Homework Hours Spent: 10 hours

1. Considering the estimator $s_{xy} := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$ and we are given that each measure $\bar{X}$ and $\bar{Y}$ are sample-mean estimators, we show that the estimator is both an unbiased and consistent estimator for the covariance of $Cov(X, Y)$.
   a. An estimator is consistent if it converges to the desired parameter in probability. We show this by first expanding the expression and then taking expectation: taking its expectation and confirming it's bias behavior:
   $s_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y}) = \frac{1}{n-1}\sum_{i=1}^{n} X_iY_i - \bar{Y}X_i - \bar{X}Y_i + \bar{X}\bar{Y}$
   We can now distribute the summation for each linear term within the sequence:
   $s_{xy} = \frac{1}{n-1}[\sum_{i=1}^{n} X_iY_i - \bar{Y}X_i - \bar{X}Y_i + \bar{X}\bar{Y}] = \frac{1}{n-1}[(\sum_{i=1}^{n} X_iY_i) - n\bar{Y}\bar{X} - n\bar{X}\bar{Y} + n\bar{X}\bar{Y}]$
   $s_{xy} = \frac{1}{n-1}[(\sum_{i=1}^{n} X_iY_i) - n\bar{Y}\bar{X}]$
   Since these variables are all i.i.d we can take the expectation of the expression.
   $E[s_{xy}] = \frac{1}{n-1}E[(\sum_{i=1}^{n} X_iY_i) - n\bar{Y}\bar{X}] = \frac{1}{n-1}(n\mu_{xy} - nE[\bar{Y}\bar{X}])$
   Now we can manipulate our sample means by rewriting in terms of their underlying random variable e.g. $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ as follows:
   $E[s_{xy}] = \frac{1}{n-1}(n\mu_{xy} - nE[\frac{1}{n}\sum_{i=1}^{n} Y_i \cdot \frac{1}{n}\sum_{i=1}^{n} X_i]) = \frac{1}{n-1}(n\mu_{xy} - \frac{1}{n}[\sum_{i=1}^{n}\sum_{j=1}^{n} Y_iX_j])$
   Combining the double-summation now yields two indexes, one in i and one in j which leads to disjoint treatment in the sum - namely when $i = j$ we have the joint expectation $E[X_iY_i]$ and when $i \neq j$ we have the product of the marginals $E[X_i]E[Y_j]$. We have n instances where $i = j$ and the residual $n^2 - n$ iterations where the marginal distribution persists:
   $E[s_{xy}] = \frac{1}{n-1}(n\mu_{xy} - \frac{1}{n}[n\mu_{xy} + (n^2 - n)\mu_x\mu_y])$
   $E[s_{xy}] = \frac{1}{n-1}(n\mu_{xy} - \mu_{xy} - n\mu_x\mu_y + \mu_x\mu_y) = \frac{1}{n-1}((n-1)\mu_{xy} - (n-1)\mu_x\mu_y)$
   $E[s_{xy}] = \frac{n-1}{n-1}(\mu_{xy} - \mu_x\mu_y) = \mu_{xy} - \mu_x\mu_y = \sigma_{xy}$
   Since we show that our estimator is unbiased in absolute, we know by strong law of large numbers (SLLN) that we are consistent and converge in probability to σ_{xy}
   b. As illustrated above, we show that the estimator is indeed unbiased as our expression converges absolutely (AS) to the desired quantity $s_{xy} \Rightarrow \sigma_{xy}$. Written another way we have that $E[s_{xy}] - \sigma_{xy} = 0$, indicating it's unbiased.

2. Given our random variables X_i have mean μ, we can leverage the first moment and the sample mean to estimate our broader population mean. As such, the method-of-moment estimator for $\hat{\mu}$ can be represented as the simple average of the X_i sequence.

$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

Now, when calculating the population variance of the dataset, we leverage the definition for variance, which defines $\sigma^2 = E[X^2] - (E[X])^2$. We know from the above that our first moment $E[X]$ is simply the average of our X_i sequence. Our second moment $E[X^2]$ follows this same pattern, albeit with each term in the sequence being squared. This then follows as a substitution of our existing moments to the expression:

$(\hat{\sigma})^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2$

Hence, using the method of moments we have that our estimators for μ and σ^2 are:

$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$(\hat{\sigma})^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2$

3. To solve for the two θ parameters, we must have two equations i.e. a first and second moment. We note, however, that either probability density function resembles that of a normal distribution, with $f_1 \sim N(\theta_1, \sigma_1^2)$ and $f_2 \sim N(\theta_2, \sigma_2^2)$, with means of θ_1 and θ_2, respectively. Using the method of moments we can then equate the first moment i.e. sample $E[f_x] = \frac{1}{2}E[f_1] + \frac{1}{2}E[f_2] = \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2$. We then move to calculate the second moment of the sample $Var[f_x] = \frac{1}{2}[E[f_1^2] + E[f_2^2]] - (\frac{1}{2}(E[f_1] + E[f_2]))^2$. We must then recognize that the second power for either f_x can be written as a sum of the variance and mean, as noted below in sample

