COMS W4701: Artificial Intelligence

Lecture 6b: Probabilistic Models

Tony Dear, Ph.D.
Department of Computer Science
School of Engineering and Applied Sciences

Today

- Probability, random variables, and distributions
- Joint and conditional probabilities and distributions

- Product rule, chain rule
- Bayes’ theorem, independence

- Markov chains

Uncertainty

- So far: Planning and decision making in fully observable environments
- How do we reason in uncertain and *partially observable* environments?

- **Belief state**: A probability distribution over the entire state space
- Represent both uncertainty in the problem as well as *degree of belief*
- We can avoid the hard requirements of logic-based approaches

- Recall 90s AI resurgence relied heavily on *probabilistic approaches*
  - Diagnosis, speech and image recognition, tracking, mapping, error correction, etc.

Probabilities

- **Sample space**: Set Ω of all possible outcomes of a random experiment
- **Event**: Subset of a sample space (often described by a logical proposition)

- **Probability model (function)**: $P: \Omega \to [0,1]$ s.t. $\sum_{\omega \in \Omega} P(\omega) = 1$
- Probability of an event ϕ: $P(\phi) = \sum_{\omega \in \phi} P(\omega)$
  - Properties: $P(\emptyset) = 0$, $P(\Omega) = 1$, $P(\bar{\phi}) = 1 - P(\phi)$

- *Uniform probability model*: $P(\omega) = 1/|\Omega| \; \forall \omega$ and $P(\phi) = |\phi|/|\Omega|$
- Probabilities may represent *frequencies* or subjective *degrees of belief*

Random Variables

- A **random variable** $X: \Omega \rightarrow R$ maps sample space outcomes to some *range* R
- Ranges may be discrete/continuous, finite/infinite, ordered/unordered

- The **probability distribution** of a RV X enumerates range value probabilities
- *Categorical distributions* describe discrete and finite RVs in a table or vector
- Can use logical operators to combine different outcomes

- $P(W = sun) = P(sun) = 0.6$
- $P(sun \text{ OR } rain) = 0.6 + 0.1 = 0.7$
- $P(cloud \text{ OR } {\sim}snow)$
  $= P(cloud) + P({\sim}snow) - P(cloud\ AND\ {\sim}snow) = 0.29 + 0.99 - 0.29 = 0.99$

W	P(W)
sun	0.6
rain	0.1
cloud	0.29
snow	0.01

Joint Probability Distributions

- **Joint distributions** enumerate probabilities of *combinations* of multiple RVs together
- Size of full categorial joint distribution $= |X_1| \times |X_2| \times \cdots \times |X_n|$

- Given a joint distribution, we can also find distributions over *subsets* of RVs
- **Marginalization**: Sum out irrelevant RVs

$$\boxed{P(x) = \sum_{y \in Y} P(x, y)}$$

T	W	Pr(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(w) = \sum_t P(t, w) \longrightarrow$$

W	P(W)
sun	0.6
rain	0.4

Conditional Probability Distributions

- **Conditional probability:** Probability of an event *given* that another one occurred
- Ratio between joint probability and marginal probability of known event

T	W	Pr(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(sun|hot) = \frac{P(sun, hot)}{P(hot)} = \frac{0.4}{0.5} = \frac{4}{5}$$

$$P(sun|cold) = \frac{P(sun, cold)}{P(cold)} = \frac{0.2}{0.5} = \frac{2}{5}$$

$$P(a|b) = \frac{P(a, b)}{P(b)}$$

- A **conditional distribution** contains the probabilities of an *unobserved* variable, all conditioned on one outcome
- Equivalent to *normalizing* all joint probabilities with the conditioned outcome values

W	P(W|hot)
sun	0.8
rain	0.2

W	P(W|cold)
sun	0.4
rain	0.6

Product Rule

- The **product rule** yields joint probability $P(x,y)$ from a marginal $P(y)$ and conditional $P(x|y)$

$$P(y)P(x|y) = P(x,y)$$

- We can also follow with marginalization to find the "other" marginal $P(x)$

$P(W)$

W	Pr
sun	0.8
rain	0.2

$P(D|W)$

D	W	Pr
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

D	W	Pr
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

$P(D)$

D	Pr
wet	0.22
dry	0.78

Chain Rule

- The product rule can be extended to more than two RVs
- Idea: Successively build up larger joint probabilities

$$P(x_1)P(x_2|x_1)P(x_3|x_1,x_2) = P(x_1,x_2)P(x_3|x_1,x_2)$$
$$= P(x_1,x_2)\frac{P(x_1,x_2,x_3)}{P(x_1,x_2)} = P(x_1,x_2,x_3)$$

