Advanced Vector Calculus

Angle between two curves

Let $\vec{r_1} = \vec{r_1}(t)$ and $\vec{r_2} = \vec{r_2}(t)$ be two curves in space. Then the angle between the curves at the point of intersection is defined as the tangential vector at the common point.

The tangential vector of $\vec{r_1} = \vec{r_1}(t)$ is

$$\frac{d\vec{r_1}}{dt} = \vec{T_1}$$

Then tangential vector of $\vec{r_2} = \vec{r_2}(t)$ is

$$\frac{d\vec{r_2}}{dt} = \vec{T_2}$$

If θ be the angle between curves

$$\cos\theta = \frac{\vec{T_1} \cdot \vec{T_2}}{|\vec{T_1}| \times |\vec{T_2}|}$$

Exercise 4.1

If $\vec{r} = e^{at}\vec{u} + e^{-at}\vec{v}$ where $\vec{u}$ & $\vec{v}$ are constant vector show that $\frac{d^2\vec{r}}{dt^2} - a^2\vec{r} = 0$

$$\vec{r} = e^{at}\vec{u} + e^{-at}\vec{v}$$

$$\frac{d\vec{r}}{dt} = a e^{at}\vec{u} - a e^{-at}\vec{v}$$

$$\frac{d^2\vec{r}}{dt^2} = a^2 e^{at}\vec{u} + a^2 e^{-at}\vec{v}$$

$$\frac{d^2\vec{r}}{dt^2} = a^2\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} - a^2\vec{r} = 0$$

b)

Solution

Given,

$$\vec{r} = e^{at}\vec{u} + e^{bt}\vec{v}$$

To prove:

$$\frac{d^2\vec{r}}{dt^2} - (a+b)\frac{d\vec{r}}{dt} + ab\,\vec{r} = 0$$

Then,

$$\frac{d\vec{r}}{dt} = a e^{at}\vec{u} + b e^{bt}\vec{v}$$

$$\frac{d^2\vec{r}}{dt^2} = a^2 e^{at}\vec{u} + b^2 e^{bt}\vec{v}$$

2) Find the unit tangent vector at any point on the curve.

a) $x = 3\cos t$ $y = 3\sin t$ $z = 4t$.

Let $\vec{r}$ be the position vector on the curve

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= 3\cos t\,\hat{i} + 3\sin t\,\vec{j} + 4t\,\vec{k}$.

Diff. both sides

$\frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\vec{j} + 4\vec{k}$.

Let $\vec{T}$ be tangent vector.

$\vec{T} = -3\sin t\,\vec{i} + 3\cos t\,\vec{j} + 4\vec{k}$.

Then,

Unit tangent vector is:

$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{-3\sin t\,\hat{i} + 3\cos t\,\vec{j} + 4\vec{k}}{\sqrt{9+16}}$

$= \frac{-3}{5}\sin t\,\vec{i} + \frac{3}{5}\cos t\,\vec{j} + \frac{4}{5}\vec{k}$

2.

b) $x = t^2 - 1$ $\quad y = 4t - 3$ $\quad z = 2t^2 - 6t$

Let $\vec{r}$ be the position vector on the curve.

$$\vec{r} = (t^2-1)\vec{i} + (4t-3)\vec{j} + (2t^2-6t)\vec{k}$$

$$\frac{d\vec{r}}{dt} = 2t\vec{i} + 4\vec{j} + (4t-6)\vec{k}$$

Let $\vec{T}$ be tangential vector.

$$\vec{T} = 2t\vec{i} + 4\vec{j} + (4t-6)\vec{k}$$

Unit tangent vector is

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|}$$

$$= \frac{2t\vec{i} + 4\vec{j} + (4t-6)\vec{k}}{\sqrt{4t^2+16+16t^2-48t+36}}$$

3) Find the angle between the tangents to the curve at the point $t = \pm 1$

a) $\vec{r} = t^2 \hat{i} + 2t \hat{j} - t^3 \vec{k}$

$$\frac{d\vec{r}}{dt} = 2t \vec{i} + 2 \vec{j} - 3t^2 \vec{k}$$

At $t = 1$,

$$\frac{d\vec{r}}{dt} = 2 \hat{i} + 2 \vec{j} - 3 \vec{k} = \vec{T_1}$$

At $t = -1$,

$$\frac{d\vec{r}}{dt} = -2 \vec{i} + 2 \vec{j} - 3 \vec{k} = \vec{T_2}$$

Now,

$$\cos\theta = \frac{\vec{T_1} \cdot \vec{T_2}}{|\vec{T_1}| \times |\vec{T_2}|}$$

$$= \frac{-4 + 4 + 9}{\sqrt{4+4+9}\ \sqrt{4+4+9}}$$

$$= \frac{9}{17}.$$

$$\therefore \theta = \cos^{-1}\left(\frac{9}{17}\right).$$

4.a. The particle moves along the curve $\vec{r} = (t^3 - 4t)\,\vec{i} + (t^2 + 4t)\,\vec{j} + (8t^2 - 3t^3)\,\vec{k}$, where t is the time. Find the velocity and magnitude of tangential component of its accⁿ at time $t = 2$ sec.

→

Solution

Given,

$$\vec{r} = (t^3 - 4t)\vec{i} + (t^2 + 4t)\vec{j} + (8t^2 - 3t^3)\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (3t^2 - 4)\vec{i} + (2t + 4)\vec{j} + (16t - 9t^2)\vec{k}.$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6t\,\vec{i} + 2\vec{j} + (16 - 18t)\vec{k}.$$

at $t = 2$,

$$\vec{v} = 8\vec{i} + 8\vec{j} - 4\vec{k}$$

$$\vec{a} = 12\vec{i} + 2\vec{j} - 20\vec{k}.$$

Component of $\vec{a}$ in the tangential vector $= \frac{\vec{v}}{|\vec{v}|} \cdot \vec{a}$

$$= \frac{(8, 8, -4) \cdot (12, 2, -20)}{\sqrt{64 + 64 + 16}}$$

$$= \frac{96 + 16 + 80}{12}$$

$$= 16$$

Solution

Given,

$x = 2t^2$, $y = t^2 - 4t$ and $z = 3t - 5$

Let $\vec{r}$ be the position vector on the curve Then,

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r} = 2t^2\vec{i} + (t^2 - 4t)\vec{j} + (3t - 5)\vec{k}$

And,

Let $\vec{u} = \vec{i} - 3\vec{j} + 2\vec{k}$

Diff. $\vec{r}$ w.r.t. t,

At t = 1

$\vec{\omega} = \frac{d\vec{r}}{dt} = 4t\,\vec{i} + (2t - 4)\,\vec{j} + 3\vec{k} = 4\vec{i} - 2\vec{j} + 3\vec{k}$

At t = 1,

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\vec{i} + 2\vec{j} + 0\vec{k} = 4\vec{i} + 2\vec{j}$

component of velocity in direction of $\vec{u}$ is

$\frac{\vec{u}}{|\vec{u}|} \cdot \vec{\omega} = \frac{(\vec{i} - 3\vec{j} + 2\vec{k}) \cdot (4\vec{i} - 2\vec{j} + 3\vec{k})}{\cancel{\sqrt{16+4+9}}\ \sqrt{1+9+4}}$

$= \frac{4 + 6 + 6}{\sqrt{14}}$

$= \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}$

component of $\vec{a}$ in direction of $\vec{u}$ is

$\frac{(\vec{i} - 3\vec{j} + 2\vec{k})\,(4\vec{i} + 2\vec{j})}{\sqrt{14}} = \frac{4 - 6 + 0}{\sqrt{14}}$

$= \frac{-2}{\sqrt{14}} = -\frac{\sqrt{14}}{7}$

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Scalar point function

A scalar point function is denoted by $\phi = \phi(x, y, z)$ to mean ϕ is a scalar point function of the variable x, y, z.

Vector point function

A vector point function is denoted by $\vec{\phi} = \vec{\phi}(x, y, z)$ to mean $\vec{\phi}$ is a vector point function.

Vector Differential operator

A vector differential operator is defined ~~denote~~

$$\nabla = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

where ∇ is read as nabla or del.

Gradient of scalar point function

let $\phi = \phi(x, y, z)$ be the scalar point function then its gradient is defined

$$\text{Grad } \phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

The gradient of scalar point function is vector function.

Directional Derivative
Let $\phi = \phi(x,y,z)$ be the scalar point function. The directional derivative of ϕ in the direction $\vec{a}$ is denoted by $D_{\vec{a}}(\phi)$ and is given as,

$$D_{\vec{a}}(\phi) = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

Angle between Surface
The angle between two surface is defined as the angle between the normals at point of intersection.

If $\phi_1(x,y,z) = 0$ and $\phi_2(x,y,z) = 0$ are any two surfaces then,

$$\vec{n_1} = \nabla\phi_1 \quad , \vec{n_2} = \nabla\phi_2$$

Let θ be the angle between surfaces

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

1) Find the grad ϕ at given points

a) $\phi = \ln(x^2 + y^2 + z^2)$ at $(1,1,1)$.

$\Rightarrow$

$$\text{Grad } \phi = \nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= \vec{i}\left(\frac{1}{x^2+y^2+z^2} \times 2x\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right)$$

$$+ \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

At $(1,1,1)$

$$\nabla \phi = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

2. b) $z = x^2 + y^2$ at $(-1,-2,5)$.

$$\phi = x^2 + y^2 - z = 0.$$

$$\nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2-z) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2-z) + \vec{k}\frac{\partial}{\partial z}(x^2$$

$$= \vec{i}\,2x + \vec{j}\,2y - \vec{k}.$$

At $(-1,-2,5)$

$$\vec{n} = \nabla \phi = -2\vec{i} + -4\vec{j} - \vec{k}$$

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Unit normal vector is:

$$-\frac{2}{\sqrt{21}}\vec{i} - \frac{4}{\sqrt{21}}\vec{j} - \frac{\vec{k}}{\sqrt{21}}$$

$\vec{?} = x\vec{i} + y\vec{j} + 2\vec{k}$

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Constant Magnitude Theorem

The necessary and sufficient condition for a vector function $\vec{r} = \vec{r}(t)$ of scalar variable to have constant magnitude is $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

Proof: Necessary condition,

Let $\vec{r} = \vec{r}(t)$ has constant magnitude, then to show $\vec{r}(t)$ satisfies $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

we have,

$$\vec{r} \cdot \vec{r} = r^2 = \text{constant}, \quad r = |\vec{r}|$$

or, $\vec{r} \cdot \frac{d\vec{r}}{dt} + \vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

or, $2\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

$\therefore \vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

Sufficient condition, let $\vec{r} = \vec{r}(t)$ satisfies $\vec{r} \cdot \frac{d\vec{r}}{dt}$ then to prove $\vec{r}$ has const. magnitude.

We know, for any vector $\vec{r} = \vec{r}(t)$

$\vec{r} \cdot \vec{r} = r^2$

Diff. both sides w.r.t. t, we get.

$$\vec{r} \cdot \frac{d\vec{r}}{dt} + \vec{r} \cdot \frac{d\vec{r}}{dt} = 2r\frac{dr}{dt}$$

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = r\frac{dr}{dt}$$

But by supposition,

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$$

$$\therefore r\frac{dr}{dt} = 0$$

Since $r \neq 0$, i.e $\vec{r} = \vec{r}(t)$ is a non-zero vector

$$\therefore \frac{dr}{dt} = 0.$$

which shows derivative of r is 0. Hence magnitude of r ~~MUS~~ of $\vec{r} = \vec{r}(t)$ must constant

Constant direction Theorem:

The necessary and sufficient condition for a vector function $\vec{r} = \vec{r}(t)$ to have constant direction is $\vec{r} \times \frac{d\vec{r}}{dt} = 0$.

Proof: Necessary condition.

Let $\vec{r} = \vec{r}(t)$ be a vector function of scalar variable and has const. direction, then to prove: $\vec{r} \times \frac{d\vec{r}}{dt} = 0$,

For any vector $\vec{r}$, we have,

$$\vec{r} = r\hat{r} \quad \text{--- (1)}$$

where, $r = |\vec{r}|$ and $\hat{r}$ is unit vector along $\vec{r}$.

