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A STOCHASTIC MODEL FOR INDIVIDUAL CHOICE BEHAVIOR [1]

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This paper presents a stochastic model which is concerned with the interrelations of the response variables observed in choice situations. The model is not a complete theory, because it involves no assumptions about the relations between stimulus and response variables. However, for given stimulus conditions, the parameters of the stochastic process do provide a convenient summary of many aspects of behaviour in a choice situation. Furthermore, the most elementary assumptions about the way in which these parameters might vary with changed stimulus conditions lead to predictions which are in qualitative agreement with experimental findings. In a sense, therefore, the stochastic model can be regarded as a rudimentary theory of certain aspects of choice behaviour.

Descriptors of Choice Behavior

A wide variety of experiments require the use of a situation involving a choice between two or more alternatives. There are several variables which may be employed in a descriptive summary of the behavior which appears in these situations. These variables can be of two kinds. Firstly, there are descriptors of the primary response to the situation, and, secondly, there are descriptors of the responses which the *S* makes to his primary choices. Those of the first kind are most commonly used and the three principal ones are: (*a*) Response time—the time taken for a definite choice to be made. (*b*) Relative response frequency—the proportion of occasions on which a particular choice response is made. (*c*) The number of vicarious trial and error responses (VTEs)—the number of vacillations between the various alternatives before a definite choice occurs. In the second group, where the descriptor is usually a verbal statement by the *S*, there are such variables as: (*a*) confidence in the correctness of a given choice and (*b*) an assessment of the subjective difficulty of the choice task.

Clearly, the extent to which these various descriptors can be employed will depend upon the specific details of an experiment. But, for many choice situations, all three descriptors of the first kind can be employed. Also in most studies with human *S*s the second kind are also available. In fact, this paper will be mainly concerned with the first kind of descriptor, but some suggestions will be advanced

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which permit those of the second kind to be also included in a unitary stochastic description of choice behavior.

Particular Choice Situations Which Are Considered

It is believed that the underlying hypotheses upon which the stochastic description is based are applicable to most choice situations. However, the derivation of a mathematical model from these hypotheses which can be readily applied to experimental data without additional assumptions is more conveniently achieved for a certain class of situations. This class consists of experiments where knowledge of the outcome or correctness of a response is not available to the *S* until after the choice has been made. Thus, for example, most ordinary disjunctive reaction time studies are *not* considered because the *S* in these experiments can *match* his response with a known requirement. Nevertheless, the class of situations which can be considered is not a trivial one. It includes among others (*a*) Discrimination experiments, including most conventional psychophysical procedures in this category. (*b*) Studies of preference and conflict. (*c*) Investigations of learning in choice situations.

The next section of the paper is mainly concerned with the events supposed to be taking place during a single experimental trial.

THE STOCHASTIC MODEL

The notions upon which the model is based are very simple and involve only two assumptions:

Assumption 1. It is first assumed that, *for given stimulus and organismic conditions, there is associated with each possible choice response a single parameter. This parameter determines the probability that in a small interval of time* $(t, t + \Delta t)$, *there will occur an "implicit" response of the kind with which the parameter is associated.*

No specific interpretation is given to the term "implicit response." It may, in certain circumstances, be taken to be equivalent to the partical response usually classified as a VTE. But there are some situations in which VTEs are not observed and would seem unlikely to be present. In these cases the "implicit" response may be regarded as a tendency to make a given response, or might perhaps be given some physiological interpretation.

The probabilities of the various kinds of "implicit" responses occurring are considered to be independent of one another. So that for given conditions, implicit responses of each kind are appearing at random intervals unaffected by the appearance of other implicit responses. It follows from the first assumption that the distribution of the intervals between successive implicit responses *of a given kind* is exponential and is determined entirely by the response parameter [e.g., see Feller, 1950, p. 220].

Assumption 2. It is assumed that *a final choice response is made when a run of* K *implicit responses of a given kind appears, this run being uninterrupted by occurrences of implicit responses of other kinds. K* may either be assumed to take a particular value or can be regarded as a further parameter, which can be estimated from experimental data.

Assumption 1 has been employed before. Mueller (1950) has used this approach to describe the intervals between bar-presses in an operant conditioning experiment where only one response is involved. For the same situation, Estes (1950) and Bush & Mosteller (1951) have used an assumption which is very similar, the only difference being that their models used a discontinuous rather than a continuous distribution of responses in time. Christie (1952) in discussing the determination of response probabilities in a discrimination experiment, has used the same assumption for situations where two responses are competing. Finally, the author of the present paper (Audley: 1957, 1958) has previously used the same notions to combine response times and response probabilities in a stochastic description of individual learning behavior. However, in all these examples, it has been assumed that $K = 1$. Bush and Mosteller (1955), in an analysis of response times in a runway situation, have considered a continuous model with $K > 1$, but this generalization does not appear to have been previously employed in a situation involving choice.

