BRACHISTOCHRONE PROBLEM

A PROJECT REPORT
SUBMITTED IN COMPLETE FULFILMENT OF THE REQUIREMENTS
FOR THE AWARD OF DEGREE
OF
BACHELOR OF TECHNOLOGY
IN
ELECTRICAL ENGINEERING

SUBMITTED BY :
Aditya Raj
(2K20/B15/15)
Kanishk Yadav
(2K20/B15/37)
UNDER THE SUPERVISION OF:
Dr. Ashish Kumar

ELECTRICAL ENGINEERING

DELHI TECHNOLOGICAL UNIVERSITY

(FORMERLY DELHI COLLEGE OF ENGINEERING)

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CANDIDATE’S DECLARATION

We, (ADITYA RAJ (2K20/B15/15) and KANISHK YADAV (2K20/B15/37)) students of Bachelor of Technology (Electrical Engineering) hereby declare that the dissertation titled “BRACHISTOCHRONE PROBLEM” which is submitted by us to the Department of Applied Physics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Bachelor of Technology, is original and is not copied from any source without proper citation. This work has not previously formed the basis of any award of any degree, diploma associateship, fellowship or any other similar title or recognition.

PLACE: DELHI KANISHK YADAV(2K20/B15/37)

ADITYA RAJ(2K20/B15/15)

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CERTIFICATE

I, hereby certify that the project dissertation named "BRACHISTOCHRONE PROBLEM", which is submitted by Aditya Raj(2K20/B15/15) (Electrical Engineering) and Kanishk Yadav (2K20/B15/37) (Electrical Engineering), Delhi Technological University, Delhi in complete fulfilment of the requirement for the award of the degree of the Bachelor of Technology, is a record of the project work carried out by the students under my supervision. To the best of my knowledge, this work has not been submitted in part or full for any degree or diploma to this University or elsewhere.

Place : Delhi Dr. Ashish Kumar

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ABSTRACT

This report is a study based on a problem first posed and solved by John Bernoulli and later investigated and solved by Newton. The problem is formally known as the brachistochrone curve or the curve of shortest descent and it has a very rich and fascinating history. In this report, we have first of all studied the problem and its solution.We have also plotted graphs for different curves in MATLAB which have really seemed to verify the result that we expected mathematically. Overall, our aim for this project was to learn about the problem and draw experimental graphs to understand it in depth.

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INTRODUCTION

The classical problem in calculus of variation is the so-called brachistochrone posed (and solved) by Bernoulli in 1696. Given two points A and B, find the path along which an object would slide (disregarding any friction) in the shortest possible time from A to B, if it starts at A in rest and is only accelerated by gravity.

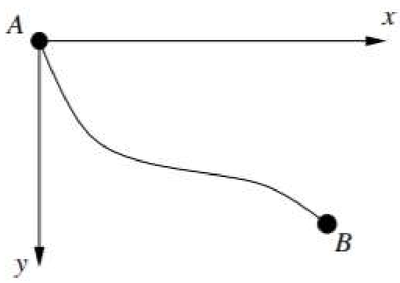


FIGURE 1. Sketch of the brachistochrone problem.

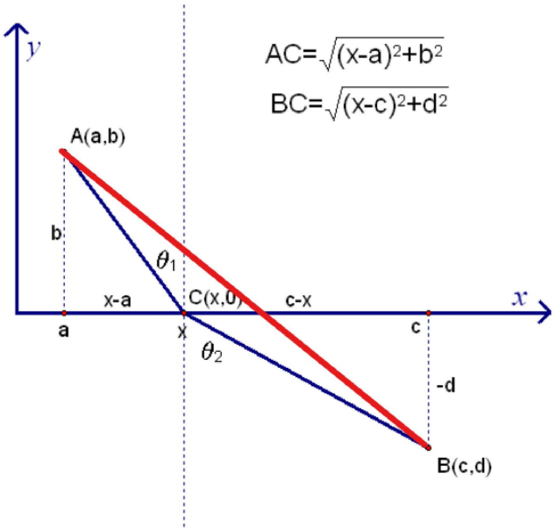
For matters simple, we will consider the point A be on origin of our coordinate axis and take the downward direction to be positive and also we will denote the path along which the object moves as s(x)

Since the object is only under the influence of gravity it's energy is converted, that is, the sum of potential energy and kinetic energy is the same at any point along the trajectory of the object. Thus the energy difference between any two points A and B is 0.

