Joint Probability Distribution of Continuous random variable

Let X and Y are two continuous random variables with joint probability function is dentoted by $f(x,y)$ $\forall x \& y$

Then, the joint prob. function $f(x,y)$ is said to be joint prob. density fu^n which satisfies the following conditions:

i) $f(x,y) \geq 0$, (non-negative).

ii) $\int_x \int_y f(x,y)\, dy\, dx = 1$

Marginal Density Function (Marginal Distribution)

Let X and Y are two continuous Random variables with joint pdf. $f(x,y)$, $\forall x$ and y.

The marginal density function of X is denoted by $f(x)$ or $f_x(x)$ and is given by:

$$\boxed{f(x) = \int_y f(x,y)\, dy}$$

"ly, the marginal density function of Y is denoted by $f(y)$ or $f_Y(y)$ and is given by:

$$f(y) = \int_x f(x,y)\, dx$$

Conditional density function

let X and Y are two continuous random variables with joint prob. density function ~~$\forall x$~~ $f(x,y)$ $\forall$ x and y. Also, let $f(x)$ and $f(y)$ are marginal density functions of X and Y respectively.

Then, conditional density function of X given Y is denoted by $f(x/y)$ or $f_{X/Y}(x/y)$ and is given by:

$$f(x/y) = \frac{f(x,y)}{f(y)}\ ; \text{ provided that } f(y) \neq 0.$$

"ly, conditional density function of Y given X is denoted by $f(y/x)$ or $f_{Y/X}(y/x)$ and is given by:

$$f(y/x) = \frac{f(x,y)}{f(x)}\ ; \text{ provided that } f(x) \neq 0.$$

Condition for independency

Two continuous random variables are said to be independent if

$$f(x,y) = f(x) \cdot f(y)$$

Otherwise not independent.

In other words, X and Y are independent if:

$$f(x/y) = f(x)$$

OR,

$$f(y/x) = f(y)$$

otherwise not independent.

Joint Distribution Function of Continuous Random Variables

Let X and Y are two continuous R.V with joint pdf $f(x,y)$; $\forall x$ and y. Then the joint distribution fun of X and Y is denoted by $F(x,y)$ and is calculated as:

$$F(x,y) = P(X \le x, Y \le y)$$
$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dx\, dy.$$

Marginal Distribution Function:

Let X and Y are two continuous R.V with joint prob. density function; $\forall$ x and y.

Then, the marginal distribution function of X is denoted by $F(x)$ and is calculated as:

$$F(x) = F(x, \infty) = P(X \leq x, Y \leq \infty)$$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f(x,y)\, dy \right] dx$$

-lly, the marginal distribution function of Y is denoted by $F(y)$ and is calculated as:

$$F(y) = F(\infty, y) = P(x \leq \infty, Y \leq y)$$

$$= \int_{-\infty}^{y} \left[\int_{-\infty}^{\infty} f(x,y)\, dx \right] dy.$$

Q) Suppose X and Y are two continuous R.V. with the following joint prob. density function:

$f(x,y) = Axy \; ; \; 0<x<1, \; 0<y<1$
$= 0 \; ; \; \text{otherwise}$

i) Find A
ii) Find marginal density function of X and Y.
iii) Find the conditional density function of x given y, and y given x
iv) Check whether X and Y are independent or not.
v) $P(X \le 0.1, \; Y \le 0.2)$
vi) Find Mean and Variance of X.
vii) Find mean and variance of Y.
viii) Find covariance of X and Y.

⇒

Solution

Given joint prob. density function is:

$f(x,y) = Axy \; ; \; 0<x<1, \; 0<y<1$
$= 0 \; ; \; \text{Otherwise.}$

We know,

i) Total probability $= 1$

or, $\int_0^1 \int_0^1 f(x,y)\, dx\, dy = 1$

or, $A\int_0^1\int_0^1 xy \, dx\, dy = 1$

or, $A\int_0^1 \left[\frac{x^2}{2} y\right]_0^1 dy = 1$

or, $\frac{A}{2}\int_0^1 [y-0]\, dy = 1$

or, $\frac{A}{2}\left[\frac{y^2}{2}\right]_0^1 = 1$

or, $\frac{A}{4} = 1$

$\therefore A = 4.$

$\therefore f(x,y) = 4xy \; ; \quad 0<x<1 \, , \quad 0<y<1$

$\quad\quad\quad\;\; = 0 \; ; \quad$ otherwise.

ii) Marginal density function of X

$f(x) = \int_0^1 f(x,y)\, dy$

$= \int_0^1 4xy\, dy$

$= 4x\left[\frac{y^2}{2}\right]_0^1$

$= 2xy[1-0]$

$\therefore \boxed{f(x) = 2x}$

The marginal density function of Y

$f(y) = \int_0^1 f(x,y)\, dx.$

$$= \int_0^1 4xy \, dx$$

$$= 4y \left[\frac{x^2}{2}\right]_0^1$$

$$= 2y [1 - 0]$$

$$\therefore \boxed{f(y) = 2y}$$

iii) Conditional density function of X given Y is:

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{4xy}{2y}$$

$$\therefore f(x/y) = 2x$$

Conditional density function of Y given X is:

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{4xy}{2x}$$

$$\therefore f(y/x) = 2y$$

(iv) To check independency of X and Y,

$f(x/y) = 2x$

$f(x) = 2x$

Since, $f(x|y) = f(x)$. Hence, X and Y are independent.

v) $P(X \le 0.1,\ Y \le 0.2)$

$\Rightarrow$

$$\int_0^{0.1}\int_0^{0.2} f(x,y)\,dy\,dx$$

$$= \int_0^{0.1}\int_0^{0.2} (4xy)\,dy\,dx$$

$$= 4\int_0^{0.1} x\left[\frac{y^2}{2}\right]_0^{0.2} dx$$

$$= 2\int_0^{0.1} [0.2^2 - 0]x\,dx.$$

$$= 2 \times 0.04 \left[\frac{x^2}{2}\right]_0^{0.1}$$

$$= 0.04 \times [0.1^2 - 0]$$

$$= 0.04 \times 0.01$$

$$= 4 \times 10^{-4}$$

vi) Mean and Variance of X

Mean of X is

$$E(x) = \int_0^1 x \cdot f(x)\, dx$$

$$= \int_0^1 x \cdot 2x \, dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

$$\therefore \boxed{E(x) = \frac{2}{3}}$$

For variance of X

$$V(X) = E(X^2) - [E(X)]^2.$$

$$E(X^2) = \int_0^1 x^2 \cdot 2x \, dx$$

$$= 2 \times \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2} \times [1-0]$$

$$= 1/2.$$

$$\therefore V(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} - \frac{4}{9}$$

$$= \frac{9-8}{18} = \frac{1}{18}$$

vii) Mean and variance of Y.

Mean of Y is:

$$E(Y) = \int_0^1 y \cdot f(y)\, dy$$

$$= \int_0^1 y \cdot 2y\, dy$$

$$= 2\left[\frac{y^3}{3}\right]_0^1$$

$$\therefore \boxed{E(Y) = \frac{2}{3}}$$

For variance of Y,

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$E(Y^2) = \int_0^1 y^2 \cdot 2y\, dy$$

$$= 2\left[\frac{y^4}{4}\right]_0^1$$

$$\therefore E(Y^2) = \frac{1}{2}$$

$$\therefore Var(Y) = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} - \frac{4}{9}$$

$$= \frac{9-8}{18}$$

$$= \frac{1}{18}.$$

viii) Covariance of X and Y is.

$$\text{cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$

Since, X and Y are independent
$\text{cov}(X,Y) = 0$.

By calculation,

$$E(XY) = \int_0^1 \int_0^1 xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 xy \times 4xy\, dx\, dy$$

$$= \int_0^1 \int_0^1 4x^2y^2\, dx\, dy$$

$$= \int_0^1 4y^2 \left[\frac{x^3}{3}\right]_0^1 dy$$

$$= \frac{4}{3}\int_0^1 y^2\, dy$$

$$= \frac{4}{3} \times \frac{1}{3}$$

$$= \frac{4}{9}$$

Now,

$$\text{cov}(X,Y) = \frac{4}{9} - \frac{2}{3} \times \frac{2}{3}$$

$$= 0$$

Theoretical Distribution

① Discrete Probability Distribution
- → Binomial Distribution
- → Poisson Distribution
- → Hyper Geometric Distribution
- → Negative Binomial Distribution.

② Continuous Probability Distribution
- → Uniform (or Rectangular) Distribution
- → Exponential Distribution
- → Gamma Distribution
- → Beta Distribution
- → Normal Distribution ☆

Binomial Distribution

This is an important and widely used discrete probability Distribution developed by Swiss Mathematician James Bernoulli.

Bernoulli Trial: Trial having only two mutually exclusive outcomes categorized as success and Failure.

Success:- The occurance of an event (or outcome)

- Req^d outcome.

Failure: The non-occurance of an event (or outcome)

OR, absence of certain characteristics.

IMP

Conditions for applying Binomial Distribution

1) The number of trials (or sample size) n is fixed and finite. ($n \leq 20$).

2) Each trial has only two mutually exclusive outcomes categorized as: success and failure

3) The probability of success is denoted by p and remains constant for every trial.

-lly, The probability of failure is denoted by q and remains constant for every trial

Therefore,

$$p+q = 1$$

4) Every trials are independent.

Probability function

Let X be a discrete Random variable represents the number of success and having Binomial Distribution with parameters n and p. Then its pmf is given by

$$P(X=x) = {}^nC_x \, p^x \, q^{n-x} \quad ; \quad x = 0, 1, 2, \ldots n$$

and $0 < p < 1$

where,

n = no. of trials. ↙ Given, $n \leq 20$

p = probability of success → Given. $p > 0.05$

q = probability of failure
$= 1-p$

x = no. of success

Note:

① If, X follows binomial distribution with parameters n and p. Then, we can write as.

$$X \sim B(n, p)$$

② If n and p are given, then the Binomial Distribution is completely determined.

③ The random variable of binomial distribution is the no. of success.

i.e. X = No. of success.

④ Prob. of success, p is very high for using binomial distribution i.e.

$p > 0.05$

Properties

i) It is a discrete prob. function with two parameter n and p.

ii) The mean of binomial distribution is np i.e $\boxed{E(x) = np}$

iii) The variance of Binomial distribution is npq [or $np(1-p)$]

i.e $V(x) = npq$

iv) Standard deviation is $\sqrt{npq}$

v) The mean is greater than the variance.

Q) If the mean and standard deviation of binomial distribution are 6 and 2

i) Determine the binomial distribution

ii) Find the prob. of getting exactly two success.

iii) Find the prob. of getting at most 2 success.

iv) Find the prob. of getting at least 2 success.

Solution

Given,

Mean $E(X) = np = 6$

Standard Deviation $(S.D) = 2 = \sqrt{npq}$

Let, random variable X represents number of success.