There are several reasons which can be advanced for assuming that $K > 1$. Firstly, when $K = 1$, but not if $K > 1$, the distributions of response times for all alternatives can be shown to be identically the same, and are exponential (e.g., see Audley, 1958). Neither of these properties is in agreement with experimental findings. Secondly, when $K > 1$, the sequence of "implicit" responses occurring before a final choice is made offer a possible means of including VTE's within the description of choice behavior. Thirdly, classification of the various sequences of "implicit" choice suggests an approach to descriptors of the second kind. For example, "perfect confidence" in a choice might be identified with sequences consisting of "implicit" responses of one kind only.

Derivation of the Stochastic Model

No further assumptions are required in the derivation of the model, which can be applied to situations involving any number, m, of choices. However, in order to keep the exposition as brief as possible, consideration in this paper will be limited to situations involving a choice between only two alternatives, i.e., $m = 2$. Furthermore, the mathematical problem is relatively simple when $K = 2$, so that only this special case will be presented. Results for the more general case have been derived and will be elaborated elsewhere.

The two-choice situation with K = *2.* The two possible responses will be called A and B, and implicit responses of the two kinds will be labelled a and b respectively. Let the parameters associated with the two responses be α and β. Assumption 1 means that $p(a)$, the probability of an a occurring in a small time interval $(t, t + \Delta t)$ is given by:

$$p(a) = \alpha \Delta t \qquad [1a]$$

Similarly

$$p(b) = \beta \Delta t \qquad [1b]$$

The probability $p(a \text{ or } b)$, of an implicit response of either kind but not both, occurring in the small time interval is

$$\begin{aligned} p(a \text{ or } b) &= p(a) + p(b) - 2p(a)p(b) \\ &= (\alpha+\beta)\Delta t - 2\alpha\beta(\Delta t)^2 \end{aligned}$$

Hence

$$p(a \text{ or } b) = (\alpha+\beta)\Delta t \qquad [1c]$$

if terms of order $(\Delta t)^2$ are ignored. This becomes possible if a transition is made to the continuous case when the distribution in time of implicit responses follows that of a Poisson process (e.g., see Feller, 1950, p. 220). Therefore the probability, $p(n, t)$, of obtaining n implicit responses in the time interval (o, t) is (e.g., again see Feller, 1950, p. 221):

$$p(n, t) = \frac{(\alpha + \beta)^n t^n e^{-(\alpha+\beta)t}}{n!} \qquad [2]$$

In particular the probability, $p(o, t)$, of obtaining no implicit response of either kind in time t is given by:

$$p(o, t) = e^{-(\alpha+\beta)t} \qquad [3]$$

The probability: P_a, that the first implicit response to occur is an a is

$$P_a = \int_{t=o}^{\infty} p(o, t)\alpha dt$$

$$= \int_{t=o}^{\infty} e^{-(\alpha+\beta)t}\alpha dt = \frac{\alpha}{\alpha+\beta} \qquad [4a]$$

$$= \text{say, } p$$

Similarly, for implicit b responses

$$P_b = \frac{\beta}{\alpha+\beta} = \text{say, } q = 1 - p \qquad [4b]$$

Since occurrences of implicit responses follow a Poisson process, Equations 4a and 4b also give the probability that, starting at any given moment, the next implicit response to occur will be an a or b respectively. Therefore, ignoring for the moment questions concerning the time intervals between successive implicit responses, the sequence of events leading to a final choice can be treated as a sequence of independent binomial trials, with the probabilities, P_a and P_b, of the two types of event given by Equations 4a and 4b.

The Probability, P_A*, That the Final Choice is an* A *Response*

The possible sequences which terminate with the occurrence of an A can be easily classified when $K = 2$. For they must all be simple alternations between a and b, until two successive a's occur. The early members of this class of sequences are: *aa*, *baa*, *abaa*, *babaa*, etc. The respective probabilities of these various sequences is clearly: p^2, p^2q, p^3q, p^3q^2 etc. The over-all probability, P_A, that the final choice is an A, is the sum of this infinite series of sequence probabilities. Thus,

$$P_A = p^2 + p^2q + p^3q + p^3q^2 + \cdots \qquad [5]$$

Whence, simplifying, and substituting for p and q from Equations 4a and 4b

$$P_A = \frac{\alpha^2[\alpha + 2\beta]}{[\alpha+\beta][(\alpha+\beta)^2 - \alpha\beta]} \qquad [6a]$$

Similarly

$$P_B = \frac{\beta^2[2\alpha + \beta]}{[\alpha+\beta][(\alpha+\beta)^2 - \alpha\beta]} \qquad [6b]$$

Equation 6a may be written in the following form:

$$P_A = \frac{\alpha}{\alpha+\beta} \cdot \frac{[(\alpha+\beta)^2 - \beta^2]}{[(\alpha+\beta)^2 - \alpha\beta]}$$

so that when $\alpha > \beta$, $P_A > \frac{\alpha}{\alpha+\beta}$ and $P_B < \frac{\beta}{\alpha+\beta}$

Thus the difference between the probabilities of the various implicit responses occurring is accentuated in the expressions for the probabilities of overt choice responses. The accentuation increases with K and implies that there is more certainty in the overt choices than in the underlying processes which determine them. This is believed to be a property which many organisms exhibit.

Vicarious Trial and Error

If we identify alternating appearances of the "implicit" responses, a and b, with VTEs, the moments of the distribution of VTEs can readily be obtained. Attention here will be confined to the mean number of VTEs preceding (a) any choice (b) a particular choice.