Diff. (1) w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt}$$

Taking cross product $\vec{r} = r\hat{r}$ on both sides.

$$\vec{r} \times \frac{d\vec{r}}{dt} = (r\hat{r}) \times \left(0 + \hat{r}\frac{dr}{dt}\right)$$

[$\because$ $\hat{r}$ has const. direction so it is const. vector.]

$$\vec{r} \times \frac{d\vec{r}}{dt} = r\frac{dr}{dt}(\hat{r} \times \hat{r})$$

$$\therefore \vec{r} \times \frac{d\vec{r}}{dt} = 0 \quad [\because \hat{r} \times \hat{r} = 0]$$

Sufficient condition,

Suppose $\vec{r} = \vec{r}(t)$ satisfies $\vec{r} \times \frac{d\vec{r}}{dt} = 0$, then to prove $\vec{r}$ has const direction.

we have,

$$\vec{r} \times \frac{d\vec{r}}{dt} = 0$$

or, $$r\hat{r} \times \frac{d r\hat{r}}{dt} = 0$$

or, $$r\hat{r} \times \left(r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt} \right) = 0$$

or, $$r^2 \hat{r} \times \frac{d\hat{r}}{dt} + r(\hat{r} \times \hat{r}) \frac{dr}{dt} = 0$$

or, $$r^2 \hat{r} \times \frac{d\hat{r}}{dt} = 0$$

or, $$\hat{r} \times \frac{d\hat{r}}{dt} = 0 \quad \text{--- (2)}$$

Since $\hat{r}$ has const magnitude 1. so by const magnitude theorem;

$$\hat{r} \cdot \frac{d\hat{r}}{dt} = 0 \quad \text{--- (3)}$$

Since same vector $\hat{r}$ and $\frac{d\hat{r}}{dt}$ satisfy both (2) & in (2) we have sine of angle btn $\hat{r}$ & $\frac{d\hat{r}}{dt}$ whereas cosine of angle betn ther

But we have no common angle for bot

to be zero. Hence.

$$\frac{d\hat{r}}{dt} = 0$$

i.e

derivative of $\hat{r}$ is 0, so it should be a const. vector and hence has both magnitude and direction const. But $\vec{r}$ and $\hat{r}$ have same direction, so $\vec{r}$ must have const. direction.

08/03 Monday

Exercise 4.2

4. Find the directional derivatives

a) $\vec{f} = xyz$ at $P(-1,1,3)$ along $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

→ Solution

Given,

$\vec{f} = xyz$. at $P(-1,1,3)$

along $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Now,

let $\vec{r} = \nabla\vec{f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\vec{r} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\vec{r} = 3\hat{i} - 3\hat{j} - \hat{k}$

The directional derivative is:

$\vec{r} \cdot \frac{\vec{a}}{|\vec{a}|}$

$\cancel{(yz\hat{i}}$

$= (3\hat{i} - 3\hat{j} - k) \cdot \frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1+4+4}}$

$= \frac{3+6-2}{3}$

$= 7/3$ Ans

3) Solution

Given,

$$\phi_1 = xy^2z - 3x - z^2$$

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\hat{i} + \frac{\partial\phi_1}{\partial y}\hat{j} + \frac{\partial\phi_1}{\partial z}\hat{k}$$

$$\cancel{= y^2z\,\hat{i} +}$$

$$= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}$$

At $(1,-2,1)$

Let $\vec{n}_1 = (4\times1 - 3)\hat{i} + (2(-2))\hat{j} + (4-2)\hat{k}$

$$= \hat{i} - 4\hat{j} + 2\hat{k}$$

And,

$$\phi_2 = 3x^2 - y^2 + 2z - 1$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} + \frac{\partial\phi_2}{\partial y}\hat{k}$$

$$= 6x\,\hat{i} + (-2y)\hat{j} + 2\hat{k}$$

$$= 6x\,\hat{i} - 2y\,\hat{j} + 2\hat{k}$$

at $(1,-2,1)$

Let $\vec{n}_2 = 6\hat{i} + 4\hat{j} + 2\hat{k}$

Now,

$$\cos\theta = \frac{(1,-4,2)\cdot(6,4,2)}{\sqrt{1+16+4}\,\sqrt{36+16+4}}$$

$$= \frac{6 - 16 + 4}{\sqrt{21}\,\sqrt{56}} = \frac{-6}{\sqrt{21\times56}}$$

b)

Solution

Given,

$f = x^2 y^2 z^2$ at $(1, 1, -1)$.

and

$(x, y, z) = (e^t, \sin 2t + 1, 1 - \cos t)$.

Let $\vec{a} = e^t \hat{i} + (\sin 2t + 1) \hat{j} + (1 - \cos t) \hat{k}$

$\frac{d\vec{a}}{dt} = e^t \hat{i} + 2\cos 2t \hat{j} + \sin t \hat{k}$.

at $t = 0$

Let, $\vec{p} = \frac{d\vec{a}}{dt} = \hat{i} + 2\hat{j} + 0\hat{k}$

Now,

$f = x^2 y^2 z^2$

Let $\vec{r} = \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$.

$= 2xy^2z^2 \hat{i} + 2yx^2z^2 \hat{j} + 2z$

Putting $(x, y, z) = (1, 1, -1)$

$\vec{r} = 2\hat{i} + 2\hat{j} - 2\hat{k}$

The dir. derivative of f in the dire of $\vec{p}$ is

$\vec{r} \cdot \frac{\vec{p}}{|\vec{p}|}$

$= (2, 2, -2) \cdot \frac{(1, 2, 0)}{\sqrt{1+4}}$

$= \frac{2 + 4 + 0}{\sqrt{5}} = \frac{6}{\sqrt{5}}$

Solution

Given surface is

Let $\phi = x\log z - y^2 + 4 = 0$

The normal to the surface is given by:

$$\vec{n} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= \ln z\,\hat{i} + (-2y)\,\hat{j} + \left(\frac{x}{z}\right)\hat{k}$$

at $(-1, 2, 1)$.

$$\vec{n} = \ln(-1)\,\hat{i} - 4\hat{j} - \hat{k}$$

Now,

$$f = xy^2 + yz^3$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= (y^2)\hat{i} + (2xy + z^3)\hat{j} + 3z^2y\,\hat{k}$$

at $(2, -1, 1)$

Let,

$$\vec{r} = \hat{i} + (-4+1)\hat{j} + 3(-1)\hat{k}$$

$$= \hat{i} - 3\hat{j} - 3\hat{k}$$

The directional derivative is:

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{a}|} = \frac{(1, -3, -3)\cdot(0, -4, -1)}{\sqrt{16+1}}$$

$$= \frac{0 + 12 + 3}{\sqrt{17}} = \frac{15}{\sqrt{17}}.$$

Divergence of a vector field

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be the vector field on $\mathbb{R}^3$. The divergency of $\vec{F}$ is denoted by $\operatorname{div}\vec{F}$ or $\nabla\cdot\vec{F}$ and defined as:

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note: If $\operatorname{div}\vec{F} = 0$. Then $\vec{F}$ is said to be solonoidal

Curl of vector field

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a vector field on $\mathbb{R}$. Then curl of vector is denoted by $\operatorname{curl}\vec{F}$ or $\nabla\times\vec{F}$ defined as:

$$\operatorname{curl} F = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If $\nabla\times\vec{F} = 0$, then $\vec{F}$ is said to be Irrotational.

5) Find the divergence of

a) $\vec{v} = x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}$

Solution

$\vec{v} = x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}$

The divergence of $\vec{v}$ is:

$$\nabla\cdot\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}\right)$$

$$= 2xyz + 2xyz + 2xyz$$

$$= 6xyz.$$

8 b. If $\phi = x^3 + y^3 + z^3 - 3xyz$

Find:

i) $div(grad\ \phi)$ ii) $curl\ (grad\ \phi)$

Given,

$$\phi = x^3 + y^3 + z^3 - 3xyz$$

Now,

$$grad\ \phi = \nabla \phi$$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

Then,

i)

$$div\ (grad\ \phi) = \nabla \cdot (grad\ \phi)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left((3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}\right)$$

$$= 6x + 6y + 6z$$

$$= 6(x+y+z)$$

ii)

$$curl\ (grad\ \phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$= \hat{i}(-3x - (-3x)) - \hat{j}(-3y - (-3y)) + \hat{k}(-3z - (-3z))$

$= 0\hat{i} - 0\hat{j} + 0\hat{k}$

$= 0$

2081/08/04 Tuesday

Line Integral

Any Integral which is evaluated over a curve C is called line integral.

The integration notation $\int_c \vec{F} \cdot d\vec{r}$ is commonly used for line integral.

Exercise 4.3

1. Evaluate $\int_c \vec{F} \cdot d\vec{r}$ for following vector fields.

~~So~~

a) $\vec{F} = [(x-y)^2, (y-x)^2]$, $C: xy = 1, \; 1 \le x \le 4.$

$\Rightarrow$

Let,

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} \quad \text{--- (1)}$$

$$xy = 1$$

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$dy = -\frac{1}{x^2}dx$$

and,

$$\vec{F} = (x-y)^2\hat{i} + (y-x)^2\hat{j}$$

Now,

$$\vec{F} \cdot d\vec{r} = \left[\left(x - \frac{1}{x}\right)^2\hat{i} + \left(x - \frac{1}{x}\right)^2\hat{j}\right] \cdot (dx\hat{i} +$$

$$= (x-\tfrac{1}{x})^2\,dx + (x-\tfrac{1}{x})^2\,dy$$

$$= (x-\tfrac{1}{x})^2\,dx + (x-\tfrac{1}{x})^2 \times -\tfrac{1}{x^2}\,dx$$

$$\int \vec{F}\cdot d\vec{r} = \int_1^4 \left(x^2-2+\frac{1}{x^2}\right)dx + \left(x^2-2+\frac{1}{x^2}\right)\left(\frac{-1}{x^2}\right)dx$$

$$= \int_1^4 \left(x^2-2+\frac{1}{x^2}\right)dx + \int_1^4 \left(-1+\frac{2}{x^2}-\frac{1}{x^4}\right)dx$$

$$= \left(\frac{x^3}{3}-2x-\frac{1}{x}\right)_1^4 + \left(-x-2x^{-1}+\frac{1}{3x^3}\right)_1^4$$

$$= \left[\frac{64}{3}-8-\frac{1}{4}-\left(\frac{1}{3}-2-1\right)\right] + \left[\left(-4-\frac{2}{4}+\frac{1}{3\times 64}\right)\right.$$

$$\left. -\left(-1-2+\frac{1}{3}\right)\right]$$

$$= \frac{891}{64}$$

f. $\vec{F} = (e^{x}, e^{-y}, e^{z})$ $c: (t, t^2, t)$

~~fro~~

c) $\vec{F} = y^2\hat{i} + 2xy\hat{j}$, $c: y^2 = x$ from $(0,0)$ to (

⇒

Let,

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

Since $y^2 = x$

$$2y\frac{dy}{dx} = 1$$

$$dx = 2y\,dy$$

Here,

$$\vec{F} = y^2\hat{i} + 2xy\hat{j}$$

$$\vec{F}\cdot d\vec{r} = (y^2\hat{i} + 2xy\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_c y^2\,dx + 2xy\,dy$$

$$= \int_c y^2 \cdot 2y\,dy + 2 \times y^2 y\,dy$$

$$= \int_0^1 4y^3\,dy$$

$$= 4\left[\frac{y^4}{4}\right]_0^1$$

$$= 1$$

d) $\vec{F} = (2z, x, -y)$, $C: \vec{r} = (\cos t, \sin t, 2t)$ from $(0,0,0)$ to $(1, 0, 4\pi)$

→ Solution

Here,

$$\vec{F} = (2z, x, -y)$$

$$\vec{r} = (\cos t, \sin t, 2t)$$

$x = \cos t$, $y = \sin t$, $z = 2t$.

$$\frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 2\hat{k}$$

$$d\vec{r} = (-\sin t\,\hat{i} + \cos t\,\hat{j} + 2\hat{k})\,dt.$$

Now,

$$\vec{F}\cdot d\vec{r} = (2z\,\hat{i} + x\,\hat{j} - y\,\hat{k})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j} + 2\hat{k})dt$$

$$= (4t\,\hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j} + 2\hat{k})dt$$

$$= (-4t\sin t + \cos^2 t - 2\sin t)\,dt.$$

Since the particle moves from $(0,0,0)$ to $(1, 0, 4\pi)$

We have, $z = 2t$, when $z = 0$, $t = 0$

when $z = 4\pi$, $t = 2\pi$

Now,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-4t\sin t + \cos^2 t - 2\sin t)\,dt$$

$$= \int_0^{2\pi} \left(-4t\sin t + \frac{\cos 2t + 1}{2} - 2\sin t \right) dt$$

$$= \int_0^{2\pi} -4t \sin t \, dt + \frac{1}{2}\left[\frac{\sin 2t}{2} + t\right]_0^{2\pi} - 2\,[-\cos$$

$$= -4\left[t(-\cos t) - \int (-\cos t)\, dt \right]_0^{2\pi} + \frac{1}{2}\left[0 + 2\pi - 0\right.$$

$$+ 2[1-1]$$

$$= -4\left[-t\cos t + \sin t \right]_0^{2\pi} + \frac{1}{2} \times 2\pi + 0$$

$$= -4[-2\pi \cdot 1 + 0 - (0+0)] + 2\pi$$

$$= -4[-2\pi] + 2\pi$$

$$= 8\pi + 2\pi$$

$$= 10\pi.$$

Ans

Imp

Example 24.

9. $\vec{F} = (2xy - z)\hat{i} + yz\hat{j} + x\hat{k}$

$C : (t, 2t, t^2 - 1)$ from $t = 0$ to 1.

