I say, “I am lying.”

Am I?

Commentary: Consider the case if the statement is true. Then the statement claims that it is not the fact. On the other hand if the statement is false. Then, the negation of the statement is true and I am not lying. Therefore, the statement must be true. Therefore, both the cases leads to absurdity. For a similar fun, https://en.wikipedia.org/wiki/Paradox_of_the_Court

CS228 Logic for Computer Science 2021

Lecture 1: Introduction and logistics

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Topic 1.1

What is logic?

What is logic?

- Have you ever said to someone, "be logical"?
  - whatever your intuition was that is logic
- Mathematization/Formalization of the intuition is mathematical logic
- Two streams of studying logic
  - use of logic : logic as a tool to study something else, e.g., math
  - properties of logic: since logic has mathematical structure, we may study its mathematical properties using logic

The self reference will haunt us!!

Why a CS student should study logic?

Differential equations
are the calculus of
Newtonian physics

Logic
is the calculus of
computer science

Logic provides tools to define/manipulate computational objects

Defining logic

Logic is about inferring conclusions from given premises

Example 1.1

1. *Humans are mortal*
2. *Socrates is a human*

Socrates is mortal ✓

1. *Apostles are twelve*
2. *Peter is an apostle*

Peter is twelve ✗

What went wrong here?

Intuitive Pattern:

1. α*s are* β
2. γ *is an* α

γ *is* β

Very easy to ill-define.
Logic needs rigorous definitions!!

Commentary: We think that the first reasoning is correct. Therefore, we may believe in the pattern below. If we interpret the pattern literally, we must admin the second derivation, which is clearly faulty. The above was one of the mistakes in the Aristotle's syllogism(inference rules), which dominated European thought until late middle ages. `https://en.wikipedia.org/wiki/Syllogistic_fallacy` Clearly understood formalization arrived in early 20th century.

Topic 1.2

Course logistics

Evaluation and website

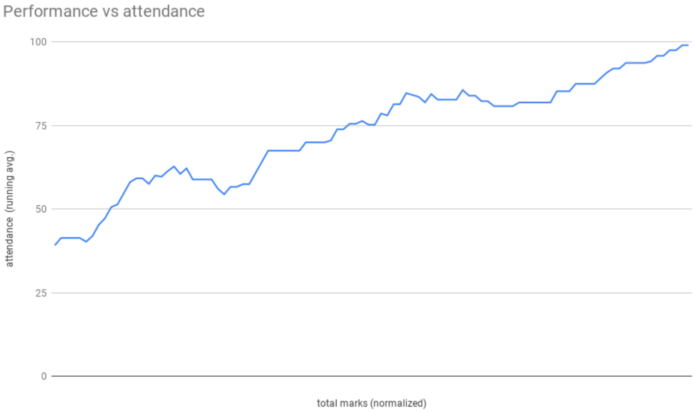
- Quizzes : 22.5% (3 quizzes)
  - Quiz 1 : dates to be decided
- Programming assignment : 7.5%
- Midterm : 25% (2 hours)
- Final : 40% (3 hours)
- Weekly participation : 15% (mini SAFE quizzes)
  - Any coping detected $\Rightarrow$ Grade drop and on repeat Fail

For the further information

`http://www.cse.iitb.ac.in/~akg/courses/2021-logic`

All the content will be posted on the website.

Why attend? – regular participation/attendance ⇒ better grade



X-axis : students sorted by their marks

Y-axis: running average of attendance of 20 students

Commentary: The data is from a 2018 course.

Topic 1.3

Course contents

The course

We will study the following topics

- Propositional logic
- First-order logic

Foundations

Midterm

- Logic for verification
- MSO logic
- Temporal logic

Applications in computer science

Propositional logic

Propositional logic

- deals with propositions,
  - Example: the shirt is cheap, the shoe is expensive
- only infers from the structure over the propositions, and
  - Example: the shirt is cheap and the shoe is expensive
- does not look inside the propositions.
  - Example: the shirt is cheap and the shirt is made of gold

Example: Propositional argument

Example 1.2

Is the following argument valid?

If the seed catalog is correct then if seeds are planted in April then the flowers bloom in July. The flowers do not bloom in July. Therefore, if seeds are planted in April then the seed catalog is not correct.