The Mean Number of VTEs Preceding Any Choice, $\bar{V}$

There are no VTEs if the sequence of implicit responses is *aa* or *bb*.

There is 1 VTE if the sequence is *baa* or *abb*.

There are 2 VTEs if the sequence is *abaa* or *babb*, and so on.

Dividing the sequences of implicit responses into those with an odd number and those with an even number of VTEs, the following probabilities are found (letting $P(V = n)$ be the probability of obtaining n VTEs):

$$\begin{aligned} p(V = 0) &= p^2 + q^2 \\ P(V = 2) &= p^3q + pq^3 \\ P(V = 4) &= p^4q^2 + p^2q^4 \\ &\text{etc.} \end{aligned}$$

$$\begin{aligned} P(V = 1) &= p^2q + pq^2 \\ P(V = 3) &= p^3q^2 + p^2q^3 \\ P(V = 5) &= p^4q^3 + p^3q^4 \\ &\text{etc.} \end{aligned}$$

Now

$$\bar{V} = P(V = 1) + 2P(V = 2) + 3P(V = 3) + \cdots$$

and after some algebraic manipulation and again substituting for p and q from Equation 4a and 4b.

$$\bar{V} = \frac{3\alpha\beta}{(\alpha + \beta)^2 - \alpha\beta} \qquad [7]$$

If $\gamma = \frac{\beta}{\alpha}$, then Equation 7 may be rewritten as

$$\bar{V} = \frac{3\gamma}{(1 + \gamma)^2 - \gamma}$$

Thus $\bar{V}$ is dependent only on the ratio of β to α, and becomes a maximum when $\gamma = 1$, i.e., $\alpha = \beta$. Therefore the number of VTEs would be a maximum when $P_A = P_B = \frac{1}{2}$.

The Mean Number of VTE*s Preceding* A *and* B *Responses,* $\bar{V}_A$ *and* $\bar{V}_B$

Separate consideration of the mean number of VTEs preceding an A and B choice yields the following results:

$$\bar{V}_A = \frac{2\alpha\beta}{(\alpha + \beta)^2 - \alpha\beta} + \frac{\beta}{\alpha + 2\beta} \qquad [8a]$$

$$\bar{V}_B = \frac{2\alpha\beta}{(\alpha + \beta)^2 - \alpha\beta} + \frac{\alpha}{2\alpha + \beta} \qquad [8b]$$

Since $\frac{\beta}{\alpha + 2\beta}$ and $\frac{\alpha}{2\alpha + \beta}$ may be rewritten as $\frac{1}{\frac{\alpha}{\beta} + 2}$ and $\frac{1}{\frac{\beta}{\alpha} + 2}$ respectively, it can be seen that on the average there would be fewer VTEs preceding the response which is dominant at any given moment, i.e., if $P_A > P_B$, $\bar{V}_A < \bar{V}_B$.

The Time Distribution of Final Choice

It is possible to determine all the moments of the time distribution of final responses. Here, however, consideration will be limited to the mean latency, $\bar{L}$, of all responses and the mean latencies for A and B responses taken separately, $\bar{L}_A$ and $\bar{L}_B$ respectively.

The Mean Latencies for A *and* B *Responses,* $\bar{L}_A$ *and* $\bar{L}_B$

Let $P(a, t)$ be the probability that, at time t, no two consecutive a's or b's have appeared, and that the last implicit response was an a. Let $P(a, t; n)$ be the probability that, at Line t, no two consecutive a's or b's have appeared, and that the last implicit response was an a, and also that there have been exactly n implicit responses. Thus

$$P(a, t) = \sum_{n=1}^{\infty} P(a, t; n)$$

To determine $P(a, t; n)$, Equation 2 and the method employed to find P_A are combined.

Let $P(a; n)$ be the probability that a sequence of n events ends with an a, no two consecutive a's or b's having occurred. Clearly,

$$P(a;1) = \frac{\alpha}{\alpha+\beta},$$

$$P(a;2) = \frac{\alpha\beta}{(\alpha+\beta)^2},$$

$$P(a;3) = \frac{\alpha^2\beta}{(\alpha+\beta)^3}, \text{ etc.}$$

these probabilities being respectively associated with the sequences; a, ba, aba, etc.

Now $P(a, t; n) = P(n, t)$. $P(a; n)$, and Equation 2 gives $P(n, t)$, so that

$$\begin{aligned} P(a, t; 1) &= P(1, t)\cdot P(a; 1) \\ &= (\alpha+\beta)te^{-(\alpha+\beta)t}\cdot\frac{\alpha}{\alpha+\beta} \\ &= \frac{\alpha te^{-(\alpha+\beta)t}}{1!} \end{aligned}$$

$$\begin{aligned} P(a, t; 2) &= \frac{(\alpha+\beta)^2t^2e^{-(\alpha+\beta)t}}{2!}\cdot\frac{\alpha\beta}{(\alpha+\beta)^2} \\ &= \frac{\alpha\beta t^2e^{-(\alpha+\beta)t}}{2!} \end{aligned}$$