$X \sim B(n, p)$

We know,

$np = 6$ —(i)

$\sqrt{npq} = 2$ —(ii)

or, $\sqrt{6 \times q} = 2$ [$\because np = 6$ from (i)]

or, $6 \times q = 4$

$\therefore q = \frac{2}{3}$

Also, $p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

From (i)

$np = 6$

$n \times \frac{1}{3} = 6$

$\therefore n = 18$

Now,

i) $X \sim B(n = 18, p = 1/3)$

Then,

$$P(X=x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$
$$= {}^{18}C_{x}\, p^{x}\, q^{18-x} \quad ; \; x = 0, 1, 2, \ldots 18$$

ii) Probability of getting exactly two success is

$$P(X=2) = {}^{18}C_{2} \times \left(\frac{1}{3}\right)^{2} \left(\frac{2}{3}\right)^{16}$$

$$P(X=2) = \frac{18!}{16! \times 2!} \times \frac{1}{9} \times \left(\frac{2}{3}\right)^{16}$$

$$= 0.0258$$

iii) Probability of getting at most two success is

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= {}^{18}C_{0} \times \left(\frac{1}{3}\right)^{0} \times \left(\frac{2}{3}\right)^{18} + {}^{18}C_{1} \times \left(\frac{1}{3}\right)^{1} \times \left(\frac{2}{3}\right)^{17}$$

$$+ 0.0258$$

$$= 6.76 \times 10^{-4} + 6.08 \times 10^{-3} + 0.0258$$
$$\simeq 0.032$$

iv) Probability of getting at least two success is

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - [P(X=0) + P(X=1)]$$
$$= 1 - [6.76 \times 10^{-4} + 6.08 \times 10^{-3}]$$
$$= 0.993$$

Q) A food packing apparatus underfilled 10% of the container. Find the prob. that for any 5 containers, the ~~there~~ no. of underfilled containers will be:

a) exactly 3
b) 0
c) at least one
d) at most two.

⇒

Solution

Here,

let X be random variable that represents no. of underfilled containers.

Now,

probability ~~of~~ that container is underfilled $(p) = 0.1$

$q = 1 - p$
$= 0.9$

$n = 5$

Here,

$X \sim B(n, p)$

Then, $P(X = x) = {}^{n}C_{x} \, p^{x} \, q^{n-x}$; $x = 0, 1, \ldots 5$

$= {}^{5}C_{x} \, (0.1)^{x} \, (0.9)^{5-x}$.

a) probability that 3 container will be underfilled:

$$P(X=3) = {}^5C_3\ (0.1)^3\ (0.9)^{5-3}$$
$$= 8.1\times10^{-3}$$

b) prob. that 0 container will be underfilled

$$P(X=0) = {}^5C_0\ (0.1)^0\ (0.9)^5$$
$$= 0.59049$$

c) prob. that at least one container is underfilled.

$$P(X\geq1) = 1-P(X=0)$$
$$= 0.40951$$

d) prob. that at most two container is underfilled.

$$P(X\leq2) = P(X=0) + P(X=1) + P(X=2)$$
$$= 0.59049 + {}^5C_1\ (0.1)^1\ (0.9)^4 + {}^5C_2\ (0.1)^2\ (0.9)^3$$
$$= 0.59049 + 0.32805 + 0.0729$$
$$= 0.99144$$

Note:

If a random experiment with n trials are repeated for large no. of times, say N. Then, the expected (or approximated) no. or frequency is given by

$$F_e = N \times P(X=x)$$

where,

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

N = No. of times that a random exp. is repeated.

IMP

Example:

6 dice are thrown 729 times. How many times would you expect that at least 3 dice show 5 or 6?

⇒ Solution

Let, x = no. of dice showing 5 or 6.

Given,

$n = 6$

$N = 729$

For each die,

Prob. that die shows 5 or 6,

$$p = \frac{2}{6} = \frac{1}{3}$$

$$\therefore q = 2/3$$

Here,

$X \sim B(n,p)$. Then,

$$P(X=x) = {}^{n}C_{x}\, p^{x}\, q^{n-x} \quad ; x = 0,1,2,\ldots 6$$

$$= {}^{6}C_{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{6-x}$$

At least 3 dice showing 5 or 6

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= {}^{6}C_{3} \left(\frac{1}{3}\right)^{3} \left(\frac{2}{3}\right)^{3} + {}^{6}C_{4} \left(\frac{1}{3}\right)^{4} \left(\frac{2}{3}\right)^{2} + {}^{6}C_{5} \left(\frac{1}{3}\right)^{5} \left(\frac{2}{3}\right)^{1} + {}^{6}C_{6} \left(\frac{1}{3}\right)^{6} \left(\frac{2}{3}\right)^{1}$$

$$= 0.219 + 0.082 + 0.016 + 1.37 \times 10^{-3}$$

$$= 0.38$$

Now,

expected no. of times that at least 3 dice show 5 or 6 is 729×0.38

$= 277.02$

≈ 277

Q) Out of 8000 families in a certain city with four children each, what would be the expected no. of families having
i) 1 boy and 3 girls
ii) 2 boys and 2 girls
iii) at least 1 boy.
iv) at most 2 boys

Assuming that boys and girls are equally likely.

⇒ Solution

Here, $p = \frac{1}{2}$, $q = \frac{1}{2}$, $N = 8000$

let X = no. of boy. ~~family having 1 boy and 3 girls.~~

We know,

$X \sim B(n, p)$

$P(X=x) = {}^nC_x \, p^x \, q^{n-x}$

$= {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x} = {}^4C_x \left(\frac{1}{2}\right)^n$

i) $P(X=1) = {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1}$

$= \frac{4!}{3! \times 1} \times \left(\frac{1}{2}\right)^4$

$= 4 \times \frac{1}{16} = \frac{1}{4}$.

Now,

expected no. of families that have 1 boy and 3 girls is.

$= N \times P(X=1)$

$= 8000 \times \frac{1}{4}$

$= 2000$

ii) $P(X=2) = {}^4C_2 \left(\frac{1}{2}\right)^4$

$= \frac{4!}{2! \times 2!} \times \frac{1}{16}$

$= \frac{\cancel{4} \times 3 \times 2 \times 1}{\cancel{4}} \times \frac{1}{16}$

$= \frac{3}{8}$

expected no. of families that have 2 boys and 2 girls is $N \times P(X=2)$

$= 8000 \times \frac{3}{8}$

$= 3000.$

iii) $P(X \geq 1) = 1 - P(X=0)$

$= 1 - {}^4C_0 \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{16}$

$= \frac{15}{16}$

expected no. of families having at least 1 boy is $8000 \times \frac{15}{16}$

$= 7500$

iv) $P(X \leq 2) = 1 - [P(X=3) + P(X=4)]$

$= 1 - \left[{}^4C_3 \left(\frac{1}{2}\right)^4 + {}^4C_4 \left(\frac{1}{2}\right)^4 \right]$

$= 1 - \left[\frac{4}{16} + \frac{1}{16} \right]$

$= \frac{16-5}{16}$

$= \frac{11}{16}$.

$\therefore$ Expected no. of families having at most 2 boys $= 8000 \times \frac{11}{16}$

$= 5500$.

Poisson Distribution

Another important discrete probability distribution, developed by French Mathematician Simeon Dewis Poisson.

Poission Distribution is considered as a limiting case of Binomial distribution.

i.e Binomial Distribution tends to Poisson Distribution under following conditions:

i) The no. of trails, n is very large: i.e $n \to \infty$ $(n > 20)$

ii) The prob. of success p is very small i.e $p \to 0$ $(p < 0.05)$

iii) $np = \lambda$ (finite value)

where, λ is parameter of poisson distribution.

Probability Function

Let X be a discrete random variable represents the no. of success and having poisson distribution with parameter λ. Then, its prob. mass function is given by;

$$\boxed{P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}} \quad ; \quad x = 0, 1, 2, \ldots$$

where, e = const.

λ = parameter.

x = reqd no. of success

Properties:

i) It is a discrete probability distribution with single parameter λ.

ii) The mean of Poisson distribution is λ.

i.e $\boxed{E(x) = \lambda}$

iii) The variance of poisson distribution is λ.

i.e $\boxed{V(x) = \lambda}$.

Note:

① If x follows poisson distribution with parameter λ. Then, we can write as.

$\boxed{X \sim P(\lambda)}$

② If the value of λ is given, then all the probabilities of poisson distribution can be obtained.

③ The random variable of poisson distribution is the no. of success

i.e. X = No. of success.

④ The occurance of an event (ie success) is defined within a given specified unit of time or space.

e.g.

i) Number of accidents per day

ii) number of printing mistakes in a page

iii) no of suicide reported per days, etc.

(5) The occurance of an even (i.e success) is the proportionate with unit of time period.

Q: An office switch board receives telephone calls at the rate of 3 calls per min. What is the prob. of receiving

i) no call in 1 minute interval
ii) exactly 3 calls in 2 minutes interval
iii) at most 3 calls in 5 minutes interval.

⇒ solution

let X = no. of calls received.

Given,

For 1 min, $(\lambda) = 3$

Here,

$X \sim P(\lambda)$. Then,

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; \quad x = 0, 1, 2, \ldots$$

i) no call in 1 minute interval

$$P(X=0) = \frac{e^{-3}\, 3^{0}}{0!} = e^{-3}$$

ii) Exactly 3 calls in 2 minutes.

For 2 minutes, $\lambda = 3 \times 2$
$= 6$.

$$P(X=3) = \frac{e^{-6}\, 6^3}{6} = 0.0892.$$

iii) At most 3 calls in 5 minutes.

For 5 minutes, $\lambda = 5 \times 3$ ← λ = for 1 min.
$= 15$

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{e^{-15}\, 15^0}{0!} + \frac{e^{-15}\, 15^1}{1!} + \frac{e^{-15}\, 15^2}{2!} + \frac{e^{-15}\, 15^3}{3!}$$

$$= 3.05 \times 10^{-7} + 4.58 \times 10^{-6} + 3.44 \times 10^{-5} + 1.72 \times 10^{-4}$$

$$= 4.38 \times 10^{-4}$$

Q) At a counter customer arrive at an avg. of 1.5 per minute. find the prob. that

i) At most 4 will arrive in any given minute

ii) At least 3 will arrive during an interval of 2 minutes.

iii) At most 2 will arrive during an interval of 6 minutes.

Solution

Given,

$\lambda = 1.5$ per minute.

let $x =$ no. of customers.

i) $P(x \leq 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= \frac{e^{-1.5} \times 1.5^0}{0!} + \frac{e^{-1.5} \times 1.5^1}{1!} + \frac{e^{-1.5} \times 1.5^2}{2} + \frac{e^{-1.5} \times 1.5^3}{3!}$

$+ \frac{e^{-1.5} \times 1.5^4}{4!}$

$= 0.223 + 0.334 + 0.251 + 0.125 + 0.047$

$= 0.98$.

ii) At least 3 will arrive at an interval of 2 minutes.

For 2 minutes,

$\lambda = 1.5 \times 2 = 3$

$P(x \geq 3) = 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - \left[\frac{e^{-3} \lambda^0}{0!} + \frac{e^{-3} \cdot 3^1}{1!} + \frac{e^{-3} \, 3^2}{2!} \right]$

$= 1 - [0.0497 + 0.149 + 0.224]$

$= 1 - 0.4227$

$= 0.5773.$

iii) At most 2 will arrive during an interval of 6 minutes.

$\Rightarrow$ For 6 minutes,

$\lambda = 9$

$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$

$= \frac{e^{-9} 9^0}{0!} + \frac{e^{-9} 9^1}{1!} + \frac{e^{-9} \, 9^2}{2!}$

$= 1.23 \times 10^{-4} + 1.11 \times 10^{-3} + 4.99 \times 10^{-3}$

$= 6.231 \times 10^{-3}.$

Example : ① Find the probability that at most 4 defective fuses in a box of 200 fuses past experience shows that 2% of such fuses are defective.

