

Controllability of Coarsely Measured Networked Linear Dynamical Systems

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Abstract—We consider the controllability of large-scale linear networked dynamical systems when complete knowledge of network structure is unavailable and knowledge is limited to coarse summaries. We provide conditions under which average controllability of the fine-scale system can be well approximated by that of the (synthesized, reduced-order) coarse-scale system. Our analysis indicates the necessity of some inherent parametric structure of the fine-scale network (e.g., in-built communities) to be able to quantify accurately the controllability from coarsely characterized networked dynamics. Therefore, we assume that the fine-scale network is generated by the stochastic block model (SBM) often studied in community detection. We then provide an algorithm that directly estimates the average controllability of the fine-scale system using a coarse summary of SBM. We also compare our method to that of the reduced-order method and highlight the regimes where both can outperform each other. Finally, we provide simulations to confirm our theoretical results for different network size and density, and the parameter that captures how much community-structure is retained in the coarse summary.

I. INTRODUCTION

In this paper we study controllability of networks with discrete-time linear dynamics (hereafter, LTI systems) when our knowledge of system structure is limited to coarse summaries. We are motivated by myriad real-world settings, ranging from power and water systems to brain networks, where system identification must be performed based upon data collected by low-resolution instruments unable to probe fine-scale structure. Our motivating real-world example is the human brain. While efforts are under way to produce a map of a canonical human brain, our knowledge of the brain as an interconnected, network system is not yet to the level of the individual neurons of the entire brain [1]. And yet, modern medical technologies aim to control the brain using coarse MRI based network connectivity [2], [3]. For example, novel brain implants designed for epilepsy patients *steer* the brain away from states

that correspond to seizures [4]. The inaccessibility of fine-scale network information may result in wrong control decisions that diverge greatly from those derived from the fine-scale network.

Our goal is to quantify the controllability of a fine-scale networked LTI system using its coarse summaries and a few of the fine-scale network properties such as community structure (see below). In the presence of full knowledge of the fine-scale system, existing works address this goal by first obtaining a lower-dimensional (coarse-scale) system using various methods under the umbrella term of model order reduction (MOR). Then, they relate the controllability of this coarse-scale system to that of the original fine-scale system [5]. But this approach is inapplicable in settings wherein one does not have access to the fine-scale network. Nonetheless, we may fit a dynamical system for coarse network data and study the controllability of this system as a proxy for the controllability of the fine-scale system. However, it is not clear when such an approximation is sufficiently accurate. In Section III, we study this approximation in detail. The second contrast between classical MOR and our setting is best understood in terms of *active* versus *passive* MOR. Classical MOR is active in that it determines how to coarsen the system to yield the best reduction. But for us, our knowledge is restricted by the precision of our observations (e.g., owing to the resolution of measurement devices or to anatomical considerations), thereby limiting our ability to devise best reduction schemes. So, passively collected network data is our starting point.

To realize our goal, we consider a suitable controllability Gramian based metric. It is well known that the Gramian based metrics help compare the control effort exerted either by two individual nodes or groups of nodes [6]–[9]. We focus on the latter to address the following question. For certain groups of nodes that we can control globally (i.e., not actuating each node individually) suppose that we have access only to their *coarse summary*; that is, certain combinations of weights of network edges in and across the groups of nodes. Then, we ask which groups influence (or require minimum energy) in controlling the fine-scale system? This joint measuring and actuating is required in epilepsy applications to determine the device (instrument) position to collapse the unstable brain-state oscillations that can lead to seizures [10], [11].

To leverage fine-scale network properties (e.g., community structure, sparsity, and node degree) along with coarse summaries, we assume that the fine-scale network is generated by a stochastic block model (SBM). These models find applications in studying communities in social networks and recently a few

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groups have considered SBMs for network dynamics [12], [13]. It is known that SBMs naturally allow for generating networks with different community structures (e.g., assortative and dis-assortative) [14], [15]. Interestingly, we show that they also naturally allow to study the relationship between controllability and the *synchronization*—a measure that quantifies how well the coarse network data are synchronized (or overlapped) with communities. The higher the synchronization, the less information the coarse summary contains regarding the fine-scale network (see Section II).

To quantify the controllability of the fine-scale LTI system, we consider the *average controllability* metric given by the trace of the controllability Gramian of a discrete-time system with non-negative system matrices [16]. Specifically, we characterize the average controllability vector of a set of systems. Each entry of the vector corresponds to the average controllability of a system with specific group of input (control) nodes as described above. We then devise two competing schemes (see below) to estimate this average controllability vector and compare the true and estimated vectors. Our main contributions are as follows.

- 1) In Section III, we consider an auxiliary, fictitious, linear dynamical system based on the coarse data, which we term the *passively reduced-order model (PROM)*.¹ Each node in the fictitious network corresponding to PROM is associated with a group of nodes in the fine-scale network that map to the same coarse measurement. We use the average controllability vector of the PROM to approximate that of the fine-scale system. Notice that this implicitly couples measurement and actuation. We derive a tight upper bound on the *approximation-error* which goes to zero as the coarse and fine network sizes grow; the network becomes dense; the coverage of the fine network by coarse measurements expands; and the synchronization between the coarse summary data and the underlying community structure increases. If synchronization is not sufficiently high, the error may not approach zero even as the network size increases.
- 2) In Section IV, we learn the average controllability vector of the fine-scale system directly from the coarse data without considering an auxiliary system. We develop a novel learning-based algorithm with its roots in a mixed-membership method proposed by [17] for unsupervised learning of the parameters and the community structure in an SBM. We derive a tight upper bound on the estimation error and characterize its convergence in terms of network parameters. Using this bound, we show that the estimation error approaches zero as long as the community structure parameter estimates are accurate and the coverage of the fine network by coarse measurements is sufficiently rich. Although the learning-based technique's estimate quality is implicitly dependent on synchronization, it has more resilience than the PROM method in low-synchronized scenarios.

¹The prefix "passive" in PROM emphasizes that the auxiliary system is not a reduction of the fine-scale model. Rather the auxiliary system is a low-dimensional fictitious model that we fit to the coarse data.

- 3) In Section V, we present multiple numerical simulations to show that the estimation errors and their qualitative behavior (in terms of the various network and coarsening parameters) are comparable to the tight error bounds and their qualitative nature we derived in Sections III and IV. Importantly, consistent with the theory, simulations show that the synchronization has less impact on the estimation error of the learning-based method. Finally, our empirical investigation reveals that both the approaches studied can outperform each other in terms of estimation error. Specifically, the learning-based approach outperforms its counterpart in regions for which the network density and the coarse network size are not too small, and the coarse measurements are not fully synchronized with communities.

A. Related work

Controllability of Complex Networks: Controllability of dynamical systems has a rich, long history since its inception in Kalman's work [18]. Several articles and textbook entries were written about its characterizations and manifestations in several applications [19]. In studies concerning network controllability, the two areas that received great attention are determining (a) the minimal set of control nodes to steer a network to a desired state [20]–[24]; and (b) the minimum control energy associated with a set of (driver) nodes to steer a network to a desired state [6], [8], [9], [25], [26]. The control energy is often quantified using certain scalar metrics of the spectrum of controllability Gramian matrix. Our work falls into the second category and we focus on the average controllability metric. Capitalizing on the sub-modularity property of several control energy metrics, including average energy, a few studies developed greedy algorithms to design network topology that has minimum control energy (see [27], [28]). Further, in [2], [29], the authors studied which group of nodes, when actuated as inputs, can be used to steer the network to an arbitrary target state, and at what cost. Different from the aforementioned works, which focus only on a metric for a group or full set of nodes, we consider a vector of such scalar metrics (specifically, average energy) to study the comparative influence of different groups of nodes. Ours is the first work to characterize this type of error bounds for the controllability of coarse networks, and we contribute to the nascent field of analyzing controllability/observability metrics for random linear systems [30], [31].

MOR for Complex Networks: There is a matured theory for MOR methods for dynamical systems [5], [32], [33]. However, MOR methods for network dynamics are limited. We briefly review some of the recent developments and recommend [34] for further reading. Methods based on reducing the dimension of individual subsystems in a network were described in [35], [36], where as methods based on reducing the number of nodes in a network using clustering and projection were described in [37]–[40]. Specifically, the latter methods typically rely on Petrov-Galerkin approximation, where projection matrices are constructed to retain certain structural aspects (e.g., clusters) in the reduced network. Although our coarsening operation relies on a given projection matrix, our goal is not to obtain a

reduced-order system that well-approximate, or retains some spatial structure of, the high-dimensional system. Rather we quantify the average controllability of the fine-scale network using a coarse network only. Finally, in the literature, the model reduction error is quantified using the system H_2 and H_∞ norms [37]. Instead, we use the ℓ_1 -norm to compare the average controllability of a group of nodes in fine-scale and their counterpart in the coarse network.

Coarsened SBM as a generative process: The SBM and its variants provide a powerful modeling framework to facilitate fundamental understanding of graph community organization and have found several applications, including social and power networks (see [14], [41], [42] and references therein). Several studies assume the knowledge of fine-scale network, which is a reasonable assumption for networks with a few thousand nodes. Nonetheless, there are several networks (eg., brain networks) with millions of nodes, and not all the network connections are available for a analyst to do inference (e.g., control or estimation). However, analyst can still have access to summaries of connections (for eg., restrictions in brain signal acquisition from macro-contacts, and imaging data [43], [44]). While the study of recovering graph structures from coarsened measurements has received some attention in the past [45]–[47], the study of extracting community structure from coarse summaries is recent. In [48], the authors extended the SBM for community detection [14], to lay out a framework for a coarsened, weighted variant of the SBM. We build off those results in this paper.

Notation: We denote vectors and matrices using bold faced small and upper case letters. For $\mathbf{Z} = [\mathbf{Z}_{uv}] \in \mathbb{R}^{n \times n}$, define $\|\mathbf{Z}\|_\infty = \max_{1 \leq u \leq n} \sum_{v=1}^n |\mathbf{Z}_{uv}|$; $\|\mathbf{Z}\|_{\max} = \max_{u,v} |\mathbf{Z}_{uv}|$; $\|\mathbf{Z}\|_{1,1} = \sum_{u=1}^n \sum_{v=1}^n |\mathbf{Z}_{uv}|$; and $\|\mathbf{Z}\|_2 = \sqrt{\lambda_{\max}(\mathbf{Z}^\top \mathbf{Z})}$, where λ_{\max} is the maximum eigenvalue. Denote the spectral radius of \mathbf{Z} by $\rho(\mathbf{Z}) = \max_i \{|\lambda_i(\mathbf{Z})|\}$, where λ_i is the i -th eigenvalue; $\text{diag}(\mathbf{Z}) = [\mathbf{Z}_{11}, \dots, \mathbf{Z}_{nn}]^\top \in \mathbb{R}^n$; and $\text{Diag}(\mathbf{Z})$ sets the off-diagonal entries of \mathbf{Z} to zero. The matrix inequality $\mathbf{Z}_1 \leq \mathbf{Z}_2$ implies element wise inequality. Denote n dimensional all ones and all zero vectors by $\mathbf{1}_n$ and $\mathbf{0}_n$. For $\mathbf{z} = [\mathbf{z}_u] \in \mathbb{R}^n$, define $\text{supp}(\mathbf{z})$ as the set of indices i for which $\mathbf{z}_i \neq 0$; $\|\mathbf{z}\|_1 = \sum_{u=1}^n |\mathbf{z}_u|$. For a positive integer n , we denote $[n] \triangleq \{1, \dots, n\}$. The cardinality of a set \mathcal{V} is denoted by $|\mathcal{V}|$. Vector \mathbf{e}_i is a standard basis vector containing all 0's except for its i -th element that is 1. The Hadamard (element-wise) product is denoted by \odot . We write $f(n) = \mathcal{O}(h(n))$ iff there exist positive reals c_0 and n_0 such that $|f(n)| \leq c_0 h(n)$ for all $n \geq n_0$. Conversely, $f(n) = \Omega(h(n))$ iff there exist $c_0, n_0 > 0$ such that $|f(n)| \geq c_0 h(n)$ for all $n \geq n_0$. $[a, b]$ shows the interval (set of real numbers) between scalar values a and b .

II. PRELIMINARIES AND PROBLEM STATEMENT

In this section, we introduce a class of stable LTI systems that evolve on networks with randomly generated edge weights. We then describe how we use the SBM to generate random *fine* networks with community structure, and how we model the coarse measurements of these networks. We then formulate our problem statement.

A. Dynamical systems and controllability

A network is an un-directed graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, with node set $\mathcal{V} \triangleq \{1, \dots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Each edge $(u, v) \in \mathcal{E}$ has a weight $\mathbf{A}_{uv} = \mathbf{A}_{vu} \in \mathbb{R}$. The *weighted symmetric adjacency matrix* of \mathcal{G} is $\mathbf{A} \triangleq [\mathbf{A}_{uv}]$, where $\mathbf{A}_{uv} = \mathbf{A}_{uv} = 0$ whenever $\mathbf{A}_{uv} \notin \mathcal{E}$. A random network is an un-directed graph where the $[\mathbf{A}_{uv}]$ are random variables. For a network \mathcal{G} with adjacency matrix \mathbf{A} , associate state $x_i[t] \in \mathbb{R}$ to the i -th node at time t . The states evolve according to LTI dynamics as

$$\mathbf{x}[t+1] = \mathbf{A}_{\text{nom}} \mathbf{x}[t] + \mathbf{B} \mathbf{u}[t], \quad \forall t \in \mathbb{N}. \quad (1)$$

The state $\mathbf{x}[t] = [x_1[t], \dots, x_n[t]]^\top$ is steered by the input $\mathbf{u}[t] \in \mathbb{R}^n$. The normalized state adjacency matrix $\mathbf{A}_{\text{nom}} \triangleq \mathbf{A}/(\epsilon + \rho(\mathbf{A}))$ is such that the spectral radius $\rho(\mathbf{A}_{\text{nom}}) < 1, \forall \epsilon > 0$; see [3]. The input matrix $\mathbf{B} = \text{Diag}(\mathbf{b})$ with $\mathbf{b} \in \{0, 1\}^n$, where we call $\mathcal{K} = \text{supp}(\mathbf{b}) \subset [n]$ the *control node set*, determines which nodes receive control input. Note that $\mathbf{B} \mathbf{u}[t]$ is equal to $\mathbf{Z} \mathbf{u}_{\mathcal{K}}[t]$, where \mathbf{Z} is a rectangular matrix composed of columns of \mathbf{B} indexed by \mathcal{K} , and $\mathbf{u}_{\mathcal{K}}$ is a \mathcal{K} -length vector. We stick with the former notation in (1) to enhance readability of notations in the next sections. For instance, for $\mathbf{B} = \text{Diag}(\mathbf{1}_{n_1}, \mathbf{0}_{n-n_1})$, the input enters the network through control nodes set $\mathcal{K} = \{1, \dots, n_1\}$.

Average controllability is a widely studied metric of controllability that quantifies the difficulty of controlling a network [2], [6], [49] defined as $\text{tr}[\mathcal{C}_T(\mathbf{A}_{\text{nom}}, \mathbf{B})]$, where

$$\mathcal{C}_T(\mathbf{A}_{\text{nom}}, \mathbf{B}) = \sum_{\ell=0}^{T-1} (\mathbf{A}_{\text{nom}})^\ell \mathbf{B} \mathbf{B}^\top (\mathbf{A}_{\text{nom}})^\ell \quad (2)$$

is the T -step controllability Gramian of (1). By construction $\mathcal{C}_T(\mathbf{A}_{\text{nom}}, \mathbf{B}) \succeq 0$, and hence, $\text{tr}[\mathcal{C}_T(\mathbf{A}_{\text{nom}}, \mathbf{B})] \geq 0$. When the context is clear, we drop the arguments \mathbf{A}_{nom} and \mathbf{B} from $\mathcal{C}_T(\mathbf{A}_{\text{nom}}, \mathbf{B})$ and write \mathcal{C}_T . For ease of exposition, we work with the average controllability for the infinite time horizon Gramian $\mathcal{C} = \lim_{T \rightarrow \infty} \mathcal{C}_T$, which exists when $\rho(\mathbf{A}_{\text{nom}}) < 1$.

The inverse of the average controllability is a lower bound on the minimum energy (in expectation) required to drive the system in (1) from the origin to any target state on the unit sphere. This energy would be $\int_S \mathbf{x}^\top \mathcal{C}^{-1} \mathbf{x} d\mathbf{x} / \int_S d\mathbf{x} = n^{-1} \text{tr}(\mathcal{C}^{-1})$, where $S = \{\mathbf{x} : \|\mathbf{x}\|_2 = 1\}$ [8], [49]. It follows that $\text{tr}(\mathcal{C}^{-1}) \geq 1/\text{tr}(\mathcal{C})$. The tightness of the bound is numerically explored in [3], [50]. The higher $\text{tr}(\mathcal{C})$ is for a given set of control nodes (indexed by the non-zero columns in \mathbf{B}), the smaller the average energy, and the lower the influence that set has on the network. In this light, $\text{tr}(\mathcal{C})$ can be interpreted as the overall network controllability proxy in all directions in the state space [51]. Owing to the above discussion and given the ill-posedness of \mathcal{C} for high-dimensional networks, the average controllability provides an easily computable proxy for the average energy [28], [52], [53]. Furthermore, because $\text{tr}(\mathcal{C})$ equals the energy of the impulse response (or equivalently, the H_2 -norm of the network system), the average controllability quantifies the capability of grouped input nodes to drive the network to easy-to-reach state with minimum effort [54]–[56].

