Laplace Transformation

Definition

Let $t > 0$ and $f(t)$ be a continuous function of t. Then the integral $\int_0^\infty e^{-st} f(t)\, dt$ is defined as the laplace transform of $f(t)$. It is denoted by $F(s)$.

$$\therefore F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Transformation of Elementary functions

1) $L(1) = \frac{1}{s} \quad (s > 0)$

2) $L(e^{at}) = \frac{1}{s-a} \quad (s > a)$

3) $L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$ where, $n = 0, 1, 2, \ldots$

4) $L(\sin at) = \frac{a}{s^2+a^2} \quad (s > 0)$

5) $L(\cos at) = \frac{s}{s^2+a^2} \quad (s > 0)$

6) $L(\sinh at) = \frac{a}{s^2-a^2} \quad (s > |a|)$

7) $L(\cosh at) = \frac{s}{s^2-a^2} \quad (s > |a|)$

Imp

Linearity of Laplace transformati

Show that laplace operator is a linear operator.

⇒

Proof:

Let $f(t)$ and $g(t)$ are two functions whose laplace transformation exist. Let a and b are constant.

Then we have,

$$L(a f(t) + b g(t)) = a L(f(t)) + b L(f(t))$$

By definition of Laplace transformation,

$$L[a f(t) + b g(t)] = \int_0^\infty e^{-st} (a f(t) + b g(t)) dt$$

$$= \int_0^\infty a e^{-st} f(t) dt + \int_0^\infty b e^{-st} g(t) dt$$

$$= a\int_0^\infty e^{-st} f(t) dt + b\int_0^\infty e^{-st} g(t) dt$$

$$= a L\{f(t)\} + b L\{g(t)\}$$

$$\therefore L\{a f(t) + b g(t)\} = a L\{f(t)\} + b L\{g(t)\}$$

Hence, laplace operator is a linear operator.

$$F(s) = \int e^{-st} f(t)\,dt = L\{f(t)\}$$

$$L^{-1} F(s) = f(t)$$

Date ______

Page ______

Imp

First shifting Theorem [Multiplication of a function by e^{at}]

Statement : If $L\{f(t)\} = F(s)$. Then,

$$L(e^{at} f(t)) = F(s-a)$$

OR, $$L\{e^{at} f(t)\} = [F(s)]_{s \to s-a}$$

⇒ Proof:

By definition of laplace transformation.

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \cancel{\int_0^\infty e^{-(a-s)}}$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$\therefore L\{e^{at} f(t)\} = F(s-a) \#$$

Enercise 3.1

1)

c) $f(t) = e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$.

$\Rightarrow$ Now,

$$L(f(t)) = L\{e^{2t}\} + 4\,L(t^3) - 2\,L(\sin 3t) + 3\,L(\cos 3t)$$

$$= 1 \cdot \frac{1}{s-2} + 4 \times \frac{\Gamma 4}{s^4} - 2 \times \frac{3}{s^2+3^2} + 3 \times \frac{s}{s^2+3^2}$$

$$F(s) = \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+3^2} + \frac{3s}{s^2+9}$$ Ans

$$L^{-1}\{F(s)\} = L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\left(\frac{24}{s^4}\right) - L^{-1}\left(\frac{6}{s^2+3^2}\right) + 3\,L^{-1}\left(\frac{s}{s^2+3^2}\right)$$

$$= e^{2t} + L^{-1}\left(\frac{1}{s^{3+1}}\,\Gamma 3+1 \times \frac{24}{3!}\right) - L^{-1}\left(\frac{3}{s^2+3^2} \times \frac{6}{3}\right) + 3\cos 3t$$

$$= e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t.$$

ISMT

09/02. Tuesday

Laplace Transform of the Integral of the function.

If $L\{F(t)\} = F(s)$. Then $L\left[\int_0^t F(u)\,du\right]$

$$= \frac{1}{s} F(s)$$

Multiplication of function by t^n

If $L\{F(t)\} = F(s)$. Then $L\left[t^n F(t)\right]$.

$$= (-1)^n \frac{d^n}{ds^n} F(s)$$

Division of a function: t

If $L\{F(t)\} = F(s)$. Then $L\left\{\frac{F(t)}{t}\right\}$

$$= \int_s^{\infty} F(s)\,ds$$

Laplace transformation of Derivative of a Function

If $L\{F(t)\} = F(s)$, Then $L\{F'(t)\}$

$$= s\,L\{F(t)\} - F(0)$$

Q) Laplace transformation of

a) $t \cdot \sinh at$

$\Rightarrow$ $L\{t \cdot \sinh at\} = (-1)^1 \frac{d}{ds} F(s)$

$= \cancel{(-1)^1}$

$F(s) = L\{\sinh at\}$

$F(s) = \frac{a}{s^2 - a^2}$

$\therefore L\{t \cdot \sinh at\} = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$

$= -a \frac{d}{ds} (s^2-a^2)^{-1}$

$= a \times 2s \times (s^2-a^2)^{-2}$

$= \frac{2as}{(s^2-a^2)^2}$.

b) $\underbrace{t e^{-3t} \cos 2t}$.