2. In a factory turning out optical lenses, there is a small chance $\frac{1}{500}$ for any lenses to be defective. The lenses are supplied in a packet of 10.
Find the approximate no. of packets out of 10000 packets, having

① No defective ② 1 defective
③ 2 defective ④ 3 defective
⑤ At most 2 defectives.

3. The no. of road accidents in a year attributed to the taxi driven follows possion distribution with mean 3. Out of 1000 taxi drivers, find the approximate no. of drivers with

a) no accidents in a year.
b) more than 3 accidents in a year.

Solutions

3) Given,

$\lambda = 3$

Let X = no. of road accidents in a year

Here,

$X \sim P(\lambda)$. Then,

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; \quad x = 0,1,2,3\ldots$$

Also,

total no. of taxi drivers = 1000

Now,

i) $$P(X=0) = \frac{e^{-3}\,3^{0}}{0!}$$

$$= e^{-3}$$

Then,

Expected no. of taxi drivers with no accidents = $1000 \times \frac{1}{e^{3}}$

$$= 49.7$$

$$\approx 50.$$

ii) $$P(X>3) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - \left[\frac{e^{-3}\,3^{0}}{0!} + \frac{e^{-3}\,3^{1}}{1!} + \frac{e^{-3}\,3^{2}}{2!} + \frac{e^{-3}\,3^{3}}{3!}\right]$$

$$= 0.352$$

Expected no of drivers with more than 3 accidents a year is

1000×0.352

$= 352$

Solution

Here,

No. of fuses, $n = 200$

prob. of defective fuse; $(p) = 0.02$

Since, $n > 20$ & $p < 0.02$ we have to use possion distribution,

Then,

$\lambda = np = 200 \times 0.02$

$= 4.$

Here,

$X \sim P(\lambda)$ Then,

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Probability that at most 4 defective fuses in a box

$P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= \frac{e^{-4} 4^0}{0!} + \frac{e^{-4} 4}{1!} + \frac{e^{-4} 4^2}{2!} + \frac{e^{-4} \times 4^3}{3!} + \frac{e^{-4} \times 4^4}{4!}$$

$= 0.018 + 0.073 + 0.146 + 0.195 + 0.195$

$= 0.627.$

2) Solution

Let, X = no. of defective lenses.

$$p = \frac{1}{500} = 0.002$$

no. of lenses in a packet = 10.

Now,

$$\lambda = np = 10 \times 0.002 = 0.02$$

Total no. of packets = 10000

Since, $p < 0.05$. Hence, we have to apply poisson distribution,

$$X \sim P(\lambda).$$

$$\therefore P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-0.02}\, 0.02^{x}}{x!}$$

i) For no defective,

$$P(X=0) = \frac{e^{-0.02}\, 0.02^{0}}{0!} = 0.98$$

Approximate no. of packets with no defective ~~chips~~ lens = 0.98 × 10000
= 9800

ii) 1 defective,

$$P(X=1) = \frac{e^{-0.02}\, 0.02^{1}}{1!} = 0.0196$$

Approximate no. of packets with 1 defective lens = 10000 × 0.0196
= 196

iii) 2 defective,

$$P(X=2) = \frac{e^{-0.02}\, 0.02^2}{2!}$$

$$= 1.96 \times 10^{-4}.$$

Approximate no. of packets with 2 defective lens $= 10000 \times 1.96 \times 10^{-4}$

$$= 1.96$$

$$\approx 2.$$

iv) 3 defective,

$$P(X=3) = \frac{e^{-0.02}\, 0.02^3}{3!}$$

$$= 1.306 \times 10^{-6}.$$

Approximate no. of packets with 3 defective len

$$= 10000 \times 1.306 \times 10^{-6} = 0.01 \approx 0$$

At most 2 defective,

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= 0.98 + 0.0196 + 1.96 \times 10^{-4}$$

$$= 0.9997$$

Now,

Approximate no. of lenses with at most 2 defective lenses $= 10000 \times 0.9997$

$$= 9997.$$

Hyper Geometric Distribution

Probability function:

A discrete random variable X is said to follow hyper geometric distribution with 3 parameters N, M and n. if its pmf is given by

$$P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \quad ; \quad x = 0, 1, 2, \ldots m \le n$$

1. Mean $E(X) = \frac{n \cdot M}{N}$

2. Variance $V(X) = \frac{nM(N-M)(N-n)}{N^2(N-1)}$

where,

N = Total no. of observations in propopulation (Population size).

M = no. of success in population.

n = no. of observations in sample (sample size)

x = No. of success.

n - (selected)
success : n.

Q) A box contains 20 bulbs among them 6 are defective. If 5 bulbs are drawn at random without replacement. What is the prob. that

a) exactly 4 bulbs are defective.

b) no. defective

c) at least one bulb is defective.

d) at most one bulb is defective.

Solution

Let X = no. of defective bulbs

Given,

total no. of bulbs $(N) = 20$

no. of defective bulbs $(M) = 6$,

no. of selected bulbs, $(n) = 5$

Here,

$X \sim HG(N, M, n)$

Then,

$$P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

a) Prob. that 4 bulbs are defective

$$P(X=4) = \frac{\binom{6}{4}\binom{14}{1}}{\binom{20}{5}} = \frac{35}{2584}$$

b) no. defective,

$$P(X=0) = \frac{\binom{6}{0}\binom{14}{5}}{\binom{20}{5}} = \frac{1001}{7752}$$

c) $P(X \geq 1) = 1 - P(X=0)$

$$= 1 - \frac{1001}{7752} = \frac{6751}{7752}$$

d) At most one bulb is defective.

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{1001}{7752} + \frac{1001}{2584}$$

$$= \frac{1001}{1938}$$

Q) A committee of 5 members is to be selected from a group of 9 engineers and 6 doctors. If the selection is made at random, what is the prob that the committee consists of
a) at least 1 engineer
b) exactly 3 engineers.
c) at most 2 engineers.

=) Solution
let X = no. of engineers.

Total no. of people $(N) = 15$
no. of members to be selected $(n) = 5$.
no. of engineers $(M) = 9$
no. of doctors = 6.
$X \sim HG(M, N, n)$.

Then,
$$P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

a) Prob. that committee consists of at least 1 engineer.
$$P(X \geq 1) = 1 - P(X=0)$$
$$= 1 - \frac{\binom{9}{0}\binom{6}{5}}{\binom{15}{5}} = 1 - \frac{2}{1001} = \frac{999}{1001}$$

b) Exactly 3 engineers,

$$P(X=3) = \frac{\binom{9}{3}\binom{6}{2}}{\binom{15}{5}} = \frac{60}{143}$$

c) at most 2 engineers:

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{2}{1001} + \frac{\binom{9}{1}\binom{6}{4}}{\binom{15}{5}} + \frac{\binom{9}{2}\binom{6}{3}}{\binom{15}{5}}$$

$$= \frac{2}{1001} + \frac{45}{1001} + \frac{240}{1001}$$

$$= \frac{287}{1001}$$

Q) Three defective items are mixed with 7 good items. If 3 items are drawn at random, find the prob. distribution of defective items. Also find the mean and the variance of the distribution.

→ Solution

Let X = no. of defective items.

Total no. of items = 3+7 = 10 = (N)

no. of non-defective items = 7

no. of defective items (M) = 3

no. of items selected (n) = 3.

The probability distribution of defective items is

$$P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

such that $X \sim HG(M, N, n)$.

→ $P(X=0) = \dfrac{\binom{3}{0}\binom{7}{3}}{\binom{10}{3}}$

$= \dfrac{7}{24}$.

→ $P(X=1) = \dfrac{\binom{3}{1}\binom{7}{2}}{\binom{10}{3}}$

$= \dfrac{21}{40}$

$\rightarrow P(X=2) = \dfrac{\binom{3}{2}\binom{7}{1}}{\binom{10}{3}} = \dfrac{7}{40}.$

$\rightarrow P(X=3) = \dfrac{\binom{3}{3}\binom{7}{0}}{\binom{10}{3}} = \dfrac{1}{120}.$

Again,

Mean $= \dfrac{n}{N} \times M = \dfrac{3}{10} \times 3$

$= \dfrac{9}{10}$

Also,

variance $= \dfrac{nM(N-M)(N-n)}{\cancel{N^2-1}\ N^2(N-1)}$

$= \dfrac{9}{10} \dfrac{(10-3)(10-3)}{N(N-1)}$

$= \dfrac{\cancel{9}}{10} \times \dfrac{7 \times 7}{10 \times \cancel{9}}$

$= \dfrac{49}{100}$

Negative Binomial Distribution

Probability function:

A discrete Random Variable X is said to follow Negative Binomial Distribution with two parameters r and p.

Then, its pmf is given by;

$$P(X=x) = \binom{x+r-1}{r-1} p^r q^x \; ; \; x = 0, 1, 2, \ldots$$

where,

$P(X=x)$ is prob. of getting 'x' failures to get r success.

r = no. of success (fixed)

p = prob. of success (const.)

$q = 1-p$

x = No. of failures.

1) Mean $E(X) = \frac{rq}{p}$

2) Variance $V(X) = \frac{rq}{p^2}$

Q) An item is produced in large number. The machine is known to produce 5% defective. A quality control engineer is examining the items by taking them at random. What is the prob. that at least 4 items are to be examined in order to get 2 defective.

⇒

Solution

Given, let X = no. of non-defective item.

prob. of defective items(p) = 0.05

$q = 0.95$

no. of success (r) = 2

Here,

$X \sim NB(r, p)$.

$\therefore P(X=x) = \binom{x+r-1}{r-1} p^r q^x ; \quad x = 0, 1, 2, \cdots$

At least 4 items are selected to get 2 defective.

i.e.

at least 2 not defectives.

$P(X \geq 2) = 1 - [P(X=0) + P(X=1) \cancel{+P(X=2)}]$

$= 1 - [C(1,1)\, 0.05^2\, 0.95^0 + C(1,1)\, 0.05^2\, 0.95^1]$

$= 1 - \left[\frac{1}{400} + \frac{19}{8000}\right]$

$= 0.995$

Q) An oil company conducts a geological study indicates that an exploratory oil well should have a 20% chance of striking oil.

What is the prob. that

a) first strike comes on third well drilled

b) the third strike comes in seventh well drilled.

c) What is the mean and variance of no. of wells that must be drilled if the oil company wants to setup 3 producing wells.

Solution

Given,

Let X = no. of well that not strike oil.

r = no. of success

p = 0.2 , x = no. of ~~succ~~ failures.

q = 0.8

a) r = 1 , x = 2

Now,

$$P(X=2) = \binom{x+r-1}{r-1} p^r q^x$$

$= {}^{2}C_0 \; 0.2^1 \; 0.8^2$

$= 0.128$

b) Here,

r = no. of success = 3

x = no. of failures = $7-3 = 4$.

Now,

$P(X=x) = {}^{6}C_2 \; 0.2^3 \; 0.8^4$

$= 0.04915$

c) For producing 3 wells,

$r = 3$

$x = 4$.

Mean ~~(μ)~~ $E(x) = \frac{rq}{p} = \frac{3 \times 0.8}{0.2} = 12$.

Variance $V(x) = \frac{rq}{p^2} = \frac{3 \times 0.8}{0.2^2} = 60$

Normal Distribution
(Gaussian)

→ It is an important continuous probability distribution.