We can interpret the average controllability using the nuclear norm which is the sum of singular values, i.e., the trace of a positive semi-definite matrix [57]. The nuclear norm is the

convex envelope of the rank function (that is, the number of rank-one matrices needed to obtain a given matrix [58, page 71]). The higher the nuclear norm, meaning the large average controllability, the harder will be to approximate the Gramian or metrics of the Gramian by those of its reduced-order model. This is one reason why our learning-based approach (see Section IV) that directly estimates the average controllability can perform better.

B. The stochastic block model

The stochastic block model (SBMs) is a probabilistic model that produces random graphs with communities. We use this model, as hinted in the introduction, to reveal the role of community structure in the quality of controllability estimation from coarse measured graphs. Formally, let $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ be the un-directed graph (i.e., the fine graph) with n nodes and random edge weights generated according to the general GSBM($n, \mathbf{Q}, \mathbf{d}$):

Definition 1. (General SBM) Given a natural number K , vector $\mathbf{d} \in [0, 1]^K$, and a matrix $\mathbf{Q} \in [0, 1]^{K \times K}$, the GSBM($n, \mathbf{Q}, \mathbf{d}$) is a distribution on a graph \mathcal{G} of size n such that 1) the \mathcal{G} 's node set, denoted by \mathcal{V} where $|\mathcal{V}| = n$, is partitioned into K disjoint communities $\mathcal{V} = \bigcup_{k=1}^K \mathcal{V}_k$. The communities have relative sizes $\mathbf{d} = [\mathbf{d}_1, \dots, \mathbf{d}_K]$ where $|\mathcal{V}_k| = n\mathbf{d}_k$; and 2) two nodes $u \in \mathcal{V}_k$ and $v \in \mathcal{V}_{k'}$ are joined by an edge with weight $\mathbf{A}_{uv} \in [0, 1]$ drawn with probability $\mathbf{Q}_{kk'}$ independently from other edges, for all $k, k' \in [K]$. \square

The GSBM($n, \mathbf{Q}, \mathbf{d}$) generates a random un-directed graph with K communities with non-identical intra- and cross-community connection probabilities specified by $\mathbf{Q} \in [0, 1]^{K \times K}$. An operational interpretation of the General SBM is provided by the community membership matrix $\mathbf{P} \in \mathbb{R}^{K \times n}$:

$$\mathbf{P}_{kv} = \begin{cases} 1 & \text{if } v \in \mathcal{V}_k \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

For any $k, k' \in [K]$, it follows that

$$\mathbf{A}_{uv} \sim \text{Bernoulli}(\mathbf{Q}_{k,k'}), \quad \text{if } \mathbf{P}_{ku} = \mathbf{P}_{k'v} = 1. \quad (4)$$

It is immediate that $\mathbf{P}\mathbf{P}^\top = \text{Diag}(|\mathcal{V}_1|, \dots, |\mathcal{V}_K|)$, where $|\mathcal{V}_k|$ is the size of the k -th community. Let $\mathbf{D} \triangleq \frac{1}{n}\mathbf{P}\mathbf{P}^\top = \text{Diag}(\mathbf{d})$ be the diagonal matrix of relative community sizes.

To facilitate analysis and to harness the power of SBM generated networks, we work with high-dimensional networks. We make some assumptions on the scaling of the parameters of the underlying SBM as $n \rightarrow \infty$.

Assumption 1. (SBM scaling [14]) There exists a sequence $\rho_n \in (0, 1)$ and a constant $K \times K$ matrix $\mathbf{Q}_o \geq 0$ such that

$$\mathbf{Q} = \rho_n \mathbf{Q}_o, \quad (5)$$

where $0 \leq [\mathbf{Q}_o]_{kk'} \leq 1$, and $\rho_n \sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$.

The parameter ρ_n in (5) allows us to regulate the density (or sparsity) of connections in and across communities. For several real-world networks, ρ_n typically decreases with n [14].

Remark 1. The relative community sizes of the GSBM in Def. 1 are set to \mathbf{d} . Another common model is to choose community

memberships in an i.i.d. manner from some distribution \mathbf{p} . In this model, relative community sizes become random with $\mathbb{E}[\mathbf{d}] = \mathbf{p}$. The analysis of this paper will hold for such a model except that, when evoking the expectation of the fine matrix \mathbf{A} , an outer expectation should be added to account for the randomness in community sizes.

C. Fine Graph and Coarse Measurements

In this part, we describe the modeling choice and notations for the coarse measuring process. We assume that the fine graph $\mathcal{G}_{\text{fine}}$ with adjacency matrix \mathbf{A} is drawn from the General SBM distribution in Definition 1. Moreover, (1) is the fine-scale system dynamics, and we refer to it as $\mathcal{S}_{\text{fine}}$, i.e.

$$\mathcal{S}_{\text{fine}} : \mathbf{x}[t+1] = \mathbf{A}_{\text{nom}}\mathbf{x}[t] + \mathbf{B}\mathbf{u}[t], \quad \forall t \in \mathbb{N}. \quad (6)$$

We define the coarse-scale summary of \mathbf{A} as

$$\tilde{\mathbf{A}} \triangleq \mathbf{W}\mathbf{A}\mathbf{W}^\top \in \mathbb{R}^{m \times m}, \quad (7)$$

where $\mathbf{W} \in \mathbb{R}^{m \times n}$ is the coarse-measurement matrix whose rows indicate which parts of the graph are measured. In this paper we assume (a) the i -th row of the coarse-measurement matrix \mathbf{W} has r_i equal non-zero terms which sum to 1. These non-zero indices are the fine nodes that are measured as a group; and (b) $\mathbf{W}\mathbf{W}^\top = \text{Diag}([1/r_1, \dots, 1/r_m])$, meaning the sets of fine nodes corresponding to each coarse measurement are disjoint. The coarsened matrix $\tilde{\mathbf{A}}$ is the starting point of our analysis.

The matrix $\tilde{\mathbf{A}}$ can be interpreted as the adjacency matrix of an un-directed coarse graph $\mathcal{G}_{\text{coarse}}$ with m fictitious nodes—referred to as the (coarse or) c -nodes. This interpretation is helpful as we discuss the network dynamics of $\tilde{\mathbf{A}}$ in Section III. The regime where the total coverage size $\sum_{i \in [m]} r_i \ll n$ indicates that c -nodes cover the fine graph only sparsely. In other words, there may exist (several) fine nodes that do not contribute to $\tilde{\mathbf{A}}$ (see Fig. 1). Similar coarsening models have been used in the literature (see [47], [48]).

D. Problem Statement

Our objective is to quantify the average controllability of $\mathcal{G}_{\text{fine}}$ using only $\mathcal{G}_{\text{coarse}}$ (see (7)), and the knowledge of $\mathcal{G}_{\text{fine}}$'s community structure in Definition 1. Furthermore, we do not have access to the way the coarse graph is acquired at the time of decision making; that is, the coarse-measurement matrix \mathbf{W} is unknown, meaning we do not know which group of fine nodes are mapped to each c -node.

Let $\mathbf{A}_{\text{nom}} = \mathbf{A}/(\epsilon + \rho(\mathbf{A}))$ in (6) be such that \mathbf{A} is drawn from General SBM given by (4). The input matrix $\mathbf{B}_{\mathcal{K}} \triangleq \mathbf{B}$ is diagonal by definition. Let $\mathcal{K}_i = \text{supp}(\mathbf{w}_i)$, for $i \in [m]$, be the set (hereafter, group) of nodes coarsened by the i -th row \mathbf{w}_i of \mathbf{W} in (7). As a result, the control node groups $\{\mathcal{K}_1, \dots, \mathcal{K}_m\}$ match the measured groups.

Recall that \mathbf{w}_i is the coarse measuring vector mapping to the i -th c -node. For the \mathcal{K}_i -th group with $\mathbf{B}_{\mathcal{K}_i} = \text{Diag}(\mathbf{w}_i^\top)$, $i \in [m]$, consider the average controllability metric $\theta_{\text{group}, \mathbf{A}}^{(i)} \triangleq \text{tr}[\mathbf{C}(\mathbf{A}_{\text{nom}}, \mathbf{B}_{\mathcal{K}_i})]$. Define the m -dimensional group average controllability vector as

$$\boldsymbol{\theta}_{\text{group}, \mathbf{A}} \triangleq [\theta_{\text{group}, \mathbf{A}}^{(1)}, \dots, \theta_{\text{group}, \mathbf{A}}^{(m)}]^\top. \quad (8)$$

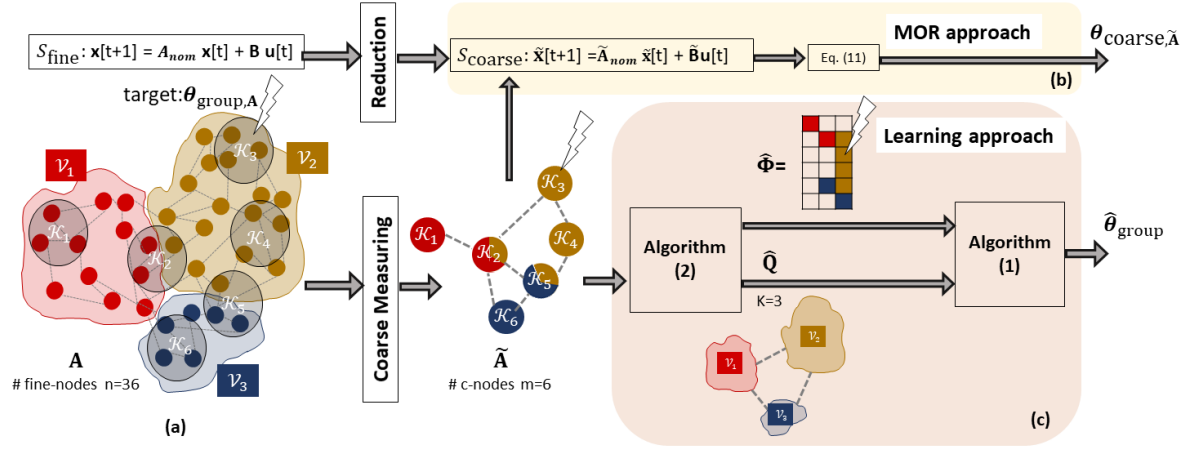


Fig. 1. Schematic of PROM- and learning-based approaches for estimating $\theta_{\text{group},A}$. (a) For $\mathcal{G}_{\text{fine}}$ consisting of $n = 36$ nodes, we have $K = 3$ communities ($\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$) with $m = 6$ coarse (c)-nodes ($\mathcal{K}_1, \dots, \mathcal{K}_6$), each having coverage size $r = 3$. The control node set is \mathcal{K}_3 . The lightening bolts show an example of where external input can be exerted. (b) In the PROM-based approach, we infer $\theta_{\text{group},A}$ via the reduced-order dynamical system S_{coarse} . (c) The learning-based approach capitalizes on the mixed-membership (MM) Algorithm 2 to estimate $\theta_{\text{group},A}$ directly from \tilde{A} . This avoids the need to consider S_{coarse} .

Note that $\theta_{\text{group},A}^{(i)} \geq 0$, due to the positive semi-definiteness of the Gramian $\mathcal{C}(\cdot)$. This vector summarizes the average controllability metrics for all control nodes sets $\{\mathcal{K}_1, \dots, \mathcal{K}_m\}$. Thus, $\theta_{\text{group},A}$ helps us deduce several quantitative properties that are key to being able to control (6) via $B_{\mathcal{K}_i}$. For example, θ_{group} tells us which \mathcal{K}_i s can drive the network to the desired target state with least control effort.

Using only the knowledge of \tilde{A} in (7), our goal is to estimate the vector $\theta_{\text{group},A}$ in (8). We will develop two approaches. The first, *Passively Reduced-Order Model (PROM)*, is based on the traditional *model order reduction (MOR)* approach. Here we rely upon a passively reduced-order auxiliary system S_{coarse} in (10), which is governed by \tilde{A} , to infer $\theta_{\text{group},A}$. Our second approach is learning-based. We directly estimate $\theta_{\text{group},A}$ using a clustering-based mixed-membership community-learning algorithm (see Section IV). In the latter case, we bypass the intermediate step of dealing with a reduced-order system. Fig. 1 illustrates the proposed approaches. We use the ℓ_1 -norm difference between the true- and the estimated controllability vectors to compare the performance of these competing methods (see (12)). Our main results in Sections III and IV are probabilistic because A is a random matrix.

We introduce the expected quantities $\bar{A} \triangleq \mathbb{E}[A]$ and $\tilde{\bar{A}} \triangleq \mathbb{E}[\tilde{A}]$. We later use these to place some mild regulations on $\mathcal{G}_{\text{coarse}}$. From (4) and (7), the expected quantities evaluate to

$$\bar{A} = P^T Q P \quad \text{and} \quad \tilde{\bar{A}} = \Phi Q \Phi^T, \quad (9)$$

where $\Phi \triangleq W P^T \in \mathbb{R}^{m \times K}$ is the *coarse community membership matrix*, and Φ_{ik} captures how much the i -th c-node overlaps with the k -th community. To make the analysis less cumbersome, in what follows, we let the coarsening matrix has fixed row support (r). The example below illustrates the definition of the matrices P , D , Φ , and W .

Example 1. For the network shown in Fig. 2-(a), the fine

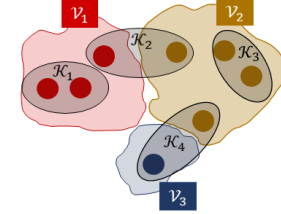


Fig. 2. A simple case of a community-structured network $\mathcal{G}_{\text{fine}}$ and its coarse measurements for Example 1. This graph contains $n = 8$ nodes, $K = 3$ communities ($\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$) and $m = 4$ coarse (c)-nodes ($\mathcal{K}_1, \dots, \mathcal{K}_4$), each having coverage size $r = 2$. Note that c-nodes 1, 3 are perfectly synchronized with one community, while c-nodes 2, 4 overlap with two communities.

community membership matrix is

$$P = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \{0, 1\}^{3 \times 8}.$$

The relative fine-scale community sizes are

$$\mathbf{d} = \left[\frac{3}{8}, \frac{1}{2}, \frac{1}{8} \right], \quad D = \text{Diag}(\mathbf{d}) = \begin{bmatrix} \frac{3}{8} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \in \mathbb{R}^{3 \times 3}.$$

The coarse-measurement matrix where $r = 2$ is

$$W = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{4 \times 8}.$$

The coarse community membership matrix tells us what fractions of each coarse node's group of fine nodes belong to each underlying community,

$$\Phi = W P^T = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{4 \times 3}.$$

Finally, we note that the rows of Φ^T determine that each community contains what portion of the coarse nodes' groups of fine nodes. Hence,

$$\frac{\Phi^T \Phi}{m} = \begin{bmatrix} \frac{5}{16} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ 0 & \frac{1}{16} & \frac{1}{16} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

is a measure of the correlation of (and cross-correlation between) the (coarse membership) representations of communities (i.e., the rows of Φ^T). Matrix $\frac{\Phi^T \Phi}{m}$ plays a role in the error bound of Theorem 1 and is discussed in Section III. Each c-node in $\{1, 3\}$ covers one community; each c-nodes in $\{2, 4\}$ overlap with two. \square

Remark 2. (Synchronization of community and coarsening): The coarsening operation in Section II-C is oblivious to the community structure of $\mathcal{G}_{\text{fine}}$. Thus, the i -th c-node may contain information about multiple communities if $|\text{supp}(\Phi_i)| > 1$ (the subscript denotes the i -th row). **Perfect synchronization** is the situation in which $|\text{supp}(\Phi_i)| = 1$. This occurs when each c-node covers nodes from only one community.