$\Rightarrow$

$x = t, \quad y = 2t, \quad z = t^2 - 1$

$\vec{r} = t\hat{i} + 2t\hat{j} + (t^2 - 1)\hat{k}$

$d\vec{r} = (\hat{i} + 2\hat{j} + 2t\hat{k})\, dt$

$\therefore \vec{F} = (4t^2 - t^2 + 1)\hat{i} + (2t^3 - 2t)\hat{j} + t\hat{k}$

$= (3t^2 + 1)\hat{i} + (2t^3 - 2t)\hat{j} + t\hat{k}$

Now,

$$\int_0^1 (3t^2 + 1,\ 2t^3 - 2t,\ t)\,(1,\ 2,\ 2t)\, dt$$

$$= \int_0^1 (3t^2 + 1 + 4t^3 - 4t + 2t^2)\, dt$$

$$= \int_0^1 (5t^2 + 4t^3 - 4t + 1)\, dt$$

$$= \left[\frac{5t^3}{3} + \frac{4t^4}{4} - \frac{4t^2}{2} + t\right]_0^1$$

$$= \left[\frac{5}{3} + 1 - 2 + 1 - 0\right]$$

$$= \frac{5}{3}$$

h) $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$

C: $x^2 = 4y$, $3x^3 = 8z$, from $0 \le x \le 2$.

$2x = 4\frac{dy}{dx}$ $\qquad 9x^2 = 8\frac{dz}{dx}$

$\frac{x\,dx}{2} = dy$ $\qquad \frac{9x^2\,dx}{8} = dz$.

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\vec{F} = 3x^2\,\hat{i} + \left(2\cdot x\cdot\frac{3x^3}{8} - \frac{x^2}{4}\right)\hat{j} + \frac{3x^3}{8}\hat{k}$.

$$\int_0^2 (\vec{F}\cdot d\vec{r})$$

$$= \int_0^2 \left(3x^2, \frac{6x^3}{8} - \frac{x^2}{4}, \frac{3x^3}{8}\right)(dx, dy, dz)$$

$$= \int_0^2 \left(3x^2, \frac{6x^4}{8} - \frac{x^2}{4}, \frac{3x^3}{8}\right)\left(dx, \frac{x\,dx}{2}, \frac{9x^2dx}{8}\right)$$

$$= \int_0^2 3x^2dx + \left(\frac{3x^4}{4} - \frac{x^2}{4}\right)\frac{x\,dx}{2} + \left(\frac{3x^3}{8}\times\frac{9x^2}{8}\right)dx.$$

$$= 3\left[\frac{x^3}{3}\right]_0^2 + \int_0^2 \frac{3x^5 - x^3}{8}\,dx + \frac{27}{64}\left[\frac{x^6}{6}\right]_0^2$$

$$= 3\times\frac{8}{3} + \frac{1}{8}\left[\frac{3x^6}{6} - \frac{x^4}{4}\right]_0^2 + \frac{27}{64}\times\frac{2^6}{6}$$

$$= 8 + \frac{1}{8}\left[\frac{3\times 2^6}{6} - \frac{2^4}{4}\right] + \frac{27}{6}.$$

$$= 16$$

08/05 wednesday

~~Evaluate $\int_c \vec{F} \cdot d\vec{r}$~~

2 Calculate $\int_c f \, ds$

a) $f = x^2 y$ $\quad C: \vec{r} = (2\cos t, 2\sin t)$

$0 \le t \le \pi/2$

$f = 4\cos^2 t \cdot 2\sin t$

$= 8 \sin t \cos^2 t$

$\vec{r} = 2\cos t \,\hat{i} + 2\sin t \,\hat{j}$

$\frac{d\vec{r}}{dt} = -2\sin t \,\hat{i} + 2\cos t \,\hat{j}$

$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}$

$= \sqrt{(-2\sin t, 2\cos t) \cdot (-2\sin t, 2\cos t)}$

$= \sqrt{4\sin^2 t + 4\cos^2 t}$

$\frac{ds}{dt} = 2$

$ds = 2\,dt$

$\therefore \int_0^{\pi/2} 8\sin t \cos^2 t \; 2\,dt$

$= 16 \frac{\Gamma 3/2 \; \Gamma 3/2}{2 \times \Gamma \frac{1+2+2}{2}}$ $\quad = \frac{8 \times 1 \times \frac{1}{2}\sqrt{\pi}}{3/2 \times 1/2 \times \sqrt{\pi}}$

$= \frac{4}{3/4} = \frac{16}{3}$

b. $f = \sqrt{16x^2 + 81y^2}$, C $\vec{r} = (3\cos t, 2\sin t)$.

$0 \le t \le \pi$

$$f = \sqrt{16 \times 9\cos^2 t + 81 \times 4\sin^2 t}$$
$$= \sqrt{144\cos^2 t + 324\sin^2 t}$$
$$= \sqrt{144 + 180\sin^2 t}$$

$$\vec{r} = 3\cos t\,\hat{i} + 2\sin t\,\hat{j}$$
$$\frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 2\cos t\,\hat{j}$$

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}$$
$$= \sqrt{9\sin^2 t + 4\cos^2 t}$$
$$\frac{ds}{dt} = \sqrt{5\sin^2 t + 4}$$
$$ds = \sqrt{4 + 5\sin^2 t}\;dt \Rightarrow \sqrt{9\sin^2 t + 4\cos^2 t}\;dt$$

$$\therefore \int_C f\,ds = \int_0^{\pi} \sqrt{144\cos^2 t + 324\sin^2 t}\;\sqrt{9\sin^2 t + 4\cos^2 t}\;dt$$

$$= \int_0^{\pi} \sqrt{1296\sin^2 t\cos^2 t + 576\cos^4 t + 2916\sin^4 t + 1296\sin^2 t\cos^2 t}$$

$$= \int_0^{\pi} \sqrt{576\cos^4 t + 2592\sin^2 t\cos^2 t + 2916\sin^4 t}$$

$$= \int_0^{\pi} \sqrt{(24\cos^2 t)^2 + 2\cdot 24\cos^2 t \cdot 54\sin^2 t + (54\sin^2 t)^2}\; dt$$

$$= \int_0^{\pi} (24\cos^2 t + 54\sin^2 t)\, dt$$

$$= 24 \int_0^{\pi} \frac{\cos 2t + 1}{2}\, dt + 54 \cdot \int_0^{\pi} \frac{1 - \cos 2t}{2}\, dt$$

$$= \cancel{24}\, 12 \left[\frac{\sin 2t}{2} + t\right]_0^{\pi} + 27 \left[t - \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$= 12 [0 + \pi - 0 - 0] + 27 [\pi - 0 - (0 - 0)]$$

$$= 12\pi + 27\pi$$

$$= 39\pi.$$

d) $F = x^2 + y^2 + z^2$ $C: \vec{r} = (\cos t, \sin t, 2t)$.

$0 < t < 4\pi$

$F = \cos^2 t + \sin^2 t + 4t^2$
$= 4t^2 + 1$

$\vec{r} = \cos t \, \hat{i} + \sin t \, \hat{j} + 2t \, \hat{k}$

$\frac{d\vec{r}}{dt} = -\sin t \, \hat{i} + \cos t \, \hat{j} + 2 \hat{k}$.

$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}$

$= \sqrt{\sin^2 t + \cos^2 t + 4}$

$ds = \sqrt{5} \, dt$.

$\therefore \int_C F \, ds = \int_0^{4\pi} (4t^2 + 1) \sqrt{5} \, dt$

$= \sqrt{5} \left[\frac{4t^3}{3} + t \right]_0^{4\pi}$

$= \sqrt{5} \left[\frac{4 \times 64\pi^3}{3} + 4\pi - 0 \right]$

$= \sqrt{5} \left[\frac{256\pi^3}{3} + 4\pi \right]$

e. $f = xyz$ C: line joining $(0,0,0)$ to $(1,2,3)$

→ Solution

Here,

Eq^n of line joining points $(0,0,0)$ to $(1,2,3)$ is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t \quad (\text{let})$$

$x = t, \quad y = 2t, \quad z = 3t$

Let,

$\vec{r}$ be the position vector, then.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= t\hat{i} + 2t\hat{j} + 3t\hat{k}$$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$\therefore f = 6t^3$

For the limit,

$x = t.$

when $x = 0, \quad t = 0$ when $x = 1, \quad t = 1.$

$0 \le t \le 1$

$\therefore$

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}} = \sqrt{1+4+9}$$
$$= \sqrt{14}$$

$$ds = \sqrt{14}\, dt$$

Now, $\int_0^1 F\, ds$

$= \int_0^1 6t^3 \times \sqrt{14}\, dt$

$= 6\sqrt{14} \times \left[\frac{t^4}{4}\right]_0^1$

$= 6\sqrt{14} \times \frac{1}{4}$

$= \frac{3\sqrt{14}}{2}$.

8. Evaluate

a. $\int_C [xy\,dx + (x+y)\,dy]$, where $C: (0,0)$ to $(1,3)$.

→ Here,

$C: (0,0)$ to $(1,3)$

i.e line joining the points is

$$\frac{x}{1} = \frac{y}{3}$$

$$y = 3x$$

$$dy = 3\,dx.$$

$$\therefore \int_0^1 [x \cdot 3x \cdot dx + (x+3x)\,3dx]$$

$$= \int_0^1 3x^2\,dx + 12\int_0^1 x\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^1 + \frac{12}{2}\left[x^2\right]_0^1$$

$$= 1 + 6[1]$$

$$= 7$$

c) $\int_c [(x-y)dx + x dy]$, where

$c: y+2 = x$ from $(4,-2)$ to $(4$

$x = y+2$

$1 = \frac{dy}{dx} + 0$

$dy = dx$.

$\int_{-2}^{2} [(y+2-y)\, dy + (y+2) \cdot dy]$

$= \int_{-2}^{2} 2\, dy + \int_{-2}^{2} (y+2)\, dy$

$= 2[y]_{-2}^{2} + \left[\frac{y^2}{2} + 2y\right]_{-2}^{2}$

$= 2[2+2] + [2+4-(2-4)]$

$= 8 + (6-(-2)]$

$= 8+6+2$

$= 16.$

Imp
Example 20

08/07 Friday

4 Prove that given vectors are irrotational and find a scalar function ϕ such that $\vec{F} = \nabla\phi$

a) $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

Now,

Curl of $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x)\hat{i} - (y-y)\hat{j} + (z-z)\hat{k}$$

$$= 0$$

$\therefore$ The vector is irrotational.

To find a scalar function ϕ

$$d\phi = \vec{F} \cdot d\vec{r}$$

$$d\phi = (yz, xz, xy) \cdot (dx, dy, dz)$$

$$d\phi = yz\,dx + xz\,dy + xy\,dz$$

Integrating we get

$$\phi = \int d\phi = \int d(xyz)$$

$$\therefore \phi = xyz + c$$

Ans

b) $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$

Solution

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= (-x + x)\hat{i} + (-y + y)\hat{j} + (-z + z)\hat{k}$$

$= 0$

$\therefore \vec{F}$ is irrotational.

Let ϕ be the scalar function such that

$\vec{F} = \nabla \phi$

$\vec{F} \cdot d\vec{r} = \nabla \phi \cdot d\vec{r}$

$(x^2 - yz, \; y^2 - zx, \; z^2 - xy) \cdot (dx, dy, dz)$

$= \cancel{(2x, 2y, 2z) \cdot (dx, dy}$

$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial y} \right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right)$

or, $(x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz$

$= d\phi.$

Integrating both

$$\int d\phi = \int (x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz$$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - \int d(xyz) + c$$

$$\therefore \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$$

Exactness condition of the vector field
let $\int (F_1 dx + F_2 dy + F_3 dz)$ be a integral

defined in domain D then the integrand (under integral sign) in exact if.

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$.

Thus ∂F $F_1 dx + F_2 dy + F_3 dz$. Exact then,

$\int_c F_1 dx + \int_c$ (terms of F_2 free from x) dy

$+ \int$ (terms of F_3 free from x & y) dz.

Exercise 4.4

a. $\int_{(0,1)}^{(2,3)} [(2x+y^3)\,dx + (3xy^2+4)\,dy]$

let, $I = (2x+y^3)\,dx + (3xy^2+4)\,dy$.

$F_1 = 2x+y^3$ $\qquad F_2 = 3xy^2 + 4$.