→ Probability function:

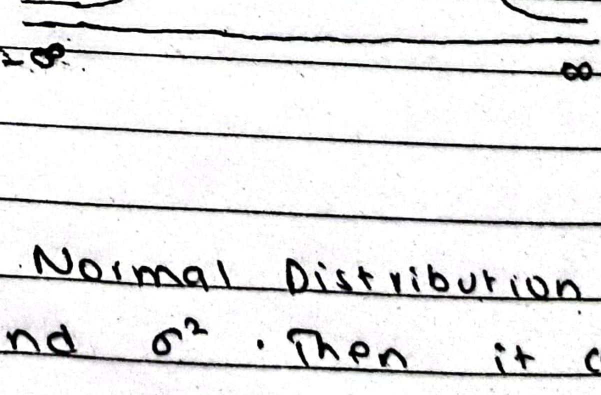
A continuous random variable X is said to follow Normal Distribution with parameters μ (mean) and σ^2 (variance), if its pdf is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; \quad -\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma^2 > 0$$

where, μ and σ^2 are two parameters of Normal distribution.



Remarks:

① If X follows Normal Distribution with parameters μ and σ^2. Then it can be written as:

$$X \sim N(\mu, \sigma^2)$$

(2) The curve of normal distribution is called Normal curve. The normal curve is bell shaped and symmetrical at $x = \mu$.

(3). The shape and size of Normal curve depends on the value of μ and σ. Since, the value of μ lies between $-\infty$ to ∞ and σ lies between 0 to ∞. Therefore, there are infinite no. of normal curve for prob. function $f(x)$.

Standard Normal Variate

Let $X \sim N(\mu, \sigma^2)$. Then,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad -\infty < x < \infty.$$

Define a new variable,

$$\boxed{Z = \frac{X-\mu}{\sigma}}$$

where,

μ = mean

σ = Standard Deviation

X = Normal variate.

Z = standard normal variate.

Here, Standard Normal variate (Z) follows standard normal distribution.

Standard Normal Distribution

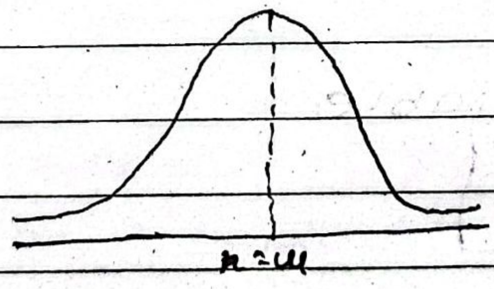
Standard Normal Distribution is a special case of Normal Distribution with mean 0 and variance 1. i.e. $N(0,1)$. Also, let $Z \sim N(0,1)$. Then its pdf is given by:

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \quad ; \quad -\infty < Z < \infty$$

Properties of Normal Distribution

i) It is a continuous prob. distribution with two parameters μ (mean) and σ^2 (variance).

2) The normal curve is bell shaped and symmetrical at $X = \mu$ ($Z = 0$)



3) The total area/probability under the normal curve is 1.

4) The normal curve is asymptotic to the X-axis.

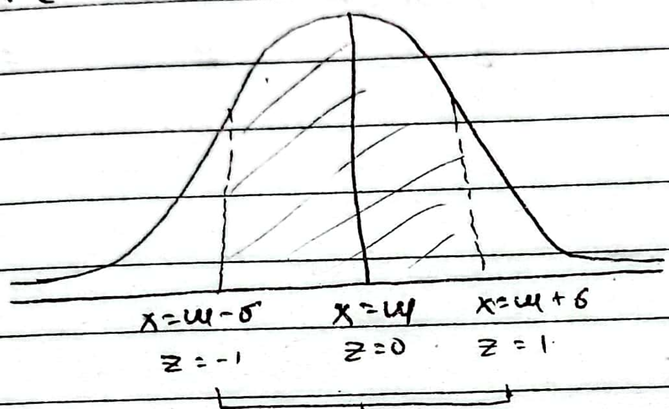
5) In normal distribution,

Mean = median = mode.

6) Area Probability

a) X lies between $\mu - \sigma$ and $\mu + \sigma$

i.e $X = \mu \pm \sigma$



Total area covered = 68.27%.

$P(\mu - \sigma < X < \mu + \sigma) = P(-1 < Z < 1)$

$= 0.6827$

b) X lies between $\mu - 2\sigma$ and $\mu + 2\sigma$ i.e.

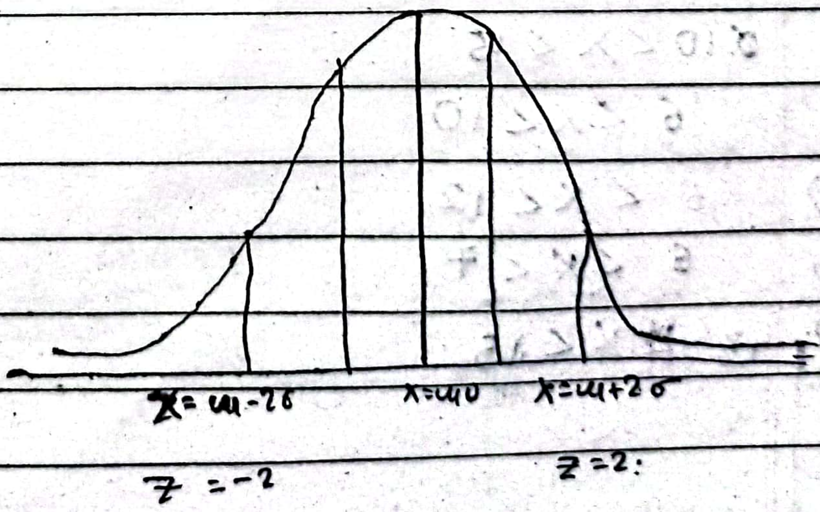
$X = \mu \pm 2\sigma$.

$P(\mu - 2\sigma < X < \mu + 2\sigma)$

$P(-2 < Z < 2)$

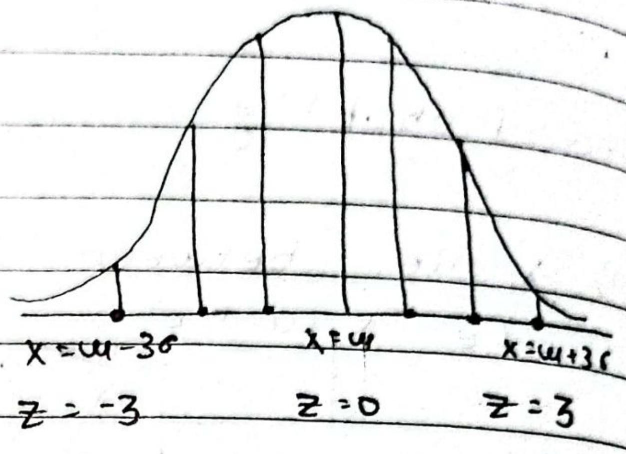
0.9544.

95.44%



c) X lies between $\mu - 3\sigma$ and $\mu + 3\sigma$

i.e. $X = \mu \pm 3\sigma \Rightarrow Z = \pm 3$



$P(\mu - 3\sigma < X < \mu + 3\sigma)$

$= P(-3 < Z < 3)$

$= 0.9973$

$= 99.73\%$

Q) Let X be a normally distributed random variable with mean 10 (μ) and S.D (σ) 2. Find the prob. that.

a) $X > 15$

b) $X > 7$

c) $X < 12$

d) $X < 6$

e) $10 < X < 15$

f) $6 < X < 10$

g) $6 < X < 12$

h) $6 < X < 7$

i) $13 < X < 15$

→ Solution

Given,

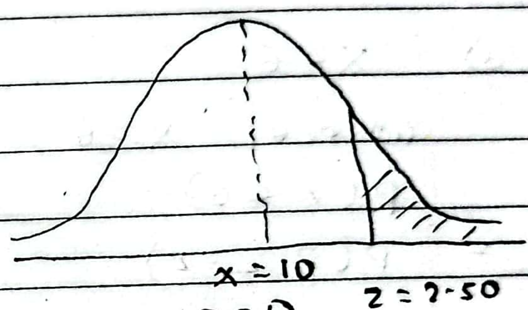
$X \sim N(\mu, \sigma^2)$

Mean $(\mu) = 10$

standard deviation $(\sigma) = 2$.

Define standard Normal variate $(z) = \frac{X-\mu}{\sigma}$

$= \frac{X-10}{2}$.



a) $X > 15$

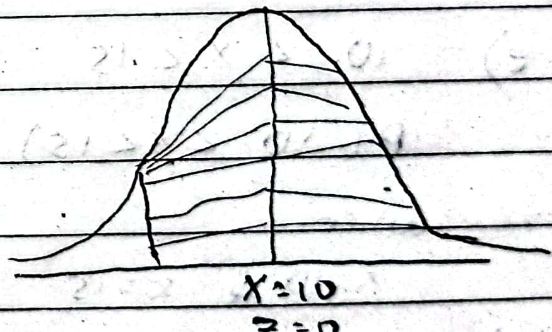
For $P(X > 15)$ where, $x = 15$

$= P(Z > 2.50)$

$= 0.5 - P(0 < z < 2.50)$

$= 0.5 - 0.4938$

$= 0.0062$



) $X > 7$

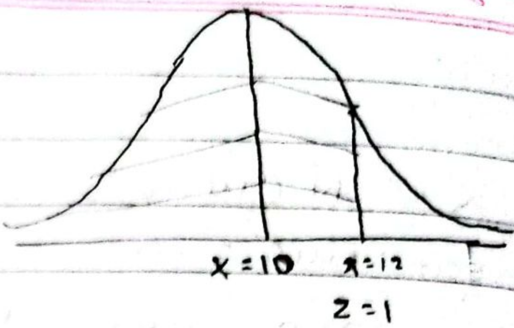
$P(X > 15)$ where $x = 7$

$= P(Z > 1.50)$ ↗ $P(-1.50 < z < 0)$ $= P(0 < z < 1.50)$ ↓ Due to symmetrical.

