Legendre Polynomials for Uniform Mixture Detection

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What is a mixture model?

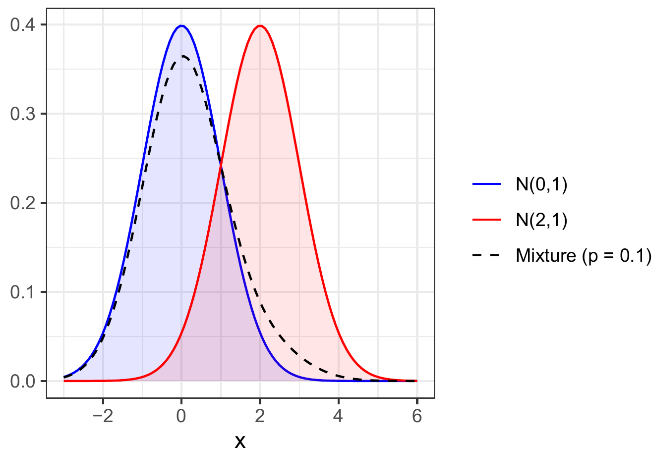
- Probability density function (pdf):

$$(1-p)f(x) + pg(x)$$

- Cumulative distribution function (CDF):

$$(1-p)F(x) + pG(x)$$

- Example:



What is a uniform mixture model?

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$$1-p+pg(x)$$

- Cumulative distribution function (CDF):

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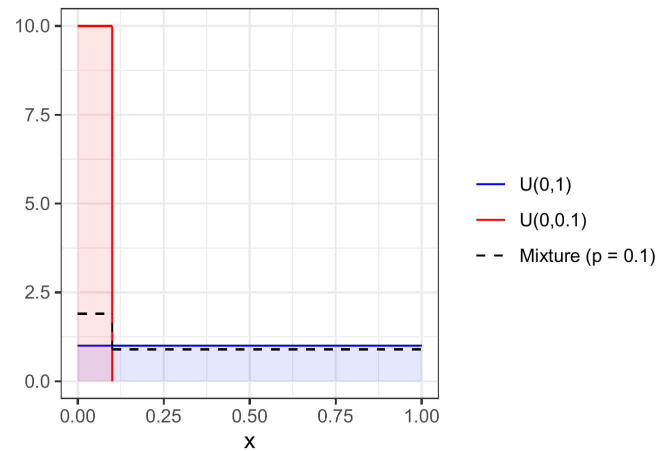


Uniform mixture detection problem

As a model for p-values

- If the null hypothesis of a test is *actually* true, then we expect the p-value to follow a $U(0, 1)$ distribution
- Now, consider a scenario where we need to conduct many identical tests
- What if, in some small proportion of cases, the null hypothesis is *actually* false?
- We can model this scenario with a uniform mixture model!

Uniform mixture detection problem

Motivational example

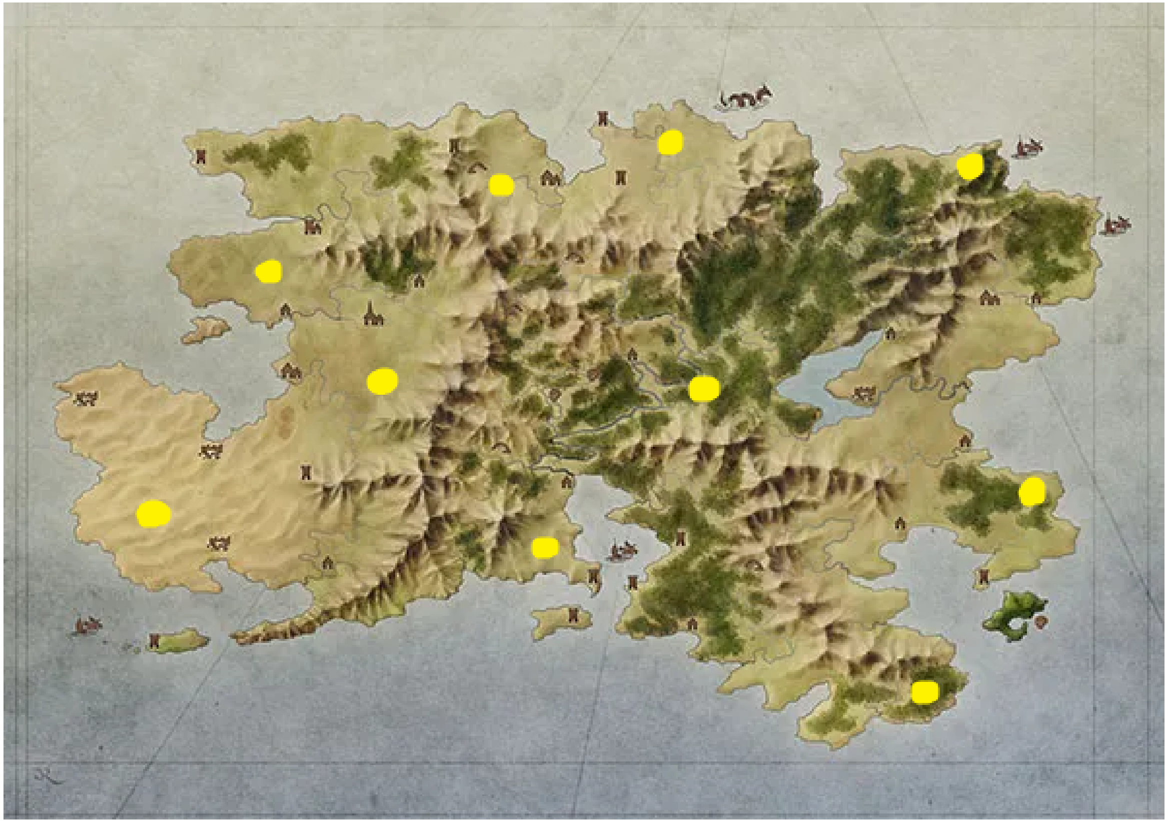
- Donoho and Jin, 2004: Higher criticism for detecting sparse heterogenous mixtures
- Bioweapon
- Causes an increase of some chemical in the blood
- Blood test returns a p-value