III. APPROXIMATING GROUP AVERAGE CONTROLLABILITY: PROM-BASED APPROACH

In this section, we define PROM (Passively Reduced-Order Model) as an auxiliary or fictitious network linear dynamics associated with $\mathcal{G}_{\text{coarse}}$. We then analyze how well the average controllability vector of this reduced system approximates $\theta_{\text{group}, \mathbf{A}}$ in (8). Formally, consider the coarse system $\mathcal{S}_{\text{coarse}}$ defined by the dynamics

$$\mathcal{S}_{\text{coarse}} : \tilde{\mathbf{x}}[t+1] = \tilde{\mathbf{A}}_{\text{nom}} \tilde{\mathbf{x}}[t] + \tilde{\mathbf{B}} \tilde{\mathbf{u}}[t], \quad (10)$$

where $\tilde{\mathbf{A}}_{\text{nom}} = \tilde{\mathbf{A}}/(\epsilon + \rho(\tilde{\mathbf{A}}))$, $\tilde{\mathbf{A}}$ is given by (7). Note that $\tilde{\mathbf{x}}[t] \in \mathbb{R}^m$ is not a compressed version of the state $\mathbf{x}[t]$ in $\mathcal{S}_{\text{fine}}$. Rather, $\tilde{\mathbf{x}}[t]$ is a fictitious state controlled by the dynamics of the (scaled) matrix $\tilde{\mathbf{A}}$. This fictitious state is controlled by the input $\tilde{\mathbf{B}} \tilde{\mathbf{u}}[t] \in \mathbb{R}^m$. We refer to this coarse-scale system modeling from the fine-scale system as PROM because the dimension of the coarse system can be much less than that of the fine system, i.e. $m \ll n$.

Since each group of nodes that receive control input maps to one c-node, $\tilde{\mathbf{B}}_i = \mathbf{e}_i$ is the coarsened input matrix. Also let $\theta_{\text{coarse}, \tilde{\mathbf{A}}}^{(i)} \triangleq \text{tr}[\mathbf{C}(\tilde{\mathbf{A}}_{\text{nom}}, \tilde{\mathbf{B}}_i)] \geq 1$ be the average controllability for the i -th node in $\mathcal{G}_{\text{coarse}}$, which is, by definition, lower-bound by 1. Define the coarse average controllability vector:

$$\theta_{\text{coarse}, \tilde{\mathbf{A}}} \triangleq \left[\theta_{\text{coarse}, \tilde{\mathbf{A}}}^{(1)}, \dots, \theta_{\text{coarse}, \tilde{\mathbf{A}}}^{(m)} \right]^T \in \mathbb{R}^m. \quad (11)$$

We show in Theorem 1 that the $\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ well approximates $\theta_{\text{group}, \mathbf{A}}$ under some conditions. To quantify the approximation, we define the ℓ_1 -error metric:

$$\Delta(\mathbf{A}, \tilde{\mathbf{A}}) \triangleq \left\| \frac{r \theta_{\text{group}, \mathbf{A}} - \mathbf{1}_m}{\|r \theta_{\text{group}, \mathbf{A}} - \mathbf{1}_m\|_1} - \frac{\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m}{\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} \right\|_1. \quad (12)$$

We prefer $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ to the usual metric $\|\theta_{\text{group}, \mathbf{A}} - \theta_{\text{coarse}, \tilde{\mathbf{A}}}\|_1$ for the following reasons. First, the factor of r in the first term of $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ in (12) accounts for the additional scaling

by $1/r$ in the fine system input matrix $\mathbf{B}_{\mathcal{K}_i} = \text{Diag}(\mathbf{w}_i^T)$ (see Lemma 2 in the appendix). Second, the shift by -1 discounts the inherent $+1$ shift in the average controllability definition (see (23) in the appendix). Third, the denominators in (12) ensure that the error is scale-invariant, i.e. the error is unaffected if all elements of either of the controllability vectors are multiplied by a common scalar factor. Moreover we chose $\|\cdot\|_1$ over other vector norms (like $\|\cdot\|_{\max}$) as it aggregates the errors when choosing a singular or any combinations of c-nodes to stimulate sequentially.

The following matrix will appear in Theorems 1 and contributes to upper bounding the controllability estimation errors

$$\Upsilon \triangleq \mathbf{D}^{-\frac{1}{2}} \left(\left[\mathbf{I} - (\iota \mathbf{D}^{\frac{1}{2}} \mathbf{Q}_o \mathbf{D}^{\frac{1}{2}})^2 \right]^{-1} - \mathbf{I} \right) \mathbf{D}^{-\frac{1}{2}} \in \mathbb{R}^{K \times K}, \quad (13)$$

where $\iota = 1/(\frac{\epsilon}{\rho_n n} + \rho(\mathbf{Q}_o \mathbf{D}))$ is the normalization factor $1/(\epsilon + \rho(\tilde{\mathbf{A}}))$ in (6) after substituting $\tilde{\mathbf{A}}$ with \mathbf{A} , with $\rho(\cdot)$ is the spectral radius, ρ_n and \mathbf{Q}_o are in Assumption 1. We set the normalized matrix $\mathbf{Z} = \iota \mathbf{D}^{\frac{1}{2}} \mathbf{Q}_o \mathbf{D}^{\frac{1}{2}}$. Hence, the term $\left[\mathbf{I} - (\iota \mathbf{D}^{\frac{1}{2}} \mathbf{Q}_o \mathbf{D}^{\frac{1}{2}})^2 \right]^{-1}$ in (13) becomes the resolvent of \mathbf{Z} , i.e. $(z\mathbf{I} - \mathbf{Z}^2)^{-1}$, evaluated at $z = 1$. Also, note that matrix $\mathbf{D}^{\frac{1}{2}} \mathbf{Q}_o \mathbf{D}^{\frac{1}{2}}$ is the intra- and cross-community probability matrix that is re-weighted according to the relative community sizes. Recall that the resolvent captures the direct and indirect effects (via the sum of the powers) of $\mathbf{D}^{\frac{1}{2}} \mathbf{Q}_o \mathbf{D}^{\frac{1}{2}}$. As a result, matrix Υ in (13) encapsulates the size-weighted community-connection probability matrix's direct and indirect impacts. For more information on the significance and the properties of resolvents see [59]. Later, in Lemma 1, we see how $[\Upsilon]_{kk}$ moves in tandem (i.e. increase and decrease together) with both $\theta_{\text{group}, \tilde{\mathbf{A}}}^{(i)}$ and $\theta_{\text{coarse}, \tilde{\mathbf{A}}}^{(i)}$ for i -th c-nodes that are perfectly synchronized with community k . The aforementioned definitions coupled with Lemma 1 in Section IV, contribute to the proof of the next Theorem.

Theorem 1. (ℓ_1 bound on $\theta_{\text{group}, \mathbf{A}} - \theta_{\text{coarse}, \tilde{\mathbf{A}}}$): Let $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ be defined as in (12) for $\mathbf{A} \in \{0, 1\}^{n \times n}$ and $\tilde{\mathbf{A}} \in \mathbb{R}^{m \times m}$, and $0 < \delta < 1$ be a constant. Then, under Assumption 1, there exists $n_0, m_0 \in \mathbb{N}$ such that for $n > n_0, m > m_0$

$$\Delta(\mathbf{A}, \tilde{\mathbf{A}}) = \mathcal{O} \left(\frac{\eta + \tilde{\eta}}{\rho_n} + \sqrt{\frac{n}{mr}} \sqrt{\frac{\log(\frac{1}{\delta})}{2}} + \frac{\|\mathbf{L}_{\Phi, \Upsilon}\|_1}{m} + \left\| \mathbf{D} - \frac{\Phi^T \Phi}{m} \right\|_{\max} \right)$$

with probability at least $1 - 3\delta$. Here $\eta = \frac{\sqrt{\log(\frac{n^2}{\delta})/(2n)}}{\|\mathbf{Q} \mathbf{D} \mathbf{Q}\|_{\min}}$;

$\tilde{\eta} = \frac{\sqrt{\log(\frac{m^2}{\delta})/(2m)}}{\|\mathbf{Q} \frac{\Phi^T \Phi}{m} \mathbf{Q}\|_{\min}}$; $\mathbf{L}_{\Phi, \Upsilon} = \Phi \text{diag}(\Upsilon) - \text{diag}(\Phi \Upsilon \Phi^T)$; $\mathbf{D} = (1/n) \mathbf{P} \mathbf{P}^T$; Υ is defined in (13); Φ is defined in (9) and \mathbf{Q} in Def. 1. \square

Theorem 1 indicates that $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ approaches zero as: 1) the total coverage proportion, i.e. mr/n , increases; 2) η and $\tilde{\eta}$, that guarantee the bound on $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ exists with large probability, become negligible; 3) $\frac{1}{m} \|\Phi \text{diag}(\Upsilon) - \text{diag}(\Phi \Upsilon \Phi^T)\|_1$ and $\|\mathbf{D} - \frac{\Phi^T \Phi}{m}\|_{\max}$ decrease. The latter two terms have to do with *synchronization (or overlap)*, i.e. the extent to which coarse measurements are synchronized (or overlapped) with communities (see Remark 2). The first term $\frac{1}{m} \|\Phi \text{diag}(\Upsilon) -$

$\text{diag}(\Phi \Upsilon \Phi^T)\|_1$ is equal to zero if perfect synchronization holds. The relationship between the second term, $\|\mathbf{D} - \frac{\Phi^T \Phi}{m}\|_{\max}$, and community overlap is less straightforward. To understand this, note that matrix $\frac{\Phi^T \Phi}{m}$ (see Example 1) quantifies a measure of correlation between (coarse membership) representations of communities. This matrix reveals two natural tradeoffs in the coarsening process. The first is *overlap*: the off-diagonal elements, $[\frac{\Phi^T \Phi}{m}]_{kk'} = \sum_{i \in [m]} \Phi_{ik} \Phi_{ik'}/m$ for $k \neq k'$, show the extent of synchronization between coarse measurements and communities (c.f., Remark 2 and Example 1). The second is *balancedness*: the diagonal elements, $[\frac{\Phi^T \Phi}{m}]_{kk} = \sum_{i \in [m]} \Phi_{ik}^2/m$ for all $k \in [K]$, reflect the relative community sizes *after* coarse measuring. Separately considering diagonal and off-diagonal elements, helps demystify the role of the error term $\|\mathbf{D} - \frac{\Phi^T \Phi}{m}\|_{\max}$ in Theorem 1:

$$\left| [\mathbf{D} - \frac{\Phi^T \Phi}{m}]_{k,k'} \right| = \begin{cases} \sum_{i \in [m]} \Phi_{ik} \Phi_{ik'}/m & \text{if } k \neq k' \\ |\mathbf{d}_k - \sum_{i \in [m]} \Phi_{ik}^2/m| & \text{if } k = k'. \end{cases} \quad (14)$$

From (14) we infer that $\|\mathbf{D} - \frac{\Phi^T \Phi}{m}\|_{\max}$ approaches zero when c-nodes overlap with fewer communities, *and* c-nodes are more evenly spread over communities. The latter means that larger communities get more coarse measurements, and vice versa.

Overall, we conclude that the PROM-based estimate $\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ will well approximate $\theta_{\text{group}, \mathbf{A}}$ when the graph density ρ_n and number of fine- and c-nodes are sufficiently large, when coarse measurements are sufficiently synchronized with the communities, and when coarse measurements are distributed across communities in a balanced way. In many settings such conditions will be (roughly) satisfied in practice, in particular when communities and coarse measurements are spatially (geographically) localized [60], [61]. This is because nearby nodes tend to be connected and form communities (e.g., social networks of friends and neighbours, regions with distinct functional and structural features in the brain). Moreover, coarse-measuring is likely local (e.g. parcellation of brain imaging data, electrophysiological recording of the brain, aggregation of geographically adjacent nodes). Also, coarse measurements tend to be uniformly distributed across a network and, hence, larger communities are typically measured more frequently.

IV. APPROXIMATING GROUP AVERAGE CONTROLLABILITY: LEARNING-BASED APPROACH

We now present our learning-based approach to estimate $\theta_{\text{group}, \mathbf{A}}$. Unlike the PROM-based approach that relies on $\mathcal{S}_{\text{coarse}}$, we directly estimate elements in $\theta_{\text{group}, \mathbf{A}}$ using a mixed-membership (MM) community-learning algorithm [17], [62], [63]. Specifically, we work with the mixed-membership algorithm of [17] which is not only numerically efficient but comes with strong theoretical guarantees.

As mentioned in Section III, the following lemma is instrumental in the proof of both of our primary results, Theorems 1 and 2. This lemma also provides motivation for our candidate estimator in (16). For these reasons, we delayed the presentation of Lemma 1 until now. In essence, this lemma characterizes the group-average controllability vector associated with the expected matrix $\tilde{\mathbf{A}}$ in (9).

Lemma 1. (Group Controllability of Expected Dynamics): Define $\theta_{\text{group}, \tilde{\mathbf{A}}}$ by replacing \mathbf{A} in (8) with the expected matrix $\tilde{\mathbf{A}}$ given by (9). Let Φ be as in (9), \mathbf{P} be in (3), and r be the coverage size of \mathbf{W} in (7). Under Assumption 1, we have

$$\theta_{\text{group}, \tilde{\mathbf{A}}} = \frac{1}{r} \left(\mathbf{1}_m + \frac{1}{n} \Phi \text{diag}(\Upsilon) \right), \quad (15)$$

where n and m are the dimensions of \mathbf{A} and $\tilde{\mathbf{A}}$; moreover Υ is defined in (13). \square

Lemma 1 gives us a formula to compute the group average controllability vector associated with the expected matrix $\tilde{\mathbf{A}}$. Recall that $\Phi_{i,k} = |\{v : v \in \mathcal{V}_k \cap \text{supp}(\mathbf{w}_i)\}|/|\text{supp}(\mathbf{w}_i)|$ ($k \in [K]$, $i \in [m]$) is the fraction of the i -th c-node's overlap with community k . From Lemma 1, it follows that $\theta_{\text{group}, \tilde{\mathbf{A}}}^{(i)} \propto \sum_{k \in [K]} \Phi_{i,k} \Upsilon_{kk}$. In view of this observation, we infer that c-nodes that have the most overlap with communities of largest Υ_{kk} are the most controllable. Moreover, Theorem 6 (see Appendix) shows that $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ can be arbitrarily small with high probability. This fact coupled with Lemma 1, leads us to propose our candidate estimator

$$\hat{\theta}_{\text{group}} \triangleq \mathbf{1}_m + \hat{\Phi} \text{diag}(\hat{\Upsilon}), \quad (16)$$

where

$$\begin{aligned} \hat{\Upsilon} &\triangleq \hat{\mathbf{D}}^{-\frac{1}{2}} \left(\left[\mathbf{I} - (\hat{\mathbf{D}}^{\frac{1}{2}} \hat{\mathbf{Q}}_o \hat{\mathbf{D}}^{\frac{1}{2}})^2 \right]^{-1} - \mathbf{I} \right) \hat{\mathbf{D}}^{-\frac{1}{2}}, \\ \hat{\iota} &= \epsilon/n + \rho(\hat{\mathbf{Q}} \hat{\mathbf{D}}). \end{aligned} \quad (17)$$

Note that the candidate estimator in (16) is similar to the true group controllability vector in (15), but there are two important differences. First, the true terms Φ , \mathbf{Q} , and \mathbf{D} are replaced with their estimated (hatted) counterparts. Second, the constant coefficients $1/r$ and $1/n$ in (15) are omitted since they affect neither the error metric (see (18) wherein $1/r$ is compensated and $1/n$ cancels out in both the numerator and denominator) nor the order of nodes when assessed based on their controllability value. The hatted quantities $\hat{\Phi}$ and $\hat{\mathbf{Q}}$ are obtained from Algorithm 2, which takes as input $\tilde{\mathbf{A}}$ and the number of communities K^2 . We obtain $\hat{\mathbf{D}}$ from Algorithm 1. Importantly, $\hat{\theta}_{\text{group}}$ in (16) is obtained from coarsened matrix $\tilde{\mathbf{A}}$ and not from the fine scale matrix \mathbf{A} . The following assumption is required by [17] for the recovery of $\hat{\Phi}$, $\hat{\mathbf{Q}}$ using their proposed Algorithm 2.

Assumption 2. (Existence of a perfectly-synchronized c-node): For all $k \in [K]$, there exist at least one coarse node $i \in [m]$ such that $\Phi_{ik} = 1$.