$L\{t e^{-3t} \cos 2t\} = (-1)^1 \frac{d}{ds} F(s)$

$= \cancel{(-1)^1}$

$F(s) = \frac{s}{s^2+4}\Big|_{s \to s-a(-3)}$

$= \cancel{\frac{s+a3}{s^2+4}}$ $\frac{s+3}{(s+3)^2+4}$.

$$\frac{d}{dx}\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

Date

Page

$+3$ -5 $+3$ -5

$$\frac{dF(s)}{ds} = \frac{d}{ds}\left(\frac{s+3}{(s+3)^2+4}\right)$$

$$= \frac{(s+3)^2+4 - (s+3)\,2(s+3)}{[(s+3)^2+4]^2}$$

$$= \frac{(s+3)^2+4 - 2(s+3)^2}{[(s+3)^2+4]^2}$$

$$\frac{dF(s)}{ds} = \frac{4-(s+3)^2}{[(s+3)^2+4]^2}$$

$$\therefore L\{t e^{-3t} \cos 2t\} = \frac{(s+3)^2-4}{[(s+3)^2+4]^2}.$$

a) $t e^{-t} \cosh t$

$$L\{t e^{-t} \cosh t\} = (-1)^{1} \frac{d F(s)}{ds} \quad -①$$

$$F(s) = L\{e^{-t} \cosh t\}$$

$$= \left.\frac{s}{s^2-1}\right|_{s \to s+1}$$

$$= \frac{(s+1)}{(s+1)^2-1}$$

Now,

$$\frac{dF(s)}{ds} = \frac{\{(s+1)^2-1\} \cancel{2(s+1)} - (s+1) \times 2(s+1)}{[(s+1)^2-1]^2}$$

$$= \frac{\cancel{2(s+1)}\{(s+1)^2-1\} - 2(s+1)^2}{[(s+1)^2-1]^2}$$

$$= \frac{(s+1)^2-1-2(s+1)^2}{[(s+1)^2-1]^2}$$

$$\therefore L\{t e^{-t} \cosh t\} = \frac{(s+1)^2+1}{[(s+1)^2-1]^2}$$

4·a $\frac{\sin at}{t}$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$= \int_s^\infty L\{\sin at\}\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= a \times \frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right).$$

09/03 wednesday

S. 1

Q) $\int_0^t \frac{\sin t}{e^{-t}} \, dt$

$= \int_0^t e^{-t} \sin t \, dt$

$= \frac{F(s)}{s}$

Here,

$F(s) = L\{e^{-t} \sin t\}$

$= \left[\frac{1}{s^2+1}\right]_{s \to s+1}$

$= \frac{1}{(s+1)^2+1}$

$\therefore \int_0^t \frac{\sin t}{e^{-t}} \, dt = \frac{1}{s} \times \frac{1}{(s+1)^2+1}$

c) $\int_0^t \frac{\sin t}{t} \, dt$

$\Rightarrow \int_s^\infty F(s) \, ds$

$F(s) = L\{\sin t\}$

$= 1$

$$\int_0^t f(u)\,du = \frac{F(s)}{s}.$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty F(s)\,ds.$$

c) $\int_0^t \frac{\sin t}{t}\,dt$

⇒ Solution

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin t\,dt}{t}\right\} = \frac{1}{s}\cot^{-1} s$$

d) $\int_0^t \frac{1-\cos t}{t}\,dt.$

⇒ Solution

$$L\left\{\frac{1-\cos t}{t}\right\} = L\left\{\frac{1}{t}\right\} - L\left\{\frac{\cos t}{t}\right\}$$

$$= \int_s^\infty \frac{1}{s}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+1}\,ds$$

$$= \left[\ln s\right]_s^\infty - \frac{1}{2}\left[\ln(s^2+1)\right]_s^\infty$$

$$= \left[\ln \frac{s}{(s^2+1)^{1/2}}\right]_s^\infty$$

$$= \ln\left[\frac{1}{\left(1+\frac{1}{s^2}\right)^{1/2}}\right]_s^\infty = \ln\frac{(1+s^2)^{1/2}}{(s^2)^{1/2}}$$

$$= \ln\left(\frac{1}{1}\right) - \ln\left(\frac{s}{(s^2+1)^{1/2}}\right) = \frac{1}{2}\ln\frac{s^2+1}{s^2}$$

$$= 0 - \ln\frac{s}{(1+s^2)^{1/2}}.$$

e) $\int_0^t \frac{1-e^{-t}}{t}\,dt$

$\Rightarrow$ $L\left\{\frac{1}{t} - \frac{e^{-t}}{t}\right\}$

$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds$

$= \left[\ln s - \ln(s+1)\right]_s^\infty$

$= \left[\ln \frac{s}{s+1}\right]_s^\infty$

$= \left[\ln \frac{1}{1+\frac{1}{s}}\right]_s^\infty$

$= \left[\ln 1 - \ln \frac{s}{s+1}\right]$

$= \ln\left(\frac{s+1}{s}\right)$

$\therefore \int_0^t \frac{1-e^{-t}}{t}\,dt = \frac{1}{s} \ln\left(\frac{s+1}{s}\right)$

f) $\int_0^t t^2 e^{-t}\,dt.$

$\Rightarrow$ $L\{e^{-t} t^2\} = \left[\frac{\Gamma 3}{s^3}\right]_{s \to s+1}$

$= \frac{2}{(s+1)^3}$

$\int_0^t t^2 e^{-t}\,dt = \frac{1}{s} \frac{2}{(s+1)^3}$

b) $\int_0^t \frac{\cos at - \cos bt}{t} dt$

$\Rightarrow$

$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$

$= \frac{1}{2}\left[\ln \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$

$= \frac{1}{2}\left[\ln \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty$

$= \frac{1}{2}\left[0 - \ln\frac{s^2+a^2}{s^2+b^2}\right]$

$= \frac{1}{2} \ln\left(\frac{s^2+b^2}{s^2+a^2}\right).$

$\therefore \int_0^t \frac{\cos at - \cos bt}{t} dt = \frac{1}{2s} \ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

7).

a. $L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2} ds$

$= a \times \frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$

$= a\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right)\right]$

$= a\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right]$

$= a \cot^{-1}\frac{s}{a}.$

Q) $te^{2t}\cosh t$.