Uniform mixture detection problem

Motivational example

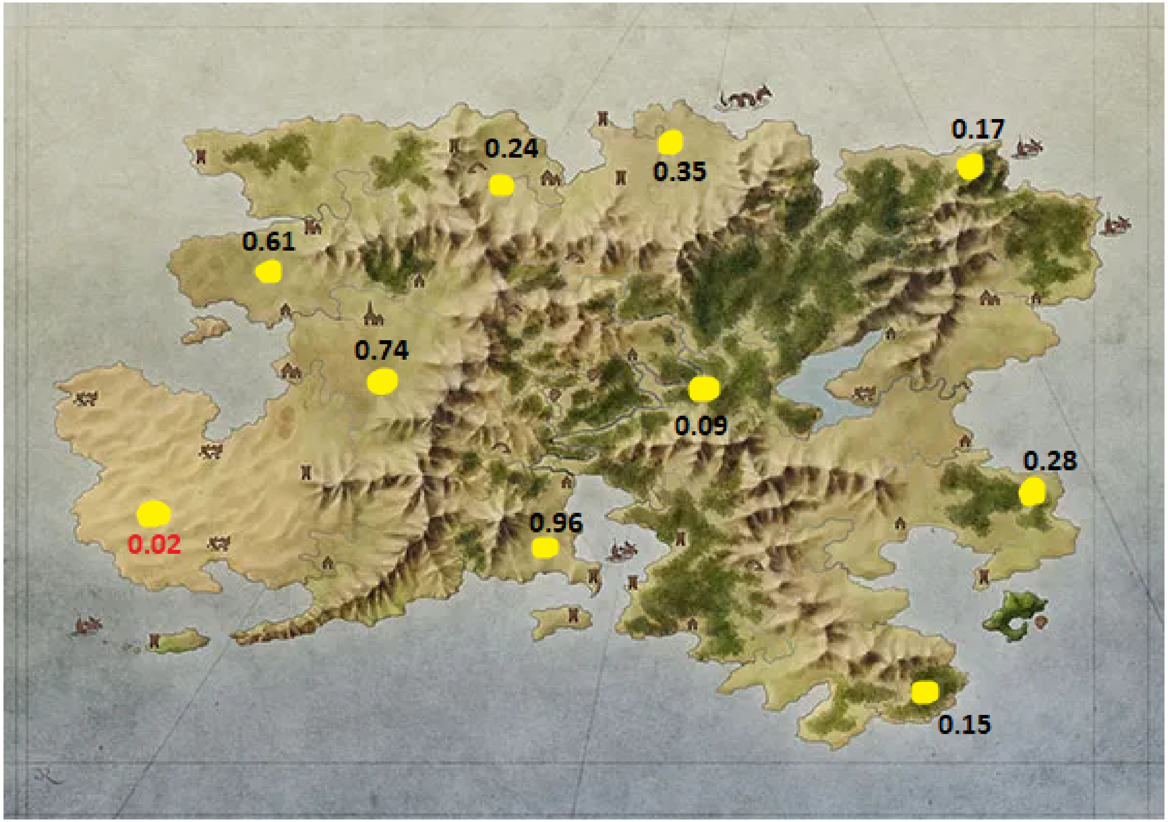
Uniform mixture detection problem

Motivational example