This means that $\mathcal{G}_{\text{coarse}}$ has at least one perfectly-synchronized (or equivalently “pure”) c-node per community $k \in [K]$. In real-world networks communities and coarse measurements are often spatially or geographically localized (see Section III for a more detailed explanation). Hence, it is reasonable to expect that there is at least one coarse measurement

²Algorithm 2, a mixed-membership algorithm, is adapted from [17]. It is a type of spectral clustering method that first performs eigen decomposition of $\tilde{\mathbf{A}}$ to find the overlapping membership (Φ_{ik}) of the fine nodes. A pruning step is included (see steps 4 and 5 in Algorithm 2) that improves algorithm performance.

per community that does not overlap with multiple communities. The next theorem bounds the error of our proposed learning-based estimation in (16) using the following metric:

$$\hat{\Delta}(\tilde{\mathbf{A}}) = \left\| \frac{\hat{\boldsymbol{\theta}}_{\text{group}} - \mathbf{1}_m}{\|\hat{\boldsymbol{\theta}}_{\text{group}} - \mathbf{1}_m\|_1} - \frac{r\boldsymbol{\theta}_{\text{group},\mathbf{A}} - \mathbf{1}_m}{\|r\boldsymbol{\theta}_{\text{group},\mathbf{A}} - \mathbf{1}_m\|_1} \right\|_1. \quad (18)$$

Theorem 2. (ℓ_1 error bound on $\hat{\boldsymbol{\theta}}_{\text{group}} - \boldsymbol{\theta}_{\text{group},\mathbf{A}}$): Let $\hat{\Delta}(\tilde{\mathbf{A}})$ be defined as in (18) for $\mathbf{A} \in \{0, 1\}^{n \times n}$ and $\tilde{\mathbf{A}} \in \mathbb{R}^{m \times m}$, and $0 < \delta < 1$ be a constant. Then, under Assumptions 1,2, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$

$$\hat{\Delta}(\tilde{\mathbf{A}}) = \mathcal{O} \left(\frac{\eta}{\rho_n} + \sqrt{\frac{n}{mr}} \sqrt{\frac{\log(\frac{1}{\delta})}{2}} + \frac{\|\mathbf{L}_{\Phi}\|_{1,1}}{m} + \|\mathbf{L}_{\mathbf{Q}}\|_{\max} + \|\mathbf{L}_{\mathbf{D}}\|_{\max} \right)$$

holds with probability at least $1 - 2\delta$, where $\mathbf{L}_{\Phi} = \hat{\Phi} - \Phi$, $\mathbf{L}_{\mathbf{Q}} = \hat{\mathbf{Q}} - \mathbf{Q}$, and $\mathbf{L}_{\mathbf{D}} = \hat{\mathbf{D}} - \mathbf{D}$; $\eta = \frac{\sqrt{\log(\frac{n^2}{\delta})/(2n)}}{\|\mathbf{Q}\mathbf{D}\mathbf{Q}\|_{\min}}$; n and m are the dimensions of \mathbf{A} and $\tilde{\mathbf{A}}$; Φ is defined in (9) and \mathbf{Q} in Def. 1; and $\mathbf{D} = (1/n)\mathbf{P}\mathbf{P}^T$. \square

Theorem 2 suggests that for sufficiently large n (smaller η which yields high probabilistic guarantees for the bound on $\hat{\Delta}(\tilde{\mathbf{A}})$) and large total coverage proportion $\frac{mr}{n}$, the estimate $\hat{\boldsymbol{\theta}}_{\text{group}}$ approximates $\boldsymbol{\theta}_{\text{group},\mathbf{A}}$ to arbitrary precision under the following conditions. First, the graph is dense enough (sufficiently large ρ_n). Second, the coarse community membership matrix is well estimated (smaller $\|\mathbf{L}_{\Phi}\|_{1,1}/m$). The third condition needed to yield small $\hat{\Delta}(\tilde{\mathbf{A}})$ is that cross-community probability estimation does not suffer from high error (smaller $\|\mathbf{L}_{\mathbf{Q}}\|_{\max}$), and the relative community sizes estimated from the coarse graph are close to the ones in the fine graph (smaller $\|\mathbf{L}_{\mathbf{D}}\|_{\max}$). The authors of [17] show that $\|\mathbf{L}_{\Phi}\|_{1,1}/m$ and $\|\mathbf{L}_{\mathbf{Q}}\|_{\max}$ in Theorem 2 approach zero under some conditions as $m \rightarrow \infty$. While it is unclear whether these theoretical conditions hold in our setting (given our coarsening process), our simulations show that $\hat{\Delta}(\tilde{\mathbf{A}})$ decreases with an increase in m . The study and characterization of the behavior of $\|\mathbf{L}_{\Phi}\|_{1,1}/m$ and $\|\mathbf{L}_{\mathbf{Q}}\|_{\max}$ with respect to graph scaling (i.e. ρ_n) is left for future work.

The result of Theorem 2 is important because one can directly infer the most influential control node groups, \mathcal{K}_i (one with high average controllability) via the most influential i -th c-nodes, and vice versa. In contrast to the PROM-based error characterization in Theorem 1, the learning-based method's error does not suffer explicitly from non-synchronization of coarse measurements and communities, although the error is implicitly influenced by the estimation quality of $\Phi, \mathbf{Q}, \mathbf{D}$. Hence, learning-based estimation of group controllability is likely to be more robust than that of the PROM-based when community overlap is present. This hypothesis is validated via simulation in Section V.

V. SIMULATIONS

We validate our theoretical results by plotting the errors $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ and $\hat{\Delta}(\tilde{\mathbf{A}})$ and show that these errors are comparable

Algorithm 1: Direct Inference of the Group Average Controllability

Require: estimates $\hat{\Phi}$ and $\hat{\mathbf{Q}}$ from Algorithm 2, and the number of communities K

- 1: compute $\hat{\mathbf{D}} = \text{Diag}(\frac{\hat{\Phi}^T \mathbf{1}_m}{\mathbf{1}_K \hat{\Phi}^T \mathbf{1}_m})$ and $\hat{t} = \frac{\epsilon}{n} + \rho(\hat{\mathbf{Q}}\hat{\mathbf{D}})$
- 2: **return** $\hat{\boldsymbol{\theta}}_{\text{group}} = \mathbf{1}_m + \hat{\Phi} \text{diag} \left(\hat{\mathbf{D}}^{-\frac{1}{2}} \left([\mathbf{I} - (\hat{\mathbf{D}}^{\frac{1}{2}} \hat{\mathbf{Q}}_o \hat{\mathbf{D}}^{\frac{1}{2}})^2]^{-1} - \mathbf{I} \right) \hat{\mathbf{D}}^{-\frac{1}{2}} \right)$

Algorithm 2: Mixed-Membership Community Estimation Algorithm [17]

Require: Coarse adjacency matrix $\tilde{\mathbf{A}}$, number of communities K

- 1: compute the highest K spectral-decomposition of $\tilde{\mathbf{A}}$ as $\hat{V}\hat{\Lambda}\hat{V}^T$ and set $\mathcal{S}_{\text{pruned}} = \text{Prune}(\hat{V})$
- 2: set $X = \hat{V}([m] \setminus \mathcal{S}_{\text{pruned}}, :)$ and compute $\mathcal{S}_{\text{pure}} = \text{Successive Projection Algorithm}(X^T)$
- 3: set $X_{\text{pure}} = X(\mathcal{S}_{\text{pure}}, :)$ and compute un-normalized $\hat{\Phi}^{\text{un-norm}} = \hat{V}X_{\text{pure}}^{-1}$
- 4: $\hat{\Phi}_{ik}^{\text{un-norm}} \leftarrow 0$ if $\hat{\Phi}_{ik}^{\text{un-norm}} < e^{-12}, \forall i \in [m], k \in [K]$
- 5: **return** $\hat{\Phi} = \text{Diag}^{-1}(\hat{\Phi}^{\text{un-norm}} \mathbf{1}_K) \hat{\Phi}^{\text{un-norm}}$ and $\hat{\mathbf{Q}} = X_{\text{pure}} \hat{\Lambda} X_{\text{pure}}^T$

to the bounds we obtained in Theorems 1 and 2³. We generate $\mathbf{A} \sim \text{GSBM}(n, \mathbf{Q}, \mathbf{d})$ and then determine $\tilde{\mathbf{A}} = \mathbf{W}\mathbf{A}\mathbf{W}^T$. For a realization of $\tilde{\mathbf{A}}$, we obtain the support of each row of \mathbf{W} independently, by following a 3-step process. First, we draw a vector of i.i.d elements $\varrho \in [K]^m$ from the distribution with probability vector $[\mathbf{d}_k]_{k=1}^K$. The i -th element $\varrho_i \in [K]$ represent the community with which the i -th c-node has the most overlap. Second, we generate a set of random vectors $\pi \sim \text{Dir}(\beta, \dots, \underbrace{1}_{\varrho_i \text{th}}, \dots, \beta) \in [0, 1]^K$, where $\text{Dir}(\cdot)$ is the Dirichlet multivariate probability distribution parameterized by $0 < \beta < 1$ [64]. $\text{Dir}(\beta, \dots, \underbrace{1}_{\varrho_i \text{th}}, \dots, \beta)$ outputs a vector of

size K whose ϱ_i -th component is the largest, the strength of other components is proportionate to β , and its elements sum to 1. The Dirichlet distribution is a distribution over a probability simplex, whose output allocates probabilities to K distinct categories (here communities). Finally, for each community k where $\pi_k > 0$, $\lfloor r \cdot \pi_k \rfloor$ fine nodes are randomly chosen from $\mathcal{V}_k^{\text{non-sel}}$ (i.e., non-selected fine indices in community k) and set as the support of \mathbf{w}_i . The chosen indices are removed from $\mathcal{V}_k^{\text{non-sel}}$. This process continues until the support of all \mathbf{w}_i s are selected. The first step ensures that the community balancedness criterion illustrated in Section III is maintained, i.e. the coverage of each community after coarse-measuring remains (in expectation) proportionate to the fine-

³All the results presented in this paper are reproducible. The theoretical findings are annotated and step-by-step elaborated in the appendix. The data generation process and the parameter values used for numerical simulations are fully explained. In addition, the Python code from which the simulation figures are generated is available upon request. The code is also on Github and the repository will go public upon submission acceptance.

scale community sizes. The second step controls the level of community overlap (or synchronization); the smaller β is chosen, the more coarse nodes get synced with communities, and vice versa (c.f., Remark 2).

We set number of fine nodes $n = 5000$, the overlap parameter $\beta = 0.05$, and the number of communities $K = 4$. Finally, for $\mathbf{Q} = \rho_n \mathbf{Q}_o \in \mathbb{R}^{K \times K}$, we set $[\mathbf{Q}_o]_{kk} = p = 0.5$, $[\mathbf{Q}_o]_{kk'} = q = 0.1$ (for $k \neq k'$). If not specified, the number of c-nodes $m = 100$, the density scaling parameter $\rho_n = 0.1$, and the coverage size per c-node $r = 10$. Fig. 3a-3c illustrate the qualitative behavior of the errors with respect to changes in the degree of (non-)synchronization in coarse nodes (i.e., β), m , and ρ_n . To fairly compare these errors, we also consider a *baseline* random vector $\boldsymbol{\mu} \in [1, 2]^m$ whose elements are generated (drawn) independently from the uniform distribution on the interval $[1, 2]$, independently of $\tilde{\mathbf{A}}$. We then define the *baseline error* as $\left\| \frac{\boldsymbol{\mu} - \mathbf{1}_m}{\|\boldsymbol{\mu} - \mathbf{1}_m\|_1} - \frac{r\boldsymbol{\theta}_{\text{group}, \mathbf{A}} - \mathbf{1}_m}{\|r\boldsymbol{\theta}_{\text{group}, \mathbf{A}} - \mathbf{1}_m\|_1} \right\|_1$. We make the following observations. First, both the learning and PROM-based errors are consistently better than the random baseline. Second, the learning-based approach has consistently smaller error than that of the PROM-based approach for large parametric regimes. Third, Fig. 3a shows that all errors monotonically decrease as m increases. Fourth, Fig. 3b shows that errors decrease as ρ_n increases. This is expected because larger values of ρ_n result in more distant intra- and cross-community edge densities. This makes community representation extraction and controllability estimation easier. Finally, Fig. 3c demonstrates the higher tolerance of the Learning approach to situations wherein coarse measurements are less synchronized (larger β), in comparison to the PROM method. From Fig. 3c we observe that in the perfectly synchronized regime on the left of the x-axis, PROM performs very well. As the overlap increases up to the point around $\beta \simeq 0.1$, measurements become less synchronized with communities and both estimation errors goes up. After this point, the overlap becomes so high that the controllability values essentially become closer and closer to one another and result in a slight decrease in the PROM estimation error. These results are consistent with our bounds in Theorems 1 and 2.

VI. CONCLUSION AND FUTURE WORK

We introduced a learning-based framework that exploits the power of community-based representation learning to infer average controllability of fine graphs from coarse summary data. We compared the performance of this approach with that of the Passively Reduced-Order Model (PROM) approach. For both these methods, we derived high probability error bounds on the deviation between the error estimate and ground truth, and validated the theory with numerical simulations. Our results highlight the role of fine- and coarse-network sizes, graph density, and measurement synchronization with communities in modulating the estimation errors. Interestingly, for the latter approach, we show that the estimation error decreases with network size albeit the synchronization bias, which is not the case with the PROM-based approach. For future, we plan to implement our theory to study the role of coarsening, community structures, and synchronization aspects on the controllability of brain networks. Extending the proposed tools

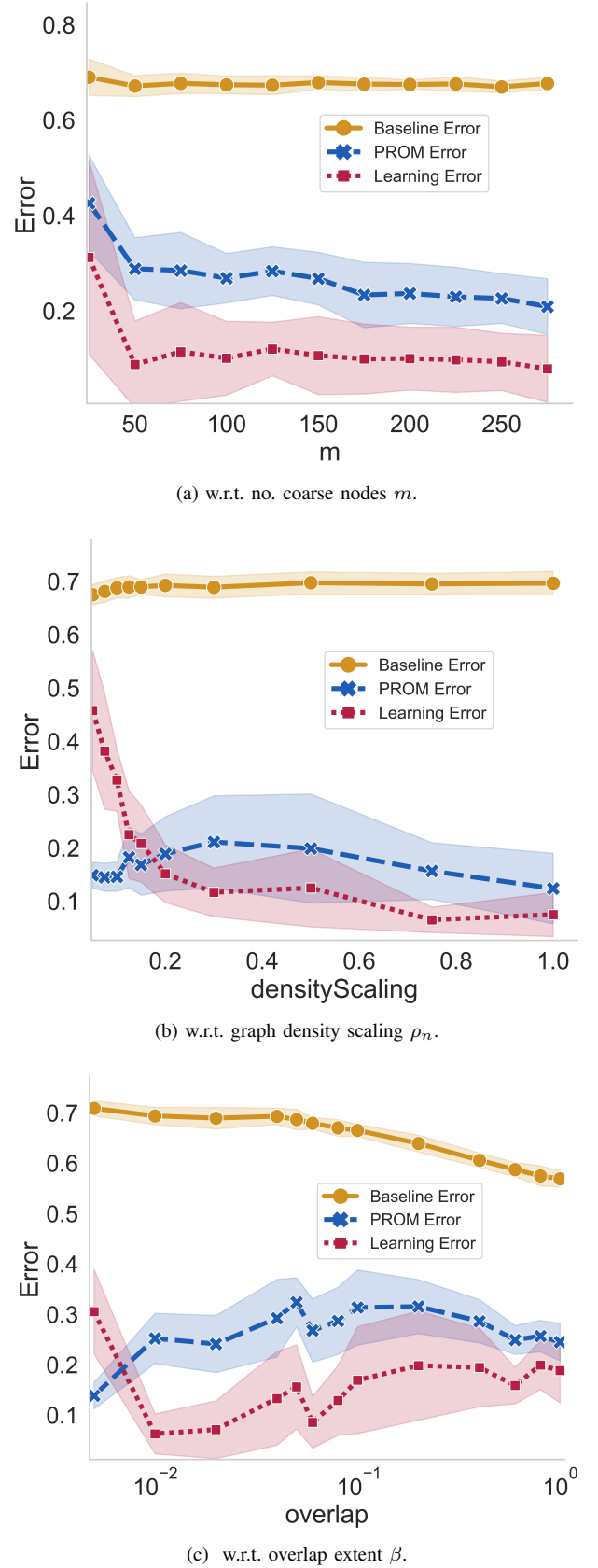


Fig. 3. Estimation error based on PROM and Learning approaches. PROM Error = $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ and Learning Error = $\Delta(\tilde{\mathbf{A}})$. The shaded region in the figures represent one standard deviation computed for 20 independent realizations.

and techniques to controllability metrics other than average controllability is left for future work.

APPENDIX

This section contains proofs for the main results in Sections III and IV. Due to space limitations, we did not include proofs of Lemma 2, Proposition 5, Eq. (26) of Lemma 4, and Theorem 6 in the appendix. Rather they could be found in the extended version of this paper posted to the Arxiv [65].

We begin with a few auxiliary results needed for analysis. In what follows, $\mathbf{1}_{K,K}$ is $K \times K$ dimensional all-one matrix. **Useful Matrix Norm Equivalence [66], [67]:** For two real, square, and symmetric matrices $\mathbf{Z}, \mathbf{Y} \in \mathbb{R}^{n \times n}$, the following equality and inequalities hold:

- 1) If \mathbf{Z}, \mathbf{Y} are invertible: $\mathbf{Z}^{-1} - \mathbf{Y}^{-1} = \mathbf{Z}^{-1}(\mathbf{Y} - \mathbf{Z})\mathbf{Y}^{-1}$.
- 2) $\text{tr}(\mathbf{Z}\mathbf{Y}) \leq \|\mathbf{Z}\|_2 \text{tr}(\mathbf{Y})$.
- 3) For all norms: if $\|\mathbf{Z}\| < 1 \Rightarrow \|(\mathbf{I} - \mathbf{Z})^{-1}\| < \frac{1}{1 - \|\mathbf{Z}\|}$.
- 4) $\|\mathbf{Z}\mathbf{Y}\|_{\max} \leq \|\mathbf{Z}\|_{\infty} \|\mathbf{Y}\|_{\max}$.
- 5) $\|\mathbf{Z}\|_{\infty} \leq n \|\mathbf{Z}\|_{\max}$.
- 6) $\|\mathbf{Z}\|_2 \leq n \|\mathbf{Z}\|_{\max}$.
- 7) $\|\mathbf{Z}\mathbf{Y}\|_{\max} \leq n \|\mathbf{Z}\|_{\max} \|\mathbf{Y}\|_{\max}$.