$= 0.5 + 0.4332$

$= 0.9332$.

c) $X < 12$



where, $x = 12$

$P(X < 12)$

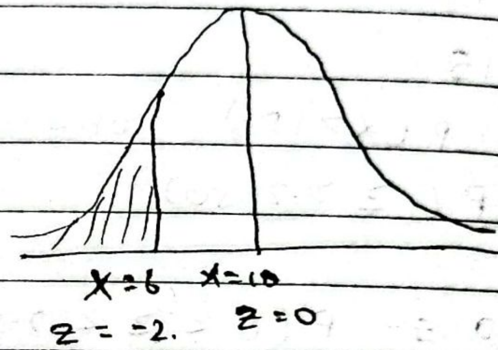
$= P(Z < 1)$

$= 0.5 + P(0 < Z < 1)$

$= 0.5 + 0.3413$

$= 0.8413.$

d) $X < 6$



where, $x = 6$

$P(X < 6)$

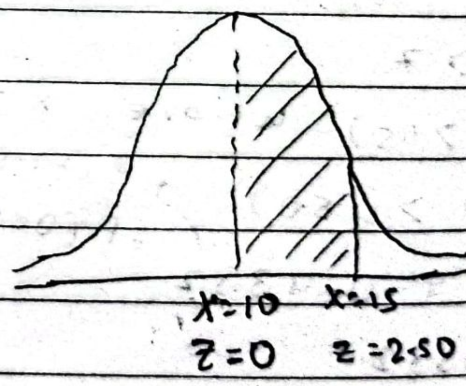
$= P(Z < -2)$

$= 0.5 - P(-2 < Z < 0)$

$= 0.5 - 0.4772$

$= 0.0228$

e) $10 < X < 15$



$P(10 < X < 15)$

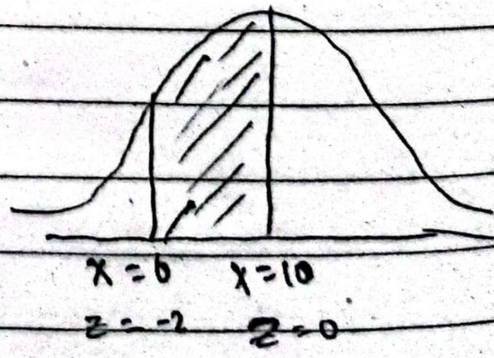
when,

$x = 10$ & $x = 15$

$\Rightarrow P(0 < Z < 2.50)$

$= 0.4938$

f) $6 < X < 10$



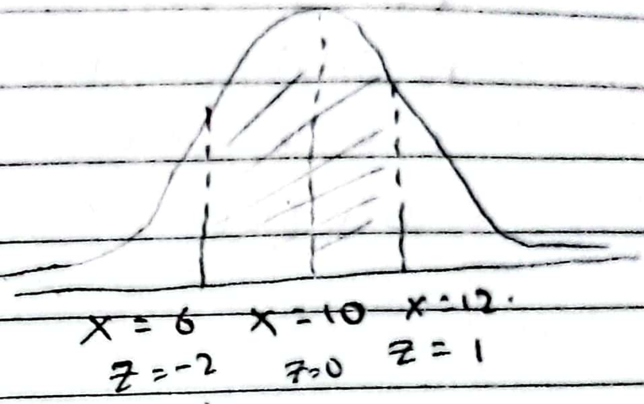
$P(6 < X < 10)$

when, $x = 6$, $x = 10$.

$P(-2 < Z \; 0)$

$= P(0 < Z < 2)$ [∵ The curve is symmetrical]

$= 0.4772.$



$6 < X < 12.$

g: $P(6 < X < 12).$

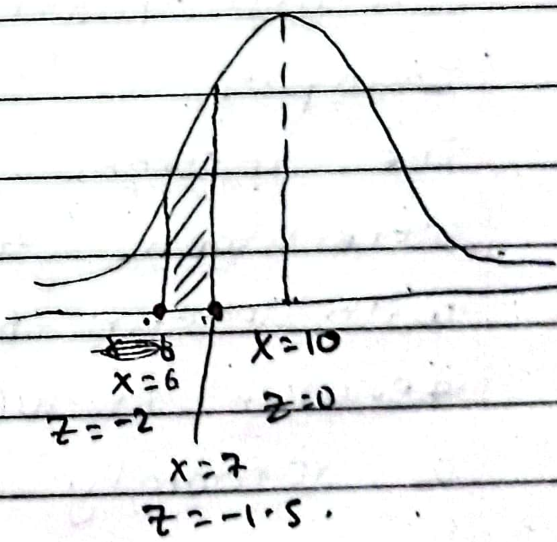
⇒ when, $X = 6$, $X = 12$

$P(\overset{-2}{\cancel{0}} < Z < 1)$

$= P(-2 < Z < 0) + P(0 \; Z < 1)$

$= \cancel{P} \; 0.4772 + 0.3413$

$= 0.8185$



h) $6 < X < 7.$

⇒ $P(6 < X < 7)$

when, $X = 6$, $X = 7.$

⇒ $P(-2 < Z < -1.5)$

⇒ $P(1.5 \; Z < 2)$. [∵ Curve is symmetrical]

⇒ $P(0 < Z < 2) - P(0 < Z < 1.5)$

⇒ $0.4772 - 0.4332$

⇒ 0.044

$$z = \frac{x-10}{2}$$

i) $13 < X < 15$

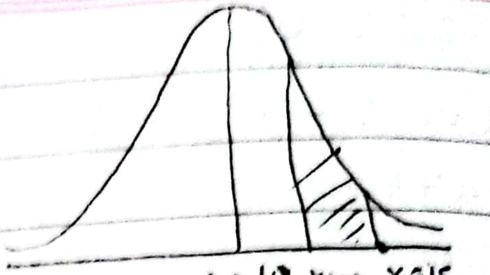
$\Rightarrow P(13 < X < 15)$ when $x = 13, x = 15$

$= P(1.5 < Z < 2.5)$

$= P(0 < Z < 2.5) - P(0 < Z < 1.5)$

$= 0.4938 - 0.4332$

$= 0.0606$



09/03 Wednesday

Normal Distribution

Example:

The incomes of a group of people were found to be normally distributed with mean Rs. 55000 and standard deviation Rs. 10000. Find the prob. that a randomly selected person has income.

i) below Rs. 40000

ii) Above Rs. 80000

iii) Between Rs. 60000 and Rs. 70000.

⇒ Solution

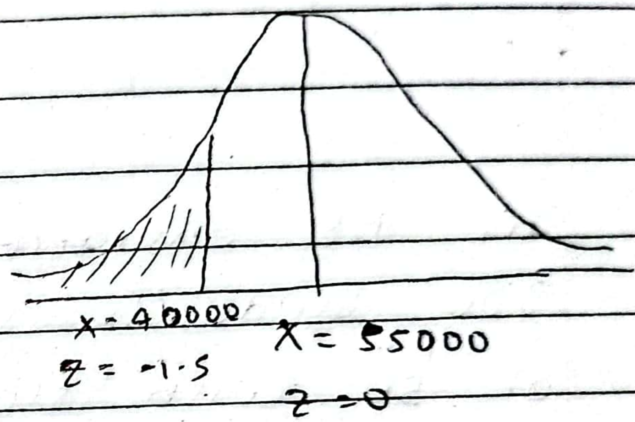
Let X = income of a person.

μ = Mean salary of people = Rs. 55000

standard deviation (σ) = Rs. 10000

Let Z be standard Normal variate.

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 55000}{10000}$$



i) $P(X < 40000)$

where,

$X = 40000$.

⇒ $P(Z < -1.5)$

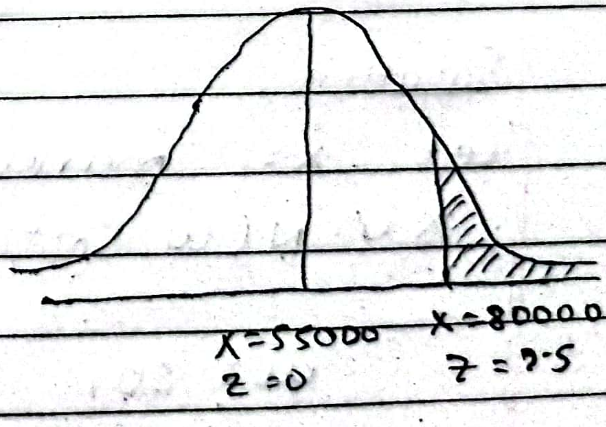
⇒ ~~0.5 − P(0 < Z < −1.5)~~

⇒ $0.5 - P(-1.5 < Z < 0)$

⇒ $0.5 - P(0 < Z < 1.5)$ [∵ curve is symmetrical]

⇒ $0.5 - 0.43319$

= 0.06681



ii) $P(X > 80000)$

where,

$X = 80000$.

⇒ $P(Z > 2.5)$

= $0.5 - P(0 < Z < 2.5)$

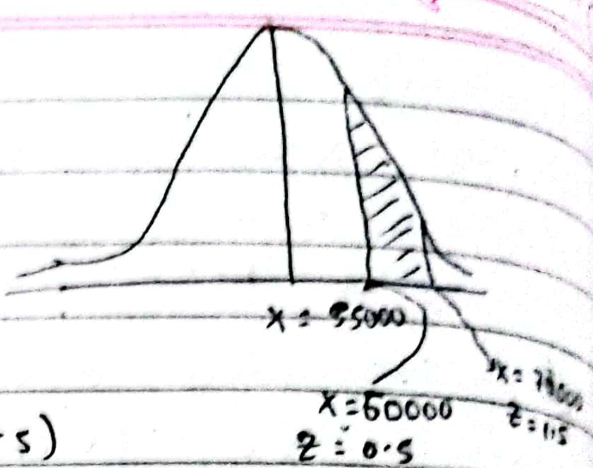
= $0.5 - 0.49379$

= 0.00621

(iii) $P(60000 < X < 70000)$

when,

$X = 60000, \quad X = 70000$



$= P(0.5 < Z < 1.5)$

$= P(0 < Z < 1.5) - P(0 < Z < 0.5)$

$= \cancel{0.34134}$

$= 0.43319 - 0.19146$

$= 0.24173$.

Q) In BE entrance examination, participated 900 studends secured mean marks of 50 with standard deviation 20.

i) Find the no. of stds securing between 30 and 70.

ii) Find the value of score exceeded by the top 90 students.

→

Solution

let x = marks secured by students.

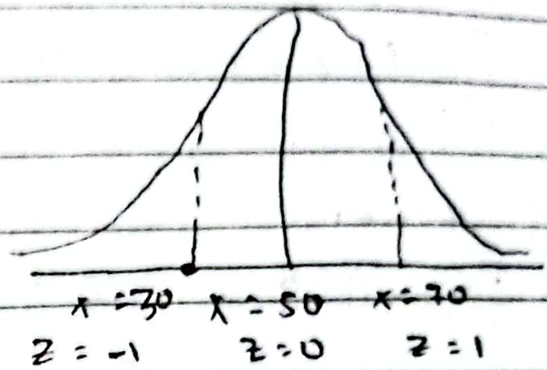
$X \sim N(\mu, \sigma^2)$. Then,

$\mu = 50, \quad \sigma = 20$

$N = 9000$

Let Z be standard Normal variate

where, $Z = \frac{X - \mu}{\sigma} = \frac{X - 50}{20}$.



i) $P(30 < X < 70)$

when,

$x = 30, \quad x = 70$.

$\Rightarrow P(-1 < Z < 1)$

$= P(-1 < Z < 0) + P(0 < Z < 1)$

$= P(-1 < Z < 0)$

$= 2P(0 < Z < 1)$ [$\because$ Due to symmetricity]

$= 2 \times 0.34134$

$= 0.68268$.

Now,

No. of students securing between 30 &

is 900×0.68268

~~= 899.31~~ $= 614.412$

~~≈ 900~~ ≈ 614.

ii) Solution

Let x be the value of score that is exceed by top 90 students.

$$P(X > x) = \frac{90}{900} = 0.1$$

when,

$$X = x \quad , \quad Z = \frac{x - 50}{20} = z_1 \text{ (say)} \quad \text{—(1)}$$

$$\therefore P(X > x) = P(Z > z_1).$$

or, $0.1 = 0.5 - P(0 < Z < z_1)$

or, $P(0 < Z < z_1) = 0.4.$

$$\therefore z_1 = 1.28$$

Then, From (1)

$$1.28 = \frac{x - 50}{20}$$

$$1.28 \times 20 + 50 = x$$

$$\therefore x = 75.6.$$

Q) The incomes of a group of people of 10000 were found to be normally distributed with mean Rs. 750 and standard deviation 50.

i) Find the lowest income of 10% richest person.

ii) Find the highest income of 2000 poorest person.

=>

Solution

let X = income of people.

$X \sim N(\mu, \sigma^2)$

$N = 10,000$

$\mu = Rs\ 750$, $\sigma = 50$.

i) let x be the income ~~the~~ that is lowest of ~~10%~~ rich 10%

lowest income of 10% richest people.

