde{x} + \tilde{B}\tilde{K}\tilde{r} \quad (11)$$

The control problem is to determine the optimal controller gain matrix such that the closed loop system has the desired characteristics.

5. INCORPORATING AN INTEGRAL ACTION IN THE LINEAR QUADRATIC (LQ) CONTROLLER BY INTRODUCING $\dot{u}$ IN PERFORMANCE INDEX INSTEAD OF U (Douglas, 1972):

The equations describing the system are considered to be,

$$\dot{x} = Ax + Bu + Ev \quad (12)$$
$$y = Cx \quad (13)$$

Where initially the system is at optimum steady state conditions, $x(0) = 0$, and at time equal to zero a number of step disturbances, v, enter in to the system.
The performance index is taken as,

$$P = \frac{1}{2}\int_0^{\theta=\infty}(x^T Qx + \dot{u}^T R\dot{u})dt \quad (14)$$

In this performance index, $\dot{u}$ is introduced instead of u to get the multivariable proportional plus integral controller. Where, Q is state weighting matrix & R is control weighting matrix.
We have to solve the following four nonlinear matrix equations simultaneously in order to get optimal multivariable proportional plus integral controller. The four nonlinear matrix equations are (Douglas, 1972),

$$LBR^{-1}B^T M = Q \quad (15)$$
$$-LBR^{-1}B^T N + K + LA = 0 \quad (16)$$
$$-NBR^{-1}B^T M + K + A^T M = 0 \quad (17)$$
$$-NBR^{-1}B^T N + M + NA + L + A^T N = 0 \quad (18)$$

The above four nonlinear matrix (15), (16), (17) & (18) equations are to be solved simultaneously to get K, L, M and N matrices. The Multivariable proportional plus integral controller is given by (Douglas, 1972),

$$u(t) = -R^{-1}B^T Nx(t) - R^{-1}B^T M\int_0^t x(t)\,dt \quad (19)$$

A comparison of steps involved in the two Methods of incorporating integral action are given in Appendix-A.

6. STATE SPACE MODEL FOR TWO CSTR'S IN SERIES

The two CSTR's in series network is chosen because its dynamic behaiour can be well described in the form of a stste space model and a comparison between two Methods of incoporating an integral action in linear quadratic controller can be easily made. Let u be the concentration at the entrance of the reactor, x_1 and x_2 be the concentration at the exit of the first and the second CSTR, and τ_1 and τ_2 be the time constants of these two CSTR's, the evolution of concentration can be written in the state-space form as in (20) & (21) (Li & Christofides, 2007),

$$\dot{x} = A \cdot x + B \cdot u \quad (20)$$
$$y = C \cdot x \quad (21)$$

Where,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} \frac{-1}{\tau_1} & 0 \\ \frac{1}{\tau_2} & \frac{-1}{\tau_2} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{\tau_1} \\ 0 \end{bmatrix}, \qquad C = [\,0\ 1\,] \quad (22)$$

For simulation study, the time constants are taken as: $\tau_1 = \tau_2 = 10$,

$$(Q) = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} \quad (23)$$

7. SIMULATION RESULTS & DISCUSSION

Here in this section, METHOD-1 is assumed for incoporating an integral action in the linear quadratic controller by augmenting the system matrix with the state variable and METHOD-2 is assumed for incorporating an integral action in the Linear Quadratic (LQ) controller by introducing $\dot{u}$ in performance index instead of u. The MATLAB code is given in Appendix-B for the solution of equations (15)-(18) to get the value of u from equation (19).

Figure 1 shows the closed loop response for the METHOD-1 and METHOD-2. For METHOD-1 and METHOD-2, control weighting factor R=10 is taken for simulations. We can observe from figure 1 that, both the Methods of incorporating an integral action achieves the steady sate at aproximately at the same time.