$$EA - EB = 0 = \Rightarrow EA = EB = \Rightarrow KA + UA = KB + UB$$

INTUITION

Intuitively it seems the answer to this problem is a straight line joining A to B, because it is the shortest path between A and B. The solution to this problem is not a straight line, rather a cycloid. It seems quite counter-intuitive because we believe what our eyes see and we have seen in the real world that the minimum time it takes to travel from point A to point B is the straight line joining these two points. It’s true but in this case, gravity is playing a major role in determining the path. The steeper the curve is, the faster any particle will accelerate and hence will reach faster. A similar example is refraction of light, the reason light refracts is to minimise its time and hence the path travelled by light ray is not a straight line.



We wish to find the point C=(x,0) on the

x axis that minimizes the time of travel of a light beam along the path A to C to B, assuming its velocity above the x axis is v1, and its velocity below the x axis is v2. Since

$$time = \frac{distance}{velocity}$$

$$T(x) = \frac{\sqrt{(x-a)^2+(x-b)^2}}{v1} + \frac{\sqrt{(x-c)^2+(x-d)^2}}{v2}$$

If we take the derivative and set it equal to 0, the equation can be rewritten as

$$\frac{x-a}{v_1\sqrt{(x-a\,^2+b^2}} + \frac{x-c}{v_2\sqrt{(x-c\,^2+d^2}} = 0$$

Where

$$sin\theta_1 = \frac{x-a}{\sqrt{(x-a\,^2+b^2}} \ \& \ sin\theta_2 = \frac{c-x}{\sqrt{(x-c)^2+d^2}}$$

The above equation can be rewritten as

$$\frac{sin\theta_1}{v_1} - \frac{sin\theta_2}{v_2} = 0$$

SOLUTION

Assuming the object starts from rest at A, we have.

$$1/2m(0)^2 + mg(0) \ = \ 1/2\ mv^2 + mg(-y) \ \Rightarrow \ v^2 = 2gy \ \Rightarrow \ v \ = \ \sqrt{2gy}$$

We can write the velocity of the object as

$$v = \frac{ds(x)}{dt} = \frac{ds(x)}{dx} . \frac{dx}{dt}$$

Therefore,

$$\sqrt{2gy} \ = \ = \frac{ds(x)}{dx} . \frac{dx}{dt} \qquad (1)$$

Using Pythagoras' theorem we can write ds(x) as

$$ds(x) \ = \sqrt{dx^2 + dy^2} \ = \sqrt{dx^2(1 + \frac{dy^2}{dx^2})} \ = \ \sqrt{1 + (\frac{dy}{dx})^2 .dx}$$

Substituting into equation (1),

$$\sqrt{2gy} = \sqrt{1 + (\frac{dy}{dx})^2 .dx} \quad \Rightarrow \text{dt} = \frac{\sqrt{1+(\frac{dy}{dx})^2}}{\sqrt{2gy}} \text{dx}$$

Taking the integral from 0 to x(B)

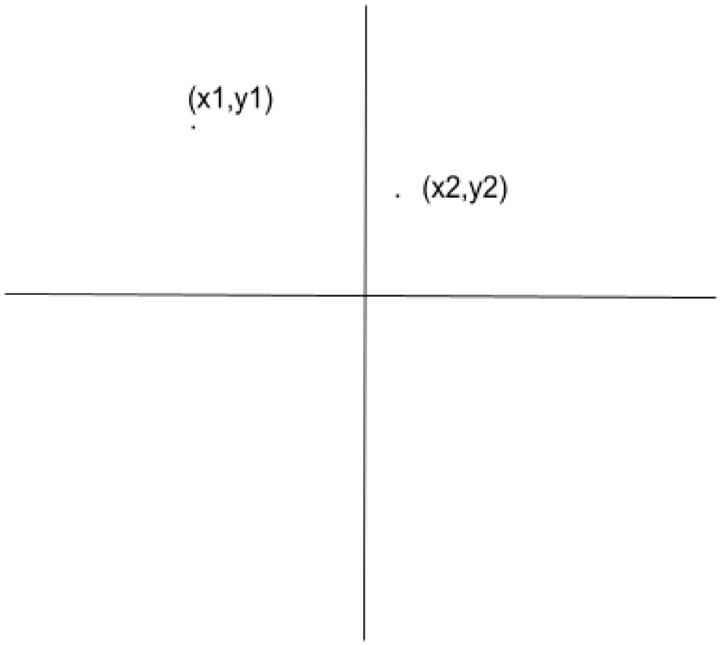
$$\int_0^{x(B)} dt \ = \ \tau \ = \frac{1}{\sqrt{2g}} . \int_0^{x(B)} \frac{\sqrt{1+(y')^2}}{\sqrt{y}} \text{dx}$$