Note that for $\mathbf{Z} = \mathbf{Z}^T$, the matrix 2-norm and the spectral radius coincide ($\rho(\mathbf{Z}) = \|\mathbf{Z}\|_2$) and are used interchangeably.

Proposition 3. (Hoeffding's Inequality) [68]: Let $\{\mathbf{z}_\ell\}_{\ell=1}^{|\mathbf{z}|}$ be bounded independent random variables. Then the following holds with probability at least $1 - \delta$, for some $\delta > 0$:

$$\left| \sum_{\ell=1}^{|\mathbf{z}|} \mathbf{z}_\ell - \mathbb{E} \left[\sum_{\ell=1}^{|\mathbf{z}|} \mathbf{z}_\ell \right] \right| \leq \sqrt{\frac{|\mathbf{z}|}{2} \log \left(\frac{1}{\delta} \right)} (\max(\mathbf{z}) - \min(\mathbf{z})) \quad (19)$$

$$\leq \sqrt{\frac{|\mathbf{z}|}{2} \log \left(\frac{1}{\delta} \right)} \frac{\max(\mathbf{z}) - \min(\mathbf{z})}{\mathbb{E} \left[\sum_{\ell=1}^{|\mathbf{z}|} \mathbf{z}_\ell \right]} \mathbb{E} \left[\sum_{\ell=1}^{|\mathbf{z}|} \mathbf{z}_\ell \right] \quad (20)$$

$$\leq \sqrt{\frac{|\mathbf{z}|}{2} \log \left(\frac{1}{\delta} \right)} \frac{\max(\mathbf{z}) - \min(\mathbf{z})}{\min \left(\mathbb{E} \left[\sum_{\ell=1}^{|\mathbf{z}|} \mathbf{z}_\ell \right] \right)} \mathbb{E} \left[\sum_{\ell=1}^{|\mathbf{z}|} \mathbf{z}_\ell \right], \quad (21)$$

where $\max(\mathbf{z})$ and $\min(\mathbf{z})$ respectively denote the maximum and minimum possible values of $\{\mathbf{z}_\ell\}_{\ell=1}^{|\mathbf{z}|}$.

Proposition 4. (Monotone Matrix Norms [69]): Suppose the element-wise inequality $\mathbf{Z} \geq \mathbf{Y}$ holds. Then, $\|\mathbf{Z}\|_p \geq \|\mathbf{Y}\|_p$, for a positive integer p .

Proposition 5. (An upper bound on the ℓ_1 norm of the difference between two ℓ_1 -normalized vectors): Let $\mathbf{z}, \mathbf{y} \in \mathbb{R}^m$ be such that $\mathbf{z} \geq \mathbf{1}_m$ and $\mathbf{y} \geq \mathbf{1}_m$, where $\mathbf{1}_m$ is the all-ones vector. Then,

$$\left\| \frac{\mathbf{z} - \mathbf{1}_m}{\|\mathbf{z} - \mathbf{1}_m\|_1} - \frac{\mathbf{y} - \mathbf{1}_m}{\|\mathbf{y} - \mathbf{1}_m\|_1} \right\|_1 \leq 2 \frac{\|\mathbf{z} - \mathbf{y}\|_1}{\max(\|\mathbf{z} - \mathbf{1}_m\|_1, \|\mathbf{y} - \mathbf{1}_m\|_1)}.$$

Proof. See Proposition 5 in the extended version at [65]. \square

A. Lemma: Relationship Between Fine- and Group-Average Controllability

Let \mathbf{Z} be an $n \times n$ system matrix that could be either \mathbf{A} or $\bar{\mathbf{A}}$ (the expected quantity). Recall the controllability Gramian of the LTI system in (1) has been defined in (2). We define

the n -dimensional *fine* average controllability vector $\boldsymbol{\theta}_{\text{fine}, \mathbf{Z}}$ similar to (8) as

$$\boldsymbol{\theta}_{\text{fine}, \mathbf{Z}}^T \triangleq [\text{tr}[\mathbf{C}(\mathbf{Z}_{\text{nom}}, \mathbf{e}_1)] \quad \dots \quad \text{tr}[\mathbf{C}(\mathbf{Z}_{\text{nom}}, \mathbf{e}_n)]] \quad (22)$$

where \mathbf{e}_i is the i -th canonical basis vector in \mathbb{R}^n . By definition in (2), the following holds:

$$\begin{aligned} \theta_{\text{fine}, \mathbf{Z}}^{(i)} &= \text{tr}[\mathbf{C}(\mathbf{Z}_{\text{nom}}, \mathbf{e}_i)] = \sum_{\tau=0}^{\infty} \text{tr}(\mathbf{Z}_{\text{nom}}^\tau \mathbf{e}_i \mathbf{e}_i^T \mathbf{Z}_{\text{nom}}^\tau) \\ &= \sum_{\tau=0}^{\infty} [\text{diag}(\mathbf{Z}_{\text{nom}}^{2\tau})]_i = 1 + [\text{diag}(\mathbf{Z}_{\text{nom}}^2)]_i + \dots \\ &= [\text{diag}((\mathbf{I} - \mathbf{Z}_{\text{nom}}^2)^{-1})]_i. \end{aligned} \quad (23)$$

The series converges to the final equality because $\rho(\mathbf{Z}) \leq 1$. Putting (23) into a vector form gives:

$$\boldsymbol{\theta}_{\text{fine}, \mathbf{Z}} = \text{diag}(\mathbf{C}(\mathbf{Z}_{\text{nom}}, \mathbf{I}_{n \times n})) = \text{diag}((\mathbf{I} - \mathbf{Z}_{\text{nom}}^2)^{-1}). \quad (24)$$

The following lemma states the group average controllability vector in (8) is a linear mapping of the fine average controllability vector defined in (24).

Lemma 2. With the notation defined above for \mathbf{Z} consider the average controllability vector in (8). Then

$$\boldsymbol{\theta}_{\text{group}, \mathbf{Z}} = \frac{1}{r} \mathbf{W} \boldsymbol{\theta}_{\text{fine}, \mathbf{Z}}. \quad \square$$

Proof. See Lemma 2 in the extended version at [65]. \square

Lemma 2 states that the i -th element in $\boldsymbol{\theta}_{\text{group}, \mathbf{Z}}$ is the weighted average of a group of elements in $\boldsymbol{\theta}_{\text{fine}, \mathbf{Z}}$; this group corresponds to the fine node indices that map to the i -th c-node.

Lemma 3. (spectral radius of $\bar{\mathbf{A}}$): Let $\mathbf{D} = \frac{1}{n} \mathbf{P} \mathbf{P}^T$ with ρ_n be defined as in (5). Then the spectral radius of $\bar{\mathbf{A}} = \mathbf{P}^T \mathbf{Q} \mathbf{P}$ in (9) is given as

$$\rho(\bar{\mathbf{A}}) = n \rho(\mathbf{Q} \mathbf{D}) = n \rho_n \rho(\mathbf{Q}_o \mathbf{D}). \quad \square$$

Proof. We recall that the eigenvalues of $\mathbf{A} \mathbf{B}$ and $\mathbf{B} \mathbf{A}$ coincide for any two square matrices \mathbf{A} and \mathbf{B} . Thus,

$$\rho(\bar{\mathbf{A}}) = \rho(\mathbf{Q} \mathbf{P} \mathbf{P}^T) = n \rho(\mathbf{Q} \mathbf{D}) = n \rho_n \rho(\mathbf{Q}_o \mathbf{D}),$$

where for the last equality we use the relation in (5). \square

B. Lemma and proof: Error between the Gramians of random and expected LTI systems

Let $\bar{\mathcal{S}}_{\text{fine}}$ denote the expected dynamics of the LTI system in (6); specifically, \mathbf{A} is replaced with the expected quantity $\bar{\mathbf{A}}$, while the rest of the parameters remain unchanged. We define $\boldsymbol{\theta}_{\text{fine}, \bar{\mathbf{A}}}$ similar to $\boldsymbol{\theta}_{\text{fine}, \mathbf{A}}$ in (24), only by replacing \mathbf{A} with $\bar{\mathbf{A}}$. Similarly, $\bar{\mathcal{S}}_{\text{coarse}}$ refers to the expected dynamics of (10) when $\bar{\mathbf{A}}$ is replaced with $\bar{\bar{\mathbf{A}}}$, without altering the rest of the parameters. $\boldsymbol{\theta}_{\text{coarse}, \bar{\bar{\mathbf{A}}}}$ is then defined similar to $\boldsymbol{\theta}_{\text{coarse}, \bar{\mathbf{A}}}$ (c.f., (11)), as the controllability vector of $\bar{\mathcal{S}}_{\text{coarse}}$, by replacing $\bar{\mathbf{A}}$ with $\bar{\bar{\mathbf{A}}}$. The following result provides an error bound on the respective differences of the fine- and coarse- average controllability vectors. The proof of this result demonstrates how this error bound is equivalent to bounding the trace of the difference between the Gramians of $\mathcal{S}_{\text{fine}}$ and $\bar{\mathcal{S}}_{\text{fine}}$, as well as between the Gramians of $\mathcal{S}_{\text{coarse}}$ and $\bar{\mathcal{S}}_{\text{coarse}}$.

Lemma 4. (Error between the Gramians of random and expected LTI systems): Let $\boldsymbol{\theta}_{\text{fine}, \mathbf{A}}, \boldsymbol{\theta}_{\text{fine}, \bar{\mathbf{A}}}, \boldsymbol{\theta}_{\text{coarse}, \bar{\bar{\mathbf{A}}}}$, and

$\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ be defined as above. Under Assumption 1, the following holds with probability at least $1 - \delta$:

$$\begin{aligned} \alpha_n &\triangleq \left\| \theta_{\text{fine}, \mathbf{A}} - \theta_{\text{fine}, \tilde{\mathbf{A}}} \right\|_1 \\ &\leq \frac{\eta \left[\frac{1/\rho_n + c_1}{c_o^2} + \left(\frac{1}{2} + 2\eta \right) \frac{c_1 + \frac{1}{\rho_n}}{c_o^2} \right] \left(1 + \sqrt{1 + \eta} \sqrt{1 + \frac{1/\rho_n}{nc_o^2} + \frac{2\epsilon/\rho_n}{nc_o}} \right)}{\left[1 - \left(\frac{c_o}{\epsilon/\rho_n + c_o} \right)^2 \right] \left[1 - \left(\frac{\rho(\mathbf{A})}{\epsilon + \rho(\mathbf{A})} \right)^2 \right]} \\ &= \mathcal{O}\left(\frac{\eta}{\rho_n}\right), \end{aligned} \quad (25)$$

and with probability at least $1 - \tilde{\delta}$:

$$\begin{aligned} \tilde{\alpha}_m &\triangleq \left\| \theta_{\text{coarse}, \tilde{\mathbf{A}}} - \theta_{\text{coarse}, \tilde{\tilde{\mathbf{A}}}} \right\|_1 \\ &\leq \frac{\tilde{\eta} \left[\frac{1/\rho_n + \tilde{c}_1}{\tilde{c}_o^2} + \left(\frac{1}{2} + 2\tilde{\eta} \right) \frac{\tilde{c}_1 + \frac{1}{\rho_n}}{\tilde{c}_o^2} \right] \left(1 + \sqrt{1 + \tilde{\eta}} \sqrt{1 + \frac{1/\rho_n}{m\tilde{c}_o^2} + \frac{2\epsilon/\rho_n}{m\tilde{c}_o}} \right)}{\left[1 - \left(\frac{\tilde{c}_o}{\epsilon/\rho_n + \tilde{c}_o} \right)^2 \right] \left[1 - \left(\frac{\rho(\tilde{\mathbf{A}})}{\epsilon + \rho(\tilde{\mathbf{A}})} \right)^2 \right]} \\ &= \mathcal{O}\left(\frac{\tilde{\eta}}{\rho_n}\right), \end{aligned} \quad (26)$$

where

$$\begin{aligned} \eta &= \frac{\sqrt{\log(\frac{n^2}{\delta})/(2n)}}{\|\mathbf{Q}\mathbf{D}\mathbf{Q}\|_{\min}}, & \tilde{\eta} &= \frac{\sqrt{\log(\frac{m^2}{\delta})/(2m)}}{\|\mathbf{Q}\frac{\Phi^T\Phi}{m}\mathbf{Q}\|_{\min}}, \\ c_o &= \rho(\mathbf{Q}_o\mathbf{D}), & c_1 &= \text{tr}\left(\left(\mathbf{D}\mathbf{Q}_o\right)^2\right), \\ \tilde{c}_o &= \rho\left(\mathbf{Q}_o\frac{\Phi^T\Phi}{m}\right), & \tilde{c}_1 &= \text{tr}\left(\left(\frac{\Phi^T\Phi}{m}\mathbf{Q}_o\right)^2\right). \end{aligned} \quad (27)$$

□

At a high level, Lemma 4 is a concentration result for the Gramians of $\mathcal{S}_{\text{fine}}$ (or $\mathcal{S}_{\text{coarse}}$) and $\tilde{\mathcal{S}}_{\text{fine}}$ (or $\tilde{\mathcal{S}}_{\text{coarse}}$). Note that $\mathcal{C}(\mathbf{Z}, \mathbf{I}) = (\mathbf{I} - \mathbf{Z}^2)^{-1}$, where \mathbf{Z} can take \mathbf{A}_{nom} , $\tilde{\mathbf{A}}_{\text{nom}}$, or $\tilde{\tilde{\mathbf{A}}}_{\text{nom}}$. Lemma 4 is effectively bounding the difference of resolvents $(z\mathbf{I} - \mathbf{Z}^2)^{-1}$ evaluated at $z = 1$. However the rate at which the difference goes to zero is different for the fine- and coarse systems. For $\mathbf{A} = \tilde{\mathbf{A}}$, we have $n = m$ and $\frac{\Phi^T\Phi}{m} = \mathbf{D}$. Hence, α_n and $\tilde{\alpha}_n$ coincide.

Proof. We begin by deriving an upper bound for α_n in (25) as the trace of difference between $\mathbf{A}_{\text{nom}}^2$ and $\tilde{\mathbf{A}}_{\text{nom}}^2$:

$$\begin{aligned} \alpha_n &\stackrel{(a)}{=} \left\| \text{diag} \left[\mathcal{C}(\mathbf{A}_{\text{nom}}, \mathbf{I}_{n \times n}) - \mathcal{C}(\tilde{\mathbf{A}}_{\text{nom}}, \mathbf{I}_{n \times n}) \right] \right\|_1 \\ &\stackrel{(b)}{=} \text{tr} \left(\left| \mathcal{C}(\mathbf{A}_{\text{nom}}, \mathbf{I}_{n \times n}) - \mathcal{C}(\tilde{\mathbf{A}}_{\text{nom}}, \mathbf{I}_{n \times n}) \right| \right) \\ &\stackrel{(c)}{=} \text{tr} \left(\left| (\mathbf{I} - \mathbf{A}_{\text{nom}}^2)^{-1} - (\mathbf{I} - \tilde{\mathbf{A}}_{\text{nom}}^2)^{-1} \right| \right) \\ &\stackrel{(d)}{=} \text{tr} \left((\mathbf{I} - \mathbf{A}_{\text{nom}}^2)^{-1} |\mathbf{A}_{\text{nom}}^2 - \tilde{\mathbf{A}}_{\text{nom}}^2| (\mathbf{I} - \tilde{\mathbf{A}}_{\text{nom}}^2)^{-1} \right) \\ &\stackrel{(e)}{\leq} \left\| (\mathbf{I} - \mathbf{A}_{\text{nom}}^2)^{-1} \right\|_2 \text{tr} \left(|\mathbf{A}_{\text{nom}}^2 - \tilde{\mathbf{A}}_{\text{nom}}^2| \right) \left\| (\mathbf{I} - \tilde{\mathbf{A}}_{\text{nom}}^2)^{-1} \right\|_2 \\ &\stackrel{(f)}{\leq} \frac{1}{1 - \|\mathbf{A}_{\text{nom}}^2\|_2} \frac{1}{1 - \|\tilde{\mathbf{A}}_{\text{nom}}^2\|_2} \text{tr} \left(|\mathbf{A}_{\text{nom}}^2 - \tilde{\mathbf{A}}_{\text{nom}}^2| \right), \end{aligned} \quad (28)$$

where (a) follows from (24); (b) follows from the equivalence of the trace of the absolute value of a matrix and the ℓ_1 norm of its diagonal entries; (d)-(f) follow respectively from items 1-3 in the matrix equivalence list at the beginning of the appendix; and finally, (c) follows by expanding the Gramian matrix in (2) using the Neumann series:

$$\begin{aligned} \mathcal{C}(\mathbf{A}_{\text{nom}}, \mathbf{I}_{n \times n}) &\triangleq \lim_{T \rightarrow \infty} \mathcal{C}_T(\mathbf{A}_{\text{nom}}, \mathbf{I}_{n \times n}) \\ &= \lim_{T \rightarrow \infty} \sum_{t=0}^{T-1} (\mathbf{A}_{\text{nom}})^{2t} = (\mathbf{I} - \mathbf{A}_{\text{nom}}^2)^{-1}, \end{aligned} \quad (29)$$

and similarly,

$$\mathcal{C}(\tilde{\mathbf{A}}_{\text{nom}}, \mathbf{I}_{n \times n}) \triangleq \lim_{T \rightarrow \infty} \mathcal{C}_T(\tilde{\mathbf{A}}_{\text{nom}}, \mathbf{I}_{n \times n}) = (\mathbf{I} - \tilde{\mathbf{A}}_{\text{nom}}^2)^{-1}.$$

The limit exists because \mathbf{A} and $\tilde{\mathbf{A}}$ are symmetric matrices and that $\|\mathbf{A}_{\text{nom}}\|_2 = \frac{\rho(\mathbf{A})}{\epsilon + \rho(\mathbf{A})} < 1$ and $\|\tilde{\mathbf{A}}_{\text{nom}}\|_2 = \frac{\rho(\tilde{\mathbf{A}})}{\epsilon + \rho(\tilde{\mathbf{A}})} < 1$.