Uniform mixture detection problem

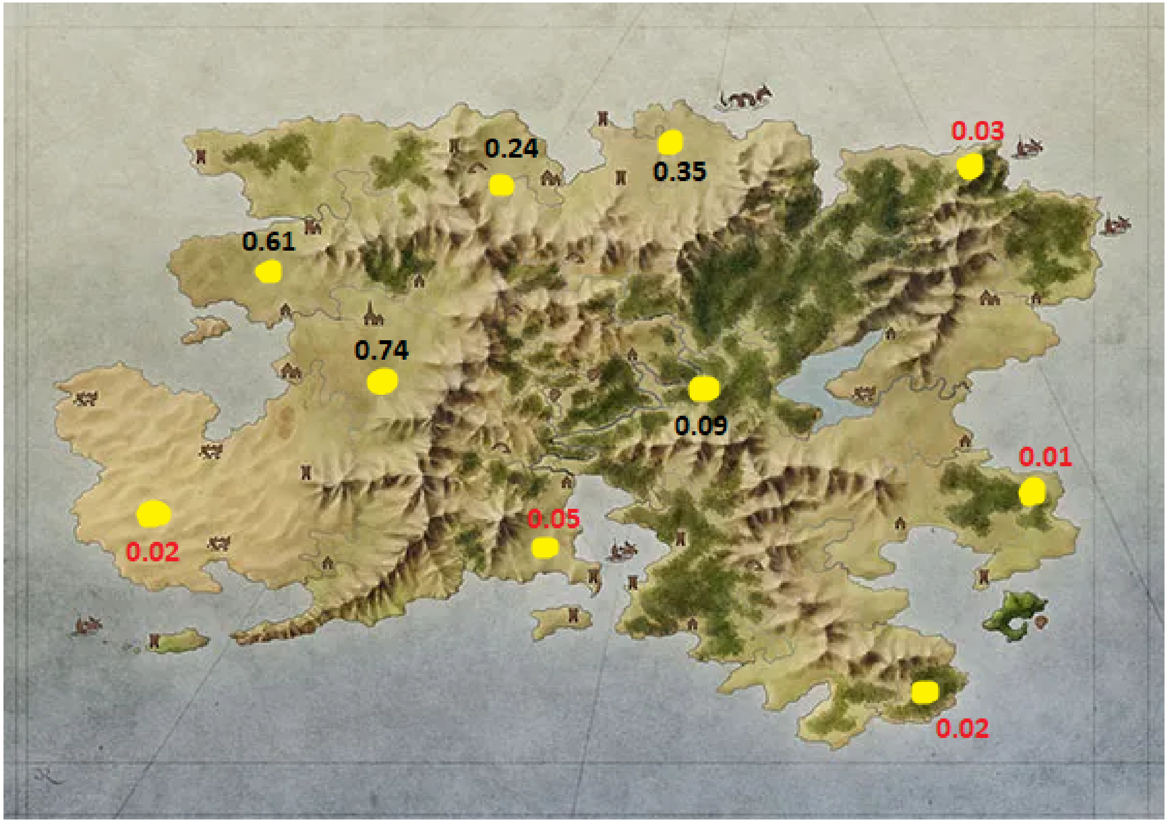
Motivational example



- Is there evidence that the bioweapon has affected some of the population?