Similarly

$$P(a, t; 3) = \frac{\alpha^2\beta t^3e^{-(\alpha+\beta)t}}{3!}$$

etc. Hence

$$\begin{aligned} P(a, t) &= \sum_{n=1}^{\infty} P(a, t; n) \\ &= \frac{\alpha te^{-(\alpha+\beta)t}}{1!} + \frac{\alpha\beta t^2e^{-(\alpha+\beta)t}}{2!} + \frac{\alpha^2\beta t^3e^{-(\alpha+\beta)t}}{3!} + \cdots \end{aligned}$$

which, upon simplification, gives

$$P(a, t) = e^{-(\alpha+\beta)t}\left[\left(\frac{e^{t\sqrt{\alpha\beta}}+e^{-t\sqrt{\alpha\beta}}}{2}-1\right) + \sqrt{\frac{\alpha}{\beta}}\left(\frac{e^{t\sqrt{\alpha\beta}}-e^{-t\sqrt{\alpha\beta}}}{2}\right)\right] \quad [9a]$$

Similarly it may be determined that

$$P(b, t) = e^{-(\alpha+\beta)t}\left[\left(\frac{e^{t\sqrt{\alpha\beta}}+e^{-t\sqrt{\alpha\beta}}}{2}-1\right) + \sqrt{\frac{\beta}{\alpha}}\left(\frac{e^{t\sqrt{\alpha\beta}}-e^{-t\sqrt{\alpha\beta}}}{2}\right)\right] \quad [9b]$$

Now

$$\begin{aligned} \bar{L}_A &= \int_{t=0}^{\infty} P(a, t)\alpha t\,dt \bigg/ \int_{t=0}^{\infty} P(a, t)\alpha\,dt \\ &= \frac{2(\alpha+\beta)}{(\alpha+\beta)^2-\alpha\beta} + \frac{\beta}{(\alpha+\beta)(\alpha+2\beta)} \quad [10a] \end{aligned}$$

and similarly

$$\bar{L}_B = \frac{2(\alpha+\beta)}{(\alpha+\beta)^2-\alpha\beta} + \frac{\alpha}{(\alpha+\beta)(2\alpha+\beta)} \quad [10b]$$

By the same kind of argument it may be demonstrated that the mean latency for all responses, $\bar{L}$, is given by

$$\bar{L} = \frac{2(\alpha+\beta)^2+\alpha\beta}{[\alpha+\beta][(\alpha+\beta)^2-\alpha\beta]} = \frac{2}{\alpha+\beta} + \frac{3\alpha\beta}{[\alpha+\beta][(\alpha+\beta)^2-\alpha\beta]} \quad [11]$$

Returning to Equations 10a and 10b it can be seen that $\frac{\beta}{(\alpha+\beta)(\alpha+2\beta)}$ and $\frac{\alpha}{(\alpha+\beta)(2\alpha+\beta)}$ may be written as $\frac{1}{(\alpha+\beta)\left(\frac{\alpha}{\beta}+2\right)}$ and $\frac{1}{(\alpha+\beta)\left(\frac{\beta}{\alpha}+2\right)}$ respectively. Thus the dominant response will, on the average, have a shorter choice time than the other, i.e., if $P_A > P_B$, $\bar{L}_A < \bar{L}_B$.

In order to compare the theoretical response time distribution to observed data, the probability $P(0, t)$ of no final response having occurred by time t is also given. This is clearly

$$P(0, t) = P(o, t) + P(a, t) + P(b, t)$$

$P(o, t)$ is given by Equation 3 and $P(a, t)$ and $P(b, t)$ by Equations 9a and 9b so that, upon some simplification,

$$P(0, t) = e^{-(\alpha+\beta)t}\left[(e^{\sqrt{\alpha\beta}t} + e^{-\sqrt{\alpha\beta}t} - 1) + \frac{\alpha+\beta}{2\sqrt{\alpha\beta}}(e^{\sqrt{\alpha\beta}t} - e^{-\sqrt{\alpha\beta}t})\right] \quad [12]$$

The Model and Descriptors of the Second Kind

At present, it is only possible to advance some speculations concerning variables such as "degree of confidence" in the correctness of a given choice. Nevertheless, it seems worth considering these since there appears to be a definite relation between the second kind of descriptor and the more conventional indices of choice behavior. Henmon (1911), whose paper will be considered in more detail later, showed that choices regarded by an S with confidence are generally quicker and more accurate than others. This result was demonstrated in a psychophysical discrimination situation where a definite correct choice existed.

There seem to be two possible ways in which "confidence" might be attributed to a particular choice. The first of these involves some classification of the various sequences of implicit responses preceding a final choice. For example, sequences which involve no vacillation at all, such as *aa*, or *bb*, might be regarded as "more confident" than sequences involving a large number of vacillations, such as *abababaa*. It will be shown that this kind of "confident" sequence has the properties required by Henmon's data.