$\frac{\partial F_1}{\partial y} = 3y^2 \qquad \frac{\partial F_2}{\partial x} = 3y^2$

$\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

Now,

$$= \int_{(0,1)}^{(2,3)} (2x+y^3)\,dx + \int_{(0,1)}^{(2,3)} 4\,dy$$

$$= \left[2\frac{x^2}{2}\right]_{(0,1)}^{(2,3)} + \left[y^3 \cdot x\right]_{(0,1)}^{(2,3)} + 4\left[y\right]_{(0,1)}^{(2,3)}.$$

$$= [2^2 - 0] + [3^3 \cdot 2 - 0] + 4[3-1]$$

$$= 4 + 27 \cdot 2 + 8$$

$$= 4 + 54 + 8$$

$$= 66.$$

d) $\displaystyle\int_{(3,3/2)}^{(4,1/2)} (2x \sin \pi y\, dx + \pi x^2 \cos \pi y\, dy).$

$\Rightarrow$

$F_1 = 2x \sin \pi y$ $\qquad$ $F_2 = \pi x^2 \cos \pi y$

$\dfrac{\partial F_1}{\partial y} = 2x \cos \pi y \times \pi$

$= 2x\pi \cos \pi y$ $\qquad$ $\dfrac{\partial F_2}{\partial x} = 2x\pi \cos \pi y$

Now,

$$\int I = \int_{(3,3/2)}^{(4,1/2)} 2x \sin \pi y\, dx + \int_{(3,3/2)}^{(4,1/2)} 0\, dy$$

$$= \left[\cancel{2} \sin \pi y \,\frac{x^2}{\cancel{2}}\right]_{(3,3/2)}^{(4,1/2)}$$

$$= \left[16 \cdot \sin \pi \times \frac{1}{2} - 9 \cdot \sin \pi \times \frac{3}{2}\right]$$

$$= 16 + 9$$

$$= 25.$$

$$\int_{(0,0,0)}^{(a,b,c)} [e^z \, dx + 2y \, dy + xe^z \, dz].$$

$F_1 = e^z$ $\qquad$ $F_2 = 2y$

$$\frac{\partial F_1}{\partial y} = 0, \quad \frac{\partial F_1}{\partial z} = e^z \qquad \frac{\partial F_2}{\partial x} = 0, \quad \frac{\partial F_2}{\partial z} = 0$$

$F_3 = xe^z$ $\qquad$ $\frac{\partial F_3}{\partial y} = 0.$

$$\frac{\partial F_3}{\partial x} = e^z$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

Now,

$$\int_{(0,0,0)}^{(a,b,c)} e^z \, dx + \int_{(0,0,0)}^{(a,b,c)} 2y \, dy + \int_{(0,0,0)}^{(a,b,c)} 0 \, dz.$$

$$= [e^z x]_{(0,0,0)}^{(a,b,c)} + [y^2]_{(0,0,0)}^{(a,b,c)} + 0$$

$$= ae^c + b^2.$$

$\therefore I = \int_{(-1,2)}^{(3,1)} [(y^2+2xy)\,dx + (x^2+2xy)\,dy]$

$\Rightarrow$

$F_1 = y^2 + 2xy$ $\qquad$ $F_2 = x^2 + 2xy$

$\dfrac{\partial F_1}{\partial y} = 2y + 2x$ $\qquad$ $\dfrac{\partial F_2}{\partial x} = 2x + 2y$

$\therefore \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial y}$

Now,

$\int I = \int F_1\,dx + \int$ (terms free from x in F_2) d

$= \int_{(-1,2)}^{(3,1)} (y^2+2xy)\,dx + \int 0\,dy$

$= \left[y^2x + 2y\dfrac{x^2}{2}\right]_{(-1,2)}^{(3,1)} + 0$

$= \cancel{[1\cdot3 + y\,1\cdot9 - (4\cdot(-1) + 4\cdot1)]}$

$= 3 + \cancel{9} + 4 + 4$

$=$

$= [1\cdot3 + 1\cdot9 - (4\cdot(-1) + 2\cdot1)]$

$= 12 - (-4+2)$

$= 12+2$

$= 14.$

$$\int_{(0,\pi)}^{(3,\pi/2)} [e^x (\cos y\, dx - \sin y\, dy]$$

$$= \int_{(0,\pi)}^{(3,\pi/2)} e^x \cos y\, dx + (-e^x \sin y)\, dy$$

~~For~~

Let, $I = e^x \cos y\, dx + (-e^x \sin y)\, dy$

Comparing with $F_1\, dx + F_2\, dy$, we get.

$F_1 = e^x \cos y$ $\qquad$ $F_2 = -e^x \sin y$

$\frac{\partial F_1}{\partial y} = -e^x \sin y$ $\qquad$ $\frac{\partial F_2}{\partial x} = -e^x \sin y$

$\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

Then,

$$\int I = \int_{(0,\pi)}^{(3,\pi/2)} F_1\, dx + \int_{(0,\pi)}^{(3,\pi/2)} 0\, dy$$

$$= \int_{(0,\pi)}^{(3,\pi/2)} e^x \cos y\, dx + 0$$

$$= [\cos y\, e^x]_{(0,\pi)}^{(3,\pi/2)}$$

$$= [0 \cdot e^3 - (1 \cdot (-1)]$$

$= 1$ Ans

e. $\int_{(0,0)}^{(1,\pi/2)} [e^x \sin y \, dx + e^x \cos y \, dy]$

$\Rightarrow$ $F_1 = e^x \sin y$ $\quad F_2 = e^x \cos y$

$\frac{\partial F_1}{\partial y} = e^x \cos y$ $\quad \frac{\partial F_2}{\partial x} = e^x \cos y.$

$\Rightarrow \int_{(0,0)}^{(1,\pi/2)} F_1 \, dx + \int_{(0,0)}^{(1,\pi/2)} 0 \, dy$

$= \int_{(0,0)}^{(1,\pi/2)} e^x \sin y \, dx + 0$

$= [e^x \sin y]_{(0,0)}^{(1,\pi/2)}$

$= [e \cdot 1 - 1 \cdot 0]$

$= e$

f. $\int_{(\pi/2,-\pi)}^{(\pi/4,0)} (\cos x \cos 2y \, dx - 2 \sin x \sin 2y \, dy)$

$\Rightarrow$ $F_1 = \cos x \cos 2y$ $\quad F_2 = -2 \sin x \sin 2y.$

$\frac{\partial F_1}{\partial y} = -2 \cos x \sin 2y.$ $\quad \frac{\partial F_2}{\partial x} = -2 \cos x \sin 2y$

$\therefore \int_{(\pi/2,-\pi)}^{(\pi/4,0)} \cos x \cos 2y \, dx + \int 0 \, dy.$

$= [\cos 2y \sin x]_{(\pi/2,-\pi)}^{(\pi/4,0)}$

$= 1 \cdot \frac{1}{\sqrt{2}} - (1 \cdot 1) = \frac{1}{\sqrt{2}} - 1$

Q.b $\int_{(4,0,3)}^{(-1,1,2)} [(yz+1)\,dx + (xz+1)\,dy + (xy+1)\,dz]$

$\Rightarrow$

$F_1 = yz+1 \qquad F_2 = xz+1 \qquad F_3 = xy+1$

$\frac{\partial F_1}{\partial y} = z \qquad \frac{\partial F_2}{\partial x} = z \qquad \frac{\partial F_3}{\partial x} = y$

$\frac{\partial F_1}{\partial z} = y \qquad \frac{\partial F_2}{\partial z} = x \qquad \frac{\partial F_3}{\partial y} = x.$

$\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_3}{\partial y} = \frac{\partial f_2}{\partial z}$

$\therefore \int_{(4,0,3)}^{(-1,1,2)} (yz+1)\,dx + \int_{(4,0,3)}^{(-1,1,2)} dy + \int_{(4,0,3)}^{(-1,1,2)} \partial z$

$= [xyz+x]_{(4,0,3)}^{(-1,1,2)} + [y]_{(4,0,3)}^{(-1,1,2)} + [z]_{(4,0,3)}^{(-1,1,2)}$

$[-1\cdot 1\cdot 2 + (-1) - (4\cdot 0\cdot 3 + 4)] + 1 + (-1)$

$-2-1-4+1-1$

-7

c) $\int_{(0,2,3)}^{(1,\pi,1)} [yz \sinh xz \, dx + \cosh xz$

c) $\int_{(0,2,3)}^{(1,\pi,1)} [yz \sinh xz \, dx + \cosh xz \, dy + xy \sinh xz$

$F_1 = yz \sinh xz \qquad F_2 = \cosh xz \qquad F_3 = xy \sinh xz$

$\frac{\partial F_1}{\partial y} = z \sinh xz$

$\frac{\partial F_1}{\partial z} = y [z \cdot x \cosh xz + \sinh xz]$

$\frac{\partial F_2}{\partial x} = z \sinh xz$

$\frac{\partial F_2}{\partial z} = x \sinh xz$

$\frac{\partial F_3}{\partial x} = y [x \; z \cosh xz + \sinh xz]$

$\frac{\partial F_3}{\partial y} = x \sinh xz$

$\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

$\Rightarrow \int_{(0,2,3)}^{(1,1,1)} yz\sinh xz\,dx + \int 0\,dy + \int 0\,dx$

$= \left[yz \frac{\cosh xz}{z} \right]_{(0,2,3)}^{(1,1,1)}$

$= [1 \cdot \cosh \cdot (1) - 2 \cdot \cosh \cdot 0]$

$= \cosh(1) - 2$

d) $\int_{(0,1,1/2)}^{(\pi/2,3,2)} [y^2 \cos x\,dx + (2y\sin x + e^{2z})\,dy + 2ye^{2z}\,dz$

$\Rightarrow$

$F_1 = y^2 \cos x \qquad F_2 = 2y\sin x + e^{2z} \qquad F_3 = 2ye^{2z}$

$\frac{\partial F_1}{\partial y} = 2y\cos x \qquad \frac{\partial F_2}{\partial x} = 2y\cos x \qquad \frac{\partial F_3}{\partial x} = 0$

$\frac{\partial F_1}{\partial z} = 0 \qquad \frac{\partial F_2}{\partial z} = 2e^{2z} \qquad \frac{\partial F_3}{\partial y} = 2e^{2z}$

$\therefore \int_{(0,1,1/2)}^{(\pi/2,3,2)} y^2 \cos x\,dx + \int_{(0,1,1/2)}^{(\pi/2,3,2)} e^{2z}\,dy + \int 0\,dz.$

$= \left[y^2 \sin x \right]_{(0,1,1/2)}^{(\pi/2,3,2)} + \left[e^{2z} y \right]_{(0,1,1/2)}^{(\pi/2,3,2)} + 0$

$= 9 \cdot 1 - (1 \cdot 0) + e^4 \cdot 3 - e \cdot 1 + 0$

$= 9 + 3e^4 - e$

e) $\int_{(\pi, \pi/2, 2)}^{(0, \pi, 1)} (-z \sin xz \, dx + \cos y \, dy - x \sin xz \, dz)$

$F_1 = -z \sin xz \qquad F_2 = \cos y \qquad F_3 = -x \sin xz$

$\frac{\partial F_1}{\partial y} = 0 \qquad \frac{\partial F_3}{\partial x} = -[xz \cos xz + \sin xz]$

$\frac{\partial F_1}{\partial z} = -[zx \cos xz + \sin xz] \qquad \frac{\partial F_3}{\partial y} = 0$

$\frac{\partial F_2}{\partial x} = 0$

$\frac{\partial F_2}{\partial z} = 0$

$\therefore \int_{(\pi, \pi/2, 2)}^{(0, \pi, 1)} \cancel{-zx} (-z \sin xz) dx + \int_{(\pi, \pi/2, 2)}^{(0, \pi, 1)} \cos y \, dy + \int 0 \, dz$

$= \left[+ \frac{\cancel{z} \cos xz}{\cancel{z}} \right]_{(\pi, \pi/2, 2)}^{(0, \pi, 1)} + [\sin y]_{(\pi, \pi/2, 2)}^{(0, \pi, 1)}$

$= (\cos 0 - \cos 2\pi) + [\sin \pi - \sin \frac{\pi}{2}]$

$(1 - 1) + (0 - 1)$

-1

Green's Theorem (Transformation between double integral line Integral)

Statement: Let R be a bounded region on xy-plane bounded by a curve C. Let $F_1(x,y)$ and $F_2(x,y)$ be two function that are continuous and their first order partial derivatives are also continuous on R. Then,

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.$$

Exercise 4-5

1. Using Green's theorem, Evaluate.

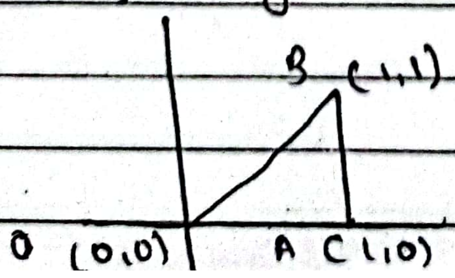
a) $\oint_C [2xy\,dx + (e^x + x^2)\,dy]$, C: triangle joining (0,0), (1,0) and (1,1)

⇒

$F_1 = 2xy$ $\qquad$ $F_2 = e^x + x^2$

$\dfrac{\partial F_1}{\partial y} = 2x$ $\qquad$ $\dfrac{\partial F_2}{\partial x} = e^x + 2x.$

Here C: Δ joining (0,0), (1,0) and (1,1).



Eq^n of OA is

$x = y$.

