COMS W4701: Artificial Intelligence

Lecture 7a: Hidden Markov Models and Inference

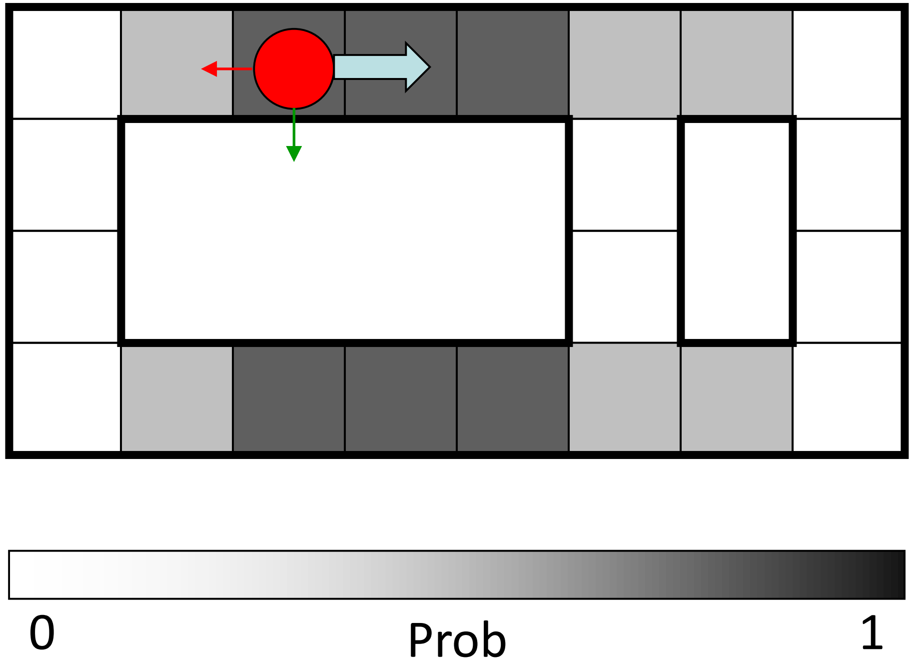
Tony Dear, Ph.D.
Department of Computer Science
School of Engineering and Applied Sciences

Today

- Markov chains
- Hidden Markov models
- State estimation (filtering): Forward algorithm

Temporal Reasoning

- Scenario: An agent's state changes over time, but not directly observable
- *Belief state:* A random variable X_t representing the agent's current state, along with a probability distribution over the state space
- A probabilistic *transition model* describes how X_t is derived from past states
- We will be interested in looking at how X_t changes over time, possibly incorporating sensor information



Markov Chains

- **Markov chain**: A sequence of RVs $X_1, X_2, \ldots$, s.t. X_t only depends on X_{t-1}
- Parameters: Initial state $P(X_1)$, **transition model** $P(X_t | X_{t-1})$

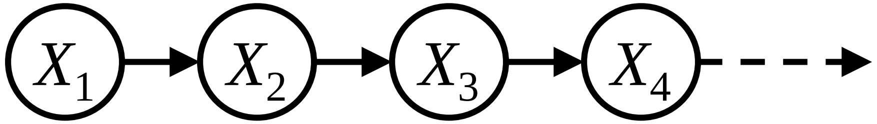
- If $|X_t| = n$, we have n^2 different $P(x_t | x_{t-1})$ transition probabilities
- Define a $n \times n$ *transition matrix* T, where $T_{ij} = P(X_t = j \mid X_{t-1} = i)$

$$T = \begin{bmatrix} P(X_t = 1 \mid X_{t-1} = 1) & \cdots & P(X_t = n \mid X_{t-1} = 1) \\ \vdots & \ddots & \vdots \\ P(X_t = 1 \mid X_{t-1} = n) & \cdots & P(X_t = n \mid X_{t-1} = n) \end{bmatrix}$$

