8th Bhadra

(Short notes)

# Different Approaches to Probability

1) classical or Mathematical Approach
2) Empirical or statistical Approach
3) Subjective Approach
4) Modern or Axiomatic Approach.

## Classical Approach

This approach is applied when the events are mutually exclusive and equally likely.

Definition: consider a random experiment with 'n' number of exhaustive, mutually exclusive and equally likely events

Out of n, m be the no of favourable cases (or events) to an event say A, then the probability of occurrance of the event A is denoted by P(A) and is calculated as:

$$P(A) = \frac{\text{favourable cases to event A}}{\text{exhaustive cases}}$$

$$\therefore P(A) = \frac{m}{n}.$$

Similarly, the probability of non-occurrance of A is denoted by $P(\bar{A})$ and is calculated as

$$P(\bar{A}) = \frac{n-m}{n}$$

$$= 1 - \frac{m}{n}$$

$$\boxed{P(\bar{A}) = 1 - P(A)}$$

Also,

$$\boxed{P(A) + P(\bar{A}) = 1}$$

i.e

$$p + q = 1$$

Example: If a coin is tossed three times. Find the probability of getting:

a) ~~getting~~ 3 head
b) 2 head
c) 1 head
d) no head
e) at least 1 head
vi) at most 1 head.

<u>Solution</u>

Random experiment:
Tossing a coin thrice.

Sample space.

$S = \{HHH, HHT, HTH, HTT,$
$THH, THT, TTH, TTT\}$

Exhaustive cases, $n = 8$

i) Let $A$ = Getting 3 heads

$\therefore$ Favourable cases, $m = 1$.

$\therefore P(A) = \frac{m}{n} = \frac{1}{8}$

ii) Let $B$ = Getting two head

$\therefore$ Favourable cases, $m = 3$.

$\therefore P(B) = \frac{m}{n} = \frac{3}{8}$

iii) Let $C$ = getting one head.

$\therefore$ Favourable cases, $m = 3$

$\therefore P(C) = \frac{3}{8}$

iv) Let $D$ = getting no head

$\therefore$ Favourable cases, $m = 1$

$\therefore P(D) = \frac{1}{8}$

v) $m = 7$ $\therefore P(E) = 7/8$.

vi. $m = 4$, $P(F) = 4/8$

If a die is thrown at random what is the probability of getting

a) Prime number
b) Even number
c) multiple of 3
d) multiple of two.

⇒ <u>Solution</u>

Sample space = {1, 2, 3, 4, 5, 6}

a) Let A = getting prime number.
$m = 3$
$\therefore P(A) = \frac{3}{6}$

b) Let B = getting even number.
$\therefore$ Favourable cases = m = 3
$\therefore P(B) = 3/6$.

c) ~~So~~ Let C = getting multiple of 3.
$\therefore$ Favourable cases = m = 2
$\therefore P(C) = 2/6$.

d) Let D = getting multiple of 2
Favourable cases = m = 3
$\therefore P(D) = \frac{3}{6}$.

If ~~1a~~ two dice are thrown simultaneously. What is the probability that the faces turn up show:

1) sum of seven
2) sum of eight or nine
3) sum of sum less than 5.
4) Same fases
5) Diff faces

Solution

Sample space = ~~{(1,1), (1,2), (1,3), (1,4), 1.~~

$$\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)$$

exhaustive cases $n = 36$

1) Sum of seven.

let $A$ = getting sum of seven.

$m$ = favourable cases = 6

$\therefore P(A) = \frac{6}{36}$

2) Sum of eight or nine

~~favourable cases = m~~

sum of eight (a) = 5, sum of nine (b) = 4

$P(a \text{ or } b) = P(a) + P(b) = \frac{5+4}{36} = \frac{9}{36}$

Let C = getting sum less than 5.

$m = 6$

$\therefore P(C) = \frac{6}{36}$.

Same faces = D

$m = 6$

$\therefore P(D) = 6/36$.

~~Diff~~ E = getting different faces

$P(E) = 1 - P(D)$

$= 1 - 6/36$

$= \frac{30}{36} = 5/6$.

07/27 Tuesday

# Counting Rules

Fundamental Rules of counting

1) Addition Rule
2) Multiplication Rule.

1) Addition Rule

When event A occurs in $n_1$ ways and event B occurs in $n_2$ ways. Also, A and B are mutually exclusive events. Then, at least one event (i.e. either A or B) can occur in $(n_1 + n_2)$ ways.

2) Multiplication Rule

When event A occurs in $n_1$ ways and event B occurs in $n_2$ ways. Then both events i.e A and B can occur in $(n_1 \times n_2)$ ways.

Methods of Counting Rule

1) Permutation

Arrangement of objects
(Selection with order)

Total no. of ditinct objects ⇒ n
Then, the permutation is given by

$$^nP_n = n!$$

If r objects are arranged. Then the permutation is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

Repeated case

$$\text{Permutation} = \frac{n!}{r!p!q!}$$

Q) In how many ways the word STATISTICS can be arranged.

⇒

Total no. of letters = 10 = n
repetition of S = r = 3
repetition of I = p = 2
repetition of T = q = 3.

2) <u>combination</u> (order doesn't matter)

Total no. of objects = n

If r no. of objects are selected. Then the combⁿ of the selection is given by.

$$^{n}C_{r} \text{ or } \binom{n}{r} = \frac{n!}{(n-r)! \times r!} \text{ or } C(n,r).$$

Q. A box contains two red, four white and 5 black marbles. In how many ways 3 marbles can be selected from the box.

⇒ <u>solution</u>

Here,

total no. of objects = 11 = n

objects to be selected = 3

∴ Total no. of ways = $^{11}C_{3}$

Q) A box contains 4 white and 6 red balls. Two balls are drawn at random from the box. What is the probability that both balls are red.

⇒ Given,

no. of white balls = 4

no. of red balls = 6

Total no. of balls = 10

Also, no. of selected balls = 2

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Out of 10 balls, two can be selected in ${}^{10}C_2$ ways.

$\therefore$ Exhaustive cases $(n) = {}^{10}C_2$

$= \frac{10!}{8! \times 2!}$

$= \frac{\cancel{10}^{5} \times 9 \times \cancel{8!}}{\cancel{8!} \times \cancel{2!}}$

$= 45$

Let $A$ = both are red.

Favourable cases $(m) = {}^{6}C_2 = \frac{6!}{4! \times 2!}$

$= \frac{6 \times 5}{2}$

$= 15$

$\therefore P(A) = \frac{m}{n} = \frac{15}{45} = \frac{1}{3}$

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Q) From a group of 3 engineers, 4 accountants and 5 statisticians, a subcommittee of 4 persons is formed. what is the probability that the sub-committee will consists of.

1) Two engineers, 1 accountant and 1 statistician
2) Two engineers and 2 accountants.
3) Two engineers.
4) All statisticians.
5) At least one engineer
6) At most two accountants.

=>

Given,

no. of engineers = 3
no. of accountants = 4
no. of statisticians = 5
∴ Total no. of persons = 12
Selecting no. of person for a committee = 4

Out of 12 persons, 4 can be selected in $^{12}C_4$ ways ~~a no~~.

∴ Exhaustive cases (n) = $^{12}C_4$

$$= \frac{12!}{8! \times 4!}$$

$$= 495$$

i) Let $E_1 = 2E, 1A$ and $1S$.

Favourable cases (m) $= {}^3C_2 \cdot {}^4C_1 \cdot {}^5C_1$

$= 60$

Then

$P(E1) = \frac{m}{n} = \frac{60}{495}$

iii) Let $E3 = 2E$ and 2 others.

Favourable cases (m) $= {}^3C_2 \cdot {}^9C_2$

$= 3 \cdot \frac{9!}{7! \times 2!}$

$= 3 \cdot \frac{9 \times 8^4}{2}$

$= 12 \cdot 9$

$= 108$

Then,

$P(E3) = \frac{108}{495}$

iv) Let $E4$ = All statistician.

Favourable cases (m) $= {}^5C_4$

$= 5$

∴ Then,

$P(E4) = \frac{m}{n} = \frac{5}{495}$

v) Let $E5$ = At least one engineer. 1E or 2E or 3E

Favourable cases (m) $= {}^3C_1 \cdot {}^9C_3 + {}^3C_2 \cdot {}^9C_2 + {}^3C_3 \cdot {}^9C_1$

$= 369$

$P(E5) = \frac{369}{495}$

vi) Let E6:

At most 2A.

2A or 1A or 0A.

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Q) A lot of empty graded ~~circuit~~ chip consists of 10 goods, 4 with minor defects and 2 with major defects. Two chips are selected randomly from the lot. What is the probability that at least on chip is good.

⇒ Solution

Given, Total no. of chips = 16.

~~Total~~ no. of good chip = 10

no. of minor defected chips = 4

no. of major defected chips = 2

no. of defected chips = 6

Selecting goods = 2.

~~no. of good~~

Exhaustive cases $(n) = {}^{16}C_2$

Let A = getting at least one good chip 1G1B or, 2G.

$(m) = {}^{10}C_1 \, {}^{6}C_1 + {}^{10}C_2$

$= 10 \times 6 + 45$

$= 105$

Q.1 From a group of 15 chess players, 8 are selected to represent a group at a convention. What is the probability that the selected include 3 of the 4 best players in the group?

Q.N.2) A bag contains 3 red and 2 green balls. Two balls are drawn at random. Find the probability that both balls have the same color.

Q.N.3) If a box contains 75 good IC chips and 25 defective chips, 12 chips are selected at random. What is the probability that at least one chip is defective.

Solution

Q.N.1

Given,

Total no. of chess players = 15

no. of best players = 4

no. of not-best players = 11

no. of players to be selected = 8

Now,

Out of 15 players, 8 can be selected in (n) = $^{15}C_8$ ways = 6435 ways

Let A = getting 3 out of 4 best players.

Favourable cases (m) = $^{4}C_{3} \cdot {}^{11}C_{5}$
$= 1848.$

$\therefore P(A) = \frac{m}{n} = \frac{1848}{6435}$

Q.N.2.

Given,

no. of red balls = 3

no. of green balls = 2

Total no. of balls = 5

no. of balls to be selected = 2

Out of 5 balls, 2 balls can be selected in (n) = $^{5}C_{2}$ ways = 10 ways.

Now,

Let A = Getting both balls of same color.