$L\{te^{2t}\cosh t\}$

$L\{\cosh t\} = \frac{s}{s^2-1}$

$L\{t\cosh t\} = (-1)^1 \frac{d}{dt}\left(\frac{s}{s^2-1}\right)$

$= -1\left[\frac{(s^2-1)-s(2s)}{(s^2-1)^2}\right]$

$= -1\left[\frac{s^2-1-2s^2}{(s^2-1)^2}\right]$

$= \frac{s^2+1}{(s^2-1)^2}$.

$L\{e^{2t}\frac{s^2+1}{(s^2-1)^2}\} = \frac{(s-2)^2+1}{\{(s-2)^2-1\}^2}$

$= \frac{s^2-4s+5}{(s^2-4s+3)^2}$

Q) $\cosh t \sin at$

$= \frac{e^{x}+e^{-x}}{2} \sin at$.

$= \frac{1}{2}\left[e^{t}\sin at + e^{-at}\sin at\right]$

$= e^{t}\frac{a}{s^2+a^2} \quad + e^{-t}\cdot\frac{a}{s^2+a^2}$

$= \frac{1}{2}\left[\frac{a}{(s-1)^2+a^2} + \frac{a}{(s+1)^2+a^2}\right]$

(2-6)

Date
Page

o) $t^2 \cos at$

$L\{t^2 \cos at\}$

$= (-1)^2 \dfrac{d^2 L(\cos at)}{ds^2}$

Now,

$L(\cos at) = \dfrac{s}{s^2+a^2}$

$\dfrac{d L(\cos at)}{ds} = \dfrac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}$

$\dfrac{d L(\cos at)}{ds} = \dfrac{a^2 - s^2}{(s^2+a^2)^2}$

or, $\dfrac{d^2 L(\cos at)}{ds^2} = \dfrac{(s^2+a^2)^2(-2s) - (a^2-s^2)\, 2(s^2+a^2)\, 2s}{(s^2+a^2)^4}$

$= \dfrac{-2s(s^2+a^2)^2 - 4s(s^2+a^2)(a^2-s^2)}{(s^2+a^2)^4}$

$= \dfrac{-2s(s^2+a^2)^2 + 4s(s^4-a^4)}{(s^2+a^2)^4}$

$= \dfrac{-2s(s^4+2a^2s^2+a^4) + 4s^5 - 4sa^4}{(s^2+a^2)^4}$

$= \dfrac{+2s^5 - 4a^2s^3 - 2a^4s + 4s^5 - 4sa^4}{(s^2+a^2)^4}$

$= \dfrac{2s^5 - 4a^2s^3 - 6sa^4}{(s^2+a^2)^4}$

$= \dfrac{+2s^3(s^2+a^2) - 6sa^2(s^2+a^2)}{(s^2+a^2)^4}$

$\frac{\sin^2 t}{t}$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty L\{\sin^2 t\}\, ds.$$

$$= \int_s^\infty L\left\{\frac{1-\cos 2t}{2}\right\} ds$$

$$= \frac{1}{2}\int_s^\infty L\{1-\cos 2t\}\, ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds.$$

$$\cancel{= \frac{1}{2}\int_s^\infty \frac{s^2+4-2s}{s(s^2+4)}}$$

$$\cancel{= \frac{1}{2}\left[\left[\ln s\right]_s^\infty - 2\left[\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty\right]}$$

$$\cancel{= \frac{1}{2}\left[\ln\right.}$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln s - \ln\left(\sqrt{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln \frac{1}{\sqrt{1+\frac{4}{s^2}}} - \ln \frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\left[0 - \ln \frac{s}{\sqrt{s^2+9}}\right]$$

$$= \frac{1}{2}\ln \frac{\sqrt{s^2+4}}{s}$$

2081/09/16
Tuesday

Convolution of Laplace Transform

Let $f(t)$ and $g(t)$ be two function defined $t > 0$. Then the convolution of two function $f(t)$ and $g(t)$ is denoted by $f * g$ and given as.

$$(f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

OR,

$$= \int_0^t f(t-u)\, g(u)\, du.$$

Find the convolution of $t * e^t$

⇒

Let, $f(t) = t$, $g(t) = e^{t}$.

$f(t-u) = t-u$ $g(u) = e^{u}$

Now,

$$(f * g)(t) = \int_0^t (t-u)\, e^{-u}\, du$$

$$= \cancel{\int_0^t t \cdot e^{-u}\, du - \int_0^t u\, du}$$

$$= \cancel{t \cdot e^{-t} [u]_0^t - \left[\frac{u^2}{2}\right]_0^t}$$

$$\cancel{= t\cdot e^{-t}[t] - \frac{t^2}{2}}$$

$$\cancel{= t^2\cdot e^{-t} - \frac{t^2}{2}}$$

$$= \int_0^t t e^{-u}\, du - \int_0^t u e^{-u}\, du$$

$$= -t[e^{-u}]_0^t - \left[\frac{u e^{-u}}{-1} - e^{-u} \right]_0^t$$

$$= -t[e^{-t} - e^0] - [-ue^{-u} - e^{-u}]_0^t$$

$$= -t[e^{-t} - 1] - [-ue^{-u} - e^{-u}]_0^t$$

$$= -t[e^{-t} - 1] - [-te^{-t} - e^{-t} - 0 + 1]$$

$$\cancel{= -te^{-t} + t + te^{-t} + e^{-t} - 1}$$

$$= te^{-t} - t - te^{t} + e^{t} - 1$$

$$= e^{t} - t - 1 \quad \text{Ans}$$

Convolution Theorem

Statement:- If $L(f(t)) = F(s)$ and $L(g(t)) = G(s)$ Then;