- In general: $P(x_1,\ldots,x_n) = P(x_1)P(x_2|x_1)\cdots P(x_n|x_1,\ldots,x_{n-1})$
$$= \prod_i P(x_i|x_1,\ldots,x_{i-1})$$

Chain Rule

- The chain rule can also be applied when all probabilities are conditioned on the same observation:

$$P(x_1|x_0)P(x_2|x_1,x_0)P(x_3|x_1,x_2,x_0)$$

$$= \frac{P(x_0,x_1)}{P(x_0)}\frac{P(x_0,x_1,x_2)}{P(x_0,x_1)}\frac{P(x_0,x_1,x_2,x_3)}{P(x_0,x_1,x_2)}$$

$$= \frac{P(x_0,x_1,x_2,x_3)}{P(x_0)} = P(x_1,x_2,x_3|x_0)$$

- In general: $P(x_1,\ldots,x_n|y_1,\ldots,y_m) = \prod_i P(x_i|x_1,\ldots,x_{i-1},y_1,\ldots,y_m)$

Example: Chain Rule

- Given: $P(a) = 0.5, P(b|a) = 0.2, P(c|a,b) = 0.7$

- Product rule: $P(a,b) = P(a)P(b|a) = 0.5 \times 0.2 = 0.1$
- (Also) product rule: $P(b,c|a) = P(b|a)P(c|a,b) = 0.2 \times 0.7 = 0.14$

- Chain rule: $P(a,b,c) = P(a)P(b|a)P(c|a,b) = 0.5 \times 0.2 \times 0.7$
  $= P(a,b)P(c|a,b) = 0.1 \times 0.7$
  $= P(a)P(b,c|a) = 0.5 \times 0.14$

- What if we were given $P(c|a)$ or $P(c|b)$ instead of $P(c|a,b)$?
- Can compute $P(a,c) = P(a)P(c|a)$, but we can't do anything with $P(c|b)$!

Bayes' Theorem

- We can combine conditional probability with the product rule to express a *posterior* probability given *evidence*:

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x) \quad \Rightarrow \quad P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- $P(x)$ is the *prior* and $P(y|x)$ is the *likelihood* of the evidence

- As with chain rule, this also holds if all terms are conditioned on another variable(s) z:

$$P(x|y,z) = \frac{P(y|x,z)P(x|z)}{P(y|z)}$$

Example: Probabilistic Inference

- Bayes' theorem can be used to *infer* hidden information given evidence

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

Binary random variables:

- M: meningitis
- S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Known probabilities}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)}$$

$$= \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999} = 0.008$$

Much smaller than $P(+s|+m)$!

Independence

- Two variables are **independent** if we can *factor* their joint distribution
- Breaks down a large joint distribution into smaller marginal ones

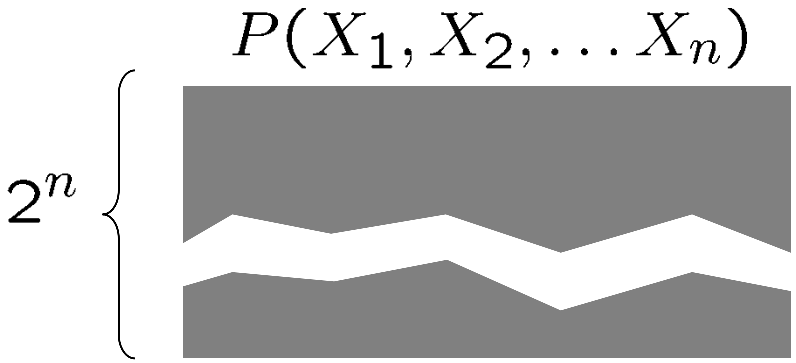
$$X \perp\!\!\!\perp Y \iff \forall x, y\colon\ P(x,y) = P(x)P(y);\quad P(x|y) = P(x)$$

- Knowing something about X tells us nothing about Y

- This is the *only case* in which we can put together marginal distributions to reconstruct a joint distribution!
- Second identity also useful for simplifying chain rule

Example: Independence

- Suppose we have N binary RVs
- Joint distribution would have size $O(2^N)$ (rows)



- What if we can assert independence?