- Sum of each row $\sum_j T_{ij} = \sum_j P(X_t = j \mid X_{t-1} = i) = 1$

Markov Assumption

- **Markov assumption**: X_t is independent of all past states given X_{t-1}



$$X_3 \perp\!\!\!\perp X_1 \mid X_2$$

$$X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$$

$$X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$

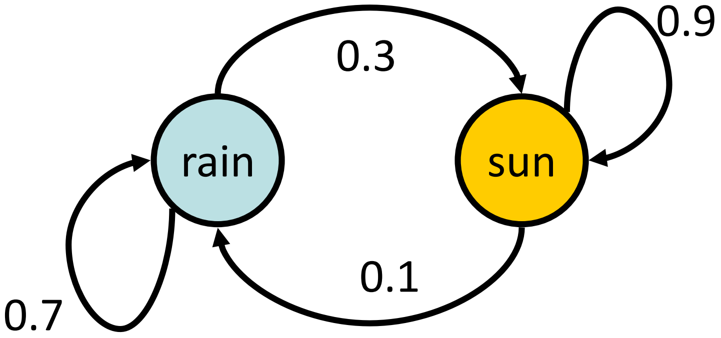
- Chain rule for joint distribution can be greatly simplified!

$$\begin{aligned} P(X_1, X_2, \ldots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1}) \\ &= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1}) \end{aligned}$$

Example: Markov Chains

$$\begin{array}{ccc} & rain & sun \\ P(X_1) = & (0.8 & 0.2) \end{array}$$

$$\begin{array}{ccc} & \begin{array}{cc} rain & sun \end{array} & \\ T = & \begin{pmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{pmatrix} & \begin{array}{c} rain \\ sun \end{array} \end{array}$$



- $P(X_2 = rain) = \sum_{x_1} P(x_1) P(X_2 = rain|x_1) \ = 0.8(0.7) + 0.2(0.1) = 0.58$
- $P(X_2 = sun) = \sum_{x_1} P(x_1) P(X_2 = sun|x_1) \ = 0.8(0.3) + 0.2(0.9) = 0.42$

- Alternatively, can compute $P(X_2) = P(X_1)T, P(X_3) = P(X_2)T, \ldots, P(X_t) = P(X_{t-1})T$
- More generally, $P(X_t) = P(X_1)T^{t-1}$

Stationary Distributions

- Observation: $\pi = (.25 \quad .75)$ satisfies $\pi = \pi \cdot T$
- π is an *eigenvector* of $T^\top$ corresponding to eigenvalue 1
- π is a **stationary distribution** of this transition matrix

$$T = \begin{pmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{pmatrix}$$

- All transition matrices have at least one stationary distribution
- Find the appropriate *eigenvector* π of $T^\top$ and rescale as $\pi / \sum_i \pi_i$ to ensure that the vector sum is 1

- Some Markov chains may have multiple stationary distributions

Markov Chain Applications

- Bioinformatics, population dynamics, epidemic modeling
- Thermodynamics, statistical mechanics, chemical reaction modeling
- Queuing theory, income and market modeling, game modeling

- Speech recognition and text generation, n-gram models
  - Unigram model: $P(word_t = i)$, bigram model: $P(word_t = i \mid word_{t-1} = j)$

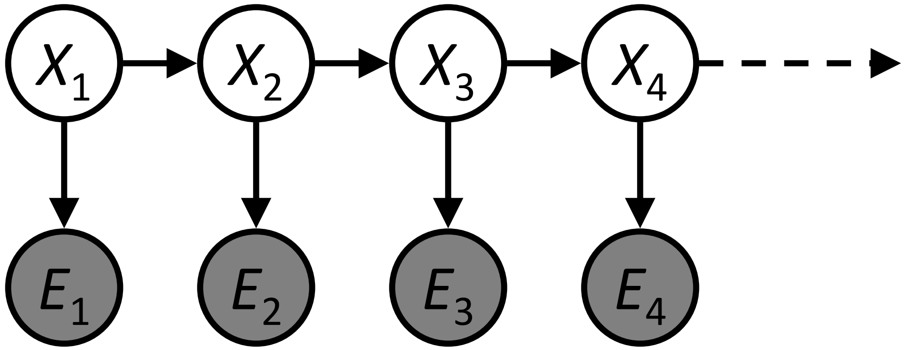
- Web browsing: PageRank algorithm to determine webpage traffic
  - Model probabilities of navigating to existing outgoing link or arbitrary webpage

Hidden Markov Models

- With Markov chains, we do not directly observe the state
- Now let's suppose we can observe indirect *evidence* of states