$\Rightarrow$ $L(f*g) = F(s) \cdot G(s)$

Proof: By definition of Laplace ~~theorem~~ transformation,

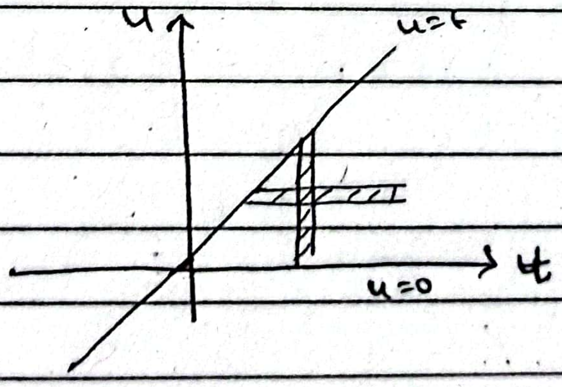
$$L(f*g) = \int_0^{\infty} e^{-st} (f*g)\, dt.$$

$$= \int_0^{\infty} e^{-st} \left[\int_0^{t} f(u)\, g(t-u)\, du \right] dt$$

$$= \int_0^{\infty} \int_0^{t} e^{-st} f(u)\, g(t-u)\, du\, dt.$$

$$= \int_0^{\infty} \int_0^{t} e^{-s(t-u)} e^{-su} f(u)\, g(t-u)\, du\, dt$$

$0 \le u \le t$. $\qquad 0 \le t \le \infty$



Changing the order,

$u \le t \le \infty \qquad 0 \le u \le \infty$

$$= \int_0^\infty \int_u^\infty e^{-s(t-u)} e^{-su} f(u) \, g(t-u) \, dt \, du.$$

$$= \int_0^\infty e^{-su} f(u) \, du \cdot \int_u^\infty e^{-s(t-u)} g(t-u) \, dt.$$

Put $t-u = p$ when $t = u$, $p = 0$

$1 = \frac{dp}{dt}$ when $t \to \infty$, $p \to \infty$.

$\therefore dt = dp$

$$= \int_0^\infty e^{-su} f(u) \, du \cdot \int_0^\infty e^{-sp} g p \, dp$$

$= F(s) \times G(s)$

$\therefore L(f * g) = F(s) \times G(s)$

Exercise 3.2

1. Find the inverse Laplace transform.

a) $\frac{3s+2}{s^2+2s+2}$

⇒ Given,

$$F(s) = \frac{3s+2}{s^2+2s+2}$$

$$= \frac{3s+2}{s^2+2s+1+1}$$

$$= \frac{3s+2}{(s+1)^2+1}$$

$$= \frac{3s+3-1}{(s+1)^2+1}$$

$$F(s) = \frac{3(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$L^{-1}(F(s)) = 3L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= \cancel{3 \sin}$$

$$= 3\cos t\, e^{-t} - \sin t\, e^{-t}$$

$$= (3\cos t - \sin t)e^{-t}.$$

b) $\frac{2s+3}{s^2+4}$

⇒ $F(s) = \frac{2s+3}{s^2+2^2} = \frac{2s}{s^2+2^2} + \frac{3}{s^2+2^2}$

$$L^{-1}(F(s)) = 2L^{-1}\left(\frac{s}{s^2+2^2}\right) + \frac{3}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$= 2 \times \cos 2t + \frac{3}{2}\sin 2t.$

c) $\frac{1}{s^2-9}$

$\Rightarrow$ $F(s) = \frac{1}{s^2-3^2} = \frac{1}{3} \times \frac{3}{s^2-3^2}.$

$L^{-1}(F(s)) = \frac{1}{3} L^{-1}\left(\frac{3}{s^2-3^2}\right) = \frac{1}{3}\sinh 3t.$

d) $\frac{s-1}{(s-1)^2+4}.$

$\Rightarrow$ $F(s) = \frac{s-1}{(s-1)^2+2^2}$

$L^{-1}(F(s)) = L^{-1}\left(\frac{s-1}{(s-1)^2+2^2}\right)$

$= e^{t} L^{-1}\left(\frac{s}{(s)^2+2^2}\right)$

$= e^{t} \times \cancel{\sin 2t}\cos 2t.$

e) $\frac{1}{s^2+2s+1+4}$

$F(s) = \frac{1}{(s+1)^2+2^2} = \frac{1}{2} \times \frac{2}{(s+1)^2+2^2}.$

$L^{-1}(F(s)) = \frac{1}{2} L^{-1}\left(\frac{2}{(s+1)^2+2^2}\right)$

$= \frac{1}{2} \times e^{-t} L^{-1}\left(\frac{2}{s^2+2^2}\right)$

$= \frac{e^{-t}}{2} \times \sin 2t.$

f. $\frac{s-4}{s^2-4}$

$$F(s) = \frac{s}{s^2-2^2} - \frac{4}{2} \times \frac{2}{s^2-2^2}$$

$$L^{-1}(F(s)) = \cosh 2t - 2 \times \sinh 2t.$$

g. $$\frac{2s^3}{s^4-1} = \frac{2s^3}{(s^2-1)(s^2+1)} = \frac{As+B}{s^2-1} + \frac{Cs+D}{s^2+1}$$

$$\frac{(s^2+1)(As+B) + (Cs+D)(s^2-1)}{(s^2-1)(s^2+1)}$$

$$\Rightarrow 2s^3 = (s^2+1)(As+B) + (Cs+D)(s^2-1)$$

Here,

$2 = A + C$ — (i)