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.$$

$$= \int_0^1\int_0^x (e^x + 2x - 2x)\,dy\,dx$$

$$= \int_0^1\int_0^x e^x\,dy\,dx$$

$$= \int_0^1 x e^x\,dx$$

$$= [xe^x - e^x]_0^1$$

$$= [e - e - (0-1)]$$

$$= 1$$

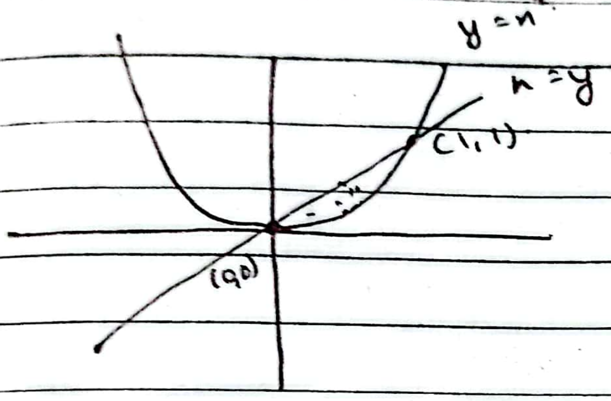
Ans

$$\oint_C [(2xy - x^2)\, dx + (x+y^2)\, dy]$$

C: boundary bounded by $y = x^2$ and $x = y$

$F_1 = 2xy - x^2$ $\qquad$ $F_2 = x + y^2$

$\frac{\partial F_1}{\partial y} = 2x$ $\qquad$ $\frac{\partial F_2}{\partial x} = 1$



Using Green's theorem,

$$\oint_C F_1\, dx + F_2\, dy = \int_0^1 \int_{x^2}^{x} (1 - 2x)\, dy\, dx$$

$$= \int_0^1 [y - 2xy]_{x^2}^{x}\, dx$$

$$= \int_0^1 [x - 2x^2 - (x^2 - 2 \cdot x^3)]\, dx$$

$$= \int_0^1 (x - 3x^2 + 2x^3)\, dx$$

$$= \left[\frac{x^2}{2} - x^3 + \frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{2} - 1 + \frac{1}{2} - 0 \qquad = 0$$

c) $\oint_C (x^2 + y^2)\, dy$, C square $2 \le x \le 4$,

$\Rightarrow$

$F_1 = 0$ $\qquad$ $F_2 = x^2 + y^2$

$\frac{\partial F_1}{\partial y} = 0$ $\qquad$ $\frac{\partial F_2}{\partial x} = 2x$

Using G. T.

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$

$$= \int_2^4 \int_2^4 [2x]\, dy\, dx$$

$$= \int_2^4 2x \cdot (4-2)\, dx$$

$$= 4 \left[\frac{x^2}{2} \right]_2^4$$

$$= 2\, [16 - 4]$$

$$= 2 \times 12$$

$$= 24 \quad \underline{\text{Ans}}$$

d) $\oint_C [2xy^3\,dx + 3x^2y^2\,dy]$ $\quad C: x^2+y^2=1$

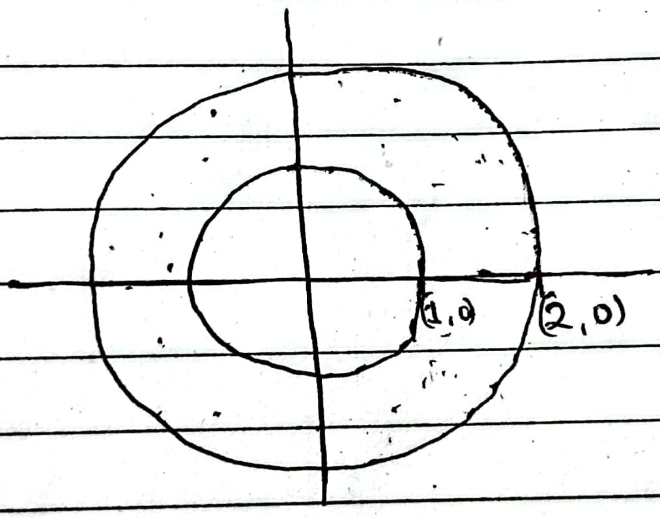
$F_1 = 2xy^3$ $\quad F_2 = 3x^2y^2$

$\frac{\partial F_1}{\partial y} = 6xy^2$ $\quad \frac{\partial F_2}{\partial x} = 6xy^2$

) $\oint_C [(x+y)\,dx + (y+x^2)\,dy]$ $\quad C:$ area betn $x^2+y^2=1,\ x^2+y^2=4$

$F_1 = x+y$ $\quad F_2 = y+x^2$

$\frac{\partial F_1}{\partial y} = 1$ $\quad \frac{\partial F_2}{\partial x} = 2x$



limit of x

$1 \le x \le 2$

For limit of y,

$y = \sqrt{1-x^2}$, $y = \sqrt{4-x^2}$

$\therefore \sqrt{1-x^2} \le y \le \sqrt{4-x^2}$

$$\int_0^{2\pi}\int_1^{2} (2x-1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{2} (2r\cos\theta - 1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{2} (2r^2\cos\theta - r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{2r^3}{3}\cos\theta - \frac{r^2}{2}\right]_1^2 d\theta$$

$$= \int_0^{2\pi} \left[\frac{16}{3}\cos\theta - \frac{4}{2} - \left(\frac{2}{3}\cos\theta - \frac{1}{2}\right)\right] d\theta.$$

$$= \int_0^{2\pi} \left(\frac{14}{3}\cos\theta - \frac{4}{2} + \frac{1}{2}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{14}{3}\cos\theta - \frac{3}{2}\right) d\theta.$$

$$= \frac{14}{3}\left[\sin\theta\right]_0^{2\pi} - \frac{3}{2}\left[\theta\right]_0^{2\pi}$$

$$= 0 - 3\pi$$

$$= -3\pi$$

F. $\oint_C [x^2 y^2 dx + (x^2 \; y^2) dy]$.

$\Rightarrow$ C: sq with vertices (0,0), (1,0), (0,1), (1,1)

Comparing $x^2y^2dx + (x^2-y^2)dy$ with F_1dx+F_2dy

we get,

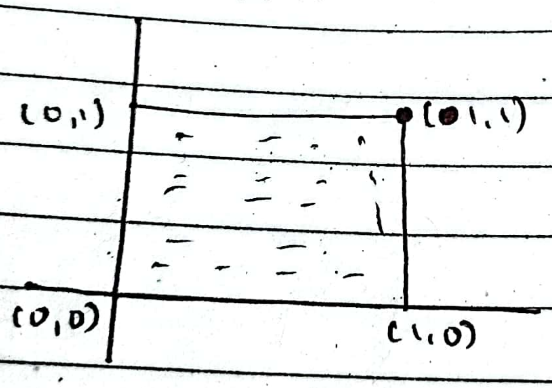
$F_1 = x^2y^2$ $\quad\quad$ $F_2 = x^2-y^2$

$\frac{\partial F_1}{\partial y} = 2x^2y$ $\quad\quad$ $\frac{\partial F_2}{\partial x} = 2x$.

Now, using Green's theorem,

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy.$$

For the limit.



limit of x is: $x=0$ to $x=1$

limit of y: $0 \le y \le 1$

Now,

$$\int_0^1 \int_0^1 (2x - 2x^2y)\, dy\, dx$$

$$\int_0^1 [2xy - x^2y^2]_0^1 \, dx$$

$$= \int_0^1 [2x - x^2] \, dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^1$$

$$= 1 - \frac{1}{3}$$

$= 2/3$ Ans

$$\oint_c \left[\frac{y^2}{1+x^2} dx + 2y \tan^{-1}x \, dy\right] \quad c: x^{2/3} + y^{2/3} = 1$$

$$F_1 = \frac{y^2}{1+x^2} \qquad F_2 = 2y \tan^{-1}x$$

$$\frac{\partial F_1}{\partial y} = \frac{2y}{1+x^2} \qquad \frac{\partial F_2}{\partial x} = \frac{2y}{1+x^2}.$$

Surface Integral:

Any Integral which is evaluated over a surface is known as the surface Integral. It is denoted by

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

where $\hat{n}$ be the unit normal vector to the surface & ϕ be the surface. Then

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

Here ds depends on the plane where surface will be projected. If surface projected on xy-plane $ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$

If surface projected on xz plane $ds = \frac{dy\,dz}{|\hat{n}\cdot\vec{j}|}$

If surface projected on yz-plane $ds = \frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

If S is given in parametric representation

$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$

Then, surface has normal vector. $\vec{N} = \vec{r_u} \times \vec{r_v}$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_R \vec{F} \cdot \vec{N} \, du \, dv.$$

Exercise 4.5

2

b) $\vec{F} = (x\cosh 2y,\ 2x^2 \sinh 2y)$

$C: x^2 \le y \le x$

$y = x^2 \qquad y = x$

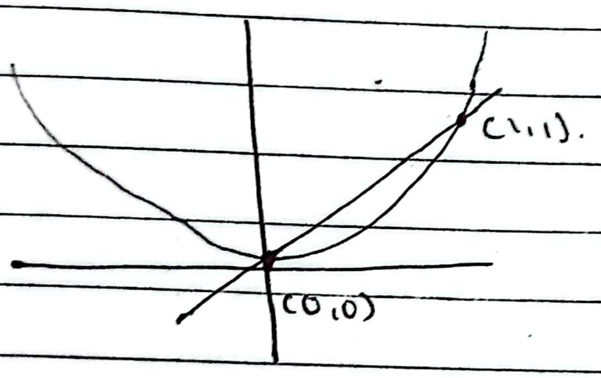
$\Rightarrow$

$\vec{F} \cdot d\vec{r} = x\cosh 2y\, dx + 2x^2 \sinh 2y\, dy$

$F_1 = x\cosh 2y \qquad F_2 = 2x^2 \sinh 2y$

$\frac{\partial F_1}{\partial y} = 2x \sinh 2y \qquad \frac{\partial F_2}{\partial x} = 4x \sinh 2y$

Now,



Using Green's theorem,

$$\oint_C F_1\, dx + F_2\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{x} (4x \sinh 2y - 2x \sinh 2y)\, dy$$

$$= \int_0^1 \int_{x^2}^{x} 2x \sinh 2y\, dy\, dx$$

$$= \cancel{2}\int_0^1 \frac{x}{\cancel{2}} [\cosh 2y]_{x^2}^{x}\, dx$$

$$= \int_0^1 x[\cosh 2x - \cosh 2x^2]\, dx$$

$$= \int_0^1 x \cosh 2x \, dx - \frac{1}{2}\int_0^1 2x \cosh 2x^2 \, dx.$$

$$= \left[\frac{x}{2} \sinh 2x - \int 1 \cdot \frac{\sinh 2x}{2}\right]_0^1 - \frac{1}{2}\left[\frac{\sinh 2x^2}{2}\right]_0^1$$

$$= \left[\frac{x}{2}\sinh 2x - \frac{1}{2}\frac{\cosh 2x}{2}\right]_0^1 - \frac{1}{4}[\sinh(2) - 0]$$

$$= \left[\frac{\sinh(2)}{2} - \frac{1}{4}\{\cosh(2) - (0 - 1)\}\right] - \frac{1}{4}\sinh(2)$$

$$= \frac{\sinh(2)}{2} - \frac{1}{4}\sinh(2) - \frac{1}{4}\cosh(2) + \frac{1}{4}.$$

$$= \frac{\sinh(2)}{4} - \frac{1}{4}\cosh(2) + \frac{1}{4}.$$

c. $\vec{F} = [e^{x+y}, e^{x-y}]$

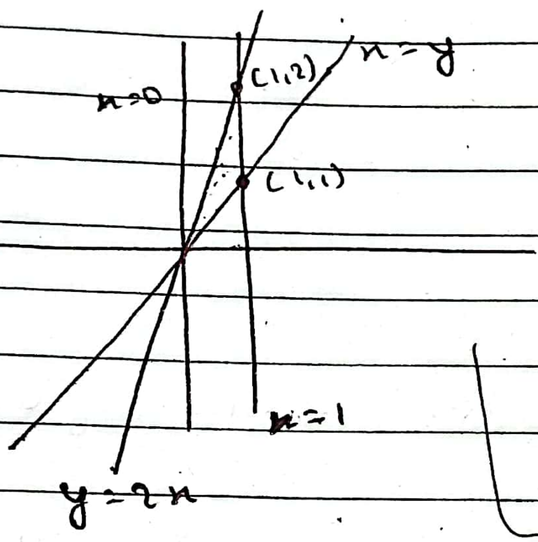
$\vec{F}\,d\vec{r} = e^{x+y}\,dx + e^{x-y}\,dy.$

$F_1 = e^{x+y}$ $\qquad$ $F_2 = e^{x-y}$

$\frac{\partial F_1}{\partial y} = e^{x+y}$ $\qquad$ $\frac{\partial F_2}{\partial x} = e^{x-y}.$