$P(X > x) = 0.1$

when,

$X = x$, $Z = \dfrac{x - 750}{50} = Z_1$ (say). — ①

$\therefore P(X > x) = P(Z > Z_1)$

or, $0.1 = 0.5 - P(0 < Z < Z_1)$

or, $P(0 < Z < Z_1) = 0.40$

$\therefore Z_1 = 1.28$.

Then from ①

$1.28 = \dfrac{x - 750}{50} \Rightarrow x = Rs.\ 814$

ii). Let x be the ~~lowest~~ highest income of 20000 poorest people.

$$\therefore P(X < x) = \frac{2000}{10000} = 0.2.$$

when,

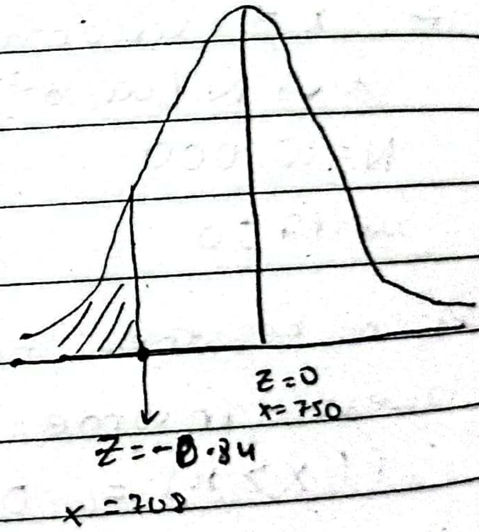
$$X = x \quad , \quad Z = \frac{x - 750}{50} = z_1 \text{ (say)}. \quad \text{—— (ii)}$$

Now,

$$\therefore P(X < x) = P(Z < -z_1)$$

or, $0.2 = 0.5 - P(-z_1 < Z < 0)$

or, $P(-z_1 < Z < 0) = 0.3$



$$\therefore z_1 = -0.84.$$

From (ii)

$$-0.84 \times 50 + 750 = x$$

$$\therefore x = 708$$

$\therefore x =$ Rs.708.

Q. In a normal distribution, 7% of the items are under 35 and 89% of the items are under 63. Find the mean and standard deviation of items.

→ Solution:

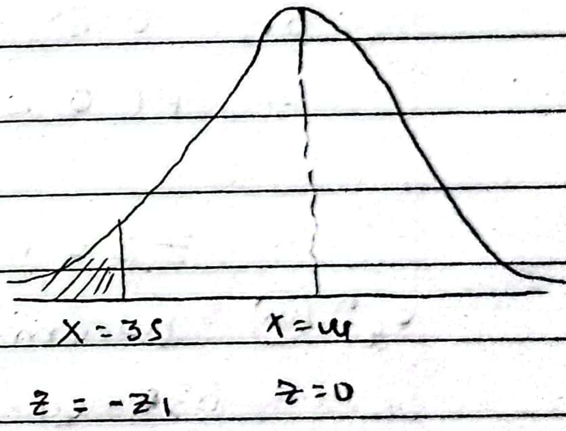
let X = Item

$\therefore X \sim N(\mu, \sigma^2)$; where μ = mean, σ = standard deviation.

Then,

let Z be standard normal variate.

$$Z = \frac{X-\mu}{\sigma} \quad \text{--- (i)}$$



Now,

$$P(X<35) = 0.07$$

when,

$$X=35, \quad Z = \frac{35-\mu}{\sigma} = -Z_1 \quad \text{--- (ii)}$$

Now,

$$P(X<35) = P(Z<Z_1)$$

$$P(X<35) = 0.5 - P(-Z_1<Z<0)$$

or, $P(0<Z<Z_1) = 0.5 - 0.7$

$\therefore P(0<Z<Z_1) = 0.43$

$\therefore Z_1 = 1.48$

$$\therefore -1.48 = \frac{35-\mu}{\sigma}$$

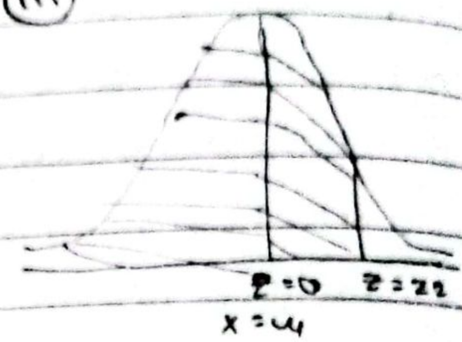
or, $35-\mu = -1.48\sigma$

0.00033

0.00065

Eq^n (ii) becomes,

$$-1.48 = \frac{35-\mu}{\sigma} \quad \text{---(iii)}$$



Again

when $x = 63$

$$P(X<63) = 0.89$$

and

$$z = \frac{63-\mu}{\sigma} = z2 \text{ (say)}.$$

$$P(X<63) = P(Z<z2)$$

or, $0.89 = 0.5 + P(0<Z<z2)$

or, $0.89 - 0.5 = P(0<Z<z2)$

$\therefore$ $P(0<Z<z2) = 0.39$

$\therefore$ $z2 = 1.23$

Eq^n (iii) becomes.

$$\frac{63-\mu}{\sigma} = 1.23$$

or, $63 - \mu = 1.23 \times \sigma$

or,

$$\frac{35-\mu}{63-\mu} = \frac{-1.48}{1.23}$$

$43.05 - \mu \times 1.23 = -93.24 + 1.48\mu$

or, $136.29 = 2.71\mu$

$\therefore$ $\mu = 50.291$

$\therefore \sigma =$

Q) In an examination 15 % of students got 60 marks or above. While 40 % of the students got below 40 marks. Assuming that the marks are normally distributed. Find mean and standard deviation.

→ Solution

Let, X = marks of student.

$\therefore X \sim N(\mu, \sigma^2)$

where,

μ = mean

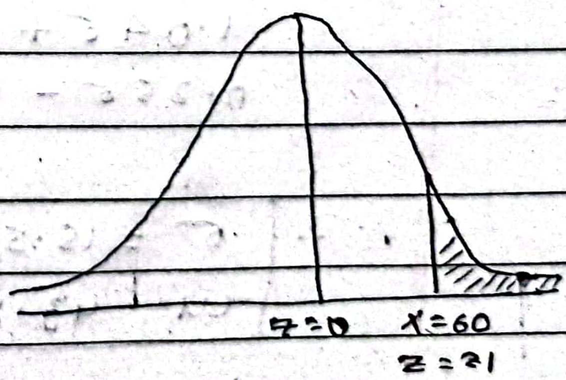
σ = Standard deviation.

Let Z be standard normal variate.

$$Z = \frac{X - \mu}{\sigma} \quad \text{—— (i)}$$

When, $X = 60$,

$$Z = \frac{60 - \mu}{\sigma} = Z_1 \text{ (let).} \quad \text{—— (ii)}$$



$P(X \geq 60) = 0.15$

or, $P(Z \geq Z_1) = 0.15$

or, $0.5 - P(0 < Z < Z_1) = 0.15$

or, $0.35 = P(0 < Z < Z_1)$

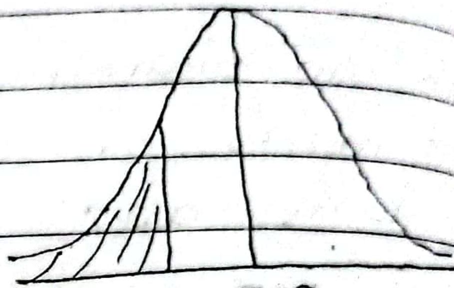
$\therefore Z_1 = 1.04$.

From (ii)

$$\frac{60-\mu}{\sigma} = 1.04 \text{ --- (a)}$$

Again,

when $x = 40$, $\frac{40-\mu}{\sigma} = -z_2$.



$P(x<40) = 0.4$

$\therefore P(x<40) = P(z<-z_2)$

or, $0.4 = 0.5 - P(-z_2 < z < 0)$

or, $P(-z_2 < z < 0) = 0.1$

$\therefore z_2 = 0.25$

$\therefore \frac{40-\mu}{\sigma} = -0.25$

or, $40-\mu = -0.25\sigma$

or, $0.25\sigma - \mu = -40$ --- (b)

From (a) and (b).

$$1.04\sigma + \mu = 60$$
$$0.25\sigma - \mu = -40$$

$\therefore$ $\boxed{\sigma = 15.503}$

$\boxed{\mu = 43.875}$

Q) In a normal distribution ~~30~~ 31% of the items are under 45 and 8% are over 64. Find the mean and variance of the distribution.

→ Solution

Let X = ~~marks~~ value of ~~student~~ item.

$X \sim N(\mu, \sigma^2)$

where,

μ = mean, σ = standard deviation.

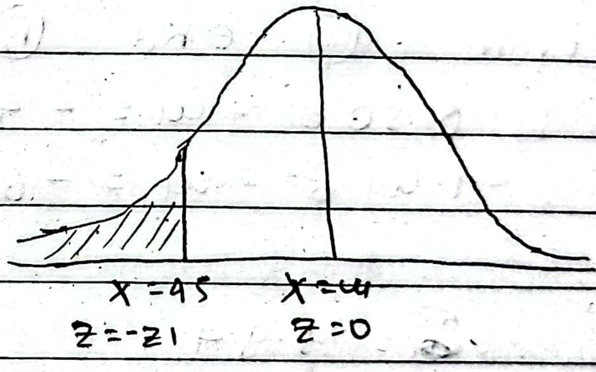
Let Z be normal standard variate.

$$Z = \frac{X - \mu}{\sigma}$$

When, $X = 45$.

$P(X < 45) = 0.31$

$\therefore Z = \frac{45 - \mu}{\sigma} = -Z_1$ —— (1)



$\therefore P(X < 45) = P(Z < -Z_1)$

or, $0.31 = 0.5 - P(-Z_1 < Z < 0)$

or, $P(-Z_1 < Z < 0) = 0.5 - 0.31$

or, $P(-Z_1 < Z < 0) = 0.19$

$\therefore Z_1 = 0.50$

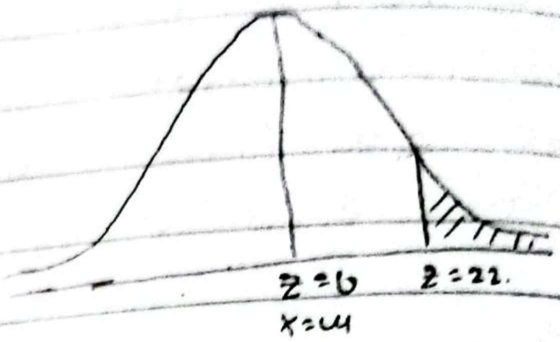
$\therefore 45 - \mu = -0.50\sigma$

$0.50\sigma - \mu = -45$ —— (a)

when,

$x = 64,$

$$Z = \frac{64 - \mu}{\sigma} = Z2$$



$P(X > 64) = 0.8$

or, $0.8 = P(Z > Z2)$

or, $0.08 = 0.5 - P(0 < Z < Z2)$

or, $P(0 < Z < Z2) = 0.42$

$\therefore Z2 = 1.41.$

$64 - \mu = 1.41\sigma$

or, $-1.41\sigma - \mu = -64$ —— (b)

From (a) and (b)

$0.50\sigma - \mu = -45$

$-1.41\sigma - \mu = -64.$

$\therefore \sigma = 9.94$

$$\boxed{\begin{gathered}\mu = 49.973 \\ \sigma^2 = 98.803\end{gathered}}$$

1\. Q) The incomes of a group of 2000 employees in a company were found to be normally distributed with mean Rs. 55000 and standard deviation Rs. 10000. Find the expected no. of employees with income

i) Below 40000

ii) Above 80000

iii) between 60000 and 70000

iv) Find the highest income of low ranked 10% of employees

v) Find the lowest income of high ranked 15% of employees.