$0 = B + D$ — (ii)

$0 = A - C$ — (iii)

$0 = B - D$ — (iv)

$\therefore B = 0, \quad D = 0.$

$2 = A + C$

$0 = A - C$

$2A = 2$ $\quad\quad$ $2 = 1 + C$

$\therefore A = 1$ $\quad\quad$ $\therefore C = 1$

$$\frac{2s^3}{(s^2-1)(s^2+1)} = \frac{s}{s^2-1} + \frac{s}{s^2+1}$$

$$\therefore \frac{2s^3}{} =$$

$$\therefore L^{-1}\left(\frac{2s^3}{s^4-1}\right) = L^{-1}\left(\frac{s}{s^2-1}\right) + L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$\therefore L^{-1}\left(\frac{2s^3}{s^4-1}\right) = \cosh t + \cos t$$

2. e)

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{s}{(s^2+2s+1)(s^2+1)} \quad \text{— (1)}$$

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{As+B}{s^2+2s+1} + \frac{Cs+D}{s^2+1}$$

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{(As+B)(s^2+1) + (s^2+2s+1)(Cs+D)}{(s^2+2s+1)(s^2+1)}$$

$$s = (As+B)(s^2+1) + (s^2+2s+1)(Cs+D)$$

$0 = A + C$ — (i)

$0 = B + D + 2C$ — (ii)

$1 = A + 2D + C$ — (iii)

$0 = B + D$ — (iv)

$1 = 2D \Rightarrow D = 1/2, \quad B = -1/2,$

$C = 0, \quad A = 0.$

from ①

$$= \frac{-1}{2(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}$$

$$\therefore L^{-1}\left(\frac{s}{(s+1)^2(s^2+1)}\right) = L^{-1}\left(\frac{-1}{2(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}\right)$$

$$= \frac{-1}{2} L^{-1}\left(\frac{1}{(s+1)^2}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{-1}{2} L^{-1}\left(\frac{1}{(s+1)^2}\right) + \frac{1}{2}\sin t$$

$$= \frac{-1}{2} e^{-t}\, t + \frac{1}{2}\sin t$$

b) $\frac{1}{(s^2+a^2)^2}$

$\Rightarrow$

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} \times \frac{1}{s^2+a^2}$$

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{1}{s^2+a^2} \times \frac{1}{s^2+a^2}\right) \quad \text{— ①}$$

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$$

By convolution theorem.

$$L^{-1}\left(F(s).\,G(s)\right) = f * g.$$

$$L^{-1}\left(\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at * \frac{1}{a}\sin at$$

$$= \frac{1}{a^2}\sin at * \sin at$$

$$= \frac{1}{a^2}\int_0^t \sin au \cdot \sin a(t-u)\,du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au \,\sin(at-au)\,du.$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2au-at) - \cos at\right] du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - \cos at\cdot u\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right]$$

4e) $\log\left(\frac{s^2-a^2}{s^2}\right)$

$F(s) = \log(s^2-a^2) - \log s^2$

$F'(s) = \frac{2s}{s^2-a^2} - \frac{2s}{s^2}$

$F'(s) = \frac{2s}{s^2-a^2} - \frac{2}{s}$

we know,

$L(t\,f(t)) = -\frac{d}{ds}F(s) = -F'(s)$

$= \frac{2}{s} - \frac{2s}{s^2-a^2}$

$t\,f(t) = L^{-1}\left(\frac{2}{s} - \frac{2s}{s^2-a^2}\right)$

$= 2(1-\cosh at)$

$\therefore f(t) = \frac{2(1-\cosh at)}{t}.$

9.D) $\frac{1}{2} \ln \left(\frac{s^2+b^2}{s^2+a^2} \right)$

$= \frac{1}{2} \left[\ln (s^2+b^2) - \ln (s^2+a^2) \right]$

$L(t\,F(t)) = -\frac{d\,F(s)}{ds}$

where,

$F(s) = \frac{1}{2} \left(\ln (s^2+b^2) - \ln (s^2+a^2) \right]$

$\frac{d\,F(s)}{ds} = \frac{1}{2} \left[\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2} \right]$

$= \left[\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2} \right]$

$L(t\,F(t)) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$t\,F(t) = \cos at - \cos bt$

$\therefore F(t) = \frac{\cos at - \cos bt}{t}$

h) $\ln\left(\frac{s(s+1)}{s^2+4}\right)$.

$= \ln(s^2+s) - \ln(s^2+4)$

$L(t\,f(t)) = -\frac{d}{ds}F(s)$

$\frac{dF(s)}{ds} = \frac{2s+1}{s^2+s} - \frac{2s}{s^2+4}$

$= \frac{2s}{s^2+s} + \frac{1}{s^2+s} - \frac{2s}{s^2+4}$

$= \frac{2}{s+1} + \frac{1}{s^2+s} - \frac{2s}{s^2+4}$

let,

$\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{(s+1)}$

or, $1 = A(s+1) + Bs$

$\therefore A+B=0$

$A=1$

$\therefore B=-1$.

$\therefore \frac{dF(s)}{ds} = \frac{2}{s+1} + \frac{1}{s} - \frac{1}{s+1} - \frac{2s}{s^2+4}$

$= \frac{1}{s+1} + \frac{1}{s} - \frac{2s}{s^2+4}$.