Since we are looking to minimize this integral we will apply the Euler-Lagrange formalism, writing the integral as

$$\tau \ = \ \int_0^{x(B)} L(y', y)dx$$

with the Lagrangian $L(y', y) \ = \ \frac{\sqrt{1+(y')^2}}{\sqrt{y}}$

Deriving Euler-Lagrange equation



Find y = F(x) such that the functional $I = \int_{x1}^{x2} F(x, y, y')dx$ is stationary.

Boundary condition: y(x_1) = y_1 , y(x_2) = y_2

Suppose y(x) makes I stationary and satisfies the above boundary conditions.

Introduce a function

$\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$

Define $\bar{y}(x) = y(x) + \varepsilon\eta(x)$

$\bar{y}$ represents the family of curves.

Now we have to find a particular $\bar{y}(x)$ which makes

$I(\varepsilon) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')dx$ stationary

$$\frac{dI}{d\varepsilon}\Big|_{\varepsilon=0} = 0$$

$$\frac{d \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')dx}{d\varepsilon}\Big|_{\varepsilon=0} = 0$$

$$\int_{x_1}^{x_2} \frac{\partial}{\partial\varepsilon} F(x, \bar{y}, \bar{y}')\Big|_{\varepsilon=0} dx = 0$$

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial\varepsilon} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial\varepsilon}\right)\Big|_{\varepsilon=0} dx = 0$$

$$\frac{\partial \bar{y}}{\partial\varepsilon} = \eta \ \& \ \frac{\partial \bar{y}'}{\partial\varepsilon} = \eta'$$

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}}\eta + \frac{\partial F}{\partial \bar{y}'}\eta'\right)\Big|_{\varepsilon=0} dx = 0$$

Integrating by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}'}\eta' dx = \frac{\partial F}{\partial \bar{y}'}\int_{x_1}^{x_2} \eta' dx - \int_{x_1}^{x_2} \left(\int \eta'\right)\frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right] dx$$

$$= \frac{\partial F}{\partial \bar{y}'}[\eta]_{x_1}^{x_2} - \int_{x_1}^{x_2} (\eta)\frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right] dx$$

$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}} \eta - (\eta) \frac{d}{dx} \left[\frac{\partial F}{\partial \bar{y}'} \right] \right) |_{\varepsilon=0} dx \ = 0$

$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}} - \frac{d}{dx} \left[\frac{\partial F}{\partial \bar{y}'} \right] \right) . \eta |_{\varepsilon=0} dx \ = 0$

At $\varepsilon = 0,\ \bar{y}(x) = y(x),$ since η is arbitrary

Thus,

$\frac{\partial F}{\partial y} - \frac{d}{dx} \left[\frac{\partial F}{\partial y'} \right]$ =0 [Euler Lagrange equation]

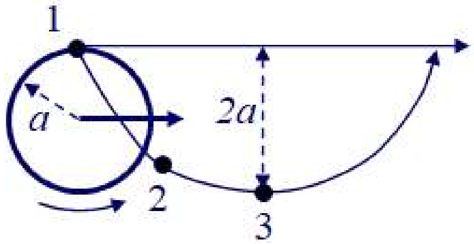
Euler-Lagrange formalism

The Euler-Lagrange equations take the form

$$0 = \frac{d}{dx}\frac{\partial L}{\partial q'_i} - \frac{\partial L}{\partial q'_i}.\ i = 1,\ 2,\ 3,\ ....N$$

with q being the general coordinate and ˙q it's derivative. In our case, it takes the form.

$$0 = \frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y}$$



With $\frac{\partial}{\partial y}\ \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = -\frac{\sqrt{1+(y')^2}}{y^{\frac{3}{2}}}$

$$\frac{\partial}{\partial y'}\ \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = \frac{y'}{(y+(y')^2y)^{1/2}}$$

Thus after taking the full derivative of the above expression, and going through the motions to get the equation in a pleasant form. We finally have.