For, suppose A be the correct and B the incorrect choice in a psychophysical situation, then generally speaking one would expect $\alpha > \beta$. The probability of the sequence *aa* would be $\frac{\alpha^2}{(\alpha+\beta)^2}$ and the probability of *bb*, $\frac{\beta^2}{(\alpha+\beta)^2}$. Hence, the probability, P_C, of being correct for this type of confident "choice," i.e., choosing A, is given by

$$P_C = \frac{\alpha^2}{\alpha^2+\beta^2} \quad [13]$$

Comparing this probability with the overall probability of an A response, P_A given by Equation 6a,

$$P_C - P_A = \frac{\alpha^2}{\alpha^2+\beta^2} - \frac{\alpha^2(\alpha+2\beta)}{[\alpha+\beta][(\alpha+\beta)^2-\alpha\beta]} = \frac{\alpha^2\beta^2(\alpha-\beta)}{[\alpha^2+\beta^2][\alpha+\beta][(\alpha+\beta)^2-\alpha\beta]} \quad [14]$$

Clearly, Equation 14 is positive when $\alpha > \beta$ and hence $P_C > P_A$.

Since for these "confident" responses only two implicit responses occur before a final choice, it is clear that their mean response time is shorter than the over-all average response time. This approach consists essentially in equating "degree of confidence" with some function of the reciprocal of the number of VTEs preceding the final choice.

The second suggested approach to judgmental confidence is based upon the fact that these appraisals of a response, under normal instructions, follow after the response itself. Degree of confidence, therefore, might be associated with implicit responses continuing to occur after an overt choice response has occurred. If, after an A response has been made, a further a occurs in the time before the statement of confidence is produced, this might be taken to lead to greater confidence than if nothing or a b appeared. Indeed, it might be possible to develop a model for the distribution of the times between making the primary choice response and giving an estimate for degree of confidence from this kind of assumption.

Other approaches to the second kind of descriptor are undoubtedly possible within the present scheme. The important point is that it is possible to test these various hypotheses quite easily. They each predict how often a given level of confidence would be employed. Also the expected distribution of descriptors of the first kind associated with each level of confidence can be determined.

The Agreement between the Properties of the Model and Empirical Data

The principal aim of this paper is to show that a set of very simple assumptions can be used to derive relations which might be expected among the variables observed in a choice situation. In an exposition of this kind it is not possible to examine, in any detail, the success of the model in describing the results of experiments which are relevant. For one thing, only the particular case arising when $K = 2$ has been presented, whereas in practice it may be more profitable to treat K as a parameter. Also, the argument so far presented is concerned with the events supposed to occur at a single experimental trial. The manner in which the model is applied to experimental data based upon a number of trials will depend very much upon the way in which separate trials resemble one another. There may be actual variations in stimulus conditions from trial to trial, or there may be a direct dependence of later upon earlier trials, as in learning experiments. For this reason, consideration of quantitative evidence will be mainly confined to an experiment by Henmon (1911), in which the conditions under which individual trials were conducted closely resemble one another and where it can reasonably be assumed that there are no systematic changes in an S's behavior. This data can therefore be regarded as appropriate for testing the model without there being any need to make further special assumptions. However, before examining Henmon's results, it seems worthwhile to exhibit the manner in which the model seems to match empirical evidence about choice behavior in general.

In effecting a general appraisal of the model, one is hindered by the general lack of individual results in the experimental literature. For reasons which cannot be examined here it seems preferable to test hypotheses about functional relations upon *individual* data. A brief argument for this point of view has been presented by Bakan (1955) and for the study of learning behavior by Audley and Jonckheere (1956). The reader is referred to these papers for further details. However, irrespective of the stand taken on this question, it is clear that the present model is concerned with *individual* results and that such results are not generally available. For this reason, the following comparison of the model with experimental evidence is largely qualitative, although, given appropriate data, quantitative comparisons would have been possible.

Psychophysical Discrimination Situations

In considering results from psychophysical experiments, say using the constant method, it is necessary to consider separately the comparison of each variable with the standard. This is so because no assumptions have thus far been made about the relation between stimulus and response variables. In spite of this, some general predictions can be made.

Consider the results obtained from the comparison of the standard with a particular variable stimulus. In this comparison, it can be supposed that the responses A and B refer to the respective statements "the variable is greater than the standard" and "the variable is smaller than the standard." α will clearly be a monotonically increasing function of the magnitude of the variable, and β a monotonically decreasing function of the same magnitude. At the PSE, $\alpha = \beta$. Within limits, and certainly for a range of stimuli close to the PSE, $(\alpha + \beta)$ can be assumed to be approximately constant. This supposition is not crucial, but simplifies the ensuing argument.

Relation of Judgment Time to the Perceived Distance between Stimuli

Equation 11 gives the mean choice time as a function of α and β. This can be rewritten in the following way:

$$\bar{L}=\frac{2}{\alpha+\beta}+\frac{3}{[\alpha+\beta]\left[\frac{(\alpha+\beta)^2}{\alpha\beta}-1\right]} \qquad [15]$$

If $(\alpha+\beta)$ is approximately constant, $\bar{L}$ will depend principally upon the product of the parameters, $\alpha\beta$. Thus $\bar{L}$ will have a maximum when $\alpha=\beta$. From Equation 6a it can be seen that the point, $\alpha = \beta$, also defines the PSE, since for these parameter values $P_A = P_B = 0.5$. It can be seen that decision time will therefore rise monotonically up to the PSE and then decrease monotonically beyond the PSE. For the range and distribution of stimuli employed in most psychophysical studies, the decrease in decision time upon either side of the PSE will be, according to the model, approximately symmetrical. These properties are in agreement with empirical data, as for example summarized by Guilford (1954).