Favourable cases (m) = 2R OR, 2G
$= {}^{3}C_{2} + {}^{2}C_{2}$
$= 3 + 1$
$= 4.$

$\therefore P(A) = \frac{m}{n} = \frac{4}{10} = \frac{2}{5}$

Q.N.3

Solution

Given,

no. of good chips = 75

no. of defective chips = 25

Total no. of chips = 100

no. of chips to be selected = 12

Out of 100 chips, 12 can be selected in ${}^{100}C_{12}$ ways (n) = ${}^{100}C_{12}$

Plet A = getting at least 1 defective chips

Favourable cases (m) = ${}^{25}C_1 \cdot {}^{100}C_{11} + {}^{25}C_2 \cdot {}^{100}C_{10} + {}^{25}C_3 \cdot {}^{100}C_9 + {}^{25}C_4 \cdot {}^{100}C_8 + {}^{25}C_5 \cdot {}^{100}C_7 + {}^{25}C_6 \cdot {}^{100}C_6 + {}^{25}C_7 \cdot {}^{100}C_5 + {}^{25}C_8 \cdot {}^{100}C_4 + {}^{25}C_9 \cdot {}^{100}C_3 + {}^{25}C_{10} \cdot {}^{100}C_2 + {}^{25}C_{11} \cdot {}^{100}C_1 + {}^{25}C_{12}$

Let $\bar{A}$ = getting all good chip.

favourable cases (m) = ${}^{75}C_{12}$

$\therefore$ ~~Favoura~~

Probability of getting all good chips

$$P(\bar{A}) = \frac{{}^{75}C_{12}}{{}^{100}C_{12}}$$

∴ Probability of getting at least one defective chip $P(A) = 1 - P(\bar{A})$

$$= 1 - \frac{^{75}C_{12}}{^{100}C_{12}}$$

$$= 0.975$$

07/28 Wednesday

Laws of Probability

1) Additive Law
2) Multiplicative Law.

1) Additive Law

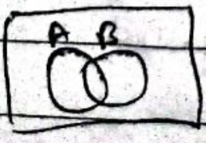
~~Law~~ Let A and B are not mutually exclusive events. Then, the probability of getting at least one of them (i.e A or B or Both) is given by
P( A or B)

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

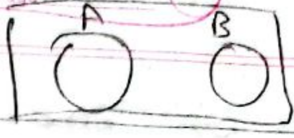
-lly,

For 3 not mutually exclusive events A, B and C.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$



* If mutually exclusive events,




$$P(A \cup B) = P(A) + P(B) \quad [\because P(A \cap B) = 0]$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \quad [\because P(A \cap B) = P(B \cap C) = P(A \cap C) = P(A \cap B \cap C) = 0]$$

Note:

a) Let A be any event, then the complimentary event of A is denoted by $\bar{A}$ or $A^c$ and its probability is given by:

$$P(A) = 1 - P(\bar{A})$$

or, $$P(\bar{A}) = 1 - P(A)$$

$$\therefore \quad P(A) + P(\bar{A}) = 1$$

b) Let A and B are two events. Then

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$P(A_{only})$

Also,

$P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$P(B_{only})$

Q) The probability that a new airport will get an award for its design and award for its efficient use of materials and both the awards is respectively 0.16, 0.24 and 0.11. Find probability that the new airport will get at least one award. Also find the probability that it will get only one of two awards.

⇒ Let,

A = getting award for design

B = getting award for material's efficiency

And,

$P(A) = 0.16$

$P(B) = 0.24$

$P(A \cap B) = 0.11$

a) Probability of getting at least one award.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.16 + 0.24 - 0.11$

$= 0.29$

b) Probability of getting only one award.

$P(A_0 \text{ or } B_0) = P(A_0) + P(B_0)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= 0.16 - 0.11 + 0.24 - 0.11$

$= 0.05 + 0.13$

$= 0.18$

Q) A construction company is bidding for two contracts A and B. The probability that the company will get A is $2/3$ and the probability that the company won't get B is $\frac{5}{9}$. If the probability of getting at least one contract is $4/5$, what is the probability that the company will get both contracts.

=)

Let,

A = getting contract A.

B = getting contract B.

And.

$P(A) = 2/3$

$P(\bar{B}) = 5/9$ $\quad P(B) = 1 - P(\bar{B}) = 4/9$

$P(A \cup B) = 4/5$

$P(A \cap B) = ?$

Then,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$

## Multiplicative law

1) Let A and B are two Independent Events. Then, the probability of getting both events (i.e. A and B) is given by

$$P(A \cap B) = P(A) \cdot P(B)$$

For 3 events (independent)

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$$

2) Let A and B are two dependent events. Then.

Read a B given A.

$$P(A \cap B) = P(A) \cdot P(B/A)$$
$$= P(B) \cdot P(A/B)$$

Now

$P(B/A)$ = conditional prob. of getting B given that A has already occurred

$P(A/B)$ = conditional prob. of getting A given that B has already occurred

Note:

a) Let A and B are two independent. Then, $\bar{A}$ & $\bar{B}$ are independent events

$\bar{A}$ & B " " "

A & $\bar{B}$ " " "

Imp

Q) A problem of statistics is given to 3 students A, B and C whose chances of solving it are $1/2$, $1/3$ and $1/4$ respectively if they solve independently. Find the probability that.

i) All of them can solve the problem.

ii) None of them can solve the problem.

iii) the problem will be solved (at least one can solve the problem)

iv) A can but B & C can't solve the problem.

v) Only one of them can solve the problem.

vi) A & B can solve but not C.

vii) Only two of them can solve the problem.

⇒ Solution

Let,

A = problem solved by A

B = problem solved by B

C = problem solved by C.

Then,

$P(A) = 1/2$ , $P(B) = 1/3$ , $P(C) = 1/4$.

$P(\bar{A}) = 1/2$ $P(\bar{B}) = 2/3$ $P(\bar{C}) = 3/4$.

Here, A, B, C are independent events.
Then probability that

i) All of them can solve

$\therefore P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$

$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}.$

ii) none of them can solve.

~~P(A~~ A can't & B can't & C can't.

$P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$

$P(\overline{A \cup B \cup C}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$

$= 1/4.$

iii) $\cancel{P(A_0) = P(A)}$ Problem will be solved.

$P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C})$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}.$

iv) A can, B and C can't

$P(A \cap \overline{B \cup C}) = P(A \cap \bar{B} \cap \bar{C})$

$= P(A) \cdot P(\bar{B}) \cdot P(\bar{C})$

v) only one of them can solved

$P(A_{only} \text{ or } B_{only} \text{ or } C_{only})$

$= P(A_{only}) + P(B_o) + P(C_o)$

$= P(A \cap \bar{B} \cap \bar{C}) + P(B \cap \bar{A} \cap \bar{C}) + P(C \cap \bar{A} \cap \bar{B})$

$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} + \frac{1}{3} \times \frac{1}{2} \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{2} \times \frac{2}{3}.$

vi) A & B but not C

$P(A \cap B \cap \bar{C}) = P(A) \cdot P(B) \cdot P(\bar{C})$

$= \frac{1}{2} \times \frac{1}{3} \times \frac{3}{4}$

$= \frac{1}{8}$

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Q) Three groups of children contain 3 girls and 2 boys, 2 girls and 2 boys, and 1 girl and 2 boys respectively. One child is selected from each group. Find the probability that the selected children will be 1 girl and two boys.

→)

| Group I | Group II | Group III |
|---|---|---|
| 3G and 2B | 2G and 2B | 1G and 2B |

One child is selected from each group (i.e. three children). Then 1G and 2B can be selected in following mutually exclusive ways.

| Cases | Group I | Group II | Group III |
|---|---|---|---|
| A | G | B | B |
| B | B | G | B |
| C | B | B | G |

Now,

$P(A) = P(G_1 \cap B_2 \cap B_3)$

$= P(G_1) \cdot P(B_2) \cdot P(B_3)$

$= \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3}$

$= \frac{1}{5}$

$P(B) = P(B_1 \cap G_2 \cap B_3)$

$= P(B_1) \cdot P(G_2) \cdot P(B_3)$

$= \frac{2}{5} \times \frac{2}{4} \cdot \frac{2}{3}$

$= \frac{2}{15}.$

$P(C) = P(B_1 \cap B_2 \cap G_3)$

$= P(B_1) \cdot P(B_2) \cdot P(G_3)$

$= \frac{2}{5} \times \frac{2}{4} \times \frac{1}{3}$

$= \frac{1}{15}$

Now,

Probability that selected children will be 1G and 2B is

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$= \frac{1}{5} + \frac{2}{15} + \frac{1}{15}$

$= \frac{3+2+1}{15}$

$= \frac{6}{15}$

$= \frac{2}{5}$

Q) The chances of wining two race horses are $\frac{1}{3}$ and $\frac{1}{6}$ resp. What is the probability that at least one will win when the horses are running.

i) in different races.

ii) in same race

Solution

Let A = winning by horse 1.

Let B = winning by horse 2.

$P(A) = 1/3 \qquad P(B) = 1/6.$

~~i) Proba~~

i) if horses are running in different races.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{3} + \frac{1}{6} - P(A) \cdot P(B)$$

$$= \frac{2+1}{6} - \frac{1}{3} \times \frac{1}{6}$$

$$= \frac{3}{6} - \frac{1}{18}$$

$$= \frac{1}{2} - \frac{1}{18}$$

$$= \frac{9-1}{18}$$

$$= \frac{8}{18}$$

For) A*...

$P(A_0) + P(B_0) + P(\bar{A} \cap \bar{B})$

ii) If horses are running in same race

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{3} + \frac{1}{6} - 0$

$= \frac{2+1}{6}$

$= \frac{3}{6}$

$= \frac{1}{2}.$

Q) The probability that campus A win a football match against campus B is 1/4. If campus A and campus B play 3 matches, what is the prob. that

1) campus A will lose all the 3 matches.
2) campus A will win at least 1 match.

Solution

Let, A = match win by campus A.

~~B = match win by campus B~~

~~P(A) = 1/4~~ , ~~$P(\bar{A}) = 3/4$.~~

~~P(B) =~~

P(A) = 1/4.

Solution

Let

A = campus A will win in 1st match

B = " " " " " 2nd match

C = " " " " " 3rd match

$P(A) = 1/4$ , $P(B) = 1/4$ , $P(C) = 1/4$.

$P(\bar{A}) = 3/4$ , $P(\bar{B}) = 3/4$ $P(\bar{C}) = 3/4$.

1) Probability that A will lose all 3 matches is. $P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\overline{A \cup B \cup C})$

$= P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$

$= \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$

2) Probability that A will win at least one match.

$\cancel{P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap A \cap \bar{B} \cap \bar{C})}$

$1 - P(\overline{A \cup B \cup C})$

$= 1 - \frac{27}{64}$

$= \frac{64-27}{64}$

$= \frac{37}{64}$.

Q) There are 3 switches in a college network namely A, B and C working independently. These switches are configured in series so that all switches should be ON to have successful transmission of data. The individual probability of being switched ON for these are $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{4}$ respectively. Find the probability that

i) There will be a successful data transfer.

ii) There won't be successful data transfer.

Answers.

Let A = A being switched ON.

B = B " " "

C = C " " "

$P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$, $P(C) = \frac{1}{4}$.

Probability that all switches are on.

i) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$= \frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{32}$.

ii) Probability that the data won't be successful data transfer. P

$P(\bar{A} \cup \bar{B} \cup \bar{C}) = 1 - P(A \cap B \cap C)$

$= 1 - \frac{3}{32} = \frac{32-3}{32} = \frac{29}{32}$.

Note:

i) If odds in favour of
Then, $P(A) = \frac{m}{m+n}$

ii) If odds against A be m:n. Then
$P(A) = \frac{n}{m+n}$.