Figure 2 shows the effect of uncertainty on the controlled variable in two CSTR's in series for METHOD-1. For simulation studies the control weighting factor (R) is assumed as 10. From figure 2, we can observe that for perturbations in each elements of the system matrix, linear quadratic controller with integral action achieves the steady state and it is robust for perturbations in the system matrix. Figure 3 shows the profile of manipulated input required to achieve the desired set point for perturbations -20%, original system and +20% in the each elements of the system matrix.

Figure 4 shows the effect of uncertainty on the controlled variable in two CSTR's in series for the METHOD-2. For simulation studies the control weighting factor (R) is assumed as 10. From figure 4, we can observe that for perturbations in each elements of the system matrix, linear quadratic controller with integral action achieves the steady state and it is robust for perturbations in the system matrix. Figure 5 shows the profile of manipulated input required to achieve the desired set point for perturbations -20%, original system and +20% in the each elements of the system matrix.

Figure 6 shows the effect of control weighting factor (R) on the closed loop response of the controlled variable in two CSTR's in series for METHOD-2. As R is increasing, the time required to reach the steady state is increasing i.e. speed of response is getting deteriorated.

Figure 7 shows the effect of control weighting factor (R) on the closed loop response of the controlled variable in two CSTR's in series for METHOD-1 & METHOD-2. Here R is varied for METHOD-1 and it is kept constant for METHOD-2. It is done because, there is no relation between these two Methods for selecting control weighting factor. As R is increased time required to reach the steady state is increased i.e. speed of response is getting deteriorate. From figure 7, we can observe that time required to reach the steady state for METHOD-1 (R=10) is approximately equal to METHOD-2 (R=0.1), we can observe this from the response of both the Methods. We can say that from the simulation results, for METHOD-2 with the control weighting factor 10 times less than that of control weighting factor of METHOD-1, the closed loop response observed is similar.

Figure 8 shows the comparison of the closed loop response of the controlled variable in two CSTR's in series for different values of Q_2 element of the state weighting matrix for METHOD-1 and one fixed value of Q_2 element of the state weighting matrix for METHOD-2. From figure 8, we can observe that METHOD-1 response and METHOD-2 response give similar behaviour and the time required to reach steady state for both the Methods is approximately the same.

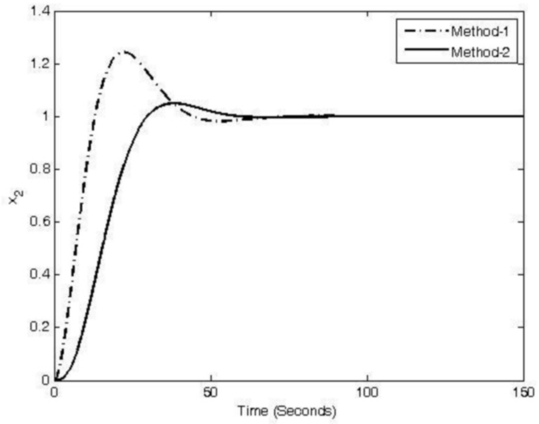

Fig 1: Response in the controlled variable in two CSTR's in series using different control schemes (For METHOD-1 & METHOD-2 : R=10).

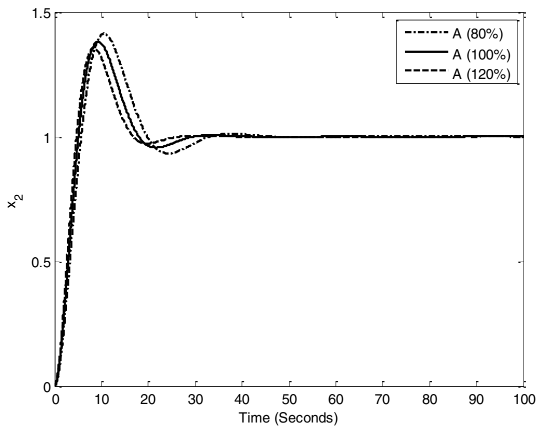

Fig 2 : Effect of uncertainty on the closed loop response of the controlled variable in two CTRS's in series (Method 1: R=10). (For -20%, original system & +20% perturbations in the each elements of the system matrix)