2) A set of examination marks is approximately normally distributed with mean of 75 and standard deviation of 5. If top 5% student get grade A and bottom 25% get grade F. what marks is the lowest of A and what marks is highest of F.

Answers

1)

Solution

Let, X = Income

$X \sim N(\mu, \sigma^2)$

Given,

$\mu = 55000$, $\sigma = 10{,}000$

$N = 2000$

Let Z be standard normal variate

$$Z = \frac{X - 55000}{10000}$$

i) $P(X < 40{,}000)$

$\Rightarrow Z = -1.5$

$P(X < 40{,}000)$

$= P(Z < -1.5)$

$= 0.5 - P(-1.5 < Z < 0)$

$= 0.5 - P(0 < Z < 1.5)$

$= 0.5 - 0.43319$

$= 0.06681$

$\therefore$ Expected no. of employees with income below 40,000 is

$2000 \times 0.06681 = 133.62$

≈ 134

ii) $P(X > 80,000)$

$\Rightarrow Z = 2.5$

$P(X > 80000)$

$= P(Z > 2.5)$

$= 0.5 - P(0 < Z < 2.5)$

$= 0.5 - 0.49379$

$= 0.00621$

$\therefore$ Expected no. of employees

$= 2000 \times 0.00621$

$= 12.42$

≈ 13

iii) $P(60000 < X < 70000)$

when, $X = 60,000$, $Z = 0.5$

$X = 70,000$, $Z = 1.5$.

$P(0.5 < Z < 1.5)$

$= P(0 < Z < 1.5) - P(0 < Z < 0.5)$

$= 0.43319 - 0.19146$

$= 0.24173$

$\therefore$ Expected no. of employees whose income is between 60000 to 70000

is 2000×0.24173

$= 483.46 \approx 484$.

iv) Highest income of low ranked 10% employees $= x$ (let)

$\Rightarrow P(X < x) = 0.1$

$z = \dfrac{x - 55000}{10000} = -z_1$ (say). — (I)

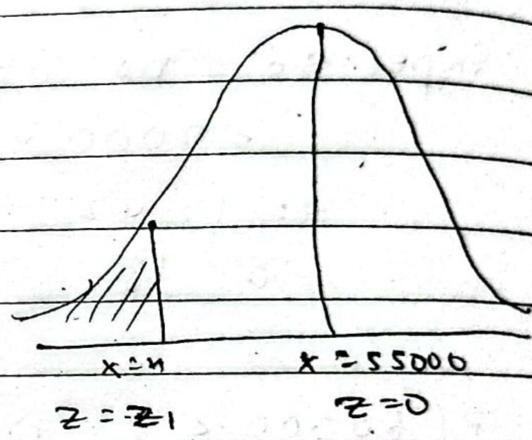
$\Rightarrow P(X < x) = P(z < -z_1)$

or, $0.1 = 0.5 - P(-z_1 < z < 0)$

or, $P(0 < z < z_1) = 0.4$

$\therefore z_1 = 1.28$

from (I)



$-1.28 = \dfrac{x - 55000}{10000}$

$\therefore x = Rs.\ 42,200$

v) lowest income of highly ranked 15% employees $= x$ (let).

$\Rightarrow P(X > x) = 0.15$

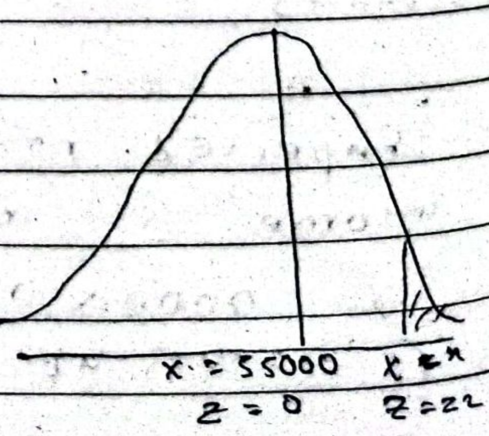
$z = \dfrac{x - 55000}{10000} = z_2$ (say) — (II)

$\Rightarrow P(X > x) = P(z > z_2)$

or, $0.15 = 0.5 - P(0 < z < z_2)$

or, $P(0 < z < z_2) = 0.35$

$\therefore z_2 = 1.04$



From (ii).

$1.04 = \frac{x - 55000}{10000}$ = Rs. 65,400

$\therefore$ The highest income of low ranked 10% employee is Rs. 47,200 and the lowest income of high ranked 15% employee is Rs. 65,400.

Q.N.2

Solution

Let, X = Marks.

$\therefore X \sim N(\mu, \sigma^2)$

Given,

$\mu = 75, \quad \sigma = 5.$

Define standard normal variate

$$(Z) = \frac{X-\mu}{\sigma} = \frac{X-75}{5}$$

① Let x be lowest of A

$P(X > x) = 0.05$

$$Z = \frac{x-75}{5} = Z_1 \quad (\text{let}) \quad \text{—①}$$

$P(X > x) = P(Z > Z_1)$

or, $0.05 = 0.5 - P(0 < Z < Z_1)$

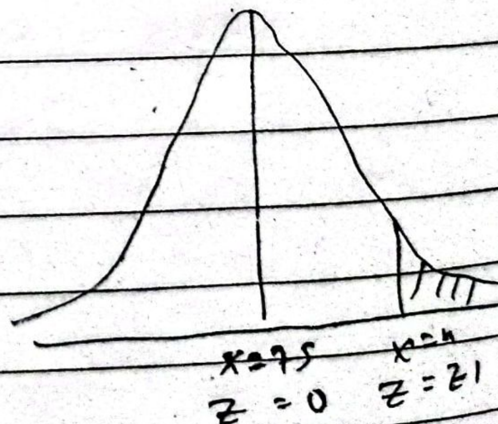
or, $P(0 < Z < Z_1) = 0.45$

$\therefore Z_1 = 1.64$

From ①;

$$1.64 = \frac{x-75}{5}$$

$\therefore x = 83.2.$



ii) let x be the marks of highest of grade F getting by bottom 25% students

$P(X < x) = 0.25$

$z = \frac{x - 75}{5} = -z_1$ (say) — (1).

$P(X < x) = P(z < -z_1)$

or, $0.25 = 0.5 - P(-z_1 < z < 0)$

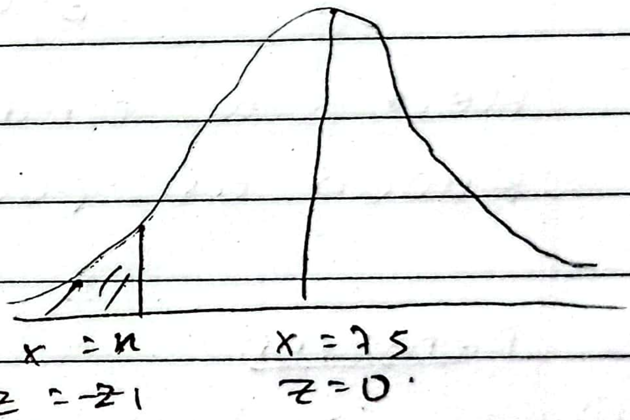
or, $P(0 < z < z_1) = 0.25$

$z_1 = 0.67$.

∴ from (1)

$-0.67 = \frac{x - 75}{5}$

∴ $x = 71.65$.



∴ The lowest mark of A is 83.2 and highest of F is 71.65.

Tuesday

Uniform Distribution
(Rectangular Distribution).

Another continuous probability Distribution

Probability Function

A continuous random variable X is said to follow uniform distribution within a given interval (a, b), if its pdf is given by;

$$f(x) = \frac{1}{b-a} \quad ; \quad a \le x \le b.$$

Here, a and b are constants known as parameters of uniform distribution.

Remarks:

① If X follows uniform distribution in an interval (a, b). Then it can be written as:

$$X \sim U(a, b).$$

② The mean of the uniform distribution is $\frac{a+b}{2}$. i.e $E(x) = \frac{a+b}{2}$.

③ The variance of uniform distribution is $\frac{(b-a)^2}{12}$. i.e $V(X) = \frac{(b-a)^2}{12}$

2024 (spring)

Example : Derive the mean and variance of uniform distribution.

$\Rightarrow$ Let $X \sim U(a, b)$

Then, its pdf is given by.

$$f(x) = \frac{1}{b-a} \; ; \quad a \le x \le b$$

Now, the Mean of X is given as:

$$E(X) = \int_a^b x \, f(x) \, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}.$$

$$\therefore \boxed{E(X) = \frac{a+b}{2}}$$

Also,

$$E(X^2) = \int_a^b x^2 \, f(x) \, dx$$

$$= \int_a^b x^2 \times \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \times \left[\frac{x^3}{3} \right]_a^b$$

$$= \frac{(b^3 - a^3)}{3(b-a)}.$$

$$= \frac{b^2 + ab + a^2}{3}$$

Now,

The variance of x is given by

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$\therefore \boxed{V(x) = \frac{(b-a)^2}{12}}$$

Q. The train on a certain route run every half hour in a day time. What is the probability that ~~the~~ a man entering the station at a random time during this period will have to wait at least 15 minutes?

⇒ let X = waiting time of a man (in minutes)

$\therefore X \sim U(a, b)$

Then, $a = 0$

$b = 30$

And its pdf is given by

$$f(x) = \frac{1}{b-a} \quad ; \quad 0 \le x \le 30.$$

$$= \frac{1}{30-0} = \frac{1}{30}.$$

$$\therefore \quad f(x) = 1/30 \quad ; \quad 0 \le x \le 30.$$

Now,

Probability that waiting time will be at least 15 min. is:

$$P(X \ge 15) = \int_{15}^{30} f(x)\,dx$$

$$= \int_{15}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30} \times [30-15]$$

$$= 15/30 = 1/2.$$

Cumulative Distribution Function of Uniform Distribution

Let $X \sim U(a, b)$. Then its pdf. is given by:

$$f(x) = \frac{1}{b-a} \quad ; \quad a \le x \le b.$$

Now, cumulative Distribution Function of x is

$$F(x) = P(X \le x) = \int_a^x f(x)\,dx$$

$$= \int_a^x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}[x]_a^x$$

$$\therefore F(x) = \frac{x-a}{b-a}$$

Q) Suppose time taken for data collection operator to fillout an electronic form for a database is uniformly distributed between 1.5 and 2.2 minutes.

a) What is the prob. that it will take less than 2 minutes to fill out the form?

b) What is the mean and variance of the time?

c) What is the cumulative distribution function of time it takes to fill out the form?

3) Let X = time taken to fill out the form in minutes.

$\therefore X \sim U(a, b)$

where $a = 1.5$, $b = 2.2$.