Uniform mixture detection problem

Motivational example



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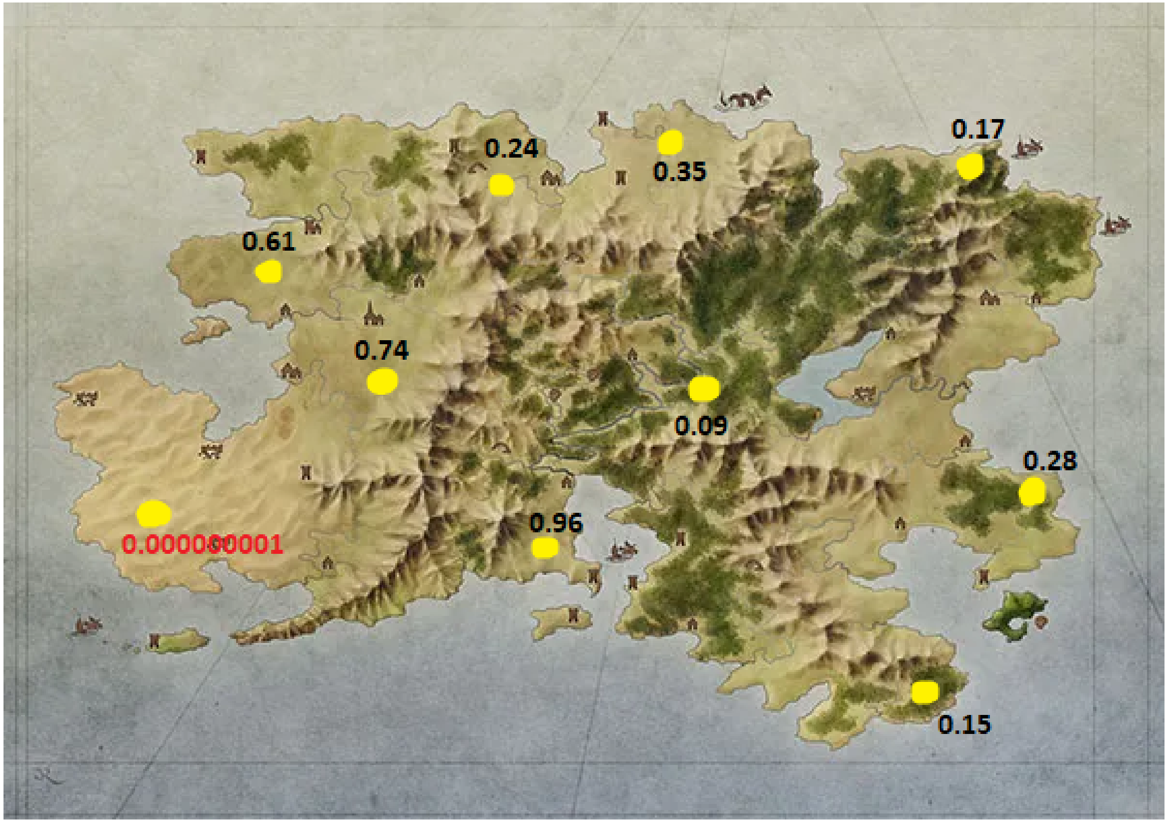
Motivational example



- Is there evidence that the bioweapon has affected some of the population?

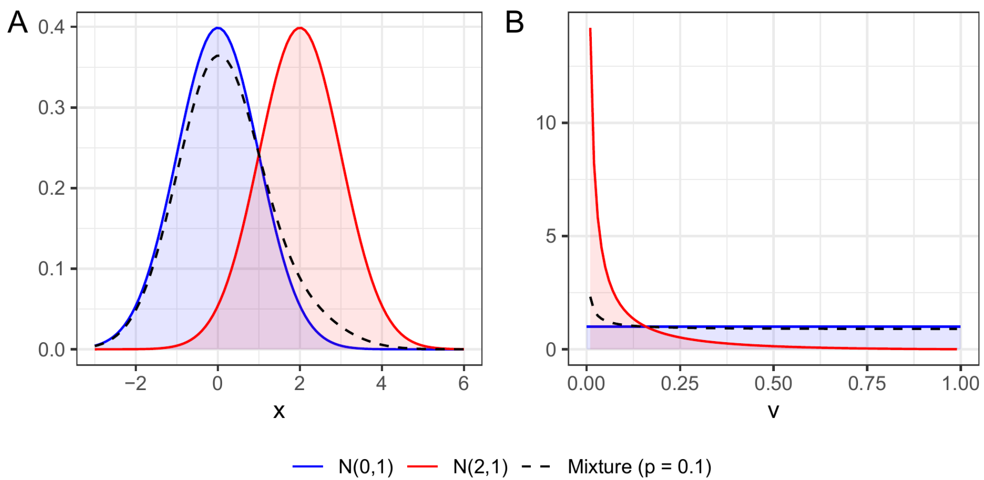
Uniform mixture detection problem

Transforming to a uniform mixture model

- We can transform a mixture model into a *uniform* mixture model
- Let the CDF of X be $(1-p)F(x) + pG(x)$ and let $V = 1 - F(X)$. Then, the CDF of V is:

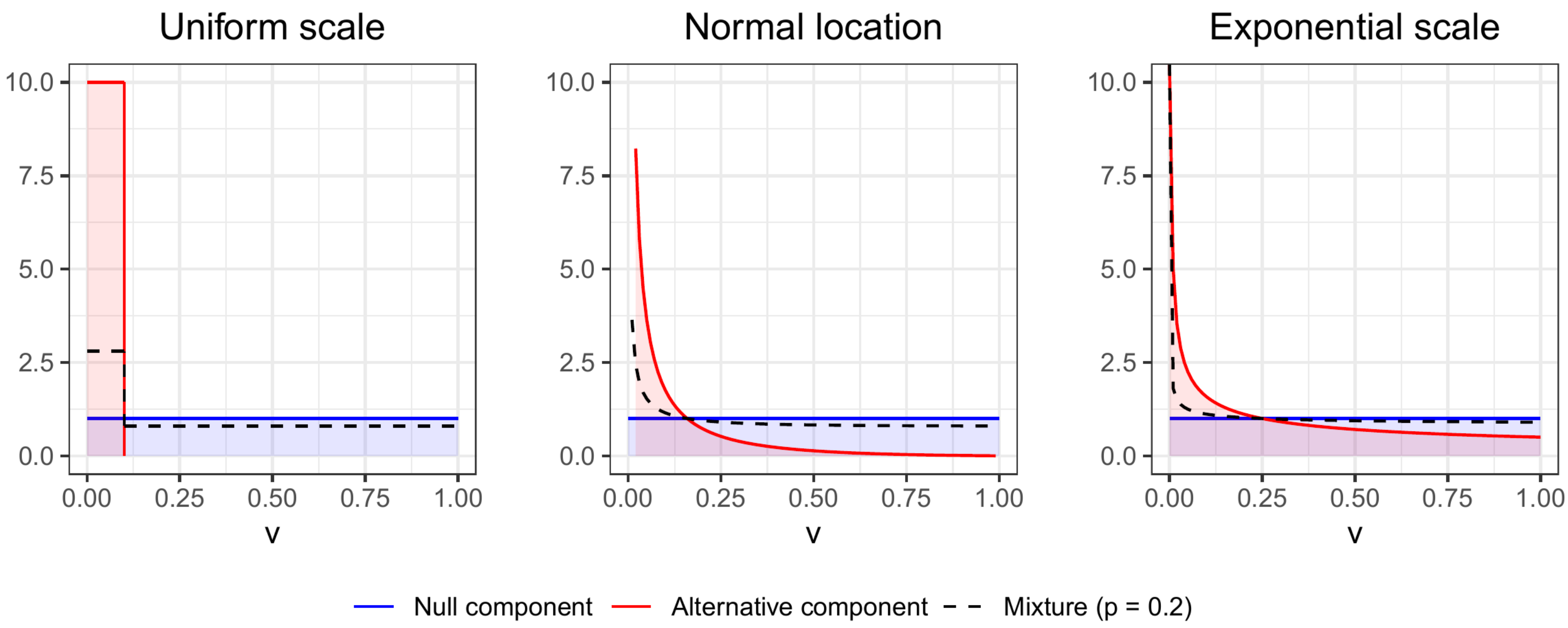
$$(1-p)v + p\left(1 - G\left[F^{-1}(1-v)\right]\right)$$

- Differentiating this gives a density of the form $1 - p + p(\ldots)$
- Example:



Uniform mixture detection problem

Common uniform mixture models



Legendre polynomials

Legendre polynomials

Definition

- The k^{th} Legendre polynomial is the coefficient of t^k in the power series expansion of the generating function given by:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Shifted Legendre polynomials

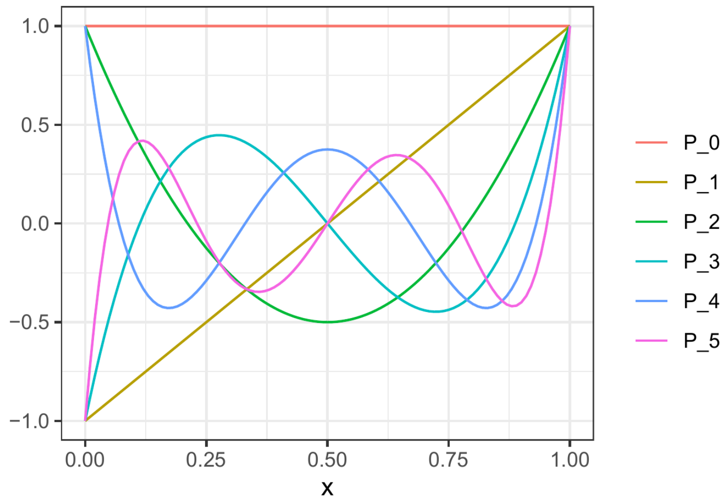
- P_k is defined on the interval $[-1, 1]$
- Uniform mixture models are defined on the interval $[0, 1]$
- Apply the following shift:

$$\tilde{P}_k(x) := P_k(2x-1)$$

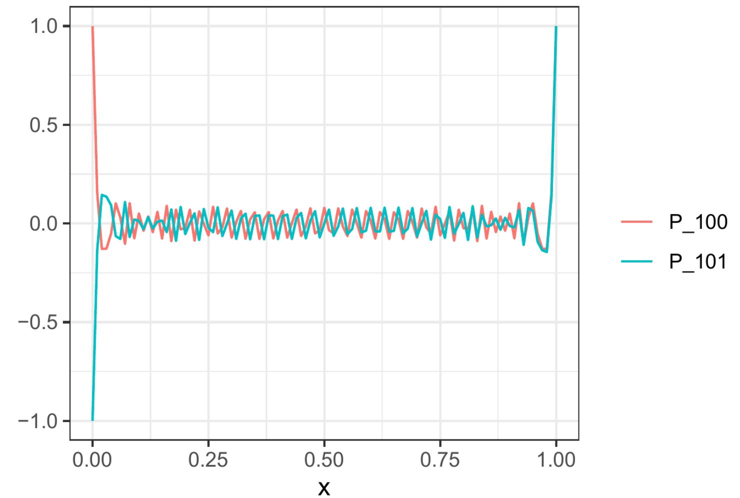
Legendre polynomials

Visualisation

- First few polynomials



- Higher-degree polynomials



Legendre polynomials

Orthogonal system

- Let $\tilde{P}_k$ and $\tilde{P}_j$ be shifted Legendre polynomials of degree k and j respectively. Then:

$$\langle \tilde{P}_k, \tilde{P}_j \rangle := \int_0^1 \tilde{P}_k(x)\tilde{P}_j(x)\,dx = \begin{cases} 0 & \text{if } k \neq j \\ \frac{1}{2k+1} & \text{if } k = j \end{cases}$$

Uncorrelated property

- If $X \sim U(0,1)$, then $\tilde{P}_k(X)$ and $\tilde{P}_j(X)$ are uncorrelated for $k \neq j$. Proof:

$$E\left[\tilde{P}_k(X)\tilde{P}_j(X)\right] = \int_0^1 \tilde{P}_k(x)\tilde{P}_j(x)\,dx = 0$$

$$E\left[\tilde{P}_k(X)\right] = \int_0^1 \tilde{P}_k(x)\,dx = \int_0^1 \tilde{P}_k(x)\tilde{P}_0(x)\,dx = 0$$

$$\mathrm{Cov}\left[\tilde{P}_k(X), \tilde{P}_j(X)\right] = \underbrace{E\left[\tilde{P}_k(X)\tilde{P}_j(X)\right]}_{=0} - \underbrace{E\left[\tilde{P}_k(X)\right]}_{=0}\underbrace{E\left[\tilde{P}_j(X)\right]}_{=0} = 0$$

Legendre polynomials

Use in uniform mixture detection

- Model data via the density $1 - p + pg(x)$
- Hypotheses are $H_0 : p = 0, H_1 : p \neq 0$
- If H_0 is actually true, i.e. $X \sim U(0,1)$, then we expect $\tilde{P}_k(X) = 0$ for all $k \geq 1$
- If H_0 is actually false, then we might expect $\tilde{P}_k(X)$ to be some other value
- Think of each polynomial as measuring the deviation from H_0 in a certain 'abstract' direction

$S_{n,k}$ statistic

$S_{n,k}$ statistic

Definition

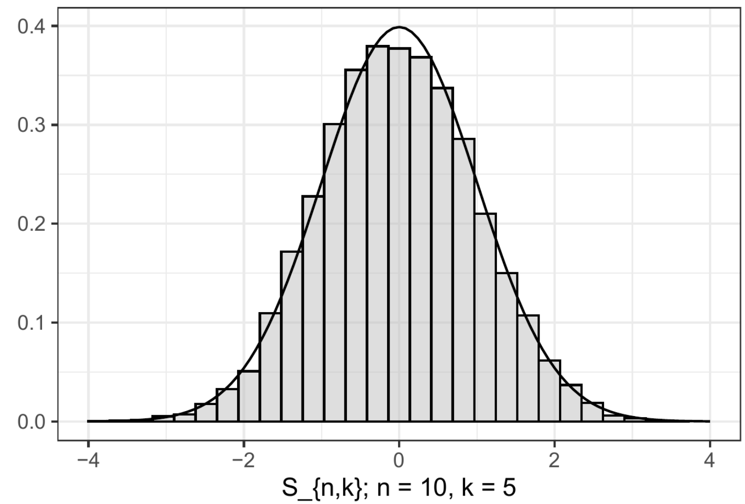
- Let $x_i, i = 1, \ldots, n$ be a sample of data. Then:

$$S_{n,k} := \sqrt{\frac{2k+1}{n}} \sum_{i=1}^{n} \tilde{P}_k(x_i)$$

- Simply a sum of $\tilde{P}_k$ over the sample, standardised to have mean 0 and variance 1
- Interpreted as the amount in which the data supports the abstract direction that $\tilde{P}_k$ is measuring

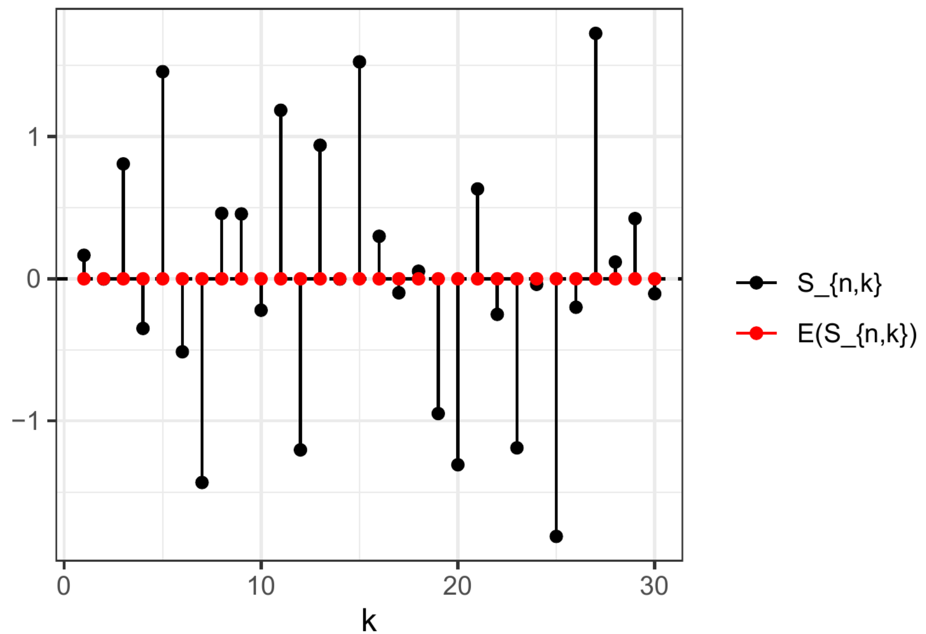
Asymptotic normality

- Central limit theorem kicks in quickly



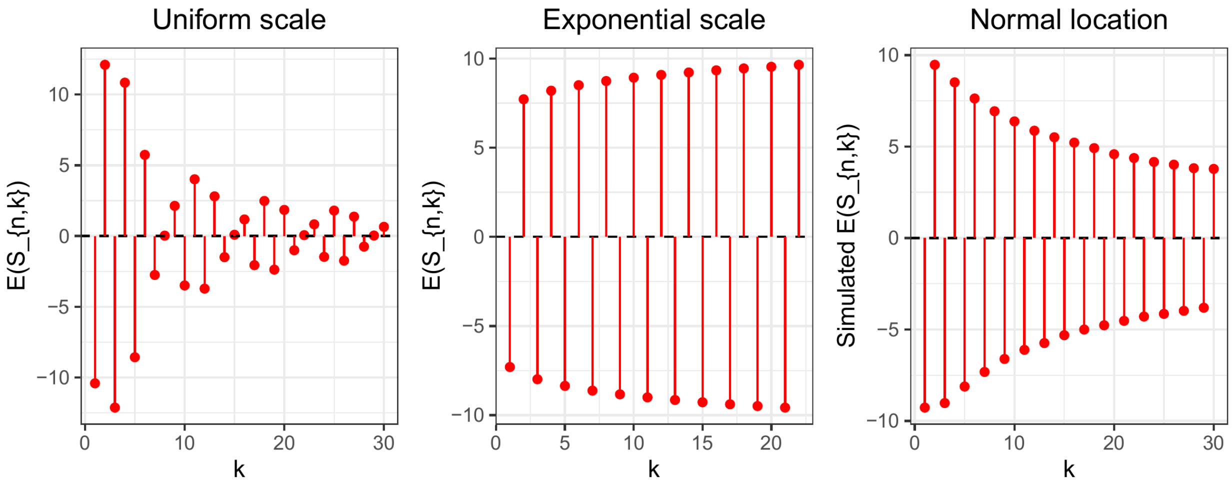
Spectrum diagrams

- Plot of $S_{n,k}$ vs k under a specific mixture model
- Under $H_0 : p = 0$, we expect $S_{n,k} = 0$ for all $k \geq 1$:



$S_{n,k}$ statistic

- Under $H_1 : p \neq 0$, we expect $S_{n,k}$ to follow the patterns below for our three different mixture models:



- These plots can be used to inspire test statistics
- It is surprising that we are able to theoretically derive $E(S_{n,k})$ for the uniform scale and exponential scale mixture models
- Example (uniform scale):

$$E(S_{n,k}) = \sqrt{n(2k+1)} \cdot p \sum_{\ell=0}^{k} (-1)^{k+\ell} \binom{k}{\ell} \binom{k+\ell}{\ell} \frac{\theta^\ell}{\ell+1}$$

Constructing test statistics

Constructing test statistics

Statistical power

- One of the main themes in mathematical statistics is to analyse the notion of power
- Power depends on the statistical test and the alternative hypothesis
- It is defined as:

$$P(\text{reject } H_0 \mid H_1 \text{ is true})$$

- We like tests that have high power across common alternative hypotheses

Constructing test statistics

The test statistics

- Our spectrum diagrams inspire our construction of test statistics
- Sum of squares:

$$\sum_{k=1}^{K} S_{n,k}^2$$

- Threshold sum of squares:

$$\sum_{k=1}^{K} S_{n,k}^2 \cdot 1_{\{|S_{n,k}| \geq c\}}$$

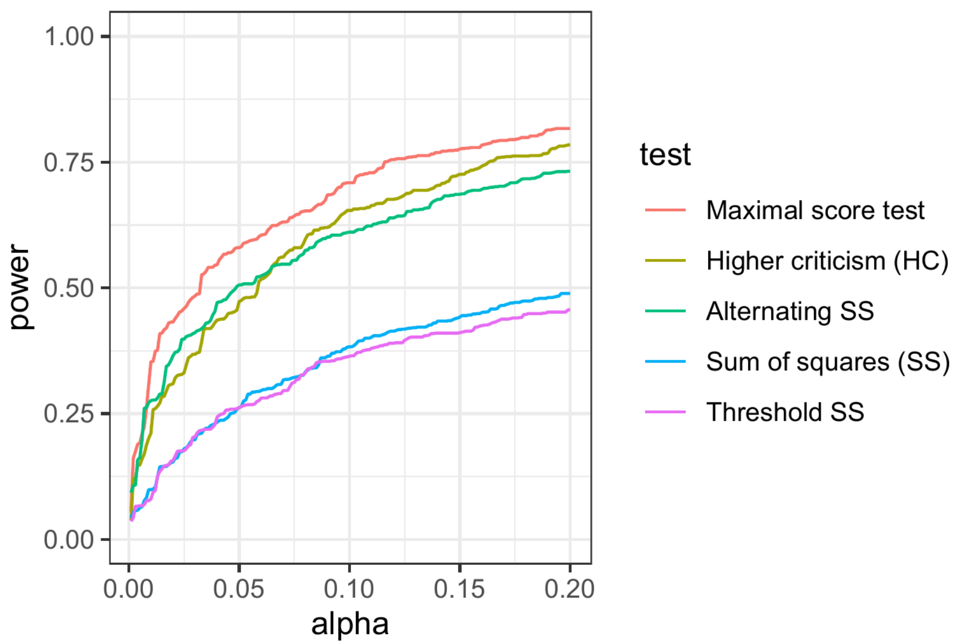
- Alternating sum of squares:

$$\sum_{k=1}^{K} \max\{0, (-1)^k \cdot S_{n,k}\}^2$$

Constructing test statistics

Simple power analysis

- A ROC curve is a plot of power vs significance level
- We simulate power under the normal location mixture model for a variety of tests:



Takeaways

Takeaways

What we looked at

- Uniform mixture detection problem
- Legendre polynomials
- $S_{n,k}$ spectrum diagrams and constructing test statistics

Questions my research looks at

- What does the spectrum diagram look like under different mixture models?
- How does the performance of tests based on the Legendre polynomials compare to other tests in the literature?
- What rate should K grow at as the sample size n increases?

End