Q) The odds against A solving a problem is 8:6 and odds in Favour of B solving the same problem is 14:10. What is the probabili that

a) both of them will solve the problem
b) at least one of them can solve the problem.
c) A solves it but not B.

Solution
Here,
Let A = problem solved by A
B = problem solved by B.

Odds against A is 8:6
$P(A) = \frac{6}{14}$, $P(\bar{A}) = \frac{8}{14}$.

odds in favour of B is 14:10
$\therefore P(B) = \frac{14}{24}$, $P(\bar{B}) = \frac{10}{24}$.

a) Probability that both will solve the problem is $P(A \cap B)$

$$= P(A) \cdot P(B)$$

$$= \frac{6}{14} \times \frac{14}{24}$$

$$= \frac{6}{24} = \frac{1}{4}$$

b) at least one of them can solve the problem is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{14} + \frac{14}{24} - \frac{1}{4} = \frac{6}{14} + \frac{14-6}{24}$$

$$= \frac{6}{14} + \frac{8}{24}$$

c) A solves but not B

$$P(A \cap \bar{B}) = P(A) \times P(\bar{B})$$

$$= \frac{6}{14} \times \frac{10}{24}$$

$$= \frac{60}{14 \times 24}$$

Q) The odds that A speaks the truth is 3:2 and the odds that B speaks the truth is 5:3. In what %age of cases are they likely to contradict each other on an identical points.

=) Solution

Let, A = truth spoken by A
B = truth spoken by B

$P(A) = \frac{3}{5}$ , $P(B) = \frac{5}{8}$.

$P(\bar{A}) = \frac{2}{5}$ $P(\bar{B}) = 3/8$

probability that they likely contradict each other is,

$P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= P(A) \cdot P(\bar{B}) + P(\bar{A}) \cdot P(B)$

$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8}$.

$= \frac{9}{40} + \frac{10}{40}$

$= \frac{19}{40}$

$\%age = \frac{19}{\cancel{40}_{2}} \times \cancel{100}^{5}\%$

$= \frac{19 \times 5}{2} \%$

$= 47.5\%$

Marginal Prob.

08/04 Tuesday

on shortnotes

# ~~Conditionality~~ Probability

Let A and B are two dependent events. Then, conditional probability of getting A given that B has already occurred is denoted by $P(A/B)$ and is calculated as:

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

provided that $P(B) \neq 0$

Similarly, the conditional probability of getting B given that A has already occurred is denoted by

$$P(B/A) = \frac{P(A\cap B)}{P(A)}$$

Provided that $P(A) \neq 0$

A manufacturer of airplane parts knows that the probability is 0.8 that an order will be ready for shipment on time, and it is 0.7 an order will be ready for shipment and will be delivered on time. What is the probability that such an order will be delivered on time given that it was also ready for shipment on time?

**Solution**

Let,

A = order will be ready for shipment

B = order will be ~~ready~~ ~~for~~ deliveryed on time.

$P(A) = 0.8$

$P(A \cap B) = 0.7$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.7}{0.8}$$

Q) A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good

⇒

Given,

no. of bad tubes = 4

no. of good tubes = 6.

no. of exhaustive cases = $^{10}C_2$

Let A = getting good tube in $1^{st}$ draw

B = getting good tube in $2^{nd}$ draw

$P(A) = 6/10$

~~$P(B) = 5/9$~~

Prob. that both tubes are good

$$P(A \cap B) = \frac{^{6}C_2}{^{10}C_2}$$

$$\therefore P(A \cap B) = \frac{5}{9}$$

Imp

# Bayes' Theorem

This is a method used for calculating the Posterior probabilities (Updated or Revised probabilities) of the prior probabilities based on the additional information.

Note: This is an extension of conditional probability

## Statement

Let $E_1, E_2, \ldots E_n$ be 'n' mutually exclusive events defined on sample space (S) with $P(E_i) \neq 0$, $i = 1, 2, 3, \ldots, n$

Also, for any orbitrary event A which is subset of sample space (S) such that $P(A) > 0$.

Then, the probability of getting $E_i$ given that event A has already occurred is calculated as:

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)}$$

where, $P(E_i)$ = Prior probabilities (Info. before random exp.)

$P(A/E_i)$ = Conditional prob (Additional info.)

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$P(E_i/A)$ = Posterior Prob.
(Updated or Revised).

Note: For two Events $E_1$ and $E_2$

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{\sum_{i=1}^{2} P(E_i) \cdot P(A/E_i)}$$

$$= \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

For $E_2$,

$$P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

Proof:

From the definition of conditional probability.

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)}$$

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{P(A)} \quad \text{—— (1)}$$

Since A is subset of sample space and the sample space is given by

$$S = E_1 \cup E_2 \cup E_3 \ldots \cup E_n$$

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Then, we can write,

$$A = A \cap S$$
$$= A \cap [E_1 \cup E_2 \cup \dots \cup E_n]$$

or, $A = (A \cap E_1) \cup (A \cap E_2) \cup \dots (A \cap E_n)$

[$\because$ Using Distributive law]

Now, taking probability on both sides we get,

$$P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n)]$$

Since $E_1, E_2, E_3, \dots, E_n$ are mutually enclusive. Then $(A \cap E_1), (A \cap E_2), \dots (A \cap E_n)$ are also mutually enclusive

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n)$$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

$$\therefore \boxed{P(A) = \sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)} \quad \text{--- (ii)}$$

Theorem of Total

From (i) and (ii)

$$\boxed{P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}}$$

Q. In a bolt factory machine A, B, C manufacture respectively 25%, 35% and 40% of the total production. and out of their outputs 1%, 3% and 2% are defective bolts. If a bolt is selected from the total production at random and is found to be defective. What is the probability that it was manufactured by

i) machine A
ii) machine B
iii) machine C

⇒

Solution

let,

$E_1$ = bolt produced from machine A
$E_2$ = " " " " B
$E_3$ = " " " " C

We know,

$P(E_1) = 0.25$
$P(E_2) = 0.35$
$P(E_3) = 0.4$

let A = bolt is defective

Then,

$P(A/E_1) = 0.01$

Prob. that bolt is defective given that bolt is produced from machine A.

$P(A/E_2) = 0.03$

$P(A/E_3) = 0.02$

Now,

Probability that the bolt is defective

$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)$

$= 0.25 \times 0.01 + 0.35 \times 0.03 + 0.4 \times 0.02$

$= 0.021.$

Then,

~~i) $P(A/E_1)$~~ Probability that bolt is produced from a given that bolt is defective

i) $P(E_1/A) = \dfrac{P(E_1) \cdot P(A/E_1)}{P(A)}$

$= \dfrac{0.25 \times 0.01}{0.021}$

$= 0.12$

ii) $P(E_2/A) = \dfrac{P(E_2) \cdot P(A/E_2)}{P(A)}$

$= \dfrac{0.35 \times 0.03}{0.021}$

$= 0.5$

iii) $P(E_3/A) = \dfrac{P(E_3) \cdot P(A/E_3)}{P(A)}$

$= \dfrac{0.4 \times 0.02}{0.021} = 0.38$

Q) A manufacturing company produces steel pipes in 3 plants with daily production volume of 500, 1000 and 2000 units resp. According to the past experience it is known that the fraction of defective outputs produced by 3 plants are resp. 0.008, 0.01, and 0.005. If a pipe is selected from the daily total production and found to be defective. From which plant the defective pipe be expected to have been produced.

⇒ Solution

let $E_1$ = pipe produced from A.
$E_2$ = pipe produced from B
$E_3$ = pipe produced from C.

$$P(E_1) = \frac{500}{3500} = \frac{1}{7} = 0.14$$

$$P(E_2) = \frac{1000}{3500} = 0.28$$

$$P(E_3) = \frac{2000}{3500} = 0.57$$

let,

A = pipe is defective.

$P(A/E_1) = 0.008$

$P(A/E_2) = 0.01$

$P(A/E_3) = 0.005$

Now,

Probability that the selected pipe is defective is

$$P(A) = \sum_{i=1}^{3} P(E_i) \cdot P(A/E_i)$$

$$= 0.14 \times 0.008 + 0.78 \times 0.01 + 0.57 \times 0.005$$

$$= 6.77 \times 10^{-3}$$

Then,

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(A)}$$

$$= \frac{0.14 \times 0.008}{6.77 \times 10^{-3}}$$

$$= 0.16$$

$$P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{P(A)}$$

$$= \frac{0.78 \times 0.01}{6.77 \times 10^{-3}} = 0.41$$

$$P(E_3/A) = \frac{0.57 \times 0.005}{6.77 \times 10^{-3}}$$

$$= 0.42$$

$\therefore$ The probability of getting the pipe

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from plant C given that it is defective
is greater

Q) Coustomers are used to evaluate preliminary product design. In the past, 95% of highly successful products received good reviews, 60% of moderately success-ful product received good reviews & 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful and 25% have been poor products.

i) What is the probability that a product attains good review?

ii) If a new design attains a good review what is the probability that it will be a highly successful product?

⇒ <u>Solution</u>

let,

$E_1$ = Product will be highly successful

$E_2$ = Product will be moderately "

$E_3$ = " " " poor.

$P(E1) = 0.4$

$P(E2) = 0.35$

$P(E3) = 0.25.$

Let, A = product will receive good review.

$P(A/E1) = 0.95$

$P(A/E2) = 0.6$

$P(A/E3) = 0.1$

i) Probability that product attains good review $P(A) = P(E1) \cdot P(A/E1) + P(E2) \cdot P(A/E2) + P(E3) \cdot P(A/E3)$

$= 0.4 \times 0.95 + 0.35 \times 0.6 + 0.25 \times 0.1$

$= 0.615$

ii) $P(E1/A) = \dfrac{P(E1) \cdot P(A/E1)}{P(A)}$

$= \dfrac{0.4 \times 0.95}{0.615}$

$\doteq 0.61$

Q) In a class of 75 students, 15 stds. were considered to be very intelligent, 45 as medium and rest were below the average. The prob. that a very intelligent student failing the exam is 0.005, the medium std failing has prob 0.05 and corresponding prob. for a below avg. is 0.15. If a student is known to have passed the exam, what is the prob. that he is below the av.

⇒ Solution

let,

$E1$ = student is intelligent

$E2$ = student is medium

$E3$ = student is below avg.

$P(E1) = \frac{15}{75} = \frac{1}{5}$ $P(E3) = \frac{15}{75} = \frac{1}{5}$

$P(E2) = \frac{45}{75} = \frac{3}{5}$

let,

A = student fail the exam.