$$\frac{d}{dx}\frac{y'}{(y+(y')^2y)^{1/2}} = \frac{2y''y-(y')^2-(y')^4}{(y+(y')^2y)^{3/2}}$$

Putting everything together, we finally get a second order differential equation for the path that minimizes the time taken by the object.

$$\frac{2y''y-(y')^2-(y')^4}{(y+(y')^2y)^{3/2}} + \frac{\sqrt{1+(y')^2}}{y^{\frac{3}{2}}} = 0 \quad \Rightarrow \quad 2y''y - (y')^2 - (y')^4 = -(1 + (y')^2)^2$$

$$y'' = -\frac{(1+(y')^2)}{2y}$$

$$\frac{d^2y}{dx^2} = -\frac{(1+(\frac{dy}{dx})^2)}{2y}$$

Euler–Lagrange second derivative equation..

$$y'' = -\frac{(1+(y')^2)}{2y}$$

$1 + 2yy'' + (y')^2 = 0$

Multiplication with y' gives

$y' + 2yy'y'' + (y')^3 = 0$

Now note that the left hand side of this equation is actually the derivative of the function $y + y(y')^2$

Thus it follows that,

$$y + y(y')^2 = C$$

for some constant C > 0. Solving for y' gives

$$y' = \sqrt{\frac{C-y}{y}}$$

With separation of variables, we now obtain,

$$\sqrt{\frac{C-y}{y}}\ dy = dx$$

which can be integrated to

$$x = \int \sqrt{\frac{y}{C-y}} dy + D$$

for some constant D ∈ R. Now we substitute

$$y = Csin^2 t$$

with 0 < t < π/2 and obtain

$$\int \sqrt{\frac{y}{C-y}} dy = \int \sqrt{\frac{Csin^{2t}}{C-Csin^2 t}} 2csintcostdt = 2C \int sin^2 tdt$$

Using the fact that

$$sin^2 t \ = \tfrac{1}{2} - \tfrac{1}{2} cos2t \ ,$$

we obtain

$$\int \ sin^2 t \ dt \ = \int \ \tfrac{1}{2} - \tfrac{1}{2} cos2t \ dt \ = \tfrac{t}{2} - \tfrac{1}{4} sin2t$$

Thus we get (in the variable t)

$$x(t) \ = Ct - \tfrac{C}{2} sin2t + D$$

with

$$y(t) \ = Csin^2 t \ = \tfrac{C}{2} - \tfrac{C}{2} cos2t$$

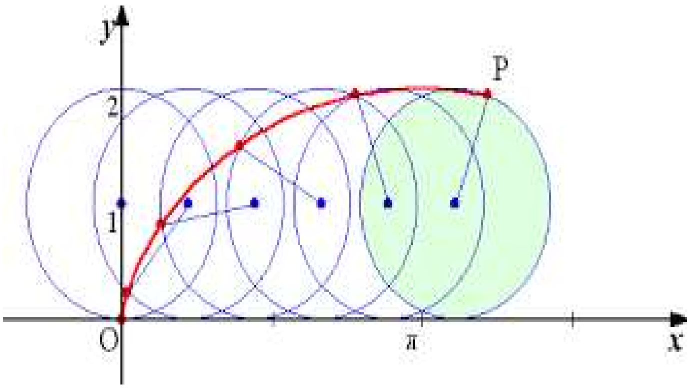
Since the path has to pass through the point (0, 0), it follows that D = 0. Thus the path can be parametrized as

$$t \rightarrow \left(\frac{x(t)}{y(t)} \right) = C \left(\frac{t - \frac{1}{2} sin2t}{\frac{1}{2} - \frac{1}{2} cos2t} \right)$$

with C > 0 being such that it passes through the point (a, b). Note that this path is actually a Cycloid.

CYCLOID

In geometry, a cycloid is a curve traced by a point on a circle as it rolls along a straight line without slipping. A cycloid is a specific form of trochoid and is an example of a roulette, a curve generated by a curve rolling on another curve.



The cycloid, with the cusps pointing upward, is the curve of fastest descent under constant gravity (the brachistochrone curve). It is also the form of a curve for which the period of an object in simple harmonic motion (rolling up and down repetitively) along the curve does not depend on the object's starting position (the tautochrone curve).