$x \le y \le 2x, \qquad 0 \le x \le 1$

$x = y$
$y = 2x$



Limit of x, $\qquad$ limit of y,

$0 \le x \le 1$ $\qquad$ $x \le y \le 2x.$

Using Green's theorem,

$$\int_0^1 \int_x^{2x} (e^{x-y} - e^{x+y})\,dy\,dx$$

$$= \int_0^1 \int_x^{2x} (e^{x}\cdot e^{-y} - e^{x}\cdot e^{y})\,dy\,dx.$$

$$= \int_0^1 (-e^{x}\cdot e^{-y} - e^{x}\cdot e^{y})_x^{2x}\,dx$$

$$= \int_0^1 (-e^{x-2x} - e^{x+2x} - (-e^{x-x} - e^{x+x}))\,dx$$

$$= \int_0^1 (-e^{-x} - e^{3x} - (-1 - e^{2x})).$$

$$= \int_0^1 -e^{-x} - e^{3x} + 1 + e^{2x}\,dx$$

$$= \left[e^{-x} - \frac{e^{3x}}{3} + x + \frac{e^{2x}}{2}\right]_0^1$$

$$= e^{-1} - \frac{e^{3}}{3} + 1 + \frac{e^{2}}{2} - \left(1 - \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{e} - \frac{e^3}{3} + \frac{e^2}{2} + \cancel{x} - \cancel{x} + \frac{1}{3}$$

$$= \frac{1}{e} - \frac{e^3}{3} + \frac{e^2}{2} - \frac{1}{6}$$

d) $\vec{F} = \text{grad}(\sin x \cos y)$

$\Rightarrow$

$\text{grad}(\sin x \cos y) = \frac{\partial(\sin x \cos y)}{\partial x}\hat{i} + \frac{\partial(\sin x \cos y)}{\partial y}\hat{j}$

$= \cos x \cos y\,\hat{i} - \sin x \sin y\,\hat{j}$

$\vec{F}\cdot d\vec{r} = \cos x \cos y\, dx - \sin x \sin y\, dy.$

$F_1 = \cos x \cos y$ $\qquad$ $F_2 = -\sin x \sin y$

$\frac{\partial F_1}{\partial y} = -\cos x \sin y$ $\qquad$ $\frac{\partial F_2}{\partial x} = -\cos x \sin y.$

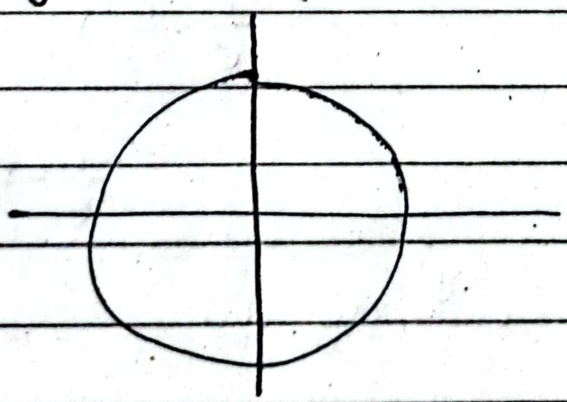
$\Rightarrow$ Ans is 0.

3. Evaluate:

a) $\oint_C (\cos x \sin y - 2xy)\,dx + \sin x \cos y\, dy,$ $\quad$ C: x

$\Rightarrow$ $F_1 = \cos x \sin y - 2xy$ $\qquad$ $F_2 = \sin x \cos y$

$\frac{\partial F_1}{\partial y} = \cos x \cos y - 2x$ $\qquad$ $\frac{\partial F_2}{\partial x} = \cos x \cos y.$



$0 \le r \le 1$ $\qquad$ $0 \le \theta \le 2\pi/2.$

Using Green's theorem,

$$\oint_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy.$$

$$= 4\int_0^1 \int_0^{\pi/2} (\cos x \cos y - \cos x \cos y + 2x)\, r\, d\theta\, dr$$

$$= 4\int_0^1 \int_0^{\pi/2} 2r^2 \cos\theta \, d\theta\, dr.$$

$$= 4\int_0^1 2r^2 [\sin\theta]_0^{\pi/2} dr$$

$$= 8\int_0^1 r^2 dr$$

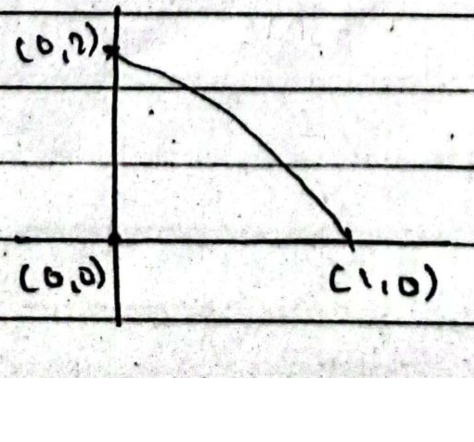
$$= \frac{8}{3}[r^3]_0^1$$

$$= 8/3.$$

b) $\oint_C [(3x^2+y)dx + 4y^2 dy]$, C is the boundary of the Δ with vertices $(0,0)$, $(1,0)$, $(0,2)$.

$F_1 = 3x^2+y$ $\qquad$ $F_2 = 4y^2$

$\frac{\partial F_1}{\partial y} = 1$ $\qquad$ $\frac{\partial F_2}{\partial x} = 0$



$0 \le x \le 1$, $0 \le y \le 2-2x$

$y - 1 = \frac{0-1}{0-\pi}(x-\pi)$

$y - 1 = \frac{x-\pi}{\pi}$

$y = \frac{x-\pi}{\pi} + 1$

$= \frac{x}{\pi}$

Date ____
Page ____

v G. T,

$$-\int_0^1\int_0^{2-2x} dy\, dx$$

$$= -\int_0^1 [y]_0^{2-2x}\, dx$$

$$= -\int_0^1 2-2x\, dx$$

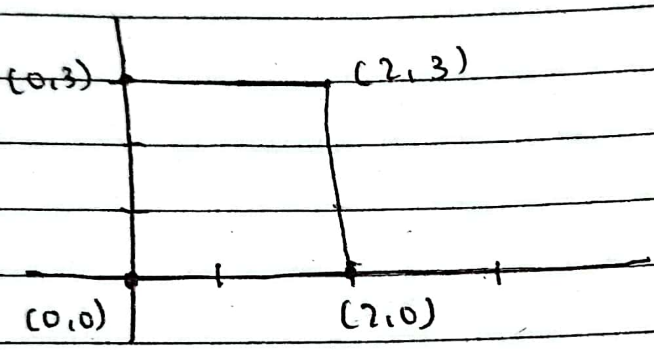
$$= \int_0^1 (2x-2)\, dx = [x^2 - 2x]_0^1 = (1-2) = -1$$

d) $\oint_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2 e^y, y^2 e^x)$.

$\vec{F} \cdot d\vec{r} = x^2 e^y \, dx + y^2 e^x \, dy$.

$F_1 = x^2 e^y$ $\qquad$ $F_2 = y^2 e^x$

$\frac{\partial F_1}{\partial y} = x^2 e^y$ $\qquad$ $\frac{\partial F_2}{\partial x} = y^2 e^x$



$0 \le x \le 2$ $\qquad$ $0 \le y \le 3$

U.G.T

$$= \iint_R (y^2 e^x - x^2 e^y) \, dx \, dy$$

$$= \int_0^3 \int_0^2 (y^2 e^x - x^2 e^y) \, dx \, dy$$

$$= \int_0^3 \left[y^2 e^x - e^y \frac{x^3}{3} \right]_0^2 dy$$

$$= \int_0^3 \left[y^2 e^2 - e^y \frac{8}{3} - (y^2) \right] dy$$

$$= \int_0^3 \left(e^2 y^2 - \frac{8}{3} e^y - y^2 \right)$$

$$= \frac{e^2}{3} \left[y^3 \right]_0^3 - \frac{8}{3} \left[e^y \right]_0^3$$

$$= \frac{27}{3} e^2 - \frac{8}{3} (e^3 - 1) -$$

$$= 9e^2 - \frac{8}{3} e^3 + \frac{8}{3} -$$

$$= 9(e^2 - 1) + \frac{8}{3} (1 - e^3$$

2081/08/17 Wednesday

1) Find $\iint (\vec{F} \cdot \hat{n})\, ds$ for

a) $\vec{F} = 3x^2 \hat{i} + y^2 \hat{j}$, $\vec{r} = x\hat{i} + y\hat{j} + (2x+3y)\hat{k}$,

$0 \le x \le 2$, $-1 \le y \le 1$

→ Solution,

$$\vec{r} = x\hat{i} + y\hat{j} + (2x+3y)\hat{k}$$

By surface integral,

$$\iint_S (\vec{F} \cdot \hat{n})\, ds = \iint_R (\vec{F} \cdot \hat{N})\, du\, dv \quad \text{--- (1)}$$

where,

$$\hat{N} = \vec{r_u} \times \vec{r_v}$$

Let $x = u$, $y = v$.

$$\vec{r} = u\hat{i} + v\hat{j} + (2u + 3v)\hat{k}$$

$$\vec{r_u} = \hat{i} + 0\hat{j} + 2\hat{k}$$

$$\vec{r_v} = 0\hat{i} + \hat{j} + 3\hat{k}$$

$$\vec{N} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$\therefore \vec{N} = -2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{F} = 3u^2\hat{i} + v^2\hat{j}$

$$\vec{F} \cdot \vec{N} = -6u^2 - 3v^2$$

from (1)

$$\iint_S (\vec{F}\cdot\hat{n})\, ds = \iint_R (-6u^2 - 3v^2)\, du\, dv$$

$$= \int_{-1}^{1}\int_{0}^{2} (-6u^2 - 3v^2)\, du\, dv.$$

$$= \int_{-1}^{1} \left[-2u^3 - 3v^2 u\right]_0^2 dv$$

$$= \int_{-1}^{1} \left[-16 - 6v^2\right] dv$$

$$= \left[-16v - 2v^3\right]_{-1}^{1}$$

$$= \left[-16 - 2 - (16+2)\right]$$

$$= -18 - 18$$

$$= -36.$$

b) $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{r} = (u\cos\theta, u\sin\theta, u^2)$;

$0 \le u \le 4$, $-\pi \le \theta \le \pi$

$\Rightarrow$

$\vec{r} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} + u^2\,\hat{k}$

$x = u\cos\theta$, $y = u\sin\theta$, $z = u^2$.

$\vec{r}_u = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 2u\,\hat{k}$

$\vec{r}_\theta = -u\sin\theta + u\cos\theta\,\hat{j} + 0\,\hat{k}$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2u \\ -u\sin\theta & u\cos\theta & 0 \end{vmatrix}$$

$= -2u^2\cos\theta\,\hat{i} - 2u^2\sin\theta\,\hat{j} + (u\cos^2\theta + u\sin$

$= -2u^2\cos\theta\,\hat{j} - 2u^2\sin\theta\,\hat{j} + u\,\hat{k}$

$\vec{F} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} + u^2\,\hat{k}$

$\vec{F}\cdot\vec{N} = -2u^3\cos^2\theta - 2u^3\sin^2\theta + u^3$

$= -2u^3 + u^3$

$= -u^3$

The surface integral is

$$\iint (\vec{F}\cdot\hat{n})\,ds = \iint (\vec{F}\cdot\vec{N})\,du\,d\theta$$

$$= \int_0^4\int_{-\pi}^{\pi} -u^3\,d\theta\,du$$

$$= -\int_{-\pi}^{\pi} \left[\frac{u^4}{4}\right]_0^4 d\theta$$

$$= -\frac{1}{4}\int_0^4 (4^4 - 0^4)\,du$$

$$= -\int_0^4\int_{-\pi}^{\pi} u^3\,d\theta\,du$$

$$= -\int_0^4 [u^3\theta]_{-\pi}^{\pi}\,du$$

$$= -\int_0^4 [\pi u^3 + \pi u^3]\,du$$

$$= -2\pi\int_0^4 u^3\,du$$

$$= -\frac{2\pi}{4}[u^4]_0^4$$

$$= -\frac{\pi}{2}\times 256$$

$$= -128\,\pi$$

Imp

c) $\iint (\vec{F} \cdot \hat{n})\, ds$ for

7) $\vec{F} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ $\quad S: 2x + 3y + 6z = 12$ in first octant.