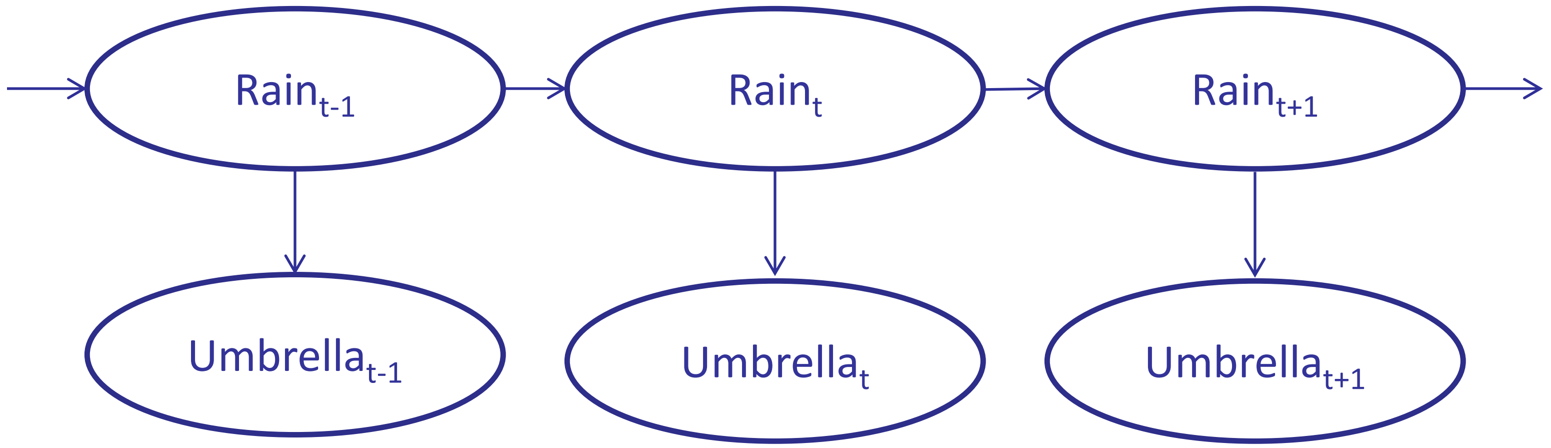
- **Hidden Markov model**: A Markov process with *hidden* states X_t and *observable* evidence variables E_t

- Initial belief state: $P(X_1)$
- Transition model: $P(X_t|X_{t-1})$
- Observation model: $P(E_t|X_t)$



Example: Weather HMM

$P(X_t \mid X_{t-1})$



$P(E_t \mid X_t)$

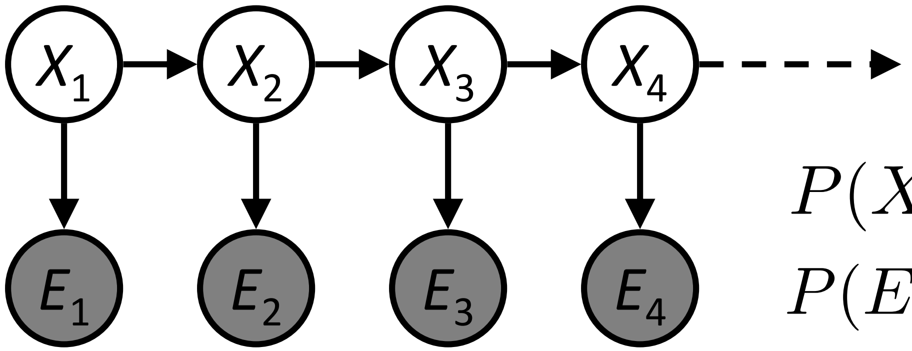
X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

X_t	E_t	$P(E_t \mid X_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Conditional Independences

- Markov chain independences:

$$X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$$



$$P(X_t \mid X_{t-1})$$
$$P(E_t \mid X_t)$$

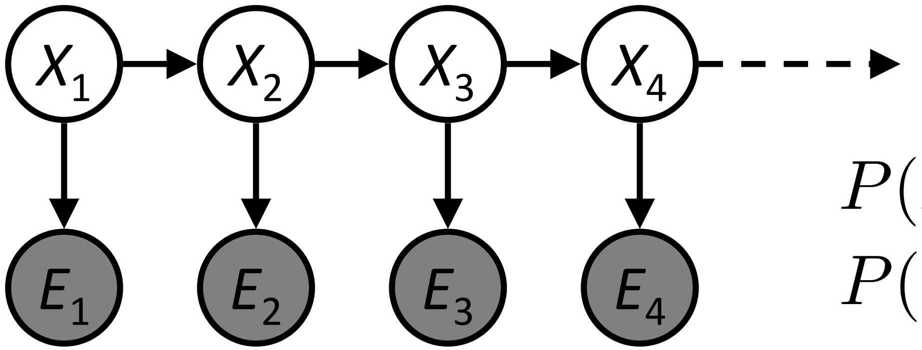
- A state is conditionally independent of past states and evidence given preceding state:

$$X_t \perp\!\!\!\perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1}$$

- An observation is conditionally independent of past states and evidence given current state:

$$E_t \perp\!\!\!\perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t$$

Joint Distribution



$P(X_t \mid X_{t-1})$
$P(E_t \mid X_t)$

- General joint distribution:

$$P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t)$$

- Marginal or smaller joint distributions can be found by summing out RVs
- For certain computations we don't even need the entire joint distribution!

Inference

- Inference tasks compute belief states or hidden states given evidence

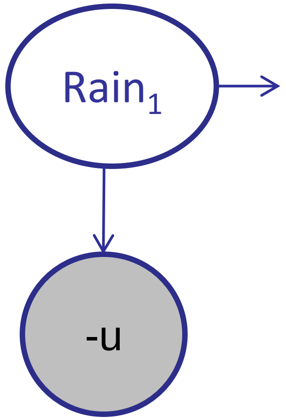
- **Filtering (state estimation)**: Find $P(X_t \mid e_{1:t})$
  - Estimate the belief state, given a sequence of past observations

- **Decoding**: Find $\operatorname{argmax}_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - Find the *sequence* of hidden states that best explains given observations

- **Smoothing**: Find $P(X_k \mid e_{1:t})$, for $1 \leq k < t$
  - Use both past and future evidence to *smooth* a belief state

Example: Weather HMM

- Want to find $P(X_1|e_1) = \frac{P(X_1,e_1)}{P(e_1)}$
- $P(e_1)$ is a *constant* for all values of X_1 since e_1 is already observed (fixed)!



$$P(X_1) = (0.5 \quad 0.5)$$

X_t	E_t	$P(E_t\|X_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

$$P(X_1|e_1) \propto P(X_1, e_1) = P(e_1|X_1) * P(X_1)$$

$$= (0.1 \quad 0.8) * (0.5 \quad 0.5) = (0.05 \quad 0.4)$$

- Since we know $P(X_1|e_1)$ sums to 1, $P(e_1)$ must be equal to $\sum_{x_1} P(x_1, e_1)$

$$P(X_1|e_1) = \frac{P(X_1, e_1)}{P(e_1)} = \frac{(0.05 \quad 0.4)}{0.05 + 0.4}$$

$$= (0.11 \quad 0.89)$$

State Estimation

- In a Markov chain, we obtained $P(X_{t+1})$ from $P(X_t)$ by multiplication with transition probabilities

- We just showed how to obtain $P(X_{t+1}|e_{t+1})$ from $P(X_{t+1})$ by multiplication with observation probabilities, followed by normalization

- To efficiently solve the state estimation problem of finding $P(X_{t+1}|e_{1:t+1})$, we need to show how to perform these steps starting from $P(X_t|e_{1:t})$
- (To simplify calculations, we will work primarily with $P(X_t, e_{1:t})$)

Forward Algorithm

- Given $P(x_t, e_{1:t})$:

Conditional independence

$$\sum_{x_t} P(X_{t+1} \mid x_t, \cancel{e_{1:t}})\boldsymbol{P(x_t, e_{1:t})} = \sum_{x_t} P(X_{t+1}, x_t, e_{1:t}) = \boldsymbol{P(X_{t+1}, e_{1:t})}$$

- So we have $P(X_{t+1}, e_{1:t}) = P(X_t, e_{1:t}) \cdot T$ (same as Markov chains)

- Product rule: $P(e_{t+1} \mid X_{t+1}, \cancel{e_{1:t}})\boldsymbol{P(X_{t+1}, e_{1:t})} = \boldsymbol{P(X_{t+1}, e_{1:t+1})}$

Conditional independence

- So $P(X_{t+1}, e_{1:t+1}) = P(X_{t+1}, e_{1:t}) * O_{t+1}$, where $O_{t+1} = P(e_{t+1} \mid X_{t+1})$ is a vector of observation probabilities and $*$ is an elementwise product