- We can represent the *same information* using N 2-row tables ($O(2N)$)

$P(X_1)$

H	0.5
T	0.5

$P(X_2)$

H	0.5
T	0.5

$\ldots$

$P(X_n)$

H	0.5
T	0.5

Conditional Independence

- Absolute / marginal independence is often difficult to assert
- It is easier to assert this relationship given some *evidence*

- Two variables can be **conditionally independent** *given* a third variable:

$$X \perp\!\!\!\perp Y | Z \iff \begin{array}{l} \forall x, y, z : P(x, y|z) = P(x|z)P(y|z) \\ \forall x, y, z : P(x|z, y) = P(x|z) \end{array}$$

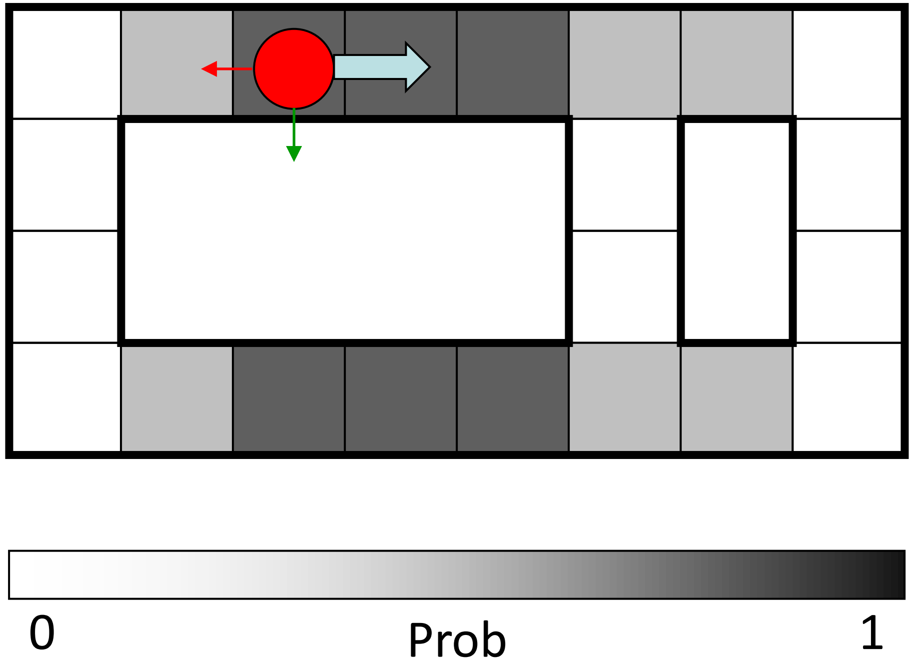
- Given Z, knowing something about X does not affect our belief about Y

Temporal Reasoning

- Scenario: An agent's state changes over time, but not directly observable
- *Belief state:* A random variable X_t representing the agent's current state, along with a probability distribution over the state space

- A probabilistic *transition model* describes how X_t is derived from past states

- We will be interested in looking at how X_t changes over time, possibly incorporating sensor information



Markov Chains

- **Markov chain**: A sequence of RVs $X_1, X_2, \ldots$, s.t. X_t only depends on X_{t-1}
- Parameters: Initial state $P(X_1)$, **transition model** $P(X_t|X_{t-1})$

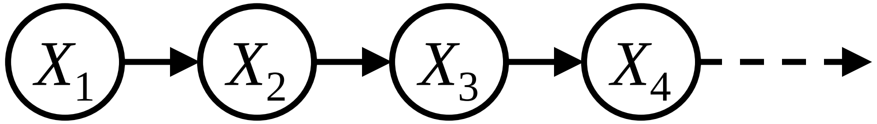
- If $|X_t| = n$, we have n^2 different $P(x_t|x_{t-1})$ transition probabilities
- Define a $n \times n$ *transition matrix* T, where $T_{ij} = P(X_t = j \mid X_{t-1} = i)$

$$T = \begin{bmatrix} P(X_t = 1 \mid X_{t-1} = 1) & \cdots & P(X_t = n \mid X_{t-1} = 1) \\ \vdots & \ddots & \vdots \\ P(X_t = 1 \mid X_{t-1} = n) & \cdots & P(X_t = n \mid X_{t-1} = n) \end{bmatrix}$$