$P(A/E1) = 0.005$

$P(A/E2) = 0.05$

$P(A/E3) = 0.15$

$$P(E_3/\bar{A}) = \frac{P(E_3)\cdot P(A/E_3)}{P(E_1)\cdot P(\bar{A}/E_1) + P(E_2)\cdot P(\bar{A}/E_2) + P(E_3)\cdot P(\bar{A}/E_3)}$$

$$\cancel{= \frac{\frac{1}{5} \times (1 - 0.15)}{\frac{1}{5} \times 0.005 + \frac{3}{5} \times 0.05 +}}$$

$$= \frac{\frac{1}{5} \times (1 - 0.15)}{\frac{1}{5}(1 - 0.005) + \frac{3}{5} \times (1 - 0.05) + \frac{1}{5}(1 - 0.15}$$

$$= \frac{\frac{1}{5} \times 0.85}{\frac{1}{5} \times 0.995 + \frac{3}{5} \times 0.95 + \frac{1}{5} \times 0.85}$$

$$= 0.18$$

Exm

Q) A given lot of IC chips contains 2% defective chips. Each chip is tested before delivering. The tester itself is not totally reliable. Probability of tester says the chip is good when it is really good is 0.95 and the chip is defective when it is actually defective is 0.94. If a tested device is indicated to be defective, what is the prob. that actually defective?

⇒ solution

let,

$E1$ = chip is defective

$E2$ = chip is good.

$\therefore P(E1) = 0.02$

$P(E2) = 0.98$

Now,

let A = tester says chip is defective.

$P(\bar{A}/E2) = 0.95 \Rightarrow P(A/E2) = 0.05$

$P(A/E1) = 0.94.$

$P(E1/A) = ?$

Then,

$$P(A) = \sum_{i=1}^{2} P(E_i) \cdot P(A/E_i)$$

$$= P(E_1) \cdot P(A/E_1) + P(E2) \cdot P(A/E2)$$

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$= 0.02 \times 0.94 + 0.98 \times 0.05$

$= 0.0678$

Again,

$$P(E1/A) = \frac{P(E1) \cdot P(A/E1)}{P(A)}$$

$$= \frac{0.02 \times 0.94}{0.0678}$$

$= 0.277$

08/07 Friday

Imp. The contents of urns I, II and III are as follows: 2 White, 1 Black and 3 Red balls
1 W, 1 B and 2 R
4 W, 3 B and 5 R

one urn is selected at random and two balls are drawn. They are found to be white and red. What is the probability that they come from
i) urn I
ii) urn II
iii) urn III

⇒ let,

$E1$ = selecting urn I
$E2$ = selecting urn II
$E3$ = selecting urn III

Then,

$P(E1) = 1/3$
$P(E2) = 1/3$
$P(E3) = 1/3$

Now,

let A = getting a white and a red ball.

$$P(A/E1) = \frac{{}^2C_1 \cdot {}^3C_1}{{}^6C_2} = \frac{2}{5}$$

$$P(A/E2) = \frac{{}^1C_1 \cdot {}^2C_1}{{}^4C_1} = \frac{1}{3}$$

$$P(A/E3) = \frac{{}^4C_1 \cdot {}^5C_1}{{}^{12}C_2} = \frac{10}{33}$$

Now,

$$P(A) = P(E1) \cdot P(A/E1) + P(E2) \cdot P(A/E2) + P(E3) \cdot P(A/E3)$$

$$= \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{10}{33}$$

$$= \frac{2}{15} + \frac{1}{9} + \frac{10}{99} = \frac{19}{55}$$

Probability that they come from urn I is.

$$P(E1/A) = \frac{P(E1) \cdot P(A/E1)}{P(A)} = \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{19}{55}} = \frac{22}{57}$$

Probability that two balls are selected from urn II is.

$$P(E2/A) = \frac{P(E2) \cdot P(A/E2)}{P(A)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{19}{55}} = \frac{55}{171}$$

Probability that two balls are selected from urn III is

$$P(E3/A) = \frac{\frac{1}{3} \times \frac{10}{33}}{\frac{19}{55}} = \frac{50}{171}$$

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Q. The chances of A, B and C becoming a chief engineer of a certain hydropower company are 4:2:3. The probabilities that the new transmission line will be introduced if A, B and C becomes chief engineer are 0.3, 0.5, 0.8 resp. If the new transmission line has been introduced, what is the prob. is that

i) A is appointed as chief Engineer
ii) B " " " " "
iii) C " " " " "

Solution

Let,

$E_1$ = A become chief engineer
$E_2$ = B " " "
$E_3$ = C " " "

Now,

$P(E_1) = \frac{4}{9}$, $P(E_2) = \frac{2}{9}$, $P(E_3) = \frac{3}{9}$

Let, A = transmission line has been introduced.

$P(A/E_1) = 0.3$
$P(A/E_2) = 0.5$
$P(A/E_3) = 0.8$

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$$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)$$

$$= \frac{4}{9} \times 0.3 + \frac{2}{9} \times 0.5 + \frac{3}{9} \times 0.8$$

$$= \frac{23}{45}$$

Now,

i) $P(E_1/A) = \dfrac{P(E_1) \cdot P(A/E_1)}{P(A)}$

$$= \frac{4/9 \times 0.3}{\frac{23}{45}}$$

$$= 6/23$$

ii) $P(E_2/A) = \dfrac{P(E_2) \cdot P(A/E_2)}{P(A)}$

$$= \frac{2/9 \times 0.5}{23/45}$$

$$= 5/23$$

iii) $P(E_3/A) = \dfrac{3/9 \times 0.8}{23/45}$

$$= \frac{12}{23}$$

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# Random Variable

Random variable (R.V) is a real value function associated with each outcomes of a random experiment.

In other words, R.V is a function defined on sample space of a random experiment.

OR,

It can ~~be~~ be any values defined on sample space.

Usually, random variables are denoted by capital letters $\overset{X}{A}, \overset{Y}{B}, \overset{Z}{C}$, etc and the value taken by r.a is denoted by small letters $\overset{x}{a}, \overset{y}{b}, \overset{z}{c}$, etc.

Example:

Consider a random experiment of tossing coin three times.

$S_1 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

let $X$ = No. of heads.

Then, $X$ can take values 0, 1, 2 and 3.

$\therefore S_2 = \{0, 1, 2, 3\}$

Als,

$P(X=0) = 1/8$ $\quad$ $P(X=2) = 3/8$

$P(X=1) = 3/8$ $\quad$ $P(X=3) = 1/8$

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Properties of Random Variable

1) If $X$ is random variable, then $aX$ and $aX+b$ are also considered as random variables

2) If $X$ and $Y$ are two random variables, then $X+Y$, $X-Y$, $X*Y$, $X/Y$ are also random variables

3) If $X$ is random variable, then $X^{-1}$, $X^2$, $X^3$, ... etc are also random variables.

Types of Random Variable

i) Discrete Random Variable

ii) Continuous Random Variable.

1) Discrete Random Variable

A random variable is said to be a discrete random variable if it takes only integers (i.e countable values)

Example
a) no. of defective items in a lot.
b) no. of printing mistakes in a page
c) no. of students enrolled every year in NCIT.

ii **Continuous Random Variable**

A random variable is said to be a continuous random variable if it takes all the possible values within a certain range or interval.

e.g.

1) Age of individuals.
ii) Temp. recorded
iii) Height of students.

**Probability Distribution**

The listing of probabilities of all possible values of a random variable is called probability distribution.

In other words, it is a mathematical / probability function which is used to calculate the probabilities of all possible values of random variable.

Types of Prob. Distribution

i) ~~Random~~ Continuous ~~Variab~~ Prob. Distribution
ii) Discrete Prob. Distribution.

Tuesday

Date
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i) Discrete Probability Distribution
The probability distribution of discrete random variable is known as discrete prob. distribution.

Also known as, probability $f^n$ of discrete R.V

Let $X$ be discrete random variable with probability function $P(x)$ or $P(X=x)$. Then the probability function $P(X=x)$ is said to be probability mass function (PMF) which satisfies the following conditions:

i). $P(X=x) \geq 0 \; ; \forall x$ (Non-negative)

ii) $\sum_{-\infty}^{\infty} P(X=x) = 1$

$\Rightarrow$ Total prob. 1.

Example: Let $X$ be a discrete R.V. with following prob. function.

| $X=x$ | $P(X=x)$ |
|---|---|
| -2 | 0.1 |
| -1 | k |
| 0 | 0.2 |
| 1 | 2k |
| 2 | 0.3 |
| 3 | k. |

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We know that,

Total prob. = 1

or, $0.1 + K + 0.2 + 2K + 0.3 + K = 1$

or, $4K + 0.6 = 1$

or, $4K = 0.4$

or, $K = \frac{4}{40}$

$\therefore K = 0.1$

Therefore, the given prob. $f^n$ becomes

| $X = x$ | $P(X = x)$ |
|---|---|
| -2 | 0.1 |
| -1 | 0.1 |
| 0 | 0.2 |
| 1 | 0.2 |
| 2 | 0.3 |
| 3 | 0.1 |

**Distribution Function of Discrete R.V**

Let X be a discrete R.V. with prob. mass $f^n$ $P(X=x)$ ; $\forall x$ then, the ~~pr.~~ distribution $f^n$ of X is denoted by $F(x)$ and is defined as

$$F(x) = P(X \le x) = \sum_{-\infty}^{x} P(X = x)$$

Also, known as cumulative distribution $f^n$ (C.D.F)

Properties

i) $0 \leq F(x) \leq 1, \forall x$

ii) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

iii) $F(\infty) = \lim_{x \to +\infty} F(x) = 1$

iv) $P(a < x \leq b) = F(b) - F(a)$..

v) $F(x) > F(y)$ if $x > y$.

aa)

Q) Let X be a discrete R-V with following prob. function.

| $X = x$ | $P(X=x)$ |
|---|---|
| -2 | 0.1 |
| -1 | 0.1 |
| 0 | 0.2 |
| 1 | 0.2 |
| 2 | 0.3 |
| 3 | 0.1 |

i) Find $P(X \leq 1)$

ii) Find $P(-2 \leq x \leq 2)$ or. $P|x| < 2$

iii) Find $P(X \geq 2)$

iv) Find $P(-2 < x < 2 / X \leq 1)$.

v) Find $F(1)$ and $F(2)$

vi) Find Distribution function of $x$.

Find CDF of $x$.

Answers

i) $P(X \leq 1) \Rightarrow P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$

$= 0.1 + 0.1 + 0.2 + 0.2$

$= 0.6$

ii) $P(-2 \leq X < 2) \Rightarrow P(X=-1) + P(X=0) + P(X=1)$

$= 0.1 + 0.2 + 0.2$

$= 0.5$

iii) $P(X \geq 2) \Rightarrow P(X=2) + P(X=3)$

$= 0.3 + 0.1$

$= 0.4$

iv) $P(-2 < X < 2 / X \leq 1)$

$$= \frac{P(-2 < X < 2 \cap X \leq 1)}{P(X \leq 1)}$$

$$= \frac{P(-2 \leq X < 2)}{P(X \leq 1)}$$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

v) $F(1) = P(X \leq 1)$

$= P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$

$= 0.1 + 0.1 + 0.2 + 0.2$

$= 0.6$

$$F(2) = P(X \leq 2)$$
$$= P(X \leq 1) + P(X = 2)$$
$$= 0.6 + 0.3$$
$$= 0.9$$

vi) The distribution $f^n$ of $X$ is given by

| $X = x$ | $P(X = x)$ | $F(x)$ |
|---|---|---|
| -2 | 0.1 | 0.1 |
| -1 | 0.1 | 0.2 |
| 0 | 0.2 | 0.4 |
| 1 | 0.2 | 0.6 |
| 2 | 0.3 | 0.9 |
| 3 | 0.1 | 1 |

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bb)

Q) A discrete random variable X has the following probability mass function

| $X=x$ | $P(X=x)$ |
|---|---|
| 1 | $K$ |
| 2 | $2K$ |
| 3 | $3K$ |
| 4 | $4K$ |
| 5 | $3K$ |
| 6 | $2K$ |
| 7 | $1K$ |

i) Find k.

ii) ~~the~~ $P(X<2)$

iii) $P(X \geq 2)$

iv) $P(0<x<4)$

v) $P(0<x<4 / x \geq 2)$

vi) Find CDF of x.