We now find an upper bound for $\text{tr}(|\mathbf{A}_{\text{nom}}^2 - \tilde{\mathbf{A}}_{\text{nom}}^2|)$ by bounding matrix \mathbf{A}^2 , using its expectation, from below and above and in an element-wise manner. Using the Hoeffding's inequality in Proposition 3, the following holds for $i, j \in [n]$ and $i \neq j$, with probability at least $1 - \delta$:

$$\begin{aligned} |[\mathbf{A}^2]_{ij} - \mathbb{E}[\mathbf{A}^2]_{ij}| &= \left| \sum_{\ell \in [n]} \mathbf{A}_{i\ell} \mathbf{A}_{j\ell} - \mathbb{E} \left[\sum_{\ell \in [n]} \mathbf{A}_{i\ell} \mathbf{A}_{j\ell} \right] \right| \\ &\leq \sqrt{\frac{n \log(\frac{1}{\delta})}{2}} \frac{\max(\mathbf{A}_{i\ell}) - \min(\mathbf{A}_{i\ell})}{\min([\mathbf{A}^2]_{ij})} [\bar{\mathbf{A}}^2]_{ij} \\ &\stackrel{(a)}{\leq} \sqrt{\frac{\log(\frac{1}{\delta})}{2n}} \frac{1}{\|\mathbf{Q}\mathbf{D}\mathbf{Q}\|_{\min}} [\bar{\mathbf{A}}^2]_{ij}, \end{aligned} \quad (30)$$

where (a) follows by the definition of $\mathbf{D} = (1/n)\mathbf{P}\mathbf{P}^T$ for \mathbf{P} in (3), coupled with the fact that $\bar{\mathbf{A}}^2 = \mathbf{P}\mathbf{Q}\mathbf{P}^T\mathbf{P}\mathbf{Q}\mathbf{P}^T = n\mathbf{P}\mathbf{Q}\mathbf{D}\mathbf{Q}\mathbf{P}^T$. Similarly for the diagonal elements, the following inequality holds with probability at least $1 - \delta$:

$$\begin{aligned} |[\mathbf{A}^2]_{ii} - \mathbb{E}[\mathbf{A}^2]_{ii}| &= \left| \sum_{\ell \in [n]} \mathbf{A}_{i\ell}^2 - \mathbb{E}[\mathbf{A}_{i\ell}^2] \right| = \left| \sum_{\ell \in [n]} \mathbf{A}_{i\ell} - \mathbb{E}[\mathbf{A}_{i\ell}] \right| \\ &\stackrel{(a)}{\leq} \sqrt{\frac{\log(\frac{1}{\delta})}{2n}} \frac{1}{\|\mathbf{Q}\mathbf{d}\|_{\min}} \mathbb{E}[\mathbf{A}^2]_{ii}, \end{aligned} \quad (31)$$

where (a) is due to the identity $\mathbb{E}[\mathbf{A}^2]_{ii} = \mathbf{P}_i \mathbf{Q} \mathbf{P} \mathbf{1}_n = \mathbf{P}_i \mathbf{Q} \mathbf{d}$ (\mathbf{P}_i is the i -th column of \mathbf{P}). Note that we will bound the difference between $\mathbb{E}[\mathbf{A}^2]$ and $\bar{\mathbf{A}}_{\text{nom}}^2$ shortly as we simplify $\text{tr}(|\mathbf{A}_{\text{nom}}^2 - \bar{\mathbf{A}}_{\text{nom}}^2|)$. By putting the element-by-element bounds in (30) and (31) into a matrix, the following element-wise inequality holds with probability at least $1 - \delta$:

$$\begin{aligned} \mathbf{A}^2 &\leq (1 + \eta) \mathbb{E}[\mathbf{A}^2] \\ &\leq (1 + \eta) (\bar{\mathbf{A}}^2 - \text{Diag}[n\mathbf{P}^T\mathbf{Q}\mathbf{D}\mathbf{Q}\mathbf{P}] + n\text{Diag}[\mathbf{P}^T\mathbf{Q}\mathbf{d}]) \\ &= (1 + \eta) n (\mathbf{P}^T\mathbf{Q}\mathbf{D}\mathbf{Q}\mathbf{P} + \text{Diag}[\mathbf{P}^T(\mathbf{Q} \odot (\mathbf{1}_{K,K} - \mathbf{Q})\mathbf{d})]), \end{aligned} \quad (32)$$

where η is defined as

$$\frac{1}{\sqrt{2n}} \max \left(\frac{\sqrt{\log(\frac{n^2}{\delta})}}{\|\mathbf{Q}\mathbf{D}\mathbf{Q}\|_{\min}}, \frac{\rho_n \sqrt{\log(\frac{n}{\delta})}}{\|\mathbf{Q}\mathbf{d}\|_{\min}} \right) \leq \sqrt{\frac{\log(\frac{n^2}{\delta})}{2n}} \frac{1}{\|\mathbf{Q}\mathbf{D}\mathbf{Q}\|_{\min}} \triangleq \eta.$$

Similarly, for the lower bound we get:

$$\mathbf{A}^2 \geq (1 - \eta) n (\mathbf{P}^T\mathbf{Q}\mathbf{D}\mathbf{Q}\mathbf{P} + \text{Diag}[\mathbf{P}^T(\mathbf{Q} \odot (\mathbf{1}_{K,K} - \mathbf{Q})\mathbf{d})]). \quad (33)$$

From (32) and (33), the following upper bound is obtained for

the $\text{tr}(|\mathbf{A}_{\text{nom}}^2 - \bar{\mathbf{A}}_{\text{nom}}^2|)$ term:

$$\begin{aligned}
& \text{tr}(|\mathbf{A}_{\text{nom}}^2 - \bar{\mathbf{A}}_{\text{nom}}^2|) \stackrel{(a)}{=} \text{tr}\left(\left|\frac{\mathbf{A}^2 \bar{\mathbf{A}}^2}{(\epsilon + \rho(\bar{\mathbf{A}}))^2} + \left(\frac{\mathbf{A}^2}{\epsilon + \rho(\mathbf{A})} - \frac{\mathbf{A}^2}{\epsilon + \rho(\bar{\mathbf{A}})}\right)\right|\right) \\
& \stackrel{(b)}{\leq} \text{tr}\left(\left|\frac{\eta \bar{\mathbf{A}}^2 + \eta n \text{Diag}[\mathbf{P}^\top \mathbf{Q} \odot (\mathbf{1} - \mathbf{Q}) \mathbf{d}]}{(\epsilon + \rho(\bar{\mathbf{A}}))^2} + \left(\frac{\mathbf{A}^2}{(\epsilon + \rho(\mathbf{A}))^2} - \frac{\mathbf{A}^2}{(\epsilon + \rho(\bar{\mathbf{A}}))^2}\right)\right|\right) \\
& \stackrel{(c)}{\leq} \eta \frac{\text{tr}(\bar{\mathbf{A}}^2)}{(\epsilon + \rho(\bar{\mathbf{A}}))^2} + \eta n \frac{\text{tr}(\text{Diag}[\mathbf{P}^\top \mathbf{Q} \odot (\mathbf{1} - \mathbf{Q}) \mathbf{d}])}{(\epsilon + \rho(\bar{\mathbf{A}}))^2} \\
& \quad + \left|\frac{1}{(\epsilon + \rho(\bar{\mathbf{A}}))^2} - \frac{1}{(\epsilon + \rho(\mathbf{A}))^2}\right| \text{tr}(\mathbf{A}^2) \\
& \stackrel{(d)}{\leq} \eta \frac{n^2 \text{tr}((\mathbf{DQ})^2)}{(\epsilon + n\rho(\mathbf{QD}))^2} + \eta \frac{n^2 \|\mathbf{Q} \odot (\mathbf{1} - \mathbf{Q}) \mathbf{d}\|_{\max}}{(\epsilon + n\rho(\mathbf{QD}))^2} + \left|1 - \frac{\rho(\mathbf{A})}{\rho(\bar{\mathbf{A}})}\right| \\
& \quad \times \left(1 + \frac{\rho(\mathbf{A}) + 2\epsilon}{\rho(\bar{\mathbf{A}})}\right) \frac{(1 + \eta)n^2 [\text{tr}((\mathbf{DQ})^2) + \|\mathbf{Qd}\|_{\max}]}{(1 - \eta)(n^2 \rho^2(\mathbf{QD}) + n \|\mathbf{Q} \odot (\mathbf{1}_{K,K} - \mathbf{Q}) \mathbf{d}\|_{\max})} \\
& \stackrel{(e)}{\leq} \eta \left[\frac{1/\rho_n + \text{tr}((\mathbf{DQ}_o)^2)}{\rho^2(\mathbf{Q}_o \mathbf{D})} + \frac{1}{2}(1 + 4\eta) \frac{[\text{tr}((\mathbf{DQ}_o)^2) + \frac{1}{\rho_n}]}{\rho^2(\mathbf{Q}_o \mathbf{D})} \right] \\
& \quad \times \left(1 + \sqrt{1 + \eta} \sqrt{1 + \frac{1}{n\rho_n \rho^2(\mathbf{Q}_o \mathbf{D})}} + \frac{2\epsilon}{n\rho_n \rho(\mathbf{Q}_o \mathbf{D})}\right), \tag{34}
\end{aligned}$$

where step (a) follows from the normalization factors defined in (1) and an additive-subtractive term; (b) is obtained from (32); (c) is due to the triangle inequality and the linearity of trace; (d) is because of (37) and (38) derived at the end of this proof, the definition of \mathbf{P} in (3), the inequality

$$\begin{aligned}
& \left| \frac{1}{(\epsilon + \rho(\mathbf{A}))^2} - \frac{1}{(\epsilon + \rho(\bar{\mathbf{A}}))^2} \right| = \left| \frac{(\epsilon + \rho(\bar{\mathbf{A}}))^2 - (\epsilon + \rho(\mathbf{A}))^2}{(\epsilon + \rho(\mathbf{A}))^2 (\epsilon + \rho(\bar{\mathbf{A}}))^2} \right| \\
& = \left| \frac{(\rho(\bar{\mathbf{A}}) + \rho(\mathbf{A}) + 2\epsilon)(\rho(\bar{\mathbf{A}}) - \rho(\mathbf{A}))}{(\epsilon + \rho(\mathbf{A}))^2 (\epsilon + \rho(\bar{\mathbf{A}}))^2} \right| \leq \frac{\left[1 + \frac{\rho(\mathbf{A}) + 2\epsilon}{\rho(\bar{\mathbf{A}})}\right] \left|1 - \frac{\rho(\mathbf{A})}{\rho(\bar{\mathbf{A}})}\right|}{(\epsilon + \rho(\mathbf{A}))^2},
\end{aligned}$$

and the equality $\text{tr}(\bar{\mathbf{A}}^2) = n^2 \text{tr}((\mathbf{DQ})^2)$. For the last equality, (9) and the invariance of trace under cyclic permutation were used; finally, (e) follows from (39), (40) (derived at the end of this proof), and two easy-to-validate inequalities $\sqrt{1 - \eta} \geq 1 - \eta/2$ and $\frac{1 + \eta}{1 - \eta} \leq 1 + 4\eta, \forall \eta < 1/2$. From (34), and using the normalization factor of the system in (6) for \mathbf{A} and its expectation, we can further simplify (28) as

$$\begin{aligned}
\alpha_n & \leq \frac{\eta \left[\frac{1/\rho_n + c_1}{c_o^2} + \left(\frac{1}{2} + 2\eta\right) \frac{[c_1 + \frac{1}{\rho_n}]}{c_o^2} \left(1 + \sqrt{1 + \eta} \sqrt{1 + \frac{1/\rho_n}{n c_o^2}} + \frac{2\epsilon/\rho_n}{n c_o}\right) \right]}{\left[1 - \left(\frac{c_o}{\epsilon/\rho_n + c_o}\right)^2\right] \left[1 - \left(\frac{\rho(\mathbf{A})}{\epsilon + \rho(\bar{\mathbf{A}})}\right)^2\right]} \\
& \stackrel{(a)}{=} \mathcal{O}\left(\frac{\eta}{\rho_n}\right). \tag{35}
\end{aligned}$$

In the above $c_o = \rho(\mathbf{Q}_o \mathbf{D})$, $c_1 = \text{tr}((\mathbf{DQ}_o)^2)$, and step (a) holds in the asymptotic scenario when \mathbf{D}, \mathbf{Q} are constant and $\sqrt{n}\rho_n$ is very large and hence, η becomes very small.

The proof of the inequality in (26) bears some similarity to the upper bound derivations for α_n in (28)-(35) and is included in the extended version of this paper at [65]. We did not include the derivation herein due to space restrictions.

Bounds on $\rho(\mathbf{A})$, $\text{tr}(\mathbf{A}^2)$, and $\frac{\rho(\mathbf{A})}{\rho(\bar{\mathbf{A}})}$: From (32), (33), and Proposition 4, we obtain several inequalities:

$$\begin{aligned}
\rho^2(\mathbf{A}) &= \rho(\mathbf{A}^2) \tag{36} \\
&\geq (1 - \eta) (\rho(\bar{\mathbf{A}})^2 + n\rho(\text{Diag}[\mathbf{P}^\top (\mathbf{Q} \odot (\mathbf{1}_{K,K} - \mathbf{Q}) \mathbf{d}])]) \\
&= (1 - \eta) (n^2 \rho^2(\mathbf{QD}) + n \|\mathbf{Q} \odot (\mathbf{1}_{K,K} - \mathbf{Q}) \mathbf{d}\|_{\max}), \\
\rho^2(\mathbf{A}) &\leq (1 + \eta)n (n\rho^2(\mathbf{QD}) + \|\mathbf{Q} \odot (\mathbf{1}_{K,K} - \mathbf{Q}) \mathbf{d}\|_{\max}) \\
&\leq (1 + \eta)n^2 \rho_n^2 \left(\rho^2(\mathbf{Q}_o \mathbf{D}) + \frac{1}{n\rho_n} \right), \tag{37}
\end{aligned}$$

where the triangle inequality on matrix norm is employed. Moreover, from (32) and the invariance of trace under cyclic permutation, we obtain

$$\begin{aligned}
\text{tr}(\mathbf{A}^2) &\leq \text{tr}[(1 + \eta)n(\mathbf{P}^\top \mathbf{QDQP} + \text{Diag}[\mathbf{P}^\top (\mathbf{Q} \odot (\mathbf{1}_{K,K} - \mathbf{Q}) \mathbf{d}])]) \\
&\leq (1 + \eta)n^2 [\text{tr}((\mathbf{DQ})^2) + \|\mathbf{Qd}\|_{\max}] \tag{38}
\end{aligned}$$

Moreover, Lemma 3 and (36) give

$$\frac{\rho(\mathbf{A})}{\rho(\bar{\mathbf{A}})} \leq \frac{\sqrt{(1 + \eta)n^2 (\rho^2(\mathbf{QD}) + \frac{1}{n\rho_n})}}{n\rho(\mathbf{QD})} \leq \sqrt{1 + \eta} \sqrt{1 + \frac{1}{n\rho_n \rho^2(\mathbf{Q}_o \mathbf{D})}}. \tag{39}$$

Similarly, by combining Lemma 3 and (37), we have

$$\frac{\rho(\mathbf{A})}{\rho(\bar{\mathbf{A}})} \geq \sqrt{(1 - \eta) + (1 - \eta) \frac{\|\mathbf{Q}_o \odot (\mathbf{1}_{K,K} - \mathbf{Q}) \mathbf{d}\|_{\max}}{n\rho_n \rho^2(\mathbf{Q}_o \mathbf{D})}} \geq \sqrt{1 - \eta}. \tag{40}$$

C. Proof of Lemma 1

We aim to find the group controllability of $\bar{\mathcal{S}}_{\text{fine}}$ defined similar to (6) after replacing \mathbf{A} with $\bar{\mathbf{A}}$. To do so, we simplify $\theta_{\text{fine}, \bar{\mathbf{A}}}$ and substitute the result in Lemma 2.