Q) $\cos \frac{}{2}$

$$t\,f(t) = L^{-1}\left(\frac{2s}{s^2+4} - \frac{1}{s+1} - \frac{1}{s}\right).$$

$$\therefore \boxed{f(t) = \frac{2\cdot\cos 2t - e^{-t} - 1}{t}}$$

Q) $\tan^{-1}\frac{2}{s^2}$

$$F(s) = \tan^{-1}\frac{2}{s^2}$$

$$\frac{d\,F(s)}{ds} = \frac{1}{1+\frac{4}{s^4}} \times 2\cdot(-2)\,s^{-3}$$

$$= \frac{s^4}{s^4+4} \times \frac{(-4)}{s^3}$$

$$= \frac{-4s}{s^4+4}$$

$$L(t\,f(t)) = \frac{4s}{s^4+4}.$$

$$t\,f(t) = L^{-1}\left(\frac{4s}{s^4+4}\right) \text{ ——— } (1)$$

$$s^4+4 = (s^2+2)^2 - 4s^2$$

$$s^2+4 = (s^2+2s+2)(s^2-2s+2)$$

$$\therefore \frac{4s}{s^4+4} = \frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}.$$

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\,ds$$

From ①

$$t\,f(t) = L^{-1}\left(\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right)$$

$$= e^{t}\sin t - e^{-t}\sin t$$

$$= 2\sin t\left(\frac{e^{t}-e^{-t}}{2}\right)$$

$$\therefore \boxed{f(t) = \frac{2\sin t\,\sinh t}{t}}$$

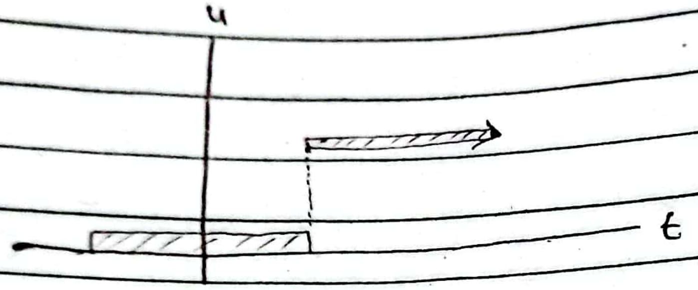
s^2

2081/09/22
Monday

Unit step function (Heaviside Function)

The unit step function is denoted by $u_a(t)$ or $u(t-a)$ and defined as:

$$u(t-a) \text{ or } u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$



Shifted Function

Let $f(t)$ be continuous function for all t then,

$$f(t-a)\,u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a. \end{cases}$$

is known as shifted function of $f(t)$ with shift $t = a$.

By defn

$e^{-\infty} = \frac{1}{e^{\infty}}$

Laplace transformation of $u_a(t)$ or $u(t-a)$

By def^n of laplace transformation.

$$L(u(t-a)) = \int_0^\infty e^{-st}\, u(t-a)\, dt.$$

$$= \int_0^a e^{-st}\, u(t-a)\, dt + \int_a^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = \left[\frac{e^{-\infty}}{-s} - \left(\frac{e^{-sa}}{-s}\right)\right]$$

$$\therefore \boxed{L(u(t-a)) = \frac{e^{-as}}{s}}$$

e.g.

$$L(u(t-10)) = \frac{e^{-10s}}{s}$$

Imp

2^{nd} Shifting Theorem

Statement: If $L(f(t)) = F(s)$.

Then, $L(f(t-a)\, u(t-a)) = e^{-as} F(s)$.

Proof: By def^n of laplace transformation,

$$L(f(t-a)\, u(t-a)) = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} F(t-a) \cdot 0 \, dt + \int_a^\infty e^{-st} F(t-a) \cdot 1 \, dt$$

$$= 0 + \int_a^\infty F(t-a) \, e^{-st} \, dt.$$

Put, $t - a = u$ when $t = \infty$, $u = \infty$

$dt = du$ when $t = a$, $u = 0$.

$t = u + a$

$$\therefore L(F(t-a) \cdot u(t-a)) = \int_0^\infty F(u) \cdot e^{-s(u+a)} \, du$$

$$= \int_0^\infty e^{-su} \cdot e^{-sa} \, F(u) \, du.$$

$$= e^{-sa} F(s)$$

$$\therefore L(F(t-a) \, u(t-a)) = e^{-sa} F(s).$$

Note: $L(F(t) \, u(t-a)) = e^{-sa} L[F(t+a)]$

$$\frac{\quad}{s^{1+1}} = \frac{\quad}{s^{2}}$$

Exercise 3.3

a) $(t-1)\,u(t-1)$.

By 2^{nd} shifting theorem,

$$L\left((t-a)\,u(t-a)\right) = e^{-as} F(s)$$

where $F(s)$ is $L(f(t))$

$$\therefore L\left((t-1)\,u(t-1)\right) = e^{-s} L(t)$$

$$= e^{-s} \times \frac{1}{s^2}$$

$$= \frac{e^{-s}}{s^2}.$$

b) $e^{-2t}\,u(t-3)$.

We know that,

$$L\left(f(t)\,u(t-a)\right) = e^{-as} L\left(f(t+a)\right)$$

$$\therefore L\left(e^{-2t}\,u(t-3)\right) = e^{-3s} L\left(f(t+a)\right)$$

$$= e^{-3s} L\left(e^{-2(t+3)}\right)$$

$$= e^{-3s} L\left(e^{-2t}\cdot e^{-6}\right)$$

$$= e^{-3s}\cdot e^{-6}\, L\left(e^{-2t}\right)$$

$$= e^{-3s-6}\cdot \frac{1}{s+2}$$

$$= \frac{e^{-3s-6}}{s+2}.$$

1-c) $t\,u(t-1)$.