Solution

i) We know that.

$$\sum_{1}^{7} P(X=x) = 1$$

or, $k+2k+3k+4k+3k+2k+k=1$

or, $16k=1$

$\therefore k = \frac{1}{16}$.

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vi. CDF of X

| $X = x$ | $P(X=x)$ | $F(X)$ | |
|---|---|---|---|
| 1 | $\frac{1}{16}$ | ~~1/16~~ | $1/16$ |
| 2 | $\frac{2}{16}$ | ~~3/16~~ | $3/16$ |
| 3 | $\frac{3}{16}$ | ~~9/16~~ | $6/16$ |
| 4 | $4/16$ | ~~13/16~~ | $10/16$ |
| 5 | $3/16$ | ~~16/16~~ | $13/16$ |
| 6 | $2/16$ | ~~18/16~~ | $15/16$ |
| 7 | $1/16$ | ~~19/16~~ | $1$ |

ii $P(x<2) = P(1)$

$= 1/16.$

iii) $P(X \geq 2) = 1 - \frac{1}{16} = \frac{15}{16}.$

iii) $P(0 \leq X < 4) = P(1) + P(2) + P(3)$

$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16}$

$= \frac{6}{16}.$

iv) $$P(0 < X < 4 / X \geq 2) = \frac{P(0<X<4 \cap X \geq 2)}{P(x \geq 2)}$$

$$= \frac{P(2 \leq X \leq 3)}{P(X \geq 2)}$$

$$= \frac{\frac{2}{16} + \frac{3}{16}}{\frac{15}{16}}$$

$$= \frac{5}{15} = \frac{1}{3}$$

H/W

(c) A random variable $X$ has the following prob. distribution.

| $X=x$ | $P(X=x)$ |
|---|---|
| 0 | $k$ |
| 1 | $3k$ |
| 2 | $2k$ |
| 3 | $5k$ |
| 4 | $7k$ |
| 5 | $4k$ |
| 6 | $8k$. |

i) Find $k$.

ii) $P(X>3)$, $P(X\leq 4)$, $P(1<X<5)$

iii) Find DF of $X$.

Solution,

i) We know that

$$\sum_{0}^{6} P(X=x) = 1$$

or, $k+3k+2k+5k+7k+4k+8k = 1$

or, $30k = 1$

$\therefore$ $k = 1/30$

(ii)

iii) CDF of X

| $X = x$ | $P(X=x)$ | $F(x)$ |
|---|---|---|
| 0 | $\frac{1}{30}$ | $1/30$ |
| 1 | $\frac{3}{30}$ | $4/30$ |
| 2 | $2/30$ | $6/30$ |
| 3 | $\frac{5}{30}$ | $11/30$ |
| 4 | $7/30$ | $18/30$ |
| 5 | $4/30$ | $22/30$ |
| 6 | $8/30$ | 1 |

ii) a) $P(X>3) \Rightarrow P(X=4) + P(X=5) + P(X=6)$

$= \frac{7}{30} + \frac{4}{30} + \frac{8}{30}$

$= \frac{19}{30}$

b) $P(X \leq 4) \Rightarrow P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{1}{30} + \frac{3}{30} + \frac{2}{30} + \frac{5}{30} + \frac{7}{30}$

$= 18/30$

c) $P(1 < X < 5) \Rightarrow P(X=2) + P(X=3) + P(X=4)$

$= \frac{2}{30} + \frac{5}{30} + \frac{7}{30}$

$= \frac{14}{30}$

08/12 Wednesday

Continuous Probability Distribution

→ Prob. Distribution of continuous random variable.

→ Prob. function of continuous R.V.

→ Prob. density $f^n$

Let $x$ be a continuous random variable with prob. $f^n$ $f(x)$, then the prob. $f^n$ $f(x)$ is said to be Prob. density function which satisfies the following conditions:

i) $f(x) \geq 0, \forall x$ (Non-negative)

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (Total prob = 1).

iii) For any arbitrary event $E$ which represents that $X$ falls in an interval $(a, b)$.

Then,

$$P(E) = P(a < x < b)$$
$$= \int_a^b f(x)\,dx.$$

## Distribution function of continuous r.v

Let X be a continuous random variable with probability density function (pdf) $f(x)$.

Then, the distribution function of X is denoted by $F(x)$ and is defined as:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx.$$

Also known as cumulative distribution function.

## Properties of CDF:

i) $0 \leq F(x) \leq 1$

ii) $F'(x) = f(x)$

$\Rightarrow \frac{d F(x)}{dx} = f(x)$

iii) $F(-\infty) = 0$

iv) $F(\infty) = 1$

v) $P(a < x < b) = P(a \leq x \leq b)$

$P(a \leq x < b) = P(a < x \leq b)$

$= F(b) - F(a)$

vi) $F(x) > F(y)$ if $x > y$.

a) A continuous random variable $X$ has a probability density function $f(x) = K(1+x)$. The range of $x$ is $2 \le x \le 5$

$$f(x) = K(1+x) \quad ; \quad 2 \le x \le 5$$
$$= 0 \quad ; \quad \text{otherwise.}$$

i) Find K.

ii) Find $P(x \le 4)$

iii) Find d.f of X

⇒ Solution

Given pdf is:

$$f(x) = K(1+x) \quad ; \quad 2 \le x \le 5$$

since, f(x) is pdf the total probability is 1. Therefore,

$$\int_2^5 f(x)\, dx = 1.$$

$$\text{or, } \int_2^5 K(1+x)\, dx = 1$$

$$\text{or, } K\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$\text{or, } K\left[5 + \frac{25}{2} - 2 - 2\right] = 1$$

$$\text{or, } K\left[\frac{35}{2} - 4\right] = 1$$

$$\text{or, } K \times \frac{35-8}{2} = 1$$

$$\text{or, } K = \frac{2}{27}$$

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$\therefore$ p.d.f $\Rightarrow$ $f(x) = \frac{2}{27}(1+x)$

ii) For $P(x \le 4)$,

$$\int_2^4 f(x)\,dx = \int_2^4 \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4.$$

$$= \frac{2}{27}[4 + 8 - 2 - 2]$$

$$= \frac{2}{27} \times 8$$

$$\therefore P(x \le 4) = \frac{16}{27}$$

For, $P(x \ge 3)$,

$$\int_3^5 f(x)\,dx = \frac{2}{27}\int_3^5 (1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_3^5$$

$$= \frac{2}{27}\left[5 + \frac{25}{2} - 3 - \frac{9}{2}\right]$$

$$= \frac{2}{27}\left[\frac{35}{2} + \frac{-6-9}{2}\right]$$

$$= \frac{2}{27}\left[\frac{35-15}{2}\right]$$

$$= \frac{2}{27} \times \frac{20}{2}$$

$$= 20/27$$

Distribution function of x,

$$F(x) = \int_2^x f(x)\, dx$$

$$= \int_2^x \frac{2}{27} (1+x)\, dx$$

$$= \frac{2}{27} \left[ x + \frac{x^2}{2} \right]_2^x$$

$$= \frac{2}{27} \left[ x + \frac{x^2}{2} - 2 - 2 \right]$$

$$= \frac{2}{27} \left[ \frac{2x + x^2}{2} - 4 \right]$$

$$= \frac{2x + x^2 - 8}{27}.$$

b)

Q) Let $X$ be a continuous random variable with density function $f(x) = \frac{1}{4}$ ; $-2 < x < 2$

$= 0$ ; otherwise

i) Find $P(x < -1)$

ii) Find $P(|x| < 1)$

iii) Find $P(2x + 3 > 5)$

i) For $P(x < -1)$;

$$\int_{-2}^{-1} f(x)\, dx = \int_{-2}^{-1} \frac{1}{4}\, dx$$

$$= \frac{1}{4} [x]_{-2}^{-1}$$

$$= \frac{1}{4} [-1 - (-2)]$$

$$= \frac{1}{4} [-1 + 2] = \frac{1}{4}.$$

ii) Find $P(-1 < x < 1)$

Imp

c) Q. A random variable X has a probability density function;

$$p(x) = ax; \quad 0 \le x < 1$$
$$= a; \quad 1 \le x < 2$$
$$= -ax + 3a; \quad 2 \le x \le 3$$
$$= 0; \quad \text{otherwise}$$

i) Find a.

ii) Find ~~P( x ≤ 1/6)~~ $P(x \le 1.5)$

⇒ Solution

Given p.d.f. is.

$$f(x) = ax; \quad 0 \le x \le 1$$
$$= a; \quad 1 \le x \le 2$$
$$= -ax + 3a; \quad 2 \le x \le 3$$
$$= 0; \quad \text{otherwise}$$

i) we know,

$$\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = 1$$

or, $\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax + 3a)\,dx = 1$

or, $\frac{a}{2}[x^2]_0^1 + a[x]_1^2 - \frac{a}{2}[x^2]_2^3 + 3a[x]_2^3$

or, $\frac{a}{2} \times 1 + a - \frac{a}{2}[9 - 4] + 3a = 1$

or, $\frac{a}{2} + a - \frac{5a}{2} + 3a = 1$

or, $\frac{3a}{2} + \frac{a}{2} = 1$

or, $4a = 2$

$\therefore a = 1/2$

$\therefore$ p.d.f is;

$$
\begin{aligned}
f(x) &= \frac{x}{2} ; \quad 0 \le x \le 1 \\
&= \frac{1}{2} ; \quad 1 \le x \le 2 \\
&= -\frac{x}{2} + \frac{3}{2} ; \quad 2 \le x \le 3 \\
&= 0 ; \quad \text{otherwise.}
\end{aligned}
$$

ii) $P(x \le 1.5)$

$$\Rightarrow \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx$$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^{1.5} \frac{1}{2}\,dx$$

$$= \frac{1}{4}[x^2]_0^1 + \frac{1}{2}[x]_1^{1.5}$$

$$= \frac{1}{4} + \frac{1}{2}\left[\frac{3}{2} - 1\right]$$

$$= \frac{1}{4} + \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}.$$

Imp. Mathematical expectation of Random variable | Mean value of R.V.
The expected val. of R.V.

let X be the Random variable. Then, the expectation of X is the mean (or expected value) of X and is denoted by E(x).