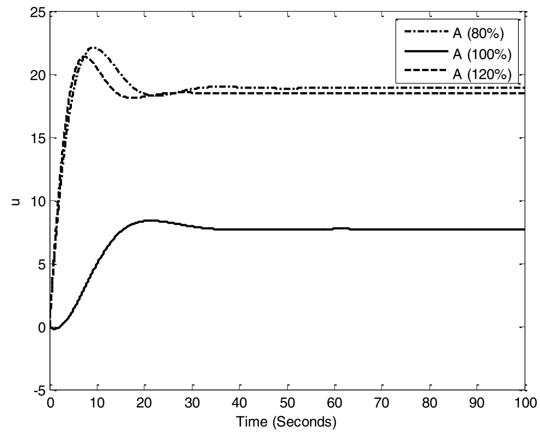

Fig 3: Profile of manipulated input required to achieve the desired set point for perturbations in the system matrix (METHOD-1: R=10). (-20%, Original system & +20% Perturbations in each elements of the system matrix)

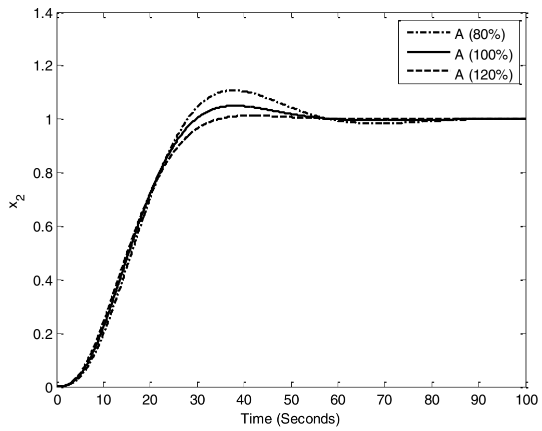

Fig 4: Effect of uncertainty on the closed loop response of the controlled variable in two CSTR's in series. (METHOD-2: R=10). (For -20%, original system & +20% perturbations in the all the elements of the system matrix)

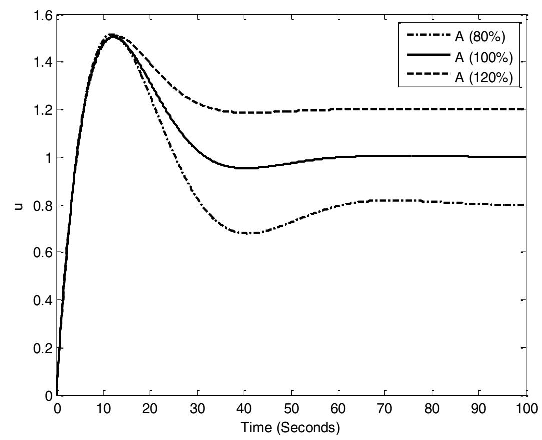

Fig 5: Profile of manipulated input required to achieve the desired set point for perturbations in the system matrix for METHOD-2: R=10. (-20%, Original system & +20% perturbations in the elements of the system matrix)

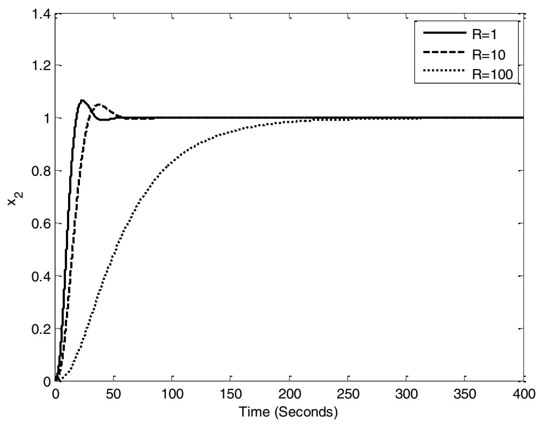

Fig 6: Closed loop response of the controlled variable in two CSTR's in series for different control weighting factors for METHOD-2.