E.g. $E[X^2] = Var(X) + (E[X])^2 = \sigma^2 + \mu^2$

This mixture distribution variance can then be expanded as follows:

$Var[f_x] = \frac{1}{2}[(\sigma_1^2 + \mu_1^2) + (\sigma_2^2 + \mu_2^2)] - \frac{1}{4}(\mu_1 + \mu_2)^2$

$Var[f_x] = \frac{1}{2}(\sigma_1^2 + \mu_1^2) + \frac{1}{2}(\sigma_2^2 + \mu_2^2)] - \frac{1}{4}(\mu_1 + \mu_2)^2$

Now we reorganize the listed terms in accordance to the relevant θ_i operators, assuming that we know both σ_1 and σ_2 from our expression. This then follows:

$\alpha_1 = \frac{1}{2}(\theta_1 + \theta_2) = \bar{X}$, (where $\bar{X}$ is equal to the sample mean)

$\alpha_2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}(\theta_1^2 + \theta_2^2)] - \frac{1}{4}(\theta_1 + \theta_2)^2 = s^2$, (where s^2 is equal to the sample variance). We now begin the arduous process of solving the system of equations.

$\frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}(\theta_1^2 + \theta_2^2)] - \frac{1}{4}(2\bar{X})^2 = s^2$

$\frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}(\theta_1^2 + \theta_2^2)] - \bar{X}^2 = s^2$, using a substitution $\theta_2 = 2\bar{X} - \theta_1$

$\frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}(\theta_1^2 + 4\bar{X}^2 - 4\bar{X}\theta_1 + \theta_1^2)] - \bar{X}^2 = s^2$

$\frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \theta_1^2 + 2\bar{X}^2 - 2\bar{X}\theta_1 - \bar{X}^2 = s^2$, we now solve for 0 the roots θ_1

$\frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \theta_1^2 + 2\bar{X}^2 - 2\bar{X}\theta_1 - \bar{X}^2 - s^2 = 0$, we now can rearrange the expression in terms of variables and constants (both given and derived) to solve the expression

$\theta_1^2 - 2\bar{X}\theta_1 + (\frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + 2\bar{X}^2 - \bar{X}^2 - s^2) = 0$

Using the quadratic-formula to solve for this lengthy expression we can finally come to terms with a potential solution and by symmetry in the expression of θ_1 and θ_2, we have that our mixture distribution yields the following method-of-moment estimator.

$\theta_1 = \bar{X} + \sqrt{s^2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)}$

$\theta_2 = \bar{X} - \sqrt{s^2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)}$

4. Given X_i i.i.d Gaussian random variables (RV) with mean μ and standard deviation σ

a. To determine whether the sample mean, as defined $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, is also a Gaussian RV we must first determine it's mean and variance, as a Gauss distribution can be described in full by two parameters, μ and σ. This follows by computing the expectation first and comparing it you our distribution:

$E[M_n] = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}(n\mu) = \mu$

Now we compute the variance of the expression which follows a similar process, since each measure is i.i.d. we share no covariance terms.

$Var[M_n] = Var[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i] = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$

Now according to the Central Limit Theorem (CLT) we know that for a given sequence of random variables with a finite mean μ and variance σ^2/n they converge in distribution to the Gaussian distribution with accompanying parameter terms for mean and variance.

b. We find the value of n such that $P(79 \le M_n(X) \le 81) = 0.99$, we first standardized our expression via the central limit theorem. More specifically, we subtract away the mean of the series and divide by the scaled standard deviation that we calculated in our above solution:

$M_n(X) \sim N(\mu, \sigma/\sqrt{n}) \Rightarrow Z = \frac{M_n - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

$P(\frac{79-80}{22/\sqrt{n}} < Z < \frac{81-80}{22/\sqrt{n}}) = P(\frac{-\sqrt{n}}{22} < Z < \frac{\sqrt{n}}{22}) = 0.99$, using a z-table we have that the probability corresponds to a score of 2.576. Solving for n in the expression then yields $n = (22 \cdot 2.576)^2 \approx 3212$

5. Please see code file entitled HW1.ipynb for full code and graphs.

a. We note that both returns are centered around zero, though BBVA exhibits a larger *spread* or wider dispersion for daily returns relative to MSFT.
b. The histogram confirms the earlier observation, that BBVA exhibits a larger variance than MSFT with greater downside, as evidenced with daily returns less than -10%.
c. Computing the mean and variance comports with the earlier expectations noted from observing the graphs, as the variance for BBVA is greater than MSFT.
d. Observing the covariance of the two-return series we note that the directionality of either stock (i.e. BBVA and MSFT) is positive, though not strongly correlated. As such, while these stocks may trend together due to relative size and stability (i.e., blue-chip status), due to the firms operating under separate sectors in different geographic regions, they are exposed to alternative idiosyncratic catalysts such as domestic growth/inflation.