Formula

i) If X is discrete R.V. Then,

$$\boxed{E(x) = \sum_{x} x \cdot P(X=x)}$$

where,

P(X=x) = Prob. Mass $f^n$

ii) If X is cont. R.V. Then

$$\boxed{E(x) = \int_{x} x \cdot f(x)\, dx}$$

where, f(x) = prob. density. $f^n$.

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HW

Continuous random variable

Q.a) Solution

Given,

$$f(x) = \frac{2}{27}(1+x) \quad ; \quad 2 \le x \le 5$$

Now, Mean value of Random variable is:

$$E(x) = \cancel{\sum_x x \cdot P(}$$

$$= \int_x x \cdot f(x)\, dx$$

$$= \int_2^5 x \cdot \frac{2}{27}(1+x)\, dx$$

$$= \frac{2}{27}\int_2^5 (x+x^2)\, dx$$

$$= \frac{2}{27}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_2^5$$

$$= \frac{2}{27}\left[\frac{25}{2}+\frac{125}{3}-\frac{4}{2}-\frac{8}{3}\right]$$

$$= \frac{11}{3}.$$

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Q.b) Solution

Given,

$$f(x) = \frac{1}{4}$$

$$-2 < x < 2.$$

$$E(x) = \int_{-2}^{2} x \cdot f(x)\, dx$$

$$= \frac{1}{4}\int_{-2}^{2} x\, dx$$

$$= \frac{1}{4} \times \left[\frac{x^2}{2}\right]_{-2}^{2}$$

$$= \frac{1}{8}\left[4 - 4\right]$$

$$= 0$$

Q.c) Solution

Given,

$$f(x) = \frac{x}{2}; \qquad 0 \leq x \leq 1$$

$$= \frac{1}{2} \qquad 1 \leq x \leq 2$$

$$= -\frac{x}{2} + \frac{3}{2}; \qquad 2 \leq x \leq 3.$$

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Now,

$$E(x) = \int_0^3 x \cdot f(x)\, dx$$

$$= \int_0^1 x \cdot f(x)\, dx + \int_1^2 x \cdot f(x)\, dx$$

$$+ \int_2^3 x \cdot f(x)\, dx$$

$$= \int_0^1 x \cdot \frac{x}{2}\, dx + \int_1^2 \frac{x}{2}\, dx + \int_2^3 x\left(\frac{3}{2} - \frac{x}{2}\right)$$

$$= \frac{1}{2}\int_0^1 x^2\, dx + \frac{1}{2}\int_1^2 x\, dx + \int_2^3 \frac{3}{2}x\, dx - \frac{1}{2}\int_2^3 x^2 dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2} \times \left[\frac{x^2}{2}\right]_1^2 + \frac{3}{2} \times \left[\frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^3}{3}\right]_2^3$$

$$= \frac{1}{2} \times \frac{1}{3} + \frac{1}{4}[4-1] + \frac{3}{4}[9-4] - \frac{1}{6}[27-8]$$

$$= \frac{1}{6} + \frac{3}{4} + \frac{3}{4} \times 5 - \frac{1}{6} \times 19$$

$$= \frac{1}{6} + \frac{3+15}{4} - \frac{19}{6}$$

$$= \frac{1}{6} + \frac{18}{4} - \frac{19}{6}$$

$$= \frac{1-19}{6} + \frac{18}{4}$$

$$= \frac{-18}{6} + \frac{18}{4} = -3 + \frac{9}{2} = \frac{-6+9}{2} = \frac{3}{2}$$

$=$

# Discrete Random variable

a)

| $X=x$ | $P(X=x)$ |
|---|---|
| $-2$ | $0.1$ |
| $-1$ | $0.1$ |
| $0$ | $0.2$ |
| $1$ | $0.2$ |
| $2$ | $0.3$ |
| $3$ | $0.1$ |

$$E(x) = \sum_{x} x \cdot P(X=x)$$

$$= \sum_{-2}^{3} x \cdot P(X=x)$$

$$= -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3$$

$$= -0.3 + 1.1$$

$$= +0.8$$

b)

| $X=x$ | $P(X=x)$ |
|---|---|
| 1 | $\frac{1}{16}$ |
| 2 | $\frac{2}{16}$ |
| 3 | $\frac{3}{16}$ |
| 4 | $4/16$ |
| 5 | $3/16$ |
| 6 | $2/16$ |
| 7 | [illegible] |

$$E(x) = \sum_{x} x \cdot P(X=x)$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16} + \frac{15}{16} + \frac{12}{16} + \frac{7}{16}$$

$$= \frac{64}{16} = 4$$

(c)

| $X=x$ | $P(X=x)$ |
|---|---|
| 0 | $\frac{1}{30}$ |
| 1 | $\frac{3}{30}$ |
| 2 | $\frac{2}{30}$ |
| 3 | $\frac{5}{30}$ |
| 4 | $7/30$ |
| 5 | $4/30$ |
| 6 | $8/30$ |

$$E(x) = \sum_{x} x \cdot P(X=x)$$

$$= 0 + \frac{3}{30} + \frac{4}{30} + \frac{15}{30} + \frac{28}{30} + \frac{20}{30} + \frac{48}{30}$$

$$= \frac{118}{30}$$

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08/17 Monday

Bivariate Distribution:
Joint Probability Distribution

The probability distribution of two dimensional random variables, (x, y). is called bivariate distribution or joint probability distribution.

Example: distribution of two random variables which are associated with the heights and the weights of the group.

Note: To study Bivariate Distribution There must be the both variables of same kind.
i.e. either both are discrete R.V. or both are continuous R.V.

## Joint Probability Distribution of Discrete Random Variables.

let X and Y are two discrete Random variables with joint probability function $P(X=x, Y=y)$ or $P(X=x \cap Y=y)$

Then,

the joint prob. function $P(X=x, Y=y)$ is said to be a joint prob. mass function which satisfies the following conditions:

i) $P(X=x, Y=y) \geq 0; \ \forall x$ and $y$

[Non-negative]

ii) $\sum_{x=-\infty}^{\infty} \sum_{y=\infty}^{\infty} P(X=x, Y=y) = 1$

Example: let X and Y be two discrete random variables with joint probability distribution

| Y \ X | 2 | 3 |
|---|---|---|
| 1 | 0.1 | 0.15 |
| 2 | 0.2 | 0.3 |
| 3 | 0.1 | 0.15. |

$P(X=2, Y=1) = 0.1$, $P(X=3, Y=1) = 0.15$

$P(X=2, Y=2) = 0.2$, $P(X=3, Y=2) = 0.3$

$P(X=2, Y=3) = 0.1$, $P(X=3, Y=3) = 0.15$

## Marginal Distribution / Marginal Prob. Distribution.

Let X and Y are two discrete random variable with joint pmf $P(X=x, Y=y)$; $\forall x$ and $y$

Then,
the marginal distribution of X is denoted by $P(X=x)$ or $P(x)$ and is calculated as

$$P(X=x) = \sum_{y} P(X=x, Y=y)$$

Similarly, the marginal distribution of Y is denoted by $P(Y=y)$ or $P(y)$ and is calculated as

$$P(Y=y) = \sum_{x} P(X=x, Y=y)$$

Q) Let X and Y be two discrete random variables with joint probability distribution

| Y \ X | 2 | 3 |
|---|---|---|
| 1 | 0.1 | 0.15 |
| 2 | 0.2 | 0.3 |
| 3 | 0.1 | 0.15 |

Q) Find the marginal distribution of X and Y.

⇒

Marginal Distribution of X

$P(X=x) = \sum_{y} P(X=x, Y=y)$

For $X=2$,

$P(X=2) = \sum_{y=1}^{3} P(X=2, Y=y)$

$= P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3)$

$= 0.1 + 0.2 + 0.1$

$= 0.4.$

For $X=3$,

$P(X=3) = \sum_{y=1}^{3} P(X=3, Y=y)$

$= 0.15 + 0.3 + 0.15$

$= 0.6$

∴ The marginal distribution of X is.

| $X=x$ | $P(X=x)$ |
|---|---|
| 2 | 0.4 |
| 3 | 0.6 |

## Marginal Distribution of Y

$$P(Y=y) = \sum_{x} P(X=x, Y=y)$$

For $Y=1$,

$$P(Y=y) = \sum_{x=2}^{3} P(X=x, Y=1)$$

$$= 0.1 + 0.15$$

$$= 0.25$$

$\therefore P(Y=1) = 0.25$

$$P(Y=2) = 0.2 + 0.3$$

$$= 0.5$$

$$P(Y=3) = 0.25$$

Marginal distribution of Y is:

| ~~P(~~ $Y=y$ | $P(Y=y)$ |
|---|---|
| 1 | 0.25 |
| 2 | 0.5 |
| 3 | 0.25 |

## Conditional Distribution

Let X and Y are two discrete random variables with joint pmf $P(X=x, Y=y)$ $\forall x$ and $y$.

Also,

Let $P(X=x)$ and $P(Y=y)$ be the marginal distribution of X and Y resp.

Then, the conditional distribution of $X=x$ given $Y=y$ is denoted by

$P(X=x/Y=y)$ and is calculated as:

$$P(X=x/Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

provided that $P(Y=y) \neq 0$.

Similarly,

the conditional distribution of $Y=y$ given $X=x$ is denoted by

$P(Y=y/X=x)$ and is calculated as:

$$P(Y=y/X=x)$$

$$\cancel{P(X=x/Y=y)} = \frac{P(X=x, Y=y)}{P(X=x)}$$

provided that $P(X=x) \neq 0$

Imp

Q) Let X and Y be two discrete random variables with joint prob. distribution

| Y\X | 2 | 3. |
|---|---|---|
| 1 | 0.1 | 0.15 |
| 2 | 0.2 | 0.3 |
| 3 | 0.1 | 0.15 |

i) Find the conditional distribution of $X=x$ given $Y=3$

ii) Find the conditional distribution of $Y=y$ given $X=2$

Solution

i) $P(X=x / Y=3) = \frac{P(X=x, Y=3)}{P(Y=3)}$

For $(X=2)$,

$P(X=2 / Y=3) = \frac{P(X=2, Y=3)}{P(Y=3)}$

$= \frac{0.1}{0.25} = \frac{1}{2.5}$

For $(X=3)$,

$P(X=3 / Y=3) = \frac{P(X=3, Y=3)}{P(Y=3)} = \frac{0.15}{0.25} = \frac{3}{5}$

ii) $P(Y=y / X=2) = \dfrac{P(Y=y, X=2)}{P(X=2)}$

For $Y=1$,

$P(Y=1 / X=2) = \dfrac{P(Y=1, X=2)}{P(X=2)}$

$= \dfrac{0.1}{0.4} = \dfrac{1}{4}$

For $Y=2$,

$P(Y=2 / X=2) = \dfrac{P(Y=2, X=2)}{P(X=2)}$

$= \dfrac{0.2}{0.4} = \dfrac{1}{2}$

For $Y=3$,

$P(Y=3 / X=2) = \dfrac{P(Y=3, X=2)}{P(X=2)}$

$= \dfrac{0.1}{0.4} = \dfrac{1}{4}$

Condition for Independency

Two discrete random variables $X$ and $Y$ are said to be independent if $P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$.