Forward Algorithm

- Given: $\boldsymbol{\alpha}_0 = P(X_0)$, or if starting with $\boldsymbol{\alpha}_1 = P(X_1)$, skip the first "elapse time" step and start by observing evidence e_1

$$\boldsymbol{\alpha}_t = P(X_t, e_{1:t})$$

$$\boldsymbol{\alpha}'_{t+1} = P(X_{t+1}, e_{1:t})$$

$$\boldsymbol{\alpha}_{t+1} = P(X_{t+1}, e_{1:t+1})$$

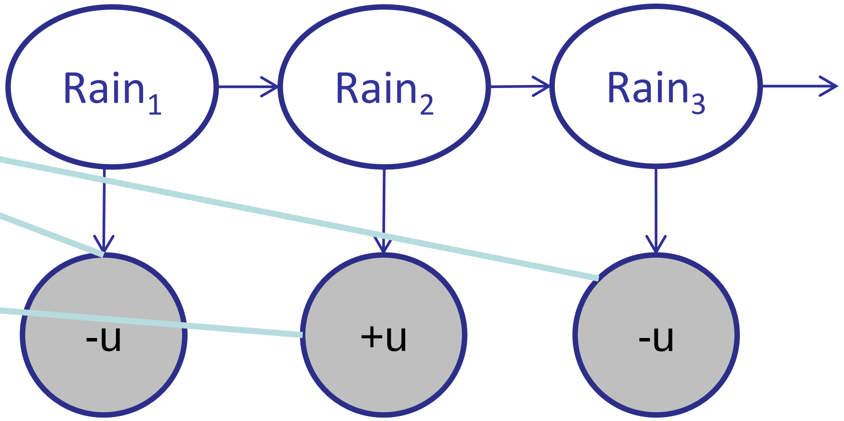
- For each timestep t:
  - Elapse time: $\boxed{\boldsymbol{\alpha}'_{t+1} = \boldsymbol{\alpha}_t T}$
  - Observe evidence e_{t+1}: $\boxed{\boldsymbol{\alpha}_{t+1} = \boldsymbol{\alpha}'_{t+1} * O_{t+1}}$
  - Normalize (as needed): $\boxed{P(X_{t+1} | e_{1:t+1}) = \boldsymbol{\alpha}_{t+1} / \sum \boldsymbol{\alpha}_{t+1}}$

Example: Weather HMM

$$T = \begin{array}{cc} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} & \begin{array}{c} +r \\ -r \end{array} \\ \begin{array}{cc} +r & -r \end{array} & \end{array}$$

$$O_1 = O_3 = \begin{pmatrix} 0.1 \\ 0.8 \end{pmatrix}^{\mathsf{T}}$$

$$O_2 = \begin{pmatrix} 0.9 \\ 0.2 \end{pmatrix}^{\mathsf{T}}$$



Suppose $\alpha_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}^{\mathsf{T}}$

$$\alpha_1' = \alpha_0 T = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}^{\mathsf{T}} \qquad \alpha_1 = O_1 * \alpha_1' = \begin{pmatrix} 0.05 \\ 0.4 \end{pmatrix}^{\mathsf{T}} \qquad P(X_1|e_1) = \begin{pmatrix} 0.11 \\ 0.89 \end{pmatrix}^{\mathsf{T}}$$

$$\alpha_2' = \alpha_1 T = \begin{pmatrix} .155 \\ .295 \end{pmatrix}^{\mathsf{T}} \qquad \alpha_2 = O_2 * \alpha_2' = \begin{pmatrix} .1395 \\ .059 \end{pmatrix}^{\mathsf{T}} \qquad P(X_2|e_{1:2}) = \begin{pmatrix} .703 \\ .297 \end{pmatrix}^{\mathsf{T}}$$

$$\alpha_3' = \alpha_2 T = \begin{pmatrix} .115 \\ .083 \end{pmatrix}^{\mathsf{T}} \qquad \alpha_3 = O_3 * \alpha_3' = \begin{pmatrix} .0115 \\ .0664 \end{pmatrix}^{\mathsf{T}} \qquad P(X_3|e_{1:3}) = \begin{pmatrix} .148 \\ .852 \end{pmatrix}^{\mathsf{T}}$$

Summary

- Temporal models are used to track partially observable environments
- Maintain and update belief states (probability distributions)

- Markov chains may have stationary distributions or steady state behavior
- Inference in HMMs compute hidden information given observed information

- State estimation: Forward algorithm iteratively computes the current state distribution given evidence to date