The cycloid through the origin, with a horizontal base given by the x-axis, generated by a circle of radius r rolling over the "positive" side of the base (y ≥ 0), consists of the points (x, y), with

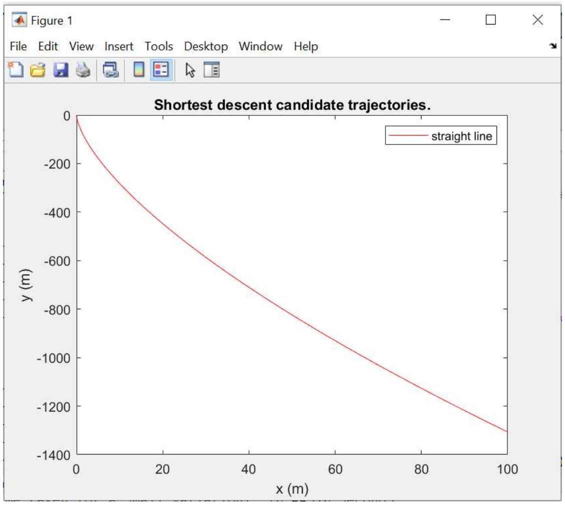
$$x = r(t - \sin t)$$

$$y = r(1 - \sin t)$$

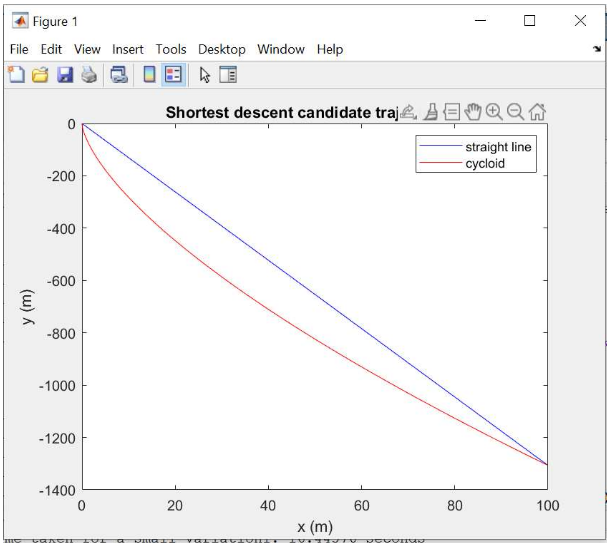
RESULTS

The plot of the results below was obtained with the following initial conditions;

$$y(0) = -0.001,\ \frac{dy}{dx}\Big|_{x=0} = -10000,\ x(0) = \frac{2}{3}\frac{y(0)}{y'(0)}\ and\ x(E) = 100$$



The time taken for this particular trajectory(cycloid) was approximately 16.3532 seconds. As a reliability test to see that indeed this is the trajectory the object can take while minimizing the time taken for traversal between A and B, I sought out to compare it with a linear curve; A straight line connecting the initial point and endpoint, and compute the time taken for such a curve. Similar initial conditions were considered, the following plot was produced.

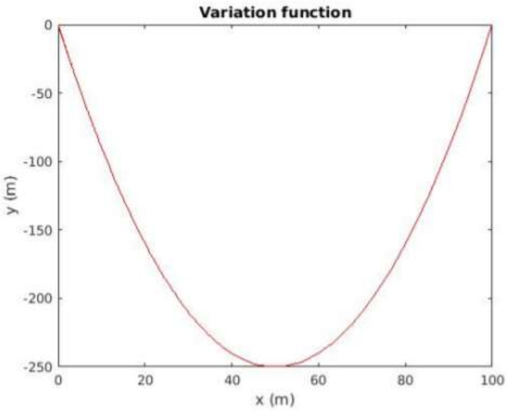


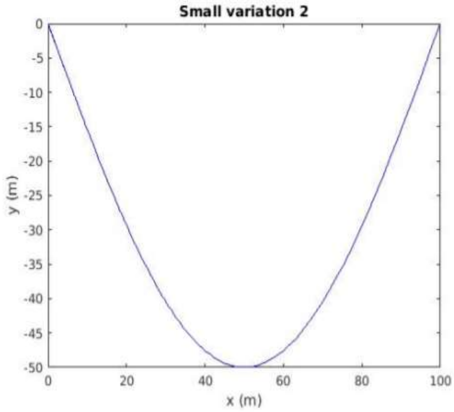
The time taken for the object to traverse along the straight-line trajectory from the initial point to the endpoint was roughly 16.3648 seconds. This is more than the time taken by a cycloid connecting the initial point and the endpoint by 0.0116 seconds!