Otherwise not independent.

Alternatively,

$X$ and $Y$ are independent if

$P(X=x / Y=y) = P(X=x)$

or,

$P(Y=y / X=x) = P(Y=y)$

Otherwise not independent.

Q) Let $X$ and $Y$ be two discrete R.V. with joint prob. distribution

| Y \ X | 2 | 3 |
|---|---|---|
| 1 | 0.1 | 0.15 |
| 2 | 0.2 | 0.3 |
| 3 | 0.1 | 0.15 |

i) Check whether $X$ and $Y$ are independent or not.

$P(X=2/Y=3) = \frac{1}{2.5} = 0.4$

$P(X=2) = 0.4$

Since, $P(X=2/Y=3) = P(X=2) = 0.4$

$\therefore$ X and Y are independent

$P(Y=2/X=2) = \frac{1}{2} = 0.5$

$P(Y=2) = 0.5$

Since, $P(Y=2/X=2) = 0.5 = P(Y=2)$

$\therefore$ X and Y are independent.

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6) Let X and Y are two discrete random variable with the following joint prob. function

| X\Y | 1 | 2 |
|---|---|---|
| 0 | 0 | 1/8 |
| 1 | 3/8 | 0 |
| 2 | 3/8 | 0 |
| 3 | 0 | 1/8 |

i) Find the marginal distribution of X and Y.

ii) Find conditional distribution of X = x given Y = 1 and Y = y given X = 2

iii) Are X and Y independent?

Solution

i) For marginal distribution of X,

$P(X=x) = \sum_y P(X=x, Y=y)$

For $X=0$,

$P(X=0) = 0 + 1/8 = 1/8$

For $X=1$,

$P(X=1) = \frac{3}{8} + 0 = 3/8$

For $X=2$,

$P(X=2) = 3/8 + 0 = 3/8$

For $x = 3$,

$P(X=3) = 0 + 1/8 = 1/8$

| $X = x$ | $P(X=x)$ |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

For marginal distribution of Y,

$P(Y=y) = \sum_x P(X=x, Y=y)$

when $y = 1$,

$P(Y=1) = 0 + \frac{3}{8} + \frac{3}{8} + 0 = \frac{6}{8}$

For $y = 2$,

$P(Y=2) = 1/8 + 0 + 0 + 1/8 = 2/8$

| $Y = y$ | $P(Y=y)$ |
|---|---|
| 1 | 6/8 |
| 2 | 2/8 |

ii) Conditional distribution of $X = x$ given $Y = 1$

$P(X=x/Y=1)$ .

For, $X = 0$,

$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{0}{6/8} = 0$

For $X = 1$,

$P(X=1/Y=1) = \frac{3/8}{6/8} = \frac{1}{2}$.

For $X = 2$,

$P(X=2/Y=1) = \frac{3/8}{6/8} = \frac{1}{2}$

For $X=3$

$$P(X=3/Y=1) = \frac{0}{6/8} = 0$$

Conditional distribution of $Y=y$ given $X=2$

$$P(Y=y/X=2) = \frac{P(X=2, Y=y)}{P(X=2)}$$

For $Y=1$

$$P(Y=1/X=2) = \frac{3/8}{3/8} = 1$$

For $Y=2$,

$$P(Y=2/X=2) = \frac{0}{3/8} = 0$$

iii) ~~let~~, To check the independency of X & Y

$$P(X=1/Y=2) = \frac{0}{2/8} = 0$$

and,

$$P(X=1) = \frac{3}{8}$$

Since,

$P(X=1/Y=2) \neq P(X=1)$. Hence, X and Y are not independent.

## Expectation

Properties:

i) $E(a) = a$, where $a$ is constant.

ii) $E(aX) = aE(X)$

iii) $E(aX+b) = aE(X) + b$ where, a and b are constants.

iv) $E(X \pm Y) = E(X) \pm E(Y)$

v) $E(XY) = E(X) \cdot E(Y)$

if X and Y are independent.

vi) Let $g(X)$ be a function of random variable X. Then,

$$E[g(x)] = \sum_{x} g(x)\, P(X=x) \quad \text{if X is discrete.}$$

$$= \int_{x} g(x)\, f(x)\, dx \quad \text{if X is continuous.}$$

e.g. let $g(x) = x^2$

$$E[g(x)] = E(X^2)$$

$$= \sum_{x} x^2\, P(X=x)$$

## Variance

It measures the variation or deviation of all the values of random variable from its expectation i.e $E(x)$.

Let $X$ be a random variable with expectation $E(x)$. Then, the variance of random variable $X$ is denoted by $V(x)$ or $Var(X)$ or $\sigma_x^2$ and is calculated as:

$$V(X) = E[X - E(X)]^2$$

On simplification, we get

$$V(X) = E(X^2) - [E(X)]^2$$

Properties:

i) $V(a) = 0$, where $a$ is constant.

e.g.

$V(5) = 0$

(ii) $V(ax) = a^2 V(X)$

e.g. $V(5X) = 25 V(X)$

iii) $V(ax+b) = a^2 V(x)$ where $a, b$ are constants

e.g.

$V(5x+6) = 25 V(x)$.

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(iv) $V(X \pm Y) = V(X) + V(Y) \pm 2\,Cov(X, Y)$

where,

$Cov(X, Y)$ = covariance of X and Y

But if X and Y are independent.

Then,

$V(X \pm Y) = V(X) + V(Y)$

since $Cov(X, Y) = 0$.

## Covariance

It measures the variation or deviation of two random variables simultaneously.

Let X and Y are two Random Variables with expectations $E(X)$ and $E(Y)$ respectively. Then,

The covariance of X and Y is denoted by $Cov(x, y)$ or $\sigma_{xy}$ and is calculated as:

$$Cov(X, Y) = E[X - E(X)][Y - E(Y)]$$

On simplification, we get

$$Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$$

where,

$$E(XY) = \sum_{x} \sum_{y} xy\, P(X = x, Y = y)$$

if X and Y are discrete R.V.

or,

$$E(XY) = \int_x \int_y xy \; f(x,y) \, dx \, dy$$

If X and Y are continuous R.V

**Imp** Example

A discrete R.V X has the following prob. mass function

| $X=x$ | $P(X=x)$ |
|---|---|
| -2 | 0.1 |
| -1 | K |
| 0 | 0.2 |
| 1 | 2K |
| 2 | 0.3 |
| 3 | K. |

i) Find the mean of X
(expected value)
(expectation of x).
$E(X)$.

ii) Find $E(2X-3)$

iii) Find variance of X
or
$V(X)$.

iv) $V(2X-3)$

v) $E(X+1)^2$.

$E(X^2) = \sum x^2 \cdot P(X=x)$

Since,

total prob. = 1.

or, $0.1 + k + 0.2 + 2k + 0.3 + k = 1$

or, $4k = 1 - 0.6$

or, $k = \frac{0.4}{4}$

$\therefore$ $k = 0.1$

Prob. distribution

| X = x | P(X = x) |
|---|---|
| -2 | 0.1 |
| ~~-1~~ | ~~0.1~~ |
| 0 | 0.2 |
| 1 | 0.2 |
| 2 | 0.3 |
| 3 | 0.1 |

i) Mean of X is:

$E(X) = \sum_x x \cdot P(X=x)$

$= -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3$

$= 0.8$

ii) $E(2X-3) = 2E(X) - 3$

$= 2 \times 0.8 - 3$

$= 1.6 - 3$

$= -1.4$

iii) Variance of X $(V(X)) = E(X^2) - [E(X)]^2$

Here,

$E(X^2) = \sum_{x=-2}^{3} x^2 \, P(X=x) = 2.8$

$[E(X)]^2 = 0.8^2$

$\therefore V(x) = 2.8 - 0.8^2$.

$= 2.16$.

iv) $V(2X-3)$

$= 2^2 V(X) - 0$

$= 4 \times V(X)$

$= 4 \times 2.16.$

$= 8.64.$

v) $E(X+1)^2$

$= E(X^2 + 2X + 1)$

$= E(X^2) + 2E(X) + 1$

$= 2.8 + 2 \times 0.8 + 1$

$= 5.4.$

Imp

Q. A continuous random variable $X$ has the following prob density function:

$f(x) = x e^{-x} \quad ; \quad x \geq 0$

$= 0; \quad$ otherwise.

Find the mean and variance of $X$.

→ Solution

Given,

Pdf is:

$f(x) = x e^{-x} \quad ; \quad x \geq 0$

$= 0 \quad ; \quad$ otherwise.

Mean of $X$

$$E(X) = \int_0^\infty x \cdot f(x)\, dx.$$

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} \, dx$$

$$= \int_0^\infty x \cdot x e^{-x} \, dx$$

$$= \int_0^\infty x^2 e^{-x} \, dx$$

$$\cancel{= \left[ x^2 (-e^{-x}) \right]_0^\infty}$$

$$= \int_0^\infty x^{3-1} e^{-x} \, dx$$

$$= \Gamma 3$$

$$= 2$$

$$\therefore \boxed{E(x) = 2}$$

ii) Also,

$$V(x) = E(x^2) - [E(x)]^2$$

Then,

$$E(x^2) = \int_0^\infty x^2 \cdot f(x) \, dx$$

$$= \int_0^\infty x^3 \cdot e^{-x} \, dx$$

$$= \Gamma 4$$

$$= 3!$$

$$= 6$$

$$\therefore V(x) = 6 - 2^2$$

$$= 6 - 4$$

$$\therefore \boxed{V(x) = 2}$$

Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$
$$= \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Q. A random variable X has the following pdf.

$$f(x) = Ax(1-x)^2; \quad 0 \leq x < 1$$
$$= 0 \quad ; \quad \text{otherwise}$$

Find mean and variance.

⇒ Solution

Given,

$$f(x) = Ax(1-x)^2; \quad 0 < x < 1$$
$$= 0 \quad ; \quad \text{otherwise.}$$

We know,

Total probability is 1

$$\therefore \int_0^1 \cancel{x}\cdot f(x)\,dx = 1$$

$$\text{or, } \int_0^1 \cancel{x}\cdot Ax(1-x)^2\,dx = 1$$

$$\cancel{\text{or, } A\int_0^1 x^{\circ}(1-2x-x^2)\,dx = 1}$$

$$\text{or, } A\int_0^1 x(1-x)^2\,dx = 1$$

$$\text{or, } A\int_0^1 x^{2-1}(1-x)^{3-1}\,dx = 1$$

$$\text{or, } A \times \frac{\Gamma 2\,\Gamma 3}{\Gamma 5} = 1$$

$$\text{or, } A \times \frac{1\times 2}{24} = 1$$

$$\therefore \boxed{A = 12}$$

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$\therefore f(x) = 12x(1-x)^2; \quad 0 < x < 1$

$= 0 \quad ; \quad$ Otherwise

Now,

Mean of $X$ is

$E(X) = \int_0^1 x \cdot f(x)\, dx$

$= \int_0^1 x \cdot 12x(1-x)^2\, dx$

$= 12 \int_0^1 x^{3-1} (1-x)^{3-1}\, dx$

$= 12\ \beta(3,3)$

$= 12 \times \frac{\Gamma 3\ \Gamma 3}{\Gamma 6}$

$= \cancel{12}^{2} \times \frac{\cancel{2} \times \cancel{2}}{5 \times \cancel{4} \times \cancel{3} \times \cancel{2}}$

$= 2/5$

Again,

Variance of $X$ is

$V(X) = E(X^2) - [E(X)]^2$.