We know,

$$L(f(t)\,u(t-a)) = e^{-as}\,L(f(t+a))$$

$$L(t \cdot u(t-1)) = e^{-s}\,L[(t+1)]$$

$$= e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$= \frac{e^{-s}}{s^2}+\frac{e^{-s}}{s}.$$

d) $u(t-\pi)\cos t$.

$\Rightarrow$ $\cos t\; u(t-\pi)$.

We know,

$$L(f(t)\,u(t-a)) = e^{-as}\,L(f(t+a))$$

$$L(\cos t\; u(t-\pi)) = e^{-\pi s}\,L(\cos(\pi+t))$$

$$= e^{-\pi s}\,L(-\cos t)$$

$$= -e^{-\pi s}\,L(\cos t)$$

$$= -e^{-\pi s} \times \frac{s}{s^2+1}$$

Exercise 3.9

1. Apply lap transformation to evaluate

a) $\int_0^\infty e^{-3t}\, t \sin t \, dt = \frac{3}{50}$

let the integral,

$$I = \int_0^\infty e^{-3t}\, t \sin t \, dt \quad \text{---} (1)$$

$$L(t \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2s}{(s^2+1)^2} = F(s)$$

$$L(t \sin t) = \int_0^\infty e^{-st}\, t \sin t \, dt \quad \text{---} (11)$$

comparing (1) & (11) we get,

$s = 3.$

$$\therefore F(s) = \frac{2s}{(s^2+1)^2} = \frac{6}{100} = \frac{3}{50}$$

$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

Application of laplace transform to solve Differential Equation

$L(y'(t)) = sL(y(t)) - y(0)$

$L(y''(t)) = s^2 L(y(t)) - s\,y(0) - y'(0)$

Exercise 3.4

2.b) $y'' + 2y' + 17y = 0$, $y(0) = 0$, $y'(0) = 12$.

Using Laplace trans. on both sides

$L(y'') + 2L(y') + 17L(y) = 0$

or, $s^2 L(y) - s(y(0)) - (y'(0)) + 2[sL(y) - (y(0))]$

$+ 17L(y) = 0$

or, $s^2 L(y) + 17L(y) \cancel{+2} - s\cdot 0 - 12 + 2[sL(y) - 0]$

$= 0$

or, $s^2 L(y) + 2sL(y) + 17L(y) - 12 = 0$

or, $L(y)[s^2 + 2s + 17] = 12$

or, $L(y) = \dfrac{12}{s^2 + 2s + 1 + 4^2}$

or, $L(y) = \dfrac{4}{(s+1)^2 + 4^2} \times 3$

$\therefore\ y = 3e^{-t}\cdot\sin 4t$

c)

$\Rightarrow y'' - 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 7.$

$L y'' - 2 L y' - 3 L(y) = 0$

or, $s^2 L(y) - s L(y(0)) - L(y'(0)) - 2[s L(y) - L(y$

$-3 L(y) = 0$

or, $s^2 L(y) - 3 L(y) - s \cdot 1 - 7 - 2 s L(y) + 2 \cdot 1 = 0$

or, $L(y)[s^2 - 3 - 2s] - s - 7 + 2 = 0$

or, $\cancel{L(y) = \frac{s+5}{s^2+6-2s}}$

$\cancel{= \frac{s}{s^2-2s+1+}}$

or, $L(y) = \frac{s+5}{s^2-2s-3}$

$= \frac{s+5}{s^2-2s+1-1-3}$

$L(y) = \cancel{\frac{s-1+1}{(s-1)^2-2^2}} + \frac{s}{(s-1)^2-2^2}$

$y = \cancel{L^{-1}\left(\frac{s}{(s-1)^2 -}\right.}$

$= \frac{s-1+6}{(s-1)^2-2^2}$

$y = L^{-1}\left(\frac{s-1}{(s-1)^2-2^2}\right) + L^{-1}\left(\frac{6 \cdot 2}{(s-1)^2-2^2}\right) \times \frac{1}{2}$

$= e^{t} \cdot \cosh 2t + 3 \cdot e^{t} \cdot \sinh 2t.$

1) $\cancel{y'' - y' - 2y}$

2) $y'' - 3y' + 2y = 4t + e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$

$L(y'') - 3L(y') + 3L(y) = L(4t + e^{3t})$

$s^2 L(y) - sL(y(0)) - L(y'(0)) - 3[sL(y) - L(y(0))]$
$+ 2L(y) = \frac{4}{s^2} + \frac{1}{s-3}$

$s^2 L(y) - s\cdot 1 - (-1) - 3[s\cdot L(y) - 1] + 2L(y)$
$= \frac{4}{s^2} + \frac{1}{s-3}$

$Ly(s^2 - 3s + 2) - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3}$

$L(y)(s^2 - 3s + 2) = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$

$L(y)(s^2 - 3s + 2) = \frac{s^2 + 4s - 12}{s^2(s-3)} + (s-4)$

$L(y) = \frac{(s+6)\cancel{(s-2)}}{s^2(s-3)(s-1)\cancel{(s-2)}} + \frac{(s-4)}{(s-2)(s-1)}$