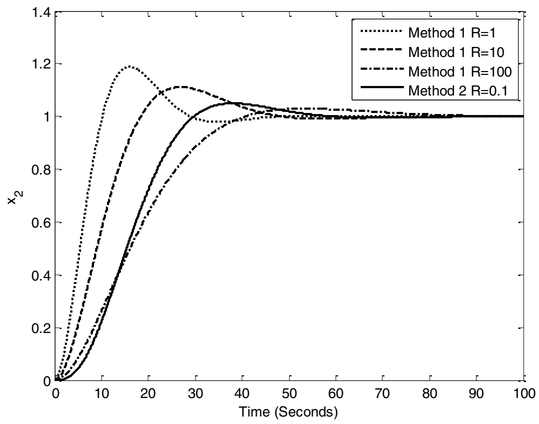

Fig 7: Comparison of closed loop responses of controlled variable in two CSTR's in series for different control weighting factors of METHOD-1 and fixed control weighting factors for METHOD-2.

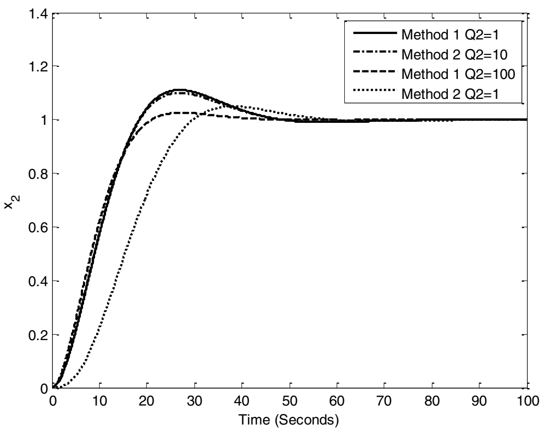

Fig 8: Comparison of closed loop responses of controlled variable in two CSTR's in series for different state weighting matrices of METHOD-1 and fixed state weighting matrix for METHOD-2.

8. CONCLUSION

Both the ways of incorporating an integral action in the LQR are found to be robust for perturbations in the elements of the system matrix. In METHOD-1 of incorporating an integral action in the linear quadratic controller we have to use augmented system matrix for calculating optimal controller gain whereas in METHOD-2 we need only original system matrix to find out optimal controller law, so for METHOD-1 computational efforts are required more than the METHOD-2. METHOD-1 and METHOD-2 closed loop responses are similar, for METHOD-1 control weighting factor more than 10 times the METHOD-2.

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APPENDIX – A: Comparison of two Methods of Incorporating an Integral Action in the linear quadratic controller