Here,

$E(X^2) = \int_0^1 x^2\ 12x(1-x)^2\, dx$

$= 12 \int_0^1 x^{4-1} (1-x)^{3-1}\, dx$

$= 12 \times B(4,3) = 12 \times \frac{\Gamma 4 \times \Gamma 3}{\Gamma 7} = \frac{1}{5}$

$$V(X) = \frac{1}{5} - \left(\frac{2}{5}\right)^2$$

$\therefore \boxed{V(X) = 0.04}$ Ans

Q) a. A continuous random variable X has the following pdf.

$$f(x) = k\, e^{-3x} \quad ; \quad x \geq 0.$$

Find mean and variance.

Q.b A discrete random variable X has the following prob. distribution

| $X = x$ | $P(X = x)$ |
|---|---|
| 0 | 0.25 |
| 1 | 0.15 |
| 2 | 2k |
| 3 | 0.2 |
| 4 | k |
| 5 | 0.16 |

Find mean and standard deviation of X

Solution

a) Given,

X is continuous Random variable

and

$f(x) = k\, e^{-3x} \quad ; \quad x \geq 0$

we know,

Total probability $= 1$

$\int_0^{\infty} f(x)\, dx = 1$

or, $k \int_0^{\infty} e^{-3x}\, dx = 1$

or, $k \left[ \frac{e^{-3x}}{-3} \right]_0^{\infty} = 1$

or, $k \left[ e^{-\infty} - e^{0} \right] = -3$

or, $k [0 - 1] = -3$

$\therefore k = 3$

$\therefore f(x) = 3\, e^{-3x}$

Now,

Mean, $E(x) = \int_0^{\infty} x \cdot f(x)\, dx$

$= \int_0^{\infty} x \cdot 3 e^{-3x} \cdot dx$

$= 3 \int_0^{\infty} x\, e^{-3x}\, dx$

$= \cancel{3 \left[ x\, e^{-3x} \right]^{\infty}}$

Put $+3x = t$, $x = +\frac{t}{3}$

$+3\,dx = dt$

$\therefore dx = \frac{dt}{+3}$

when $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$\therefore E(x) = \cancel{3}\int_0^\infty \frac{t}{\cancel{3}} \cdot e^{-t} \frac{dt}{3}$

$= \frac{1}{3}\int_0^\infty t^{2-1} e^{-t} dt$

$= \frac{1}{3} \Gamma 2$

$\therefore E(x) = \frac{1}{3}$

Again,

$V(x) = E(x^2) - [E(x)]^2$

We know,

$E(x^2) = \int_0^\infty x^2\, 3 \cdot e^{-3x} dx$

$= 3\int_0^\infty x^2 e^{-3x} dx$

Put,

$3x = t$ $\qquad x = \frac{t}{3}$

$dx = \frac{dt}{3}$

$\therefore E(x^2) = 3\int_0^\infty \frac{t^2}{9} \cdot e^{-t} \times \frac{dt}{3}$

$$= \frac{1}{9}\int_0^\infty t^{3-1} e^{-t}\, dt.$$

$$= \frac{1}{9}\Gamma 3 = \frac{2}{9}.$$

$$\therefore V(x) = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

b) <u>Solution</u>

We know,

Total prob. is 1.

$\therefore 0.25 + 0.15 + 2k + 0.2 + k + 0.16 = 1$

or, $0.76 + 3k = 1$

or, $3k = 0.24$

$\therefore k = 0.08$

$\therefore$ Prob distribution is.

| $X = x$ | $P(X=x)$ | $x^2$ | $x^2 P(X=x)$ |
|---|---|---|---|
| 0 | 0.25 | 0 | 0 |
| 1 | 0.15 | 1 | 0.15 |
| 2 | 0.16 | 4 | 0.64 |
| 3 | 0.2 | 9 | 1.8 |
| 4 | 0.08 | 16 | 1.28 |
| 5 | 0.16 | 25 | 4 |

Now,

Mean, $E(x) = \sum_{x=0}^{5} x \cdot P(X=x)$

$= 0 + 0.15 + 0.32 + 0.6 + 0.32 + 0.8$

$Var(x) = E(x^2) - [E(x)]^2$

For,

$$E(x^2) = \sum_{n=0}^{5} n^2 \, p(x=n)$$

$= 0 + 0.15 + 0.64 + 1.8 + 1.28 + 4$

$= 7.87$

$\therefore V(x) = 7.87 - 2.19^2$

$V(x) = 3.0739$

Standard deviation of $x$ is

$\sigma_x = \sqrt{V(x)}$

$= \sqrt{3.0739}$

$= 1.753$

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## Joint Distribution Function of Discrete Random variables

Let X and Y are two ~~distribution function~~ discrete random variables with joint pmf $P(X=x, Y=y)$ ; $\forall x$ and $y$.

Then,

the joint distribution function of X and Y is denoted by $F(x,y)$ and is calculated as:

$$F(x,y) = P(X \le x, Y \le y) = \sum_{-\infty}^{x} \sum_{-\infty}^{y} P(X=x, Y=y).$$

Imp **Example:** Let X and Y are two discrete random variables with joint prob. distribution.

| X \ Y | 1 | 2 |
|---|---|---|
| 0 | 0 | 1/8 |
| 1 | 3/8 | 0 |
| 2 | 3/8 | 0 |
| 3 | 0 | 1/8 |

i) Find the Marginal Distribution of X and Y.

ii) Find the conditional distribution of $X = x$ given $Y = 2$ and $Y = y$ given $X = 1$

iii) Are X and Y independent?

iv) Find $F(1,2)$ and $F(2,2)$.

v. Find $E(X)$, $V(X)$, $E(Y)$, $V(Y)$.

vi. Find covariance of $X$ and $Y$.

Solution

i) Marginal Distribution of $X$,

$$P(X=x) = \sum_y P(X=x, Y=y)$$

For $x=0$,

$$P(X=0) = 0 + 1/8 = 1/8$$

For $x=1$,

$$P(X=1) = 3/8$$

For $x=2$,

$$P(X=2) = 3/8$$

For $x=3$,

$$P(X=3) = 1/8$$

Marginal Distribution of $X$ is given by,

| $X=x$ | $P(X=x)$ |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

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Marginal Distribution of Y,

$P(Y = y) = \sum_{x} P(Y = y, X = x)$

For $(Y = 1)$,

$P(Y = 1) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8}$

For $(Y = 2)$

$P(Y = 2) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$

Marginal distribution of Y is given by,

| $Y = y$ | $P(Y = y)$ | $y^2$ | $y^2 P(Y = y)$ |
|---|---|---|---|
| 1 | 6/8 | 1 | 6/8 |
| 2 | 2/8 | 4 | 8/8 |
| | | | 14/8 |

ii) Conditional distribution of $X = x$ given $Y = 2$ is

$P(X = x / Y = 2) = \frac{P(X = x, Y = 2)}{P(Y = 2)}$.

For, $X = 0$,

$P(X = 0 / Y = 2) = \frac{P(X = 0, Y = 2)}{P(Y = 2)} = \frac{1/8}{2/8} = \frac{1}{2}$.

For $X = 1$,

$P(X = 1 / Y = 2) = \frac{0}{2/8} = 0$.

For $X = 2$,

$P(X = 2 / Y = 2) = \frac{0}{2/8} = 0$

For $X = 3$,

$P(X = 3 / Y = 2) = \frac{1/8}{2/8} = 1/2$.

Conditional distribution of $Y = y$ given $X = 1$,

$$P(Y=y/X=1) = \frac{P(Y=y, X=1)}{P(X=1)}$$

For, $Y = 1$

$$P(Y=1/X=1) = \frac{P(Y=1, X=1)}{P(X=1)} = \frac{3/8}{3/8} = 1$$

For, $Y = 2$,

$$P(Y=2/X=1) = \frac{P(Y=2, X=1)}{P(X=1)} = \frac{0}{3/8} = 0$$

iii) To check independency of $X$ and $Y$,

$$P(X=1/Y=1) = \frac{3/8}{6/8} = \frac{3}{8} \times \frac{8}{6} = \frac{1}{2}$$

$$P(X=1) = 3/8$$

Since, $P(X=1/Y=1) \neq P(X=1)$.

Hence, $X$ and $Y$ are not independent

iv) $F(1,2) = P(X \le 1, Y \le 2)$

$$= \sum_{x=0}^{1} \sum_{y=1}^{2} P(X=x, Y=y)$$

$$= 0 + \frac{1}{8} + \frac{3}{8} + 0 = \frac{4}{8} = \frac{1}{2}$$

$F(2,2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = 7/8$

v) $E(X) = \sum_{x=0}^{3} x \; P(X = x)$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$

$= \frac{12}{8}$

$E(X^2) = \sum_{x=0}^{3} x^2 \; P(X=x)$

$= \frac{24}{8} = 3$

Now,

$V(X) = E(X^2) - [E(X)]^2$

$= 3 - \left(\frac{3}{2}\right)^2$

$= 3/4.$

Also,

$E(Y) = \sum_{y=1}^{2} y \cdot P(Y=y)$

$= \frac{6}{8} + \frac{4}{8} = \frac{10}{8}$

$E(Y^2) = \frac{14}{8}$

$V(Y) = E(Y^2) - [E(Y)]^2$

$= \frac{14}{8} - \left(\frac{10}{8}\right)^2$

$= \frac{3}{16}$

Note: If two r.v.s X and Y are indep then covariance is 0 but the converse may not be true.

| X \ Y | 1 | 2 |
|---|---|---|
| 0 | 0 | 1/8 |
| 1 | 3/8 | 0 |
| 2 | 3/8 | 0 |
| 3 | 0 | 1/8. |

$Cov(X, Y) = E(XY) - E(X)\ E(Y).$

$$E(XY) = \sum_{x=0}^{3} \sum_{y=1}^{2} xy\ P(X=x, Y=y)$$

$$= 0\cdot 1 \cdot P(X=0, Y=1) + 0\cdot 2 \cdot P(X=0, Y=2) + 1\cdot 1\cdot \tfrac{3}{8} + 1\cdot 2\cdot 0 + 2\cdot 1\cdot \tfrac{3}{8} + 2\cdot 2\cdot 0 + 3\cdot 1\cdot 0 + 3\cdot 2\cdot \tfrac{1}{8}$$

$$= 0 + 0 + \frac{3}{8} + 0 + \frac{6}{8} + 0 + 0 + \frac{6}{8}$$

$$= \frac{15}{8}$$

Now,

$$Cov(X, Y) = \frac{15}{8} - \frac{12}{8} \times \frac{10}{8}$$

$$= \frac{15}{8} - \frac{120}{64}.$$

$$= 0$$

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