Let us identify the propositions in the argument

- c = *the seed catalogue is correct*
- s = *seeds are planted in April*
- f = *the flowers bloom in July*

If we replace the proposition with symbols, we obtain

If c then if s then f. not f. Therefore, if s then not c.

Propositional logic (PL) topics

We will study

- Week1: definition of PL and meaning (philosopher's view)
- Week2: Formal proofs (mathematician's view)
- Week3: Proof systems for PL and their properties (computer scientist's view)
- Week4: PL solvers *aka* SAT solvers (hacker's view)

Commentary: PL has limited expressive power. However, there are a lot of real world problems that can be *encoded* using PL. SAT solver is an effective tool to solve the problems.

First-order logic (FOL)

First-order logic

- looks inside the propositions,
- deals with parameterized propositions and quantifiers, and
- can express a lot of interesting math.

Example 1.3

Is the following argument valid?
Humans are mortal. Socrates is a human. Therefore, Socrates is mortal.

The parametric propositions in the argument.

- $H(x) = x$ *is a human*
- $M(x) = x$ *is mortal*
- $s =$ *Socrates*

FOL is not the most general logic.
Many arguments can not be expressed in FOL

If we replace the parametric propositions with symbols, we obtain

For all x *if* $H(x)$ *then* $M(x)$. $H(s)$. *Therefore,* $M(s)$.

Logical theories

In a theory, we study validity of FOL arguments under specialized assumptions (called axioms).

Example 1.4

The number theory uses symbols $0, 1, .., <, +, \cdot$ *with specialized meanings*

The following sentence has no sense until we assign the meanings to $>$ *and* $\cdot$

$$\forall x\, \exists p\, (p > x \wedge (\forall v_1 \forall v_2\, (v_1 > 1 \wedge v_2 > 1 \Rightarrow p \neq v_1 \cdot v_2)))$$

Under the meanings *it says that there are arbitrarily large prime numbers.*

> In the earlier example, we had no interpretation of predicate 'x is human'. Here we precisely know what is predicate '$x < y$'.

Commentary: The specialized meaning are defined using axioms. For example, the sentence $\forall x.\ 0 + x = x$ describes one of the properties of 0 and $+$. We will cover number theory at length.

Logical theories II

The logical theories are useful in studying specialized domains.

Logic was thought to be an immensely useful general purpose tool in studying properties of various mathematical domains.

Incompleteness results

But sadly, one can prove that

Theorem 1.1 (Gödel's incompleteness(1930))

There are theories whose assumptions can not be listed.

Proof sketch.

We assume that there is such a list for the number theory. Now, we can **construct a true sentence** that says

"This sentence can not be proven by the list".

Therefore, the list cannot imply the sentence. Contradiction. □

The incompleteness was an epic failure of logic as a tool to do math.

From the ashes of logic, rose computer science

Commentary: The incompleteness essentially shows that some of our mathematical intuitions cannot be formally characterized. The proof of the above theorem is not part of this course. We recommend that you read the proof after the course.

First-order logic (FOL) topics

We will study

- Week 5: definition of FOL and syntactic properties (philosopher's view)
- Week 6: proof systems for FOL, etc (computer scientist's view)
- Week 7: first-order theorem provers (hacker's view)

Topic 1.4

Problems

Mistake

Exercise 1.1

What is wrong with the following argument?
All supermen can fly. Therefore, there is a superman.

Commentary: One of the errors in Aristotle's logic `https://en.wikipedia.org/wiki/Existential_fallacy`

Does God exist?

Exercise 1.2

Is there a logical problem with the following argument aka *Ontological argument?*

1. *God is the greatest possible being that can be imagined.*
2. *God exists as an idea in the mind.*
3. *A being that exists as an idea in the mind and in reality is, other things being equal, greater than a being that exists only as an idea in the mind.*
4. *Thus, if God exists only as an idea in the mind, then we can imagine something that is greater than God.*
5. *But we cannot imagine something that is greater than God.*
6. *Therefore, God exists.*

(text source Wikipedia)

Fun side of the argument: `https://xkcd.com/1505/`

A puzzle from internet

Exercise 1.3

Sanjay and Salman are new friends with Madhuri, and they want to know her birthday. Madhuri gives them a list of possible dates.
March 14, March 15, March 18,
April 16, April 17,
May 13, May 15,
June 13, June 14, June 16

Madhuri then tells Sanjay and Salman separately the month and the day of her birthday respectively.