Next, we considered a small variation to the curve of shortest descent y, since the cycloid is assumed to be the curve of shortest descent any variation added to it should result in a curve increasing the time taken. We will consider a new function $y''(x)$, such that

$y''(x) = y(x) + \eta(x)$

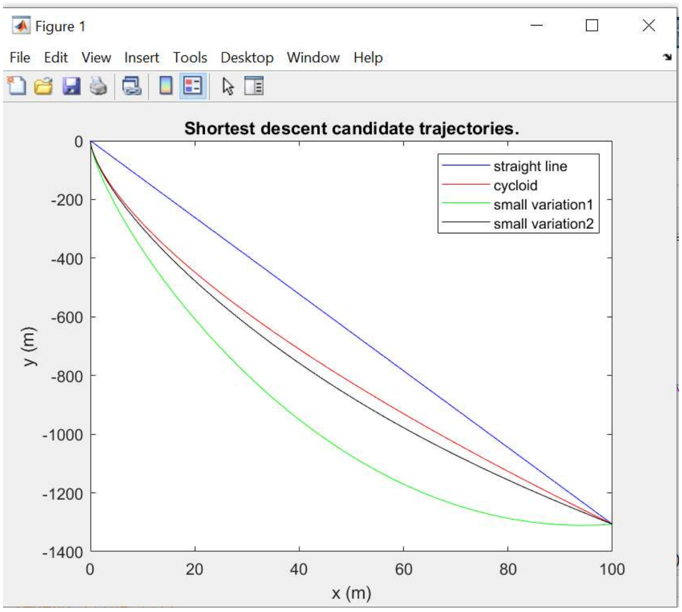
The only condition imposed on the variation function η(x) is that the initial and endpoints are zero, η(0) = 0 and also $\eta(x_e) = 0$ The following variation functions that satisfies those conditions was chosen





This first variation function is mathematically defined by $\eta(x) = x^2 - (x_e)x$ and second variation function is defined to be η(x) = sin(π x x_e) these two functions clearly satisfy the conditions for a variation function as they both have roots at x = 0 and x = x_e.

The initial conditions are still unchanged, the following plot compares the four different paths for the object considered thus far.



The first of the newly considered trajectories takes the most time out of the four, the object takes roughly 16.443 seconds to traverse this trajectory; The object takes 0.0789 seconds more than it does in traversing the straight line, takes 0.0900 more seconds to traverse the trajectory obtained from the second variation function and it takes 0.0905 seconds more than it does in traversing the cycloid. The trajectory formed by adding the second variation to the original function y(x) takes roughly 16.3537 seconds. These variations are all greater than the time taken by cycloid connecting the points A and B; these provide quantitative proof that the cycloid is indeed the trajectory of shortest descent.

DISCUSSION AND CONCLUSION

The results considered in the section above provide quantitative proof that the cycloid is indeed the trajectory of shortest descent between A and B. This may come as a surprise that the shortest path between A and B; a straight line, doesn't yield the shortest descent. We can try to explain why the cycloid minimizes the time taken by the object to descend by looking at its structure compared to the straight line and also considering the energy conversation.

As the object falls from A to B, there is a trade-off between the potential
energy and the kinetic energy of the object, such that their sum remains constant. This means that the lower the objects get close to the ground, there will be the decrease in potential energy compensated for by an increase in kinetic energy, the kinetic energy increases as a direct consequence of its speed increasing. To get from point A to point B in the shortest time there should be a combination of fast speed and a short distance to traverse, looking at the shape of the
cycloid we see there is a fast drop in height just after point A, this will result
in the rapid increase in falling speed. We see that before the curve gets to point B, and it has started to curve up and resemble a straight line. These two traits of the cycloid combine for a fast speed and a short distance to cover.

The points stated this suffice to explain why the cycloid is preferred over a
straight line, it is because it has a perfect trade-off between making the object gain speed fast enough over a short distance such that it results in the shortest descent.

REFERENCES

1. https://mathworld.wolfram.com/BrachistochroneProblem.html
2. https://mathshistory.st-andrews.ac.uk/HistTopics/Brachistochrone/
3. https://wiki.math.ntnu.no/
4. https://in.mathworks.com/help/symbolic/functional-derivative.html
5. https://en.wikipedia.org/wiki/Cycloid
6. https://www.youtube.com/watch?v=6HeQc7CSkZs
7. http://www.hep.caltech.edu/