By replacing \mathbf{Z} in (22) with $\bar{\mathbf{A}}$, we have:

$$\begin{aligned}
\theta_{\text{fine}, \bar{\mathbf{A}}} &= \text{diag}(\sum_{\tau=0}^{\infty} \bar{\mathbf{A}}_{\text{nom}}^{2\tau}) \stackrel{(a)}{=} \text{diag}(\sum_{\tau=0}^{\infty} (\frac{\iota}{n} \mathbf{P}^\top \mathbf{Q}_o \mathbf{P})^{2\tau}) \\
&= \text{diag}(\mathbf{I} + (\frac{\iota}{n})^2 \mathbf{P}^\top \mathbf{Q}_o \mathbf{P} \mathbf{P}^\top \mathbf{Q}_o \mathbf{P} + \dots) \\
&= \mathbf{1}_n + \frac{1}{n} \text{diag}(\mathbf{P}^\top \mathbf{D}^{-\frac{1}{2}} [(\iota \mathbf{D}^{\frac{1}{2}} \mathbf{Q}_o \mathbf{D}^{\frac{1}{2}})^2 + \dots] \mathbf{D}^{-\frac{1}{2}} \mathbf{P}) \\
&= \mathbf{1}_n + \frac{1}{n} \text{diag}(\mathbf{P}^\top \Upsilon \mathbf{P}) \stackrel{(b)}{=} \mathbf{1}_n + \frac{1}{n} \mathbf{P}^\top \text{diag}(\Upsilon), \tag{41}
\end{aligned}$$

where $\iota \triangleq \frac{\epsilon}{\rho_n n + \rho(\mathbf{Q}_o \mathbf{D})} = \mathcal{O}(1)$ since $\frac{\epsilon}{n\rho_n} \xrightarrow{n \rightarrow \infty} 0$, and Υ is given by

$$\begin{aligned}
\Upsilon &= \mathbf{D}^{-\frac{1}{2}} \left(\left[\mathbf{I} - (\iota \mathbf{D}^{\frac{1}{2}} \mathbf{Q}_o \mathbf{D}^{\frac{1}{2}})^2 \right]^{-1} - \mathbf{I} \right) \mathbf{D}^{-\frac{1}{2}} \\
&= \iota^2 \mathbf{Q}_o \mathbf{D} \mathbf{Q}_o (\mathbf{I} - (\iota \mathbf{Q}_o \mathbf{D} \mathbf{Q}_o)^2)^{-1}; \tag{42}
\end{aligned}$$

step (a) in (41) follows from the definition of $\bar{\mathbf{A}}$ in (9) and Lemma 3; step (b) of (41) is due to the special structure of \mathbf{P} in (3), where for an arbitrary matrix \mathbf{Z} of appropriate size the following equality holds:

$$[\text{diag}(\mathbf{P}^\top \mathbf{Z} \mathbf{P})]_i = \sum_{k, k' \in [K]} \mathbf{P}_{ki} \mathbf{Z}_{k, k'} \mathbf{P}_{k'i} = \sum_{k \in [K]} \mathbf{P}_{ki}^2 \mathbf{Z}_{k, k} \stackrel{(c)}{=} \mathbf{P}_i \text{diag}(\mathbf{Z}),$$

in which step (c) is true because \mathbf{P} is binary.

Combining Lemma 2 and (41), with Φ defined in (9), yields:

$$\theta_{\text{group}, \bar{\mathbf{A}}} = \frac{1}{r} \mathbf{W} \left(\mathbf{1}_n + \frac{1}{n} \mathbf{P}^\top \text{diag}(\Upsilon) \right) = \frac{1}{r} \left(\mathbf{1}_m + \frac{1}{n} \Phi \text{diag}(\Upsilon) \right). \tag{43}$$

D. Theorem: ℓ_1 error between the group controllability of the system with true and expected \mathbf{A}

Define the error $\Delta(\mathbf{A}, \bar{\mathbf{A}})$ similar to (12):

$$\Delta(\mathbf{A}, \bar{\mathbf{A}}) \triangleq \frac{\|r\theta_{\text{group}, \mathbf{A}} - \mathbf{1}_m\|_1}{\|r\theta_{\text{group}, \mathbf{A}} - \mathbf{1}_m\|_1} - \frac{\|r\theta_{\text{group}, \bar{\mathbf{A}}} - \mathbf{1}_m\|_1}{\|r\theta_{\text{group}, \bar{\mathbf{A}}} - \mathbf{1}_m\|_1}. \tag{44}$$

The following theorem bounds the difference between the group controllability of the true and expected \mathbf{A} in (44).

As hinted in Section IV, this difference, when small, will allow the introduction of the candidate estimator in (16). The estimator is only a function of Φ , \mathbf{Q} , and \mathbf{D} , which are retrievable from the coarse-scale matrix $\tilde{\mathbf{A}}$.

Theorem 6. (ℓ_1 error bound between $\theta_{\text{group}, \mathbf{A}}$ and $\theta_{\text{group}, \tilde{\mathbf{A}}}$): Let $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ be as in (44) for $\mathbf{A} \in \{0, 1\}^{n \times n}$ and $\tilde{\mathbf{A}} \in \mathbb{R}^{m \times m}$, and $0 < \delta_o, \delta < 1$ be two constants. Then, under Assumption 1, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$

$$\Delta(\mathbf{A}, \tilde{\mathbf{A}}) \leq \frac{2(\frac{\epsilon}{\rho_n n} + \rho(\mathbf{Q}_o \mathbf{D}))^2}{\|\mathbf{Q}_o \mathbf{D} \mathbf{Q}_o\|_{\min}} \left[1 + \sqrt{\frac{n}{mr}} \sqrt{\frac{\log(\frac{1}{\delta_o})}{2}} \right] \alpha_n \quad (45)$$

$$= \mathcal{O} \left(\frac{\eta}{\rho_n} + \sqrt{\frac{n}{mr}} \sqrt{\frac{\log(\frac{1}{\delta_o})}{2}} \right),$$

holds with probability at least $(1 - \delta_o)(1 - \delta)$, where $\eta = \frac{\sqrt{\log(\frac{n^2}{2})/(2n)}}{\|\mathbf{Q} \mathbf{D} \mathbf{Q}\|_{\min}}$; n and m are the dimensions of \mathbf{A} and $\tilde{\mathbf{A}}$; \mathbf{Q} is defined in Def. 1; and $\mathbf{D} = (1/n)\mathbf{P}\mathbf{P}^\top$. \square

Proof. See Theorem 6 in the extended version at [65]. \square

E. Proof of Theorem 1

Let $i \in [m]$, and recall that $\mathbf{B}_i = \text{diag}(\mathbf{w}_i^\top)$ in (6) and $\tilde{\mathbf{B}}_i = r\mathbf{W}\text{diag}(\mathbf{w}_i^\top)$ in (10). Consider the following bound on $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ in (12), using the triangle inequality for matrix norms:

$$\Delta(\mathbf{A}, \tilde{\mathbf{A}}) \leq \underbrace{\left\| \frac{r\theta_{\text{group}, \mathbf{A}} - \mathbf{1}_m}{\|r\theta_{\text{group}, \mathbf{A}} - \mathbf{1}_m\|_1} - \frac{r\theta_{\text{group}, \tilde{\mathbf{A}}} - \mathbf{1}_m}{\|r\theta_{\text{group}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} \right\|_1}_{=\Delta(\mathbf{A}, \tilde{\mathbf{A}})} + \underbrace{\left\| \frac{\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m}{\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} - \frac{\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m}{\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} \right\|_1}_{=\Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}})} + \underbrace{\left\| \frac{r\theta_{\text{group}, \tilde{\mathbf{A}}} - \mathbf{1}_m}{\|r\theta_{\text{group}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} - \frac{\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m}{\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} \right\|_1}_{=\Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}})}. \quad (46)$$

where $\theta_{\text{group}, \mathbf{A}}$ and $\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ are as in (8) and (11). Similarly, $\theta_{\text{group}, \tilde{\mathbf{A}}}$ and $\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ are defined by replacing \mathbf{A} with $\tilde{\mathbf{A}}$ in (8), and $\tilde{\mathbf{A}}$ with $\tilde{\mathbf{A}}$ in (11).

An upper bound on $\Delta(\mathbf{A}, \tilde{\mathbf{A}})$ has been provided in Theorem 6. We now derive an upper bound on $\Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}})$ with the help of Proposition 5. Replacing \mathbf{Z} with $\tilde{\mathbf{A}}$ in (24) yields $\theta_{\text{coarse}, \tilde{\mathbf{A}}} = \text{diag} \left((\mathbf{I} - \tilde{\mathbf{A}}_{\text{nom}}^2)^{-1} \right)$. Hence

$$\begin{aligned} \|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}\|_1 &= \text{tr} \left[(\mathbf{I} - \tilde{\mathbf{A}}_{\text{nom}}^2)^{-1} - \mathbf{I} \right] = \text{tr} \left[\tilde{\mathbf{A}}_{\text{nom}}^2 (\mathbf{I} - \tilde{\mathbf{A}}_{\text{nom}}^2)^{-1} \right] \\ &\geq \text{tr} \left[\tilde{\mathbf{A}}_{\text{nom}}^2 \right] \stackrel{(a)}{\geq} \frac{\text{tr} \left[\left(\frac{\Phi^\top \Phi}{m} \mathbf{Q}_o \right)^2 \right]}{\left(\frac{\epsilon}{m\rho_n} + \rho \left(\mathbf{Q}_o \frac{\Phi^\top \Phi}{m} \right) \right)^2}, \end{aligned} \quad (47)$$

where for step (a) (9) the normalization factor in (10), and the invariance of trace and spectral radius (for square matrices) under cyclic permutation are used. Replacing \mathbf{z} and \mathbf{y} in

Proposition 5 with $\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ and $\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ yields the following bound on $\Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}})$ in (46):

$$\begin{aligned} \Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}}) &\leq \frac{2\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \theta_{\text{coarse}, \tilde{\mathbf{A}}}\|_1}{\max \left(\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1, \|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1 \right)} \quad (48) \\ &\stackrel{(a)}{\leq} \frac{2\tilde{\alpha}_m}{\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} \stackrel{(b)}{\leq} 2 \frac{\left(\frac{\epsilon}{m\rho_n} + \rho \left(\mathbf{Q}_o \frac{\Phi^\top \Phi}{m} \right) \right)^2}{\text{tr} \left(\left(\frac{\Phi^\top \Phi}{m} \mathbf{Q}_o \right)^2 \right)} \tilde{\alpha}_m \stackrel{(c)}{=} \mathcal{O} \left(\frac{\tilde{\eta}}{\rho_n} \right), \end{aligned}$$

with probability at least $1 - \tilde{\delta}$, where in step (a), $\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \theta_{\text{coarse}, \tilde{\mathbf{A}}}\|_1$ is substituted with $\tilde{\alpha}_m$ as defined in (26); In (b), the term $\|\theta_{\text{coarse}, \tilde{\mathbf{A}}} - \mathbf{1}_m\|_1$ is replaced by its upper bound found in (47); and in (c), the scaling function in (26) is used, assuming n, m are sufficiently large.

Next, we find an upper bound for the remaining error term $\Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}})$ in (46). The proof follows similar to the derivations of (45) and (48) and, hence, detailed steps of the proof are not given to avoid redundancy. We first derive a closed-form formula for $\theta_{\text{coarse}, \tilde{\mathbf{A}}}$ similar to the one derived for $\theta_{\text{group}, \mathbf{A}}$ in Lemma 2:

$$\begin{aligned} \theta_{\text{coarse}, \tilde{\mathbf{A}}} &= \text{diag} \left(\sum_{\tau=0}^{\infty} \tilde{\mathbf{A}}_{\text{nom}}^{2\tau} \right) = \text{diag} \left(\sum_{\tau=0}^{\infty} \left(\frac{\tilde{\Gamma}}{m} \Phi \mathbf{Q}_o \Phi^\top \right)^{2\tau} \right) \\ &= \mathbf{1}_m + \frac{1}{m} \text{diag} \left(\Phi \tilde{\Upsilon} \Phi^\top \right), \end{aligned} \quad (49)$$

where

$$\tilde{\Upsilon} \triangleq \tilde{\Gamma}^2 \mathbf{Q}_o \frac{\Phi^\top \Phi}{m} \mathbf{Q}_o (\mathbf{I} - (\tilde{\Gamma} \frac{\Phi^\top \Phi}{m} \mathbf{Q}_o)^2)^{-1},$$

and $\tilde{\Gamma} \triangleq 1 / \left(\frac{\epsilon}{\rho_n m} + \rho \left(\mathbf{Q}_o \frac{\Phi^\top \Phi}{m} \right) \right)$. From (43) and (49), the upper bound on $\Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}})$ defined in (46) is simplified as

$$\begin{aligned} \Delta(\tilde{\mathbf{A}}, \tilde{\mathbf{A}}) &= \left\| \frac{\frac{1}{n} \Phi \text{diag}(\Upsilon)}{\|\frac{1}{n} \Phi \text{diag}(\Upsilon)\|_1} - \frac{\frac{1}{m} \text{diag}(\Phi \tilde{\Upsilon} \Phi^\top)}{\|\frac{1}{m} \text{diag}(\Phi \tilde{\Upsilon} \Phi^\top)\|_1} \right\|_1 \\ &\stackrel{(a)}{\leq} 2 \frac{\|\Phi \text{diag}(\Upsilon) - \text{diag}(\Phi \tilde{\Upsilon} \Phi^\top)\|_1}{\max \left(\|\Phi \text{diag}(\Upsilon)\|_1, \|\text{diag}(\Phi \tilde{\Upsilon} \Phi^\top)\|_1 \right)} \quad (50) \\ &\leq 2 \frac{\|\Phi \text{diag}(\Upsilon) - \text{diag}(\Phi (\tilde{\Upsilon} - \Upsilon + \Upsilon) \Phi^\top)\|_1}{m \|\Upsilon\|_{\min}} \\ &\leq 2 \frac{\|\Phi \text{diag}(\Upsilon) - \text{diag}(\Phi \Upsilon \Phi^\top)\|_1 + m \|\tilde{\Upsilon} - \Upsilon\|_{\max}}{m \|\Upsilon\|_{\min}} \\ &\stackrel{(b)}{=} \mathcal{O} \left(\frac{\|\Phi \text{diag}(\Upsilon) - \text{diag}(\Phi \Upsilon \Phi^\top)\|_1}{m} + \|\mathbf{D} - \frac{\Phi^\top \Phi}{m}\|_{\max} \right), \end{aligned}$$

where step (a) follows from similar logic as in Proposition 5; (b) is similar to a later elaborate derivation in Section F which results in (61), except that $\hat{\mathbf{Q}}$ in (61) is replaced with \mathbf{Q} (i.e. $\mathbf{L}_\mathbf{Q} = 0$) and $\hat{\mathbf{D}}$ in (61) is substituted by $\frac{\Phi^\top \Phi}{m}$ (i.e. $\mathbf{L}_\mathbf{D} = \mathbf{D} - \frac{\Phi^\top \Phi}{m}$). The statement of the theorem follows with probability at least $1 - \delta_o - \delta - \tilde{\delta}$ (using union bound on the probability of joint events) from (46), (45), (48), and (50). To simplify the notations, we set $\delta_o = \tilde{\delta} = \delta$. The proof is now complete. \square