Method:1 Introducing an integral action by augmenting system matrix	**Method 2 :** Introducing an integral action by $\dot{u}$ in performance index
$\dot{x} = Ax + Bu$ $y = Cx$	$\dot{x} = Ax + Bu$ $y = Cx$
In this Method an integral action is incorporated in LQ controller by augmenting system matrix with error in state variables. Defining integral state q as follows: $\dot{q} = r - y = r - Cx$ Therefore the system after augmenting integral action becomes: $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$ $y = \tilde{C}\tilde{x}$ Where, $\dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}$, $\tilde{x} = \begin{bmatrix} x \\ q \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\tilde{C} = [C \quad 0]$	In this Method an integral action is introduced in LQ controller by introducing $\dot{u}$ in performance index instead of u. The performance index is taken as, $P = \frac{1}{2}\int_0^{\theta=\infty} (x^T Q x + \dot{u}^T R \dot{u}) dt$
One Riccati equation to solve to get the optimal control law. $\tilde{A}^T P + P\tilde{A} - P\tilde{B}R^{-1}\tilde{B}^T P + Q = 0$ Where, P is the solution of above Riccati equation. Here, these $\tilde{A}$ and $\tilde{B}$ are augmented matrices with error in state variable.	Four Non-Linear Matrices equations have to solve to get the optimal control law. $LBR^{-1}B^T M = Q$ $-LBR^{-1}B^T N + K + LA = 0$ $-NBR^{-1}B^T M + K + A^T M = 0$ $-NBR^{-1}B^T N + M + NA + L + A^T N = 0$ K, L, M and N are the solution.
Optimal Control law : $u = -R^{-1}B^T P x$	Optimal Control law : $u(t) = -R^{-1}B^T N x(t) - R^{-1}B^T M \int_0^t x(t)\, dt$
Here, for MIMO systems, computational efforts are required more because we have to use augmented matrices to get an integral action in the controller.	Here, we have to use original system equations to get the optimal control law.
For 10 states variable system, In this Method to get LQR with Integral action controller settings the augmented matrix A will become (20x20).	Whereas in this Method we have to solve original system to get LQ with integral action controller.

APPENDIX – B

MATLAB code for solving equations (15) to (18) is given below,

Function file:

```
function y=areklmn_reactor(x)

% STATE SPACE COEFFICIENT MATRICES
A=[-0.1 0; 0.1 -0.1];
B=[0.1;0];

% State weighting matrix and Control
weighting matrix
Q=[0 0; 0 1];
R=0.01;  % This value will be inverse of
R

% Extracting elements from x to get
K,L,M,N
k=x(1:4);
l=x(5:8);
m=x(9:12);
n=x(13:16);

% Converting vectors into matrices
K=reshape(k,[2,2]);
L=reshape(l,[2,2]);
M=reshape(m,[2,2]);
N=reshape(n,[2,2]);

% Four simultaneous nonlinear matrix
equations
y1=L*B*R*B'*M-Q;
y2=-(L*B*R*B'*N)+K+L*A;
y3=-(N*B*R*B'*M)+K+A'*M;
y4=-(N*B*R*B'*N)+M+N*A+L+A'*N;

yt=[y1;y2;y3;y4];

y=yt(:);

end
```

Command file:

```
clc
global KP
global KI

% STATE SPACE COEFFICIENT MATRICES
A=[-0.1 0; 0.1 -0.1];
B=[0.1;0];
C=[0 1];

% State weighting matrix and Control
weighting matrix
Q=[0 0;0 1];
R=0.01; % This value will be inverse of R

% To create random initial guess to get
solution Riccati equation
```

```
x10=rand([2,2]);
x20=rand([2,2]);
x30=rand([2,2]);
x40=rand([2,2]);
x0=[x10;x20;x30;x40];
options =
optimset('MaxFunEvals',50000000,'MaxIter'
,1000000,'TolFun',1e-
10000,'FunValCheck','on');

% fsolve to solve Algebric Riccati
Equation

x=fsolve(@areklmn_reactor,x0(:),options);

% Extracting elements from x to get
K,L,M,N
k=x(1:4);
l=x(5:8);
m=x(9:12);
n=x(13:16);

% Converting vectors into matrices
K=reshape(k,[2,2])
L=reshape(l,[2,2])
M=reshape(m,[2,2])
N=reshape(n,[2,2])

% Function values for check of answer
y1=L*B*R*B'*M-Q
y2=-(L*B*R*B'*N)+K+L*A
y3=-(N*B*R*B'*M)+K+A'*M
y4=-(N*B*R*B'*N)+M+N*A+L+A'*N

% Calculating Proportional term
KP=inv(R)*B'*N
% Calculating Integral term
KI=inv(R)*B'*M
KP=-R*B'*N;
KI=-R*B'*M;
```