Even where the S is allowed three categories of response, it is the boundaries between these categories which show peak decision times (Cartwright, 1941). This would be expected if a further parameter be used to characterize "equal" or "doubtful" responses. It would be of great interest to determine whether, in fact, a further response parameter is required when a third response category is permitted. Almost by definition, the response "doubtful" implies that no decision has been reached by a certain time. Such responses would then appear to be best described by the time which the S is willing to spend in attempting to come to a decision. This would make the range of stimuli over which judgments of "doubtful" are made depend only indirectly upon differential sensitivity. The readiness of the S to continue attempting to arrive at a definite answer would also play an important role. This is in accord with the generally accepted view of the use of a third category, e.g., Woodworth (1938), Guilford (1954). On the other hand, a parameter to specify judgments of "equality" may still be required. This would allow for a time determined "doubtful" judgment of the kind discussed above, but would also introduce a true "equals" category. This would enable an analysis of the third category to be carried out in accordance with the suggestions of Cartwright (1941) and George (1917).

The Relation between Confidence, Decision Time and Perceived Distance between Stimuli

The exact nature of the relations between the variables considered in this section, will depend upon whether stimulus conditions are the same for all trials. Nevertheless, some general predictions can be advanced.

Here, "degree of confidence" will be equated with some function of the reciprocal of the number of VTEs preceding a final choice. The number of VTEs can, of course, range from zero to infinity. Generally speaking, confidence is rated upon some scale from zero to unity. Let C, be the degree of confidence associated with a given choice, and, V, the number of VTEs preceding this choice act. Determining a suitable relation between C and V would, in fact, be one of the experimental problems suggested by the present approach. For the moment, however, it will be assumed that,

$$C = \frac{1}{V+1} \qquad [16]$$

so that when $V = 0$, $C = 1$; and when $V = \infty$, $C = 0$.

It will be recalled from the section concerned with VTEs that the mean number of these will, when $K = 2$, be two less than the number of implicit responses preceding a final choice. Now it can easily be demonstrated, using Equation 1c, that the mean choice time when n implicit responses occur, T_n, is given by

$$T_n = \frac{n}{\alpha + \beta} \qquad [17]$$

Whence, since $V = n - 2$, and because n is eliminated from Equation 17, it is possible to express the mean choice time $\bar{T}$, as a function of V, given by

$$\bar{T} = \frac{V+2}{\alpha + \beta} \qquad [18]$$

Substituting for V from Equation 16 and adding an arbitrary constant, T_0, for the minimum choice time possible,

$$\bar{T} = \frac{1}{(\alpha+\beta)C} + \frac{1}{\alpha+\beta} + T_0. \qquad [19]$$

This hyperbolic function is in agreement with experimental determinations of the relation between confidence and judgment time, e.g., see again Guilford (1954).

If the stimulus conditions are varied between different sets of trials, as for example in the constant method discussed in the previous section, general conclusions are again possible. For in discussing Equation 7, it was shown that the mean number of VTEs depends only upon the ratio of α to β. Again assuming that $(\alpha + \beta)$ is approximately constant, V would be a roughly symmetrical function of the magnitude of the variable, having a maximum at the PSE. Thus the average degree of confidence, $\bar{C}$, would be a roughly **U** shaped function having a minimum at the PSE. Since choice time has been shown to have a maximum at the PSE and to decrease upon either side of this point, $\bar{C}$ and $\bar{T}$ would again vary inversely. This agrees with experimental data (see Guilford, 1954).

Preference and Conflict Situations

In this kind of situation, a number of objects are paired and the subject makes a choice indicating the preferred object of each pair. For any given pair of objects, say A and B, the parameters α and β can be taken to represent some measure of preference for A and B. Because there are a number of objects, it is more convenient to label the r objects presented to the subject as X_i, and to let the parameter associated with a kind of "absolute preference" for each, be α_i $(i = 1, 2, \cdots r)$. The α and β of the equations will now be replaced by, say α_j and α_k, for the comparison of the ith and jth objects, X_j and X_k. This, of course, is to make the very strong assumption that the α_i's are independent of the particular comparison in which they are involved. This assumption could be readily tested by using the model appropriately, and is accepted here only in order to simplify notation. The results of the following argument would be qualitatively the same, even if there were in fact, contextual effects peculiar to each comparison.