F. Proof of Theorem 2

By an application of the triangle inequality for matrix norms, we upper bound $\hat{\Delta}(\tilde{\mathbf{A}})$ in (18) as

$$\begin{aligned} \hat{\Delta}(\tilde{\mathbf{A}}) &\leq \underbrace{\left\| \frac{r\boldsymbol{\theta}_{\text{group},\tilde{\mathbf{A}}} - \mathbf{1}_m}{\|r\boldsymbol{\theta}_{\text{group},\tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} - \frac{r\boldsymbol{\theta}_{\text{group},\mathbf{A}} - \mathbf{1}_m}{\|r\boldsymbol{\theta}_{\text{group},\mathbf{A}} - \mathbf{1}_m\|_1} \right\|_1}_{=\Delta(\mathbf{A},\tilde{\mathbf{A}})} \\ &\quad + \underbrace{\left\| \frac{\hat{\boldsymbol{\theta}}_{\text{group}} - \mathbf{1}_m}{\|\hat{\boldsymbol{\theta}}_{\text{group}} - \mathbf{1}_m\|_1} - \frac{r\boldsymbol{\theta}_{\text{group},\tilde{\mathbf{A}}} - \mathbf{1}_m}{\|r\boldsymbol{\theta}_{\text{group},\tilde{\mathbf{A}}} - \mathbf{1}_m\|_1} \right\|_1}_{=\hat{\Delta}(\tilde{\mathbf{A}},\tilde{\mathbf{A}})}. \end{aligned} \quad (51)$$

The first term on the right hand side of the inequality in (51) has already been bounded in (45). In the following, we bound the second term in (51). We substitute $\hat{\boldsymbol{\theta}}_{\text{group}}$ with the output of Algorithm 1, and $\boldsymbol{\theta}_{\text{group},\tilde{\mathbf{A}}}$ with (43), and Υ defined in (13) to get:

$$\hat{\Delta}(\tilde{\mathbf{A}},\tilde{\mathbf{A}}) = \left\| \frac{\hat{\Phi}\text{diag}(\hat{\Upsilon})}{\|\hat{\Phi}\text{diag}(\hat{\Upsilon})\|_1} - \frac{\Phi\text{diag}(\Upsilon)}{\|\Phi\text{diag}(\Upsilon)\|_1} \right\|_1. \quad (52)$$

Let $\mathbf{L}_\Phi = \hat{\Phi} - \Phi$, $\mathbf{L}_Q = \hat{Q} - Q$, and $\mathbf{L}_D = \hat{D} - D$ be error matrices of appropriate sizes which we substitute into (52). We successively apply of the triangle inequality, and we use of Proposition 5 to obtain

$$\begin{aligned} \hat{\Delta}(\tilde{\mathbf{A}},\tilde{\mathbf{A}}) &\leq 2 \frac{\|\hat{\Phi}\text{diag}(\hat{\Upsilon}) - \Phi\text{diag}(\Upsilon)\|_1}{\max(\|\hat{\Phi}\text{diag}(\hat{\Upsilon})\|_1, \|\Phi\text{diag}(\Upsilon)\|_1)} \\ &\leq 2 \frac{\|(\Phi + \mathbf{L}_\Phi)\text{diag}(\hat{\Upsilon}) - \Phi\text{diag}(\Upsilon)\|_1}{\|\Phi\text{diag}(\Upsilon)\|_1} \\ &\leq 2 \frac{\|\Phi\text{diag}(\hat{\Upsilon} - \Upsilon)\|_1 + \|\mathbf{L}_\Phi\text{diag}(\hat{\Upsilon})\|_1}{m\|\text{diag}(\Upsilon)\|_{\min}} \\ &\leq 2 \frac{\|\hat{\Upsilon} - \Upsilon\|_{\max} + \|\hat{\Upsilon}\|_{\max}\|\mathbf{L}_\Phi\|_{1,1}/m}{\|\Upsilon\|_{\min}}. \end{aligned} \quad (53)$$

We simplify $\|\hat{\Upsilon} - \Upsilon\|_{\max}$ in (53) using the two error matrices, \mathbf{L}_Q and \mathbf{L}_D , introduced in the statement of the theorem:

$$\begin{aligned} (\mathbf{Q}_o + \mathbf{L}_Q)(\mathbf{D} + \mathbf{L}_D)(\mathbf{Q}_o + \mathbf{L}_Q) &= \mathbf{Q}_o\mathbf{D}\mathbf{Q}_o + \Gamma_1, \\ (\mathbf{D} + \mathbf{L}_D)(\mathbf{Q}_o + \mathbf{L}_Q) &= \mathbf{D}\mathbf{Q}_o + \Gamma_2, \\ (\mathbf{D}^{\frac{1}{2}} + \mathbf{L}_D^{\frac{1}{2}})(\mathbf{Q} + \mathbf{L}_Q)(\mathbf{D}^{\frac{1}{2}} + \mathbf{L}_D^{\frac{1}{2}}) &= \mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{\frac{1}{2}} + \Gamma_3, \\ (\mathbf{D}\mathbf{Q}_o + \Gamma_2)^{2\ell} &= (\mathbf{D}\mathbf{Q}_o)^{2\ell} + \Gamma_4. \end{aligned} \quad (54)$$

From which, we have the following:

$$\begin{aligned} \Gamma_1 &\triangleq \mathbf{Q}_o\mathbf{D}\mathbf{L}_Q + \mathbf{Q}_o\mathbf{L}_D\mathbf{Q}_o + \mathbf{Q}_o\mathbf{L}_D\mathbf{L}_Q + \mathbf{L}_Q\mathbf{D}\mathbf{Q}_o \\ &\quad + \mathbf{L}_Q\mathbf{D}\mathbf{L}_Q + \mathbf{L}_Q\mathbf{L}_D\mathbf{Q}_o + \mathbf{L}_Q\mathbf{L}_D\mathbf{L}_Q, \\ \Gamma_2 &\triangleq \mathbf{D}\mathbf{L}_Q + \mathbf{L}_D\mathbf{Q}_o + \mathbf{L}_D\mathbf{L}_Q \\ \Gamma_3 &\triangleq \mathbf{D}^{\frac{1}{2}}\mathbf{L}_Q\mathbf{D}^{\frac{1}{2}} + \mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{L}_D^{\frac{1}{2}} + \mathbf{D}^{\frac{1}{2}}\mathbf{L}_Q\mathbf{L}_D^{\frac{1}{2}} \\ &\quad + \mathbf{L}_D^{\frac{1}{2}}(\mathbf{Q}\mathbf{D}^{\frac{1}{2}} + \mathbf{L}_Q\mathbf{D}^{\frac{1}{2}} + \mathbf{Q}\mathbf{L}_D^{\frac{1}{2}} + \mathbf{L}_Q\mathbf{L}_D^{\frac{1}{2}}) \\ \Gamma_4 &\triangleq (\mathbf{D}\mathbf{Q}_o)^{2\ell-1}\Gamma_2 + \Gamma_2(\mathbf{D}\mathbf{Q}_o)^{2\ell-1} + (\mathbf{D}\mathbf{Q}_o)^{2\ell-2}\Gamma_2\mathbf{D}\mathbf{Q}_o \\ &\quad + (\mathbf{D}\mathbf{Q}_o)^{2\ell-2}\Gamma_2^2 + \Gamma_2(\mathbf{D}\mathbf{Q}_o)^{2\ell-1} + \Gamma_2(\mathbf{D}\mathbf{Q}_o)^{2\ell-2}\Gamma_2 + \dots \end{aligned} \quad (55)$$

The last inequality in (55) holds for integer $\ell > 0$. Note that for any norm, $\|\Gamma_3\| = \mathcal{O}(\|\mathbf{L}_Q\| + \|\mathbf{L}_D\|)$, $\|\Gamma_4\| = \mathcal{O}(\|\Gamma_2\|)$

when $\|\mathbf{L}_D\|, \|\mathbf{L}_Q\| \ll 1$. From (54), we also have

$$|\rho(\mathbf{Q}\mathbf{D} + \Gamma_3) - \rho(\mathbf{Q}\mathbf{D})| \stackrel{(a)}{\leq} \|\Gamma_3\|_2 \stackrel{(b)}{\leq} K\|\Gamma_3\|_{\max}, \quad (56)$$

where (a) is due to Proposition 4, and (b) uses item 6 of the list in beginning of the appendix. From (54), (56), (42), and (17) (after re-arranging terms similarly to (42)), (53) is simplified as

$$\begin{aligned} &\|\hat{\Upsilon} - \Upsilon\|_{\max} \\ &= \|\hat{\Upsilon}^2\hat{\mathbf{Q}}\hat{\mathbf{D}}\hat{\mathbf{Q}}(\mathbf{I} - (\hat{\mathbf{D}}\hat{\mathbf{Q}})^2)^{-1} - \iota^2\mathbf{Q}_o\mathbf{D}\mathbf{Q}_o(1 - (\iota\mathbf{D}\mathbf{Q}_o)^2)^{-1}\|_{\max} \\ &= \hat{\iota}^2\|(\mathbf{Q}_o\mathbf{D}\mathbf{Q}_o + \Gamma_1)(\mathbf{I} - (\hat{\mathbf{D}}\hat{\mathbf{Q}})^2)^{-1} \\ &\quad - \left(\frac{\hat{\iota}}{\iota}\right)^2\mathbf{Q}_o\mathbf{D}\mathbf{Q}_o(1 - (\iota\mathbf{D}\mathbf{Q}_o)^2)^{-1}\|_{\max} \\ &= \hat{\iota}^2\|(\mathbf{Q}_o\mathbf{D}\mathbf{Q}_o)[(\mathbf{I} - (\hat{\mathbf{D}}\hat{\mathbf{Q}})^2)^{-1} - (\mathbf{I} - (\iota\mathbf{D}\mathbf{Q}_o)^2)^{-1}] \\ &\quad + \Gamma_1(\mathbf{I} - (\hat{\mathbf{D}}\hat{\mathbf{Q}})^2)^{-1} - \frac{\hat{\iota}^2 - \iota^2}{\iota^2}\mathbf{Q}_o\mathbf{D}\mathbf{Q}_o(\mathbf{I} - (\iota\mathbf{D}\mathbf{Q}_o)^2)^{-1}\|_{\max} \\ &\stackrel{(c)}{\leq} \|\mathbf{Q}_o\mathbf{D}\mathbf{Q}_o\|_{\max}\|\Gamma_5\|_{\max} + \|\Gamma_1\|_{\max}\|(1 - (\iota\mathbf{D}\mathbf{Q}_o)^2)^{-1}\|_{\max} \\ &\quad + \left(2\rho(\mathbf{Q}\mathbf{D}) + \rho(\Gamma_3) + \frac{2\epsilon}{n}\right) \frac{\|\Gamma_3\|_2}{\left(\frac{\epsilon}{n} + \rho(\mathbf{Q}\mathbf{D})\right)^2} \\ &\quad \times \|\mathbf{Q}_o\mathbf{D}\mathbf{Q}_o\|_{\max}\|(\mathbf{I} - (\iota\mathbf{D}\mathbf{Q}_o)^2)^{-1}\|_{\max} K\hat{\iota}^2 \\ &\stackrel{(d)}{=} \mathcal{O}(\|\Gamma_5\|_{\max} + \|\Gamma_1\|_{\max} + \|\Gamma_3\|_{\max}), \end{aligned} \quad (57)$$

where $\Gamma_5 \triangleq (\mathbf{I} - (\hat{\mathbf{D}}\hat{\mathbf{Q}})^2)^{-1} - (\mathbf{I} - (\iota\mathbf{D}\mathbf{Q}_o)^2)^{-1}$; (c) follows from the triangle inequality of matrix norms, multiple use of item 7 in the list at the beginning of the appendix, (13), (17), and inequality (a) in (56); and (d) is a consequence of inequality (b) in (56), \mathbf{Q}_o, \mathbf{D} being constant values, and $\Gamma_1, \Gamma_3, \Gamma_5 \ll 1$.

We simplify (57). First, we upper bound $\|\Gamma_5\|_{\max}$ in (57) using Neumann series, (54), and the triangle inequality for matrix norms:

$$\begin{aligned} \|\Gamma_5\|_{\max} &= \left\| \sum_{\ell \geq 1} [(\hat{\mathbf{D}}\hat{\mathbf{Q}})^{2\ell} - (\iota\mathbf{D}\mathbf{Q}_o)^{2\ell}] \right\|_{\max} \\ &= \left\| \sum_{\ell \geq 1} \hat{\iota}^{2\ell}\Gamma_4 - (\hat{\iota}^{2\ell} - \iota^{2\ell})(\mathbf{D}\mathbf{Q}_o)^{2\ell} \right\|_{\max} \\ &\leq \|\hat{\iota}^2(1 - \hat{\iota}^2)^{-1}\Gamma_4\|_{\max} + \|(\mathbf{D}\mathbf{Q}_o)^2[\mathbf{I} - (\mathbf{D}\mathbf{Q}_o)^2]^{-1}\|_{\max} \max_{\ell \geq 1} |\hat{\iota}^{2\ell} - \iota^{2\ell}| \\ &\stackrel{(a)}{=} \mathcal{O}(\|\Gamma_4\|_{\max} + |\hat{\iota} - \iota|) \stackrel{(b)}{=} \mathcal{O}(\|\Gamma_2\|_{\max} + \|\Gamma_3\|_{\max}), \end{aligned} \quad (58)$$

where (a) holds because $\hat{\iota}, \hat{\mathbf{D}}, \mathbf{D}, \mathbf{Q}_o = \mathcal{O}(1)$; (b) follows from (55), $|\hat{\iota} - \iota| = \frac{|\rho(\mathbf{Q}\mathbf{D} + \Gamma_3) - \rho(\mathbf{Q}\mathbf{D})|}{(\epsilon/(\rho_n n) + \rho(\mathbf{Q}_o\mathbf{D}))(\epsilon/(\rho_n n) + \rho(\mathbf{Q}\mathbf{D}))}$, and (56). Similar to (58), $\|\Gamma_1\|_{\max}, \|\Gamma_2\|_{\max}$ can be upper bounded as

$$\begin{aligned} \|\Gamma_1\|_{\max} &\leq K^2(\|\mathbf{Q}_o\mathbf{D}\|_{\max}\|\mathbf{L}_Q\|_{\max} + \|\mathbf{Q}_o\|_{\max}\|\mathbf{L}_D\|_{\max}\|\mathbf{Q}\|_{\max} \\ &\quad + \|\mathbf{Q}_o\|_{\max}\|\mathbf{L}_D\|_{\max}\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_Q\|_{\max}\|\mathbf{D}\mathbf{Q}_o\|_{\max} \\ &\quad + \|\mathbf{L}_Q\|_{\max}\|\mathbf{D}\|_{\max}\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_Q\|_{\max}\|\mathbf{L}_D\|_{\max}\|\mathbf{Q}_o\|_{\max} \\ &\quad + \|\mathbf{L}_Q\|_{\max}\|\mathbf{L}_D\|_{\max}\|\mathbf{L}_Q\|_{\max}) \\ &\leq K^2(\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_D\|_{\max} + \|\mathbf{L}_D\|_{\max}\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_Q\|_{\max} \\ &\quad + \|\mathbf{L}_Q\|_{\max}^2 + \|\mathbf{L}_Q\|_{\max}\|\mathbf{L}_D\|_{\max} + \|\mathbf{L}_Q\|_{\max}^2\|\mathbf{L}_D\|_{\max}) \\ &= \mathcal{O}(\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_D\|_{\max}), \end{aligned} \quad (59)$$

and

$$\begin{aligned}
\|\Gamma_2\|_{\max} &= \|\mathbf{D}\mathbf{L}_Q + \mathbf{L}_D\mathbf{Q}_o + \mathbf{L}_D\mathbf{L}_Q\|_{\max} \\
&\leq K(\|\mathbf{D}\|_{\max}\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_D\|_{\max}\|\mathbf{Q}_o\|_{\max} + \|\mathbf{L}_D\|_{\max}\|\mathbf{L}_Q\|_{\max}) \\
&\stackrel{(b)}{\leq} K(\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_D\|_{\max} + \|\mathbf{L}_D\|_{\max}\|\mathbf{L}_Q\|_{\max}) \\
&= \mathcal{O}(\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_D\|_{\max}), \tag{60}
\end{aligned}$$

where (b) in (60) is a consequence of the definitions of \mathbf{D} and \mathbf{Q}_o in Assumption 1, and the explanation below (4); and the scaling behaviour in the last lines of both in (59) and (60) follows when $\|\mathbf{L}_Q\|_{\max}, \|\mathbf{L}_D\|_{\max} \ll 1$.

From (58), (59), and (60), $\|\hat{\Upsilon} - \Upsilon\|_{\max}$ of (57) becomes:

$$\|\hat{\Upsilon} - \Upsilon\|_{\max} = \mathcal{O}(\|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_D\|_{\max}), \tag{61}$$

From (61) and (53), we assert that

$$\hat{\Delta}(\tilde{\mathbf{A}}, \bar{\mathbf{A}}) = \mathcal{O}\left(\frac{\|\mathbf{L}_\Phi\|_{1,1}}{m} + \|\mathbf{L}_Q\|_{\max} + \|\mathbf{L}_D\|_{\max}\right), \tag{62}$$

where we used the facts that $\|\Upsilon\|_{\min} = \Omega(1)$ and $\|\hat{\Upsilon}\|_{\max} = \mathcal{O}(1)$ as defined in (13) and (17). The proof concludes by invoking (62) and (45) in (51). \square

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