Let $\vec{r}$ be the position vector on the surface S:

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Put,

$x = u, \quad y = v, \quad \cancel{z = 12x - 3y - 6z}$

$z = \dfrac{12 - 2x - 3y}{6}$

$\cancel{\vec{r_u}} =$

$\cancel{\vec{r_v}}$

$\therefore \vec{r} = u\hat{i} + v\hat{j} + \left(\dfrac{12 - 2u - 3v}{6}\right)\hat{k}$

$\vec{r_u} = \hat{i} + 0\hat{j} + \dfrac{-1}{3}\hat{k}$

$\vec{r_v} = 0\hat{i} + \hat{j} + \dfrac{-1}{2}\hat{k}$

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \end{vmatrix}$$

$\vec{N} = \dfrac{1}{3}\hat{i} + \dfrac{1}{2}\hat{j} + \hat{k}$

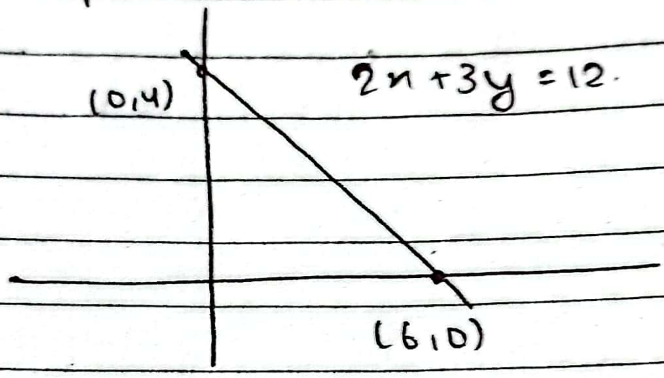
Now,

$$\vec{F}\cdot\vec{N} = \left(3(12-2u-3v)\hat{i} - 12\hat{j} + 3v\hat{k}\right)\cdot\left(\frac{1}{3}\hat{i} + \frac{1}{2}\hat{j} + \hat{k}\right)$$

$$= 12 - 2u - \cancel{3v} - 6 + \cancel{3v}$$

$$\vec{F}\cdot\vec{N} = 6 - 2u$$

For the limit of u and v taking x, y plane



$$0 \le x \le 6 \quad , \quad 0 \le y \le \frac{12-2x}{3}$$

Now,

$$\iint_R (\vec{F}\cdot\vec{N})\, du\, dv$$

$$= \int_0^6 \int_0^{\frac{12-2u}{3}} (6-2u)\, dv\, du$$

$$= \int_0^6 \left[6v - 2uv\right]_0^{\frac{12-2u}{3}} du$$

$$= \int_0^6 \left[2(12-2u) - 2u \times \frac{12-2u}{3}\right] du$$

$$= 2\int_0^6 (12-2u)\,du - \frac{2}{3}\int_0^6 (12u-2u^2)\,du.$$

$$= 2\left[12u - u^2\right]_0^6 - \frac{2}{3}\left[6u^2 - \frac{2}{3}u^3\right]_0^6$$

$$= 2\left[72 - 36\right] - \frac{2}{3}\left[216 - \frac{2}{3}\times 216\right]$$

$$= 72 - \frac{2}{3}\left[216 - 144\right]$$

$$\cancel{= 72 - 72}$$

$$\cancel{= 144}$$

$$= 24.$$

Also,
Using the method of projection

Solution

Given,

$$\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$$

$$S: \ 2x + 3y + 6z - 12 = 0$$

let,

$$\phi = 2x + 3y + 6z - 12 \quad \text{—— (I)}$$

The normal unit vector in S be

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

where,

$$\nabla\phi = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\therefore \hat{n} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}.$$

Taking the projection of the surface xy plane, Then.

$$dS = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}.$$

$$dS = \frac{dx\,dy}{\frac{6}{7}} \quad \text{—— (II)}$$

From S: $6z = 12 - 2x - 3y$

$$\therefore \vec{F} = 3(12 - 2x - 3y)\hat{i} - 12\hat{j} + 3y\hat{k}$$

$$= (36 - 6x - 9y)\hat{i} - 12\hat{j} + 3y\hat{k}$$

$$\vec{F}\cdot\hat{n} = \frac{2(36-6x-9y) - 12\times 3 + 3y\times 6}{7}$$

$$= \frac{6}{7}\left[[12-2x-3y] - 6 + 3y\right]$$

$$= \frac{6}{7}[12 - 2x - \cancel{3y} - 6 + \cancel{3y}]$$

$$= \frac{6}{7}[6-2x].$$

Now,

$$\iint(\vec{F}\cdot\hat{n})\,ds = \iint \frac{\cancel{6}}{\cancel{7}}(6-2x)\times\frac{dx\,dy}{\cancel{6/7}}$$

$$= \iint (6-2x)\,dx\,dy$$

$$= \int_0^6\int_0^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

$$= \int_0^6 [6y - 2xy]_0^{\frac{12-2x}{3}}\,dx$$

$$= 24.$$

b) $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

$S: x^2+y^2+z^2 = a^2$ in first octant.

⇒ Solution

Given,

$$\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$$

$$S: x^2 + y^2 + z^2 = a^2.$$

Let, $\phi = x^2 + y^2 + z^2 - a^2.$

Let, $\hat{n}$ be unit normal vector on the surface.

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

Where,

$$\nabla \phi = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}.$$

$$\therefore \hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\vec{F}\cdot\hat{n} = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}) \cdot \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{xyz + xyz + xyz}{a}$$

$$= \frac{3xyz}{a}$$

$$= \frac{3xy\sqrt{a^2 - x^2 - y^2}}{a}.$$

~~Taking~~ placing projection on xy plane.

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \frac{dx\,dy}{\frac{z}{a}}$$

Now,

$$\iint (\vec{F}\cdot\hat{n})\,ds = \iint_R \frac{3xyz}{a} \times \frac{dx\,dy}{\frac{z}{a}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} 3xy\,dy\,dx$$

$$= \int_0^a \frac{3}{2}\left[x y^2\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \times \frac{3}{2}\left[x(a^2-x^2)\right] dx$$

$$= \frac{3}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{3}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{3}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} - 0\right]$$

$$= \frac{3}{2} \times \frac{a^4}{4}$$

$$= \frac{3a^4}{8}.$$

08/17 Monday

Stoke's theorem (Transformation between line and surface integral)

Let $\vec{F}(x,y,z)$ be a continuity differentiable function defined on surface S. bounded by a simple closed curve C. Then.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

$$= \iint_S (\nabla\times\vec{F})\cdot\vec{N}\,dx\,dy$$

Here, $\vec{N} = \vec{r_x}\times\vec{r_y}$

$$= \iint_S (\nabla\times\vec{F})\cdot\vec{N}\,du\,d\theta.$$

Here, $\vec{N} = \vec{r_u}\times\vec{r_\theta}$

Exercise 4.3

1\. Evaluate $\oint \vec{F} \cdot d\vec{r}$ by using stokes theorem.

a) $\vec{F} = y^2\hat{i} - x^2\hat{j}$, $S: x^2 + y^2 \le 4,\ y \ge 0,\ z = 0$

Solution

$$\vec{F} = y^2\hat{i} - x^2\hat{j}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -x^2 & 0 \end{vmatrix}$$

$$= 0\hat{i} - \hat{j}(0) + \hat{k}(-2x - 2y)$$

$$= (-2x - 2y)\hat{k}$$

Let,

$\vec{r}$ be the position vector on S.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r_x} = \hat{i} + 0\hat{j} + 0\hat{k} = \hat{i}$$

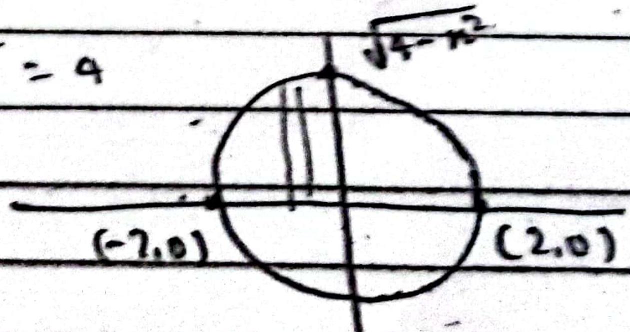
$$\vec{r_y} = 0\hat{i} + \hat{j} + 0\hat{k} = \hat{j}$$

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \hat{i} \times \hat{j} = \hat{k}$$

$$\therefore (\nabla \times \vec{F}) \cdot \vec{N} = -2x - 2y$$

Now,

$$x^2 + y^2 = 4$$



$$\therefore \oint_c \vec{F}\cdot d\vec{r} = \iint_R (-2x-2y)\,dx\,dy$$

$$= \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (-2x-2y)\,dy\,dx$$

$$= \int_{-2}^{2} \left[-2xy - y^2\right]_0^{\sqrt{4-x^2}}\,dx.$$

$$= \int_{-2}^{2} \left[-2x\sqrt{4-x^2} - 4 + x^2\right] dx$$

$$= \int_{-2}^{2} \left(-2x\sqrt{4-x^2}\right) dx + \int_{-2}^{2} \left(-4 + x^2\right) dx$$

$$= \frac{2}{3}\left[(4-x^2)^{3/2}\right]_{-2}^{2} + \left[-4x + \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{2}{3}\left[0-0\right) + \left[-8 + \frac{1\times 8}{3} - \left(8 - \frac{1\times 8}{3}\right)\right.$$

$$= -8 + \frac{1\times 8}{3} - 8 + \frac{1\times 8}{3}$$

$$= -16 + \frac{2\times 1\times 8}{3}$$

$$= -16 + \frac{16}{3}$$

$$= \cancel{\frac{-12+64}{3}} = \frac{-32}{3}$$

$$\cancel{= \frac{52}{3}}$$

b) $\vec{F} = e^{z}\hat{i} - e^{z}\sin y\,\hat{j} + e^{z}\cos y.\hat{k}$

$S: \; z = y^2, \quad 0 \le x \le 4, \quad 0 \le y \le 2.$

c)

<u>Solution</u>

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{z} & -e^{z}\sin y & e^{z}\cos y \end{vmatrix}$$

$$= \hat{i}(-e^{z}\sin y + e^{z}\sin y) - \hat{j}(0 - e^{z}) + \hat{k}(0 - 0)$$

$$= \cancel{(e^{z}\sin y - e^{z}\sin y)}$$

$$\nabla \times \vec{F} = (-e^{z}\sin y)\hat{i} + e^{z}\hat{j} + 0\hat{k} = 0\hat{i} + e^{z}\hat{j} + 0\hat{k}$$

let $\vec{r}$ be the position vector on S.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + y^2\hat{k}$

$\vec{r_x} = \hat{i} + 0\hat{j} + 0\hat{k}$

$\vec{r_y} = 0\hat{i} + \hat{j} + 2y\hat{k}$

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2y \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}\,2y + \hat{k} = 0\hat{i} - 2y\hat{j} + \hat{k}$$

$(\nabla \times \vec{F}) \cdot \vec{N} = (0, e^2, 0) \cdot (0, -2y, 1)$

$= 0 - 2y e^2 + 0$

$= -2y e^2$

$= -2y e^{y^2}$

Now,

By Stoke's theorem

$\oint \vec{F} \cdot d\vec{r} = \iint_R -2y e^{y^2} \, dx \, dy$

$= \int_0^4 \int_0^2 (-2y e^{y^2}) \, dy \, dx$

$= -\int_0^4 \int_0^2 (2y e^{y^2}) \, dy \, dx$

$= -\int_0^4 [e^{y^2}]_0^2 \, dx$

$= -\int_0^4 [e^4 - 1] \, dx$

$= -[e^4 x - x]_0^4$

$= -[e^4 \times 4 - 4]$

$= 4 - 4e^4$

i $\vec{F} = 2y^2\hat{i} + x\hat{j} - z^3\hat{k}$

$S: x^2 + y^2 = a^2 ; z = b.$

⇒ Solution

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & x & -z^3 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1-4y)$$

$$\nabla \times \vec{F} = 0\hat{i} + 0\hat{j} + (1-4y)\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= x\hat{i} + y\hat{j} + b\hat{k}$$

$$\vec{r_x} = 1\hat{i} + 0\hat{j} + 0\hat{k}$$
$$\vec{r_y} = 0\hat{i} + 1\hat{j} + 0\hat{k}$$

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + \hat{k}$$

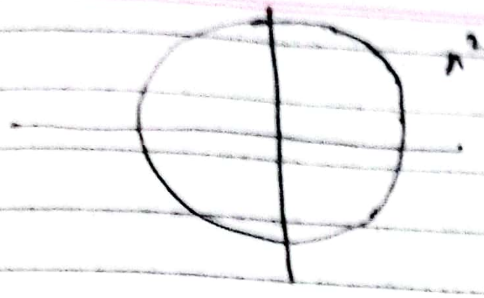
$$(\nabla \times \vec{F}) \cdot \vec{N} = (0, 0, (1-4y)) \cdot (0, 0, 1)$$
$$= 1 - 4y.$$

$y = 4 x \sin\theta$
$\theta = 0, 2\pi$
$r = 0, a$

$a - 2a^2 \sin\theta$

$[a\theta + 2a^2 \cos\theta]_0^{2\pi}$

$2\pi a$



$x^2 + y^2 = a^2$

$-a \le x \le a \qquad -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

$$\therefore \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (1-4y)\, dy\, dx$$

$$= \int_{-a}^{a} \left[y - 2y^2 \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} \left[\left(\sqrt{a^2-x^2} - 2(a^2-x^2) \right) - \left\{ -\sqrt{a^2-x^2} - 2(a^2-x^2) \right\} \right]$$

$$= \int_{-a}^{a} \left[\sqrt{a^2-x^2} - \cancel{2(a^2-x^2)} + \sqrt{a^2-x^2} + \cancel{2(a^2-x^2)} \right] dx$$

$$= 2 \int_{-a}^{a} \sqrt{a^2-x^2}\, dx$$

$$2 \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_{-a}^{a}$$

$$2 \left[0 + \frac{a^2}{2} \times \frac{\pi}{2} - \left(0 + \frac{a^2}{2} \times \frac{3\pi}{2} \right) \right]$$

$$2 \left[\frac{\pi a^2}{4} - \frac{3\pi a^2}{4} \right]$$

$$\frac{\pi a^2 - 3\pi a^2}{2} = -\pi a^2$$

$\vec{F} = (x^2+y^2, \; -2xy, \; 0)$

$x = \pm a, \quad y = 0, \quad y = b$

$\Rightarrow$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0-0) + \hat{k}(-2y-2y)$$

$$= 0\hat{i} + 0\hat{j} + (-4y)\hat{k}$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + 0\hat{k}$