Variation in choice time among different comparisons. The set of r objects, on the basis of a paired comparison technique, can usually be ranked. Let i be an individual's ranking of an object, so that we may write $X_1 > X_2 > \cdots > X_i > X_{i+1} > \cdots > X_r$, meaning X_i is preferred to X_2 and so on. This means that $\alpha_1 > \alpha_2 > \cdots > \alpha_i > \alpha_{i+1} > \cdots > \alpha_r$. Consider any pair of parameters, say α_j and α_k, and let these be the α and β of the earlier equations. Then the mean choice time is given by Equation 11, and this can now be rewritten as

$$\bar{L}_{(j,k)} = \frac{2}{\alpha_j + \alpha_k} + \frac{3\alpha_j\alpha_k}{[\alpha_j + \alpha_k][(\alpha_j + \alpha_k)^2 - \alpha_j\alpha_k]} \qquad [20]$$

Clearly $\bar{L}_{(j,k)}$ depends upon two things; the sum of the parameters $(\alpha_j + \alpha_k)$ and, secondly, the product of the parameters, $\alpha_j\alpha_k$. Other things being equal, the choice time will decrease as $(\alpha_j + \alpha_k)$ increases. Again, with $(\alpha_j + \alpha_k)$ constant, $\bar{L}_{(j,k)}$ will increase with the product, reaching a maximum when $\alpha_j = \alpha_k$. Choice time will therefore (*a*) depend upon the general level of preference for objects, being quicker for preferred objects, (*b*) will be quicker the greater the difference in preference for the two paired objects. This in agreement with experimental finding, e.g., for children choosing among liquids to drink, Barker (1942), for aesthetic preferences, Dashiell (1937).

It will be interesting to determine how far the assumption of an absence of contextual effects can be maintained. If the assumption turns out to be approximately true, then the parameters, α_i, would provide a means of scaling the stimulus objects for a given individual. In essence, such an approach would resemble that adopted by Bradley and Terry (1952), but would have the added advantage that the scale values would have an absolute rather than a relative basis, so that the scale values should be unaffected by the inclusion of new comparisons.

Number of VTE*s for different comparisons.* It was shown, in discussing Equation 7, that the mean number of VTEs in a given situation, depends entirely upon the ratio of α to β. Using the present notation this would be the ratio of α_j to α_k, for objects X_j and X_k. The number of VTEs has a maximum when $\alpha_j = \alpha_k$, and decreases as the values of the parameter become more disparate. Thus the number of VTEs should depend entirely upon the differences in preference and not upon the general level of preference for the two paired objects. Thus for adjacent objects, X_i and X_{i+1}, the number of VTEs before a final choice will not rise with choice time as one proceeds from preferred to nonpreferred objects. This is slightly complicated by differences in "preference distance" between adjacent objects, but the prediction is again found to be in agreement with experimental evidence, e.g., see Barker (1942).

Learning in choice situations. It is in considering learning behavior that the need for individual results is greatest (Audley & Jonckheere, 1956). The full advantages of the present approach to response variables can only be gained by incorporating the assumption in a stochastic model for learning. The way in which this might be contrived, when $K = 1$, has already been outlined and illustrated elsewhere (Audley: 1957, 1958). On the whole, therefore, the experimental literature does not provide results in a way which enable the predictions of the model to be falsified, even at a qualitative level. The most that can be done here is to show that the predictions might well be good approximations to the properties of learning data.

Given a particular theory of learning it would, of course, be possible to anchor the theory more closely to response variables by identifying the parameter of the choice model with an appropriate theoretical construction.

The properties of the model and simple learning behavior. Consider, for example, learning in a simple two-choice situation. Let α be associated with A, the correct response, and β with B, the incorrect response. The way in which α and β vary with reward and punishment is naturally a matter for investigation and would certainly condition the form of the prediction which would be made. Nevertheless, it is not unreasonable to assume that α will be some monotonic increasing function, and β some monotonic decreasing function of practice and of punishments and rewards.

Let it be supposed that the S has at first a strong tendency to produce the incorrect choice, i.e., α is small relative to β. Consider, firstly, what might be expected to happen to the over-all latency $\bar{L}$, and the latencies of A and B, $\bar{L}_A$ and $\bar{L}_B$ respectively. In discussing Equations 10a and 10b it was shown that the dominant response, on the average, will have the shorter choice time. Thus in the first place it will be expected that $\bar{L}_A$ will be greater than $\bar{L}\cdot$ until the probability of making the correct choice, P_A, reaches and exceeds 0.5, when $\bar{L}_A$ will be generally shorter than $\bar{L}_B$.

All of the latencies are dependent upon two factors, the sum $(\alpha + \beta)$ and the ratio of α to β. The over-all latency, $\bar{L}$, if $(\alpha + \beta)$ remains constant, will rise to a maximum until $P_A = P_B = 0.5$ (i.e., $\alpha = \beta$) and then fall again. Superimposed upon this rise and fall will be the influence of $(\alpha + \beta)$, and if the levels of, say punishment and reward, are such as to disturb the constancy of this quantity, then there will be an accentuation or flattening of the curve of latency as a function of practice. The monotonic decline in response latencies observed when an S is introduced into a learning situation for the first time does not counter this prediction. For, then, it is to be expected that $(\alpha + \beta)$ will be initially small and the effect of increasing α, and, hence, $(\alpha + \beta)$ will be reinforced by the growing difference in magnitude between α and β. In original learning, therefore, the two factors work together and produce the monotonic decrease in latency.

The number of VTEs, from Equation 7, is seen to be a function only of the ratio of α to β. Thus VTEs would be expected to rise to a maximum until $\alpha = \beta$, i.e., $P_A = P_B = 0.5$, and the decline.