$L(y) = \frac{(s+6)}{s^2(s-3)(s-1)} + \frac{(s-4)}{(s-2)(s-1)}.$

$y = L^{-1}\left(\frac{s+6}{s^2(s-3)(s-1)} + \frac{s-4}{(s-2)(s-1)}\right)$

$$\frac{s+6}{s^2(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s-3)}$$

$$s+6 = A\,s(s-1)(s-3) + B(s-1)(s-3)$$
$$+ C \cdot s^2(s-3) + D \cdot s^2(s-1)$$
$$= A((s^2-s)(s-3)) + B(s^2-3s-s+3)$$
$$+ C(s^3-3s^2) + D(s^3-s^2)$$
$$= A(s^3-3s^2-s^2+3s) + B(s^2-4s+3)$$
$$+ C(s^3-3s^2) + D(s^3-s^2)$$
$$s+6 = A(s^3-4s^2+3s) + B(s^2-4s+3)$$
$$+ C(s^3-3s^2) + D(s^3-s^2)$$

$$A+C+D = 0$$
$$-4A+B-3C-D = 0$$
$$3A-4B = 1$$
$$3B = 6$$
$$\boxed{B=2}$$

$$3A-8=1$$
$$3A=9$$
$$\therefore \boxed{A=3}$$

$$-4\cdot3+2-3C+D=0$$
$$-12+2-3C-D=0$$
$$-3C-D=10$$
$$C+D=-3$$

$$\cancel{C=\frac{13}{4}}, \quad \cancel{D=\frac{1}{4}}.$$

$$C=-\frac{7}{2}, \quad D=\frac{1}{2}.$$

$\therefore \frac{3}{s} + \frac{2}{s^2} - \frac{7}{2(s-1)} + \frac{1}{2(s-3)}.$

$\therefore L^{-1}\left\{ \frac{3}{s} + \frac{2}{s^2} - \frac{7}{2(s-1)} + \frac{1}{2(s-3)} \right\}$

$= 3 + 2t - \frac{7}{2}e^t + \frac{1}{2}e^{3t}$

Also,

let, $\frac{s-4}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}$

$s-4 = A(s-1) + B(s-2).$

~~s-4 =~~

$A+B=1$

$-A-2B=-4.$

$A+2B=4.$

$\therefore A=-2, \quad B=3.$

$L^{-1}\left(\frac{-2}{s-2} + \frac{3}{s-1} \right) = -2e^{2t} + 3 \cdot e^t$

$\therefore$ Ans $\Rightarrow$ $3 + 2t - \frac{7}{2}e^t + \frac{1}{2}e^{3t} - 2e^{2t} + 3e^t.$

$3 + 2t - \frac{e^t}{2} + \frac{1}{2}e^{3t} - 2e^{2t}.$

b) $y' - 4y = 5e^{-4t}$, $y(0) = 1$

or, $L(y') - 4L(y) = 5L(e^{-4t})$

or, $sL(y) - y(0) - 4L(y) = 5 \cdot \frac{1}{s+4}$

or, $L(y)(s-4) - 1 = \frac{5}{s+4}$.

or, $L(y) = \frac{5}{\cancel{(s-4)}(s+4)(s-4)} + \frac{1}{(s-4)}$

or, $L(y) = \frac{5}{s^2-4^2} + \frac{1}{s-4}$

$\cancel{y = L^{-1}\left(\frac{5}{s^2-4^2} + \frac{1}{s-4}\right)}$

$\cancel{= \cosh 4t + e^{4t}}$

$y = \frac{5}{4} L^{-1}\left(\frac{4}{s^2-4^2}\right) + L^{-1}\left(\frac{1}{s-4}\right)$

$= \frac{5}{4} \cdot \sinh 4t + e^{4t}$.

K. $y'' + 2y' - 3y = 6e^{-2t}$, $y(0) = 2$, $y'(0) = -14$.

$L(y'') + 2L(y') - 3L(y) = L(6e^{-2t})$.

or, $s^2 L(y) - s\,y(0) - y'(0) + 2[sL(y) - y(0)] - 3L(y) = 6\frac{1}{s+2}$.

or, $s^2 L(y) - s.2 + 14 + 2sL(y) - 2.2 - 3L(y) = \frac{6}{s+2}$.

or, $s^2 L(y) - 2s + 14 - 4 + 2sL(y) - 3L(y) = \frac{6}{s+2}$.

or, $L(y)(s^2 + 2s - 3) - 2s + 10 = \frac{6}{s+2}$.

or, $L(y) = \left(\frac{6}{s+2} - 2s + 10\right) \cdot \frac{1}{s^2+2s-3}$

or, $L(y) = \frac{6}{(s+2)(s+3)(s-1)} - \frac{2s}{(s+3)(s-1)} + \frac{10}{(s+3)(s-1)}$

$= \frac{6}{(s-1)(s+2)(s+3)} + \frac{10-2s}{(s-1)(s+3)}$.

Let,

$\frac{6}{(s-1)(s+2)(s+3)} = \frac{A}{(s-1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$

$6 = A(s+2)(s+3) + B(s-1)(s+3) + C(s-1)(s+2)$

$6 = A(s^2+5s+6) + B(s^2+2s-3) + C(s^2+s-2)$

$A + B + C = 0$

$5A + 2B + C = 0$

$6A - 3B - 2C = 0$

$A = 0 \quad , B = 0 \; , \; C = 0.$

Let,

$$\frac{10-2s}{(s-1)(s+3)} = \frac{A}{(s-1)} + \frac{B}{(s+3)}$$

$$10 - 2s = A(s+3) + B(s-1).$$

$A + B = -2$

$3A - B = 10$

$A = 2, \; B = -4.$

$$\therefore L^{-1}\left(\frac{2}{s-1} + \frac{-4}{s+3}\right)$$

$$= 2 \cdot e^{t} - 4 \cdot e^{-3t}$$