- Sum of each row $\sum_j T_{ij} = \sum_j P(X_t = j \mid X_{t-1} = i) = 1$

Markov Assumption

- **Markov assumption**: X_t is independent of all past states given X_{t-1}



$$X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$

$$X_3 \perp\!\!\!\perp X_1 \mid X_2$$

$$X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$$

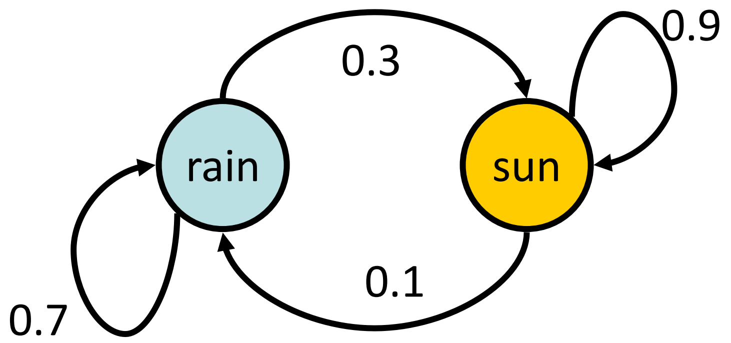
- Chain rule for joint distribution can be greatly simplified!

$$\begin{aligned} P(X_1, X_2, \ldots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1}) \\ &= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1}) \end{aligned}$$

Example: Markov Chains

$$P(X_1) = \begin{array}{cc} \text{rain} & \text{sun} \\ (0.8 & 0.2) \end{array}$$

$$T = \begin{array}{ccc} \text{rain} & \text{sun} & \\ \multicolumn{2}{c}{\begin{pmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{pmatrix}} & \begin{array}{c} \text{rain} \\ \text{sun} \end{array} \end{array}$$



- $P(X_2 = rain) = \sum_{x_1} P(x_1)P(X_2 = rain|x_1) \ = 0.8(0.7) + 0.2(0.1) = 0.58$
- $P(X_2 = sun) = \sum_{x_1} P(x_1)P(X_2 = sun|x_1) \ = 0.8(0.3) + 0.2(0.9) = 0.42$

- Alternatively, can compute $P(X_2) = P(X_1)T, P(X_3) = P(X_2)T, \ldots, P(X_t) = P(X_{t-1})T$
- More generally, $P(X_t) = P(X_1)T^{t-1}$

Stationary Distributions

- Observation: $\pi = (.25 \quad .75)$ satisfies $\pi = \pi \cdot T$
- π is an *eigenvector* of $T^\top$ corresponding to eigenvalue 1
- π is a **stationary distribution** of this transition matrix

$$T = \begin{pmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{pmatrix}$$

- All transition matrices have at least one stationary distribution
- Find the appropriate *eigenvector* π of $T^\top$ and rescale as $\pi / \sum_i \pi_i$ to ensure that the vector sum is 1

- Some Markov chains may have multiple stationary distributions

Markov Chain Applications

- Bioinformatics, population dynamics, epidemic modeling
- Thermodynamics, statistical mechanics, chemical reaction modeling
- Queuing theory, income and market modeling, game modeling

- Speech recognition and text generation, n-gram models
  - Unigram model: $P(word_t = i)$, bigram model: $P(word_t = i \mid word_{t-1} = j)$

- Web browsing: PageRank algorithm to determine webpage traffic
  - Model probabilities of navigating to existing outgoing link or arbitrary webpage

Summary

- Probability is the language of uncertainty
- Belief states are probability distributions, usually over random variables

- Given a joint distribution, we can do find marginal and conditional probs
- For inference, use conditioning, product/chain rule, Bayes' theorem

- Independence and conditional independence assert relationships between variables, can help simplify models