Sanjay: I don't know the date, but I know that Salman doesn't know too.
Salman: At first I didn't know the date, but I know now.
Sanjay: Then I also know the date.

So when is Madhuri's birthday?

Topic 1.5

Some math

Topic 1.6

Cardinality

Comparing sizes of sets

Cardinality is a measure of the number of elements of the set, which is denoted by $|A|$ for set A. Non-trivial to understand if $|A|$ is not finite.

Definition 1.1
$|A| \leq |B|$ if there is an one-to-one $f : A \to B$.

Theorem 1.2
If f is also onto then $|A| = |B|$.

Exercise 1.4
Prove theorem 1.2

Countable/uncountable

Definition 1.2

A is countable if $|A| \leq |\mathbb{N}|$.

Definition 1.3

A is uncountable if $|A| > |\mathbb{N}|$

Exercise 1.5

Show $\mathbb{Q}$ *is countable.*

Countable finite words

How to prove?
Find an one-to-one map to $\mathbb{N}$

Theorem 1.3
If Symbs *is countable,* Symbs* *is countable.*

Proof.
Since Symbs is countable, there exists one-to-one f : Symbs $\rightarrow \mathbb{N}$.
We need to find an one-to-one h : Symbs* $\rightarrow \mathbb{N}$.

Let p_i be the ith prime. Our choice of h is

$$h(a_1 \dots a_n) = \Pi_{i \in 1..n} p_i^{f(a_i)}.$$

□

Exercise 1.6
Show h is one-to-one.

Cantor's theorem

Theorem 1.4

$|A| < |\mathfrak{p}(A)|$

Proof.

Consider function $h(a) = \{a\}$. h is one-to-one function.
There is an one-to-one function in $A \to \mathfrak{p}(A)$, therefore $|A| \leq |\mathfrak{p}(A)|$.

To show strictness, we need to show that there is no one-to-one and onto function in $A \to \mathfrak{p}(A)$.

Let us suppose $f : A \to \mathfrak{p}(A)$ is one-to-one and onto.
Consider, $S \triangleq \{a | a \notin f(a)\}$. Since f is onto, there is a b such that $f(b) = S$.
Case $b \in S$: Since $f(b) = S$, $b \in f(b)$. Due to def. S, $b \notin S$. Contradiction.
Case $b \notin S$: Since $f(b) = S$, $b \notin f(b)$. Due to def. S, $b \in S$. Contradiction.

□

Exercise 1.7

For countable A, why is A^* lot smaller than $\mathfrak{p}(A)$? Give an intuitive answer.

Topic 1.7

Some problems in related mathematics

One-to-one and onto for cardinality

Exercise 1.8

Prove or disprove:

For sets A and B, if there exist one-to-one functions $f : A \rightarrow B$ and $g : B \rightarrow A$, then there exists a one-to-one and onto function in $A \rightarrow B$.

Combinatorics***

Exercise 1.9

For $i \in 0..m$, *consider* $R_i = \{h_{i1}, \dots, h_{i5}\}$, *where* $h_{ij} \in 1..m$. *For a given* t, *how many ways to choose* R_i*s such that there is a* $P \subseteq 0..m$ *for which the following holds?*

$$|\{j | \textit{unique } i \in P \textit{ such that } j \in R_i\}| < |P| = t$$

Note that it is possible that $|R_i| < 5$.

Partial orders

Exercise 1.10

Let $\leq \subseteq A \times A$ be a partial order and $S : A \hookrightarrow A$ be a partial one-to-one function, such that $a < b$ iff $S(a) \leq b$ for every $a \in dom(S)$ and $b \in A$. Prove that $\leq$ can be extended to a total order $\sqsubseteq$, such that $a \sqsubset b$ iff $S(a) \sqsubseteq b$ for every $a \in dom(S)$ and $b \in A$.

End of Lecture 1