$\vec{r}_x = \hat{i} + 0\hat{j} + 0\hat{k}$

$\vec{r}_y = 0\hat{i} + \hat{j} + 0\hat{k}$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + \hat{k}$$

$(\nabla \times \vec{F}) \cdot \vec{N} = -4y$

$$\int_{-a}^{a} \int_{0}^{b} -4y \, dy \, dx$$

$$= -\int_{-a}^{a} 2[y^2]_0^b \, dx = -2\int_{-a}^{a} b^2 \, dx$$

$$= -2b^2 [x]_{-a}^{a} = -2b^2(a+a)$$

$$= -4b^2 a$$

1) $\vec{F} = (y^2, z^2, x^2)$

$x + y + z = 1$

$\Rightarrow$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & z^2 & x^2 \end{vmatrix}$$

$$= \hat{i}(-2z) - \hat{j}(2x - 0) + \hat{k}(-2y)$$
$$= -2z\,\hat{i} - 2x\,\hat{j} - 2y\,\hat{k}$$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + (1 - x - y)\,\hat{k}$$

$$\vec{r}_x = \hat{i} + 0\,\hat{j} - \hat{k}$$
$$\vec{r}_y = 0\,\hat{i} + \hat{j} - \hat{k}$$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{i}(1) - \hat{j}(-1) + \hat{k}(1) = \hat{i} + \hat{j} + \hat{k}$$

$$(\nabla \times \vec{F}) \cdot \vec{N} = (-2z, -2x, -2y) \cdot (1, 1, 1)$$
$$= -2z - 2x - 2y$$

~~Now,~~

$$= -2(1 - x - y) - 2x - 2y$$
$$= -2 + \cancel{2x} + \cancel{2y} - \cancel{2x} - \cancel{2y}$$
$$= -2$$

$$\therefore -2\int_0^1 \int_0^{1-x} dy\, dx = -2\int_0^1 (1 - x)\, dx$$

$$= -2\left[x - \frac{x^2}{2}\right]_0^1 = -2\left[1 - \frac{1}{2}\right] = -1$$

Date
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Gauss Divergence Theorem

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dx\,dy\,dz = \iiint_V \nabla\cdot\vec{F}\, dx\,dy\,dz.$$

1) Evaluate $\iint \vec{F}\cdot\hat{n}\, ds$ by using Gauss Divergence Th.

a) $\vec{F} = (e^x, e^y, e^z)$ $\quad S: |x| \le 1,\ |y| \le 1,\ |z| \le 1$

$\Rightarrow$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)(e^x, e^y, e^z)$$

$$\therefore \nabla\cdot\vec{F} = e^x + e^y + e^z$$

Now,

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (e^x + e^y + e^z)\, dx\, dy\, dz.$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left[e^x + xe^y + xe^z\right]_{-1}^{1} dy\, dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(e + e^y + e^z - (e^{-1} - e^y - e^z)\right) dy\, dz.$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(e + e^y + e^z - e^{-1} + e^y + e^z\right) dy\, dz.$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(e - e^{-1} + 2e^y + 2e^z\right) dy\, dz.$$

$$= \int_{-1}^{1} \left[y(e - e^{-1}) + 2e^y + 2ye^z\right]_{-1}^{1} dz$$

$$= \int_{-1}^{1} \left[e - e^{-1} + 2e + 2e^z - \left[-(e - e^{-1}) + 2e^{-1} - 2e^z\right]\right] dz.$$

$$= \int_{-1}^{1} [2(e-e^{-1}) + 2e + 2e^{z} - 2e^{-1} + 2e^{z}] \, dz.$$

$$= \int_{-1}^{1} [4(e-e^{-1}) + 4e^{z}] \, dz$$

$$= [4z(e-e^{-1}) + 4e^{z}]_{-1}^{1}$$

$$= 4(e-e^{-1}) + 4e - [-4(e-e^{-1}) + 4e^{-1}]$$

$$= 4(e-e^{-1}) + 4e + 4(e-e^{-1}) - 4e^{-1}$$

$$= 8(e-e^{-1}) + 4e - 4e^{-1}$$

$$= 12(e-e^{-1})$$

b) $\vec{F} = (4xz, -y^2, yz)$

S: Cube bounded by plane; $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

$\Rightarrow$

$$\text{div } \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (4xz, -y^2, yz)$$

$$= 4z - 2y + y$$

$$= 4z - y.$$

Now,

$$\int_0^1 \int_0^1 \int_0^1 (4z - y) \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^1 [4xz - xy]_0^1 \, dy \, dz$$

$$= \int_0^1 \int_0^1 [4z - y] \, dy \, dz$$

$$= \int_0^1 \left[4yz - \frac{y^2}{2}\right]_0^1 dz$$

$$= \int_0^1 \left[4z - \frac{1}{2} \right] dz$$

$$= \left[2z^2 - \frac{1}{2} z \right]_0^1$$

$$= 2 - \frac{1}{2}$$

$$= 3/2.$$

$\sqrt{a^2-x^2}$

e) $\vec{F} = (x, 3y, 6z)$. S: is the surface of cone $\sqrt{x^2+y^2} \le z$, $0 \le z \le 3$

$\nabla \cdot \vec{F} = 1 + 3 + 6$
$= 10$.

Now,

$x^2 + y^2 = z^2$

~~x =~~

limit of x, limit of y

$-2 \le x \le 2$ $\quad -\sqrt{z^2-x^2} \le y \le \sqrt{z^2-x^2}$.

Then.

$$\int_0^3 \int_{-2}^{2} \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} 10 \, dy \, dx \, dz$$

$$= 10 \int_0^3 \int_{-2}^{2} \left(2\sqrt{z^2-x^2}\right) dx \, dz.$$

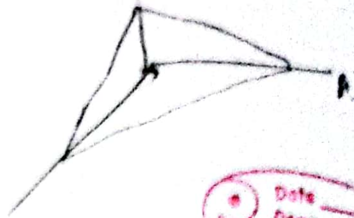
$$= 20 \int_0^3 \left[\frac{x\sqrt{z^2-x^2}}{2} + \frac{z^2}{2} \sin^{-1}\frac{x}{az} \right]_{-2}^{2} dz.$$

$$= 20 \int_0^3 \left[0 + \frac{z^2}{2} \times \frac{\pi}{2} - \left(0 + \frac{z^2}{2} \times \left(-\frac{\pi}{2}\right)\right) \right] dz.$$

$$= 20 \int_0^3 \left[\frac{z^2\pi}{4} + \frac{\pi z^2}{4} \right] dz$$

$$= 10\pi \left[\frac{z^3}{3} \right]_0^3$$

$$= 10\pi \times [9 - 0] = 90\pi$$



Q: $\vec{F} = (y^2 e^z, \ -xy, \ x\tan^{-1} y)$

S: $x + y + z = 1$

$\Rightarrow$

$\nabla \cdot \vec{F} = 0 - x + 0$

$= -x.$

S: $x + y + z = 1$

$0 \leq x \leq 1 - y - z$

$0 \leq y \leq 1 - z$

$0 \leq z \leq 1$

$$\therefore \int_0^1 \int_0^{1-z} \int_0^{1-y-z} (-x) \, dx \, dy \, dz$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-z} [x^2]_0^{1-y-z} \, dy \, dz$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-z} (1-y-z)^2 \, dy \, dz$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-z} (1 + y^2 + z^2 - 2y + 2yz - 2z) \, dy \, dz.$$

$$= -\frac{1}{2} \int_0^1 \left[y + \frac{y^3}{3} + z^2 y - y^2 + y^2 z - 2yz \right]_0^{1-z} dz$$

$$= -\frac{1}{2} \int_0^1 \left[1 - z + \frac{(1-z)^3}{3} + z^2(1-z) - (1-z)^2 + (1-z)^2 z - 2(1-z)z - (0+0+0) \right]$$

$$= -\frac{1}{2}\int_0^1 \Big[1 - z + \frac{1 - 3z + 3z^2 - z^3}{3} + z^2 - z^3$$
$$- (1 - 2z + z^2) + (1 - 2z + z^2) z$$
$$- 2z + 2z^2 \Big] dz$$

$$= -\frac{1}{2}\int_0^1 \Big[\cancel{1} - \cancel{z} + \frac{1 - \cancel{z}}{3} + \frac{\cancel{z^2} - z^3}{3} + \cancel{z^2} - \cancel{z^3}$$
$$- \cancel{1} + 2\cancel{z} - z^2 + z - 2\cancel{z^2} + \cancel{z^3} - 2z + 2z^2 \Big] d$$

$$= -\frac{1}{2}\int_0^1 \Big[\frac{1}{3} - \frac{z^3}{3} + z^2 - z \Big] dz$$

$$= -\frac{1}{2}\Big[\frac{1}{3} z - \frac{z^4}{12} + \frac{z^3}{3} - \frac{z^2}{2} \Big]_0^1$$

$$= -\frac{1}{2}\Big[\frac{1}{3} - \frac{1}{12} + \frac{1}{3} - \frac{1}{2} - 0 \Big]$$

$$= -\frac{1}{2}\Big[\frac{2}{3} - \frac{1}{2} - \frac{1}{12} \Big]$$

$$= -\frac{1}{2}\Big[\frac{4 - 3}{6} - \frac{1}{12} \Big]$$

$$= -\frac{1}{2}\Big[\frac{1}{6} - \frac{1}{12} \Big]$$

$$= -\frac{1}{2}\Big[\frac{2 - 1}{12} \Big]$$

$$= -\frac{1}{2} \times \frac{1}{12} = -\frac{1}{24}$$

f. $\vec{F} = (\cos y, \sin x, \cos z)$

$S: x^2 + y^2 \le 9, \quad 0 \le z \le 2.$

$\Rightarrow$

$\nabla \cdot \vec{F} = -\sin z.$

Now,

$$\int_0^2 \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (-\sin z)\, dx\, dy\, dz$$

$$= \int_0^2 \int_{-3}^{3} (-\sin z) \, [x]_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \, dy\, dz$$

$$= \int_0^2 \int_{-3}^{3} (-\sin z) \, (2\sqrt{9-y^2}) \, dy . dz$$

$$= \int_0^2 (-2\sin z) \left[\frac{y\sqrt{9-y^2}}{2} + \frac{9}{2} \cancel{\tan} \sin^{-1}\frac{y}{3} \right]_{-3}^{3}$$

$$= \int_0^2 -2\sin z \left[0 + \frac{9}{2} \cdot \frac{\pi}{2} - \left(0 + \frac{9}{2} \times \left(-\frac{\pi}{2}\right)\right) \right]$$

$$= -2 \int_0^2 \sin z \left[\frac{9\pi}{4} + \frac{9\pi}{4} \right] dz$$

$$= -\int_0^2 9\pi \sin z \; dz$$

$$= -9\pi \left[-\cos z \right]_0^2$$

$$= +9\pi \left[\cos(2) - 1 \right]$$

$$= 9\pi \left[\cos 2 - 1 \right]$$

1. $\vec{F} = (x, z, y)$ $\qquad$ S: $x^2 + y^2 + z^2 = 4$.

$\nabla \cdot \vec{F} = 1$

$$\therefore \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz \; dx \; dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2\sqrt{4-y^2-x^2} \; dx \; dy$$

$$= 2\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{(\sqrt{4-y^2})^2 - x^2} \; dx \; dy$$

$$2\int_{-2}^{2} \left[\frac{x\sqrt{4-y^2-x^2}}{2} + \frac{4-y^2}{2} \sin^{-1} \frac{x}{\sqrt{4-y^2}} \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy.$$

$$= 2\int_{-2}^{2} \left[0 + \frac{4-y^2}{2} \times \frac{\pi}{2} - \left(0 \times \frac{4-y^2}{2} \left(-\frac{\pi}{2} \right) \right) \right] dy$$

$$= 2\int_{-2}^{2} \frac{2\,(4-y^2)\,\pi}{4} \; dy.$$

$$\pi \int_{-2}^{2} \cancel{4\pi} \; \pi \; (4-y^2) \; dy$$

$$\pi \left[4y - \frac{y^3}{3} \right]_{-2}^{2}$$

$$\pi \left[8 - \frac{8}{3} - \left(-8 + \frac{8}{3} \right) \right]$$

$$= \pi \left[8 - \frac{8}{3} + 8 - \frac{8}{3} \right]$$

$$= \pi \left[16 - \frac{16}{3} \right]$$

$$= \pi \times \frac{32}{3}$$