These predictions are probably only applicable to the very simple two-choice situations so far considered. For discrimination studies, the problem is complicated by the way in which the relevant cues are being utilized by the organism and there is no point in reviewing the controversy over this matter. It does however seem worthwhile pointing out that, in discrimination behavior, it is very probable that there appears something like the problem of the use of the third category in psychophysical procedures. That is, a distinction seems to be necessary between, on the one hand, a definite act of choice and, on the other hand, behavior which occurs simply because something has to be done in the situation. This speculative point is raised because the size of the parameters may exert an influence upon behavior in two ways. Firstly, by determining the probability of making a particular response when a "true" choice is made and, secondly, by determining the probability that a "true" choice is made.

Henmon's experiment. The experiment conducted by Henmon (1911) is of particular interest, because it provides data from individual *S*s, in a situation where stimulus conditions can be assumed to be fairly constant from trial to trial. The observations, therefore, are important for any model concerned with the properties of choice behavior.

Henmon required *S*s, in each of 1,000 trials, to decide whether one of two horizontal lines was longer or shorter than the other. The lengths of the lines were always 20 mm and 20.3 mm respectively. In addition, *S*s were instructed to indicate their confidence in each judgment.

The model is qualitatively in agreement with Henmon's data, except in two things. Firstly, although average choice time for wrong responses is larger than that for correct choices, as predicted by the model, the wrong responses are relatively quicker in each category of confidence. The second qualitative difference appears in examining accuracy as a function of time. There is some indication for some *S*s that although there is a general decline in accuracy with longer choice times, again predicted by the model, there is also a slight rise in accuracy in going from very short to moderately short choice times. It is possible that both of these differences may be accounted for by a suitable analysis of judgments of confidence about which only a few speculations have been advanced in the present paper. The important point, it seems to the author, is that the general stochastic model is capable of dealing with this kind of issue, rather than that it succeeds in all details at the present time.

Henmon gives the distribution of all choice times for each individual. Since this can also be derived from the model, a comparison of the two distributions should give further indications as to the adequacy of the present approach to choice behavior. In testing the goodness of fit of the model in this matter, it would be usual to estimate the parameters from the distribution of choice times alone. However, it was decided that perhaps a stronger case could be made out if the only time datum used to estimate the parameters was the mean latency. Two equations are of course required if values of α and β are to be determined, and P_A, the probability of a correct response, was chosen for the second. Accordingly the present estimates are based upon Equations 6a and 11.

There must, of course, be some minimum response time before which no response can occur. This is not easy to determine from Henmon's tables of results, because the data are already grouped in intervals of 200 milliseconds. For this reason, the minimum possible time was estimated in the following way. For various assumed minimum times, estimates of α and β were determined, and the theoretical distribution of choice times computed. The value leading to the best fit was then adopted. This is not entirely a satisfactory procedure, but with K assumed to be 2, and with no

TABLE 1

Subject Bl			Subject Br		
Time interval in milliseconds	Observed Frequency	Expected Frequency	Time interval in milliseconds	Observed Frequency	Expected Frequency
100–	(2)[a]	—	100–299	(2)[a]	—
300–	57	53	300–	350	352
500–	214	229	500–	381	398
700–	220	229	700–	170	165
900–	159	168	900–	65	57
1100–	113	111	1100–	26	19
1300–	85	83	1300–	5	6
1500–	74	48	Above 1500	1	3
1700–	32	30			
1900–	18	20		1000	1000
2100–	11	10			
2300–	8	8			
Above 2500–	7	11			
	1000	1000			

[a] These observations ignored in calculations.

direct indication of the minimum time, it seemed the best available in the circumstances. The results for Henmon's (1911, Table 2, p. 194) *S*s Bl and Br are considered below.

For Bl, the minimum possible time was taken to be about 0.40 sec. On this basis $\alpha = 3.19$ and $\beta = 1.28$, these values referring to a time scale measured in seconds. For Br, the minimum time was taken to be 0.34 sec. giving $\alpha = 6.68$ and $\beta = 4.28$. A comparison of the observed and expected distributions of response times is given in Table 1. The agreement between model and data seems to be reasonably good.

Concluding Remarks

On the whole, there is a certain looseness in the way in which many contemporary theories and even local hypotheses are linked to observed response variables. It seems worthwhile, therefore, to try to determine whether these variables might not be related to one another by relatively simple laws which operate in most choice situations. In this way, not only are descriptions of choice behavior considerably simplified, but better ways of formulating and testing theories are suggested. The model itself is naturally also a theory about a certain aspect of behavior, and as such needs to be tested.

In this presentation of the general stochastic model the intention is to indicate the potentialities of the approach, rather than to make specific tests of the case arising when $K = 2$. It is not to be expected that the two simple assumptions will alone account for the relations existing between response variables in a wide diversity of situations. Each situation will undoubtedly have certain unique conditions which have to be taken into account. But the model does seem to share certain important properties with choice behavior and therefore it appears to be a reasonable initial working hypothesis. It can be tested in great detail against data, and the parameters are of a kind which could be identified with either psychological or physiological constructs.

Methods of estimating parameters and statistical tests of goodness of fit will be discussed elsewhere. For the present model, neither of these procedures involves any novel problems. For example, given the probability of occurrence of one of the alternative responses and the over-all mean response time, Equations 6 and 11 may be easily solved to give the appropriate parameter values.

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