And, the pdf is:

$$f(x) = \frac{1}{b-a} = \frac{1}{2.2-1.5}$$

$$= \frac{10}{7}; \quad 1.5 \le x \le 2.2$$

i) Time takes less than 2 minutes.

$$P(X < 2) = \int_{1.5}^{2} f(x)\, dx$$

$$= \frac{10}{7} [x]_{1.5}^{2}$$

$$= \frac{10}{7} [2 - 1.5] = \frac{10}{7} \times 0.5$$

$$= \frac{5}{7}$$

ii) Mean of time:

$$E(X) = \int_{1.5}^{2.2} x \, f(x)\, dx.$$

$$= \int_{1.5}^{2.2} x \times \frac{10}{7}\, dx$$

$$= \frac{10}{7} \times \left[\frac{x^2}{2}\right]_{1.5}^{2.2}$$

$$= \frac{5}{7} \times [2.2^2 - 1.5^2] = 1.85$$

Variance;

$$V(x) = \frac{(b-a)^2}{12} = \frac{(2.2-1.5)^2}{12}$$

$$= 0.04083$$

iii) CDF is,

$$F(x) = \frac{x-a}{b-a}$$

$$= \frac{x-1.5}{2.2-1.5} = \frac{x-1.5}{0.7}$$

Gamma Distribution

Properties of Gamma Function.

1) $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$

2) $\frac{\Gamma n}{a^n} = \int_0^\infty x^{n-1} e^{-ax} dx$

3) $\Gamma n = (n-1)\Gamma{n-1}$

4) $\Gamma n = (n-1)!$

5) $\Gamma \frac{1}{2} = \sqrt{\pi}$

6) $\Gamma 1 = 1$

Another continuous probability distribution, used to study the reliability theory and quality control and also queueing time.

Probability Function

A continuous random variable X is said to follow Gamma ~~direct~~ distribution with two parameters α and β, if its pdf is given by:

$$\boxed{f(x) = \frac{1}{\beta^{\alpha}\,\Gamma\alpha}\, x^{\alpha-1}\, e^{-x/\beta}} \quad ; \quad x > 0 \text{ and } \alpha, \beta > 0.$$

where,

α = shape parameter.

β = Inverse scale parameter.

Remarks:

① If X follows gamma distribution with two parameters α and β, then it can be written as:

$$\boxed{X \sim G(\alpha, \beta)}$$

The mean of Gamma distribution is $\alpha\beta$

i.e $E(X) = \alpha\beta$

Variance, $V(X) = \alpha/\beta^2$

Q In a certain industrial area the daily consumption of electric power (in million units) can be described as a random variable having Gamma distribution with $\alpha = 3$, $\beta = 2$. If the power plant of this area has a daily capacity of 12 Million units, what is the prob. that the power supply is insufficient on any given day?

⇒

Let X = daily consumption of electric power (in Million units)

$X \sim G(\alpha, \beta)$. Then; $\alpha = 3$, $\beta = 2$.

The pdf is:

$$f(x) = \frac{1}{\beta^{\alpha} \Gamma\alpha} x^{\alpha-1} e^{-x/\beta} \; ; \; x > 0.$$

$$= \frac{1}{2^3 \Gamma 3} x^{3-1} e^{-x/2}$$

$$\therefore f(x) = \frac{1}{16} x^{3-1} e^{-x/2}.$$

Now, prob. that the power supply is insufficient.

i.e $P(X > 12) = 1 - P(X \le 12)$

$$= 1 - \int_0^{12} \frac{1}{16} x^2 e^{-x/2} dx.$$

$$= 1 - \frac{1}{16}\left[x^2 \frac{e^{-x/2}}{-1/2} - 2x\frac{e^{-x/2}}{1/4} + 2\frac{e^{-x/2}}{-1/8}\right]_0^{12}$$

$$= 1 - \frac{1}{16}\left[-2x^2 e^{-x/2} - 8x e^{-x/2} - 16 e^{-x/2}\right]_0^{12}$$

$$= 1 + \frac{2}{16}\left[x^2 e^{-x/2} + 4x e^{-x/2} + 8 e^{-x/2}\right]_0^{12}$$

$$= 1 + \frac{1}{8}\left[144 \cdot e^{-6} + 48 e^{-6} + 8 e^{-6} - (0 + 0 + 8)\right]$$

$$= 1 + \frac{1}{8}\left[144 \cdot e^{-6} + 48 e^{-6} + 8 e^{-6} - 8\right]$$

$$= 1 + \frac{1}{8}\left[200 e^{-6} - 8\right]$$

$$= 0.0619$$

$\therefore$

Q) In a certain city the daily consumption of water in millions of gallons follows approximately gamma distribution with $\alpha = 2$ and $\beta = 3$. If the daily capacity of the city is 9 Million gallons, find the prob. that on any given day the water supply is inadequate.

⇒ Solution

Let X = daily consumption of water (in millions of gallons)

$\therefore X \sim G(\alpha, \beta)$

where, $\alpha = 2$, $\beta = 3$

The pdf is:

$$f(x) = \frac{1}{\beta^{\alpha} \Gamma\alpha} x^{\alpha-1} e^{-x/\beta}$$

$$= \frac{1}{3^2 \Gamma 2} x^1 e^{-x/3}$$

$$\therefore f(x) = \frac{1}{9} x . e^{-x/3}$$

Now probability that on any day water supply is inadequate is

$$P(X > 9) = 1 - P(X \leq 9)$$

$$= 1 - \int_0^9 \frac{1}{9} x e^{-x/3} dx$$

$$= 1 - \frac{1}{9} \left[\frac{x . e^{-x/3}}{-1/3} - \frac{e^{-x/3}}{+1/9} \right]_0^9$$

$$= 1 - \frac{1}{9}\left[-3\, x e^{-x/3} - 9 e^{-x/3} \right]_0^9$$

$$= 1 + \frac{3}{9}\left[x e^{-x/3} + 3e^{-x/3} \right]_0^9$$

$$= 1 + \frac{1}{3}\left[9 \cdot e^{-3} + 3e^{-3} - (0 + 3) \right]$$

$$= 1 + \frac{1}{3}\left[12 e^{-3} - 3 \right]$$

$$= 1 + [4e^{-3} - 1]$$

$$= 0.1991$$

Beta Distribution

Properties of Beta Function

(i) $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$

(ii) $B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Another continuous probability distribution usually used to modelling the probability (or proportion) of the occurance of event.

Probability Function

A continuous random variable x is said to follow beta distribution with two parameters α and β, if its pdf is given by:

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \; ; \quad 0 < x < 1 \text{ and } \alpha, \beta > 0$$

Remarks:

(i) If x follows Beta distribution with parameters α and β. Then it can be written as:

$$x \sim \beta(\alpha, \beta)$$

② The Mean of Beta distribution is

$$E(X) = \frac{\alpha}{\alpha+\beta}$$

③ Variance of Beta distribution is

$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)}$$

Q) If the annual proportion of erroneous income tax return filed with in land revenue department ~~with~~ can be looked as a random variable having a beta distribution with $\alpha = 2$ and $\beta = 3$, what is the prob. that in any given year there will be fewer than 10% erroneous returns.

⇒

Let X = proportion of erroneous returns (in year)

$\therefore X \sim B(\alpha, \beta)$

We have, $\alpha = 2$, $\beta = 3$.

The pdf is;

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \; ; \quad 0 < x < 1 \text{ and } \alpha, \beta > 0$$

$$= \frac{1}{B(2,3)} x^{2-1} (1-x)^{3-1}$$

$$= \frac{x \cdot (1-x)^2}{\Gamma 2\, \Gamma 3} \times \Gamma 5$$

$$= 12\, x \cdot (1-x)^2$$

Probability that in any given year there will be fewer than 10% enviromental returns;

i.e. $P(X < 0.1) = \int_0^{0.1} 12 \cdot x(1-x)^2 \, dx$

$= 12 \int_0^{0.1} x(1-x)^2 \, dx$

$= 12 \int_0^{0.1} x(x^2 - 2x + 1) \, dx$

$= 12 \int_0^{0.1} (x^3 - 2x^2 + x) \, dx$

$= 12 \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^{0.1}$

$= 12 \left[\frac{0.1^4}{4} - \frac{2 \times 0.1^3}{3} + \frac{0.1^2}{2} - 0 \right]$

$= \cancel{12}\ 0.0523$

Q) In in certain mountain, the proportion of soil lapsed. ~~In~~ 10 years is a random variable having beta distribution $\alpha = 3$, $\beta = 2$. ~~Fi~~

(i) Find the av. percentage of soil lapsed in a given 10 years

(ii) Find the prob that at most 25% of soil will lapse in any given 10 years.

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3) **Solution**

let X = proportion of soil lapsed in 10 years.

$\therefore X \sim \beta(\alpha, \beta)$

where, $\alpha = 3, \quad \beta = 2$

The pdf is:

$$f(x) = \frac{1}{\beta(3,2)} x^2 (1-x)^1$$

$$= \frac{x^2(1-x)}{\Gamma 3 \, \Gamma 2} \times \Gamma 5$$

$$f(x) = 12 x^2 (1-x) \quad ; \quad 0 < x < 1, \; \alpha, \beta > 0.$$

(i) The average percentage of soil lapsed is $E(X) = \frac{\alpha}{\alpha+\beta} = \frac{3}{5}$

(ii) Probability that at most 25% of soil will lapse in any given 10 yrs is:

$$P(X \leq 0.25) = \int_0^{0.25} 12 x^2 (1-x) \, dx$$

$$= 12 \int_0^{0.25} (x^2 - x^3) \, dx$$

$$= 12 \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^{0.25}$$

$$= 12 \left[\frac{0.25^3}{3} - \frac{0.25^4}{4} - 0 \right]$$

$$= 0.0507$$

Exponential Distribution

Another important continuous Probability Distribution used to study the Reliability theory and Queuing Theory.

Probability Function:

A continuous random variable X is said to follow exponential distribution with parameter θ, if its pdf is given by,

$$f(x) = \theta e^{-\theta x} \quad ; \quad x > 0 \text{ and } \theta > 0.$$

Similarly, if $\theta = \frac{1}{\lambda}$

$$\therefore f(x) = \frac{1}{\lambda} e^{-x/\lambda} \quad x > 0, \ \lambda > 0.$$

Remarks:

① If X Follows Exponential Distribution with parameter θ. Then, it can be written as:

$$X \sim Exp(\theta).$$

② The mean of exponential distribution is $1/\theta$

$$\therefore E(X) = 1/\theta$$

③ Variance $V(X) = \frac{1}{\theta^2}$

Imp ④ Exponential distribution has a special property known as Memoryless property.

Memorylessn Property
(Property of lack of memory).
let $X \sim Exp(\theta)$. Then,
The pdf is $f(x) = \theta e^{-\theta x}$; $x > 0$
for any positive real number say t & s

$$P\left(X > s+t / X > t\right) = P(X > s)$$

Alternatively,

$$P(X < s+t / X < t) = P(X < s)$$

Proof: let $X \sim Exp(\theta)$. Then,

$$f(x) = \theta e^{-\theta x} \quad ; \quad x > 0$$

For any positive real number t and s. We have,

$$LHS = P(X > s+t / X > t) = \frac{P(X > s+t \cap X > t)}{P(X > t)}$$

$$= \frac{P(X > s+t)}{P(X > t)}$$

Here,

$$P(X > s+t) = \int_{s+t}^{\infty} f(x)\,dx$$

$$= \int_{s+t}^{\infty} \theta e^{-\theta x}\,dx$$

$$= \frac{\theta}{-\theta}\left[e^{-\theta x}\right]_{s+t}^{\infty}$$

$$= -\left[0 - e^{-\theta(s+t)}\right]$$

$$= e^{-\theta s - \theta t}$$

And,

$$P(X>t) = \int_t^{\infty} \theta e^{-\theta x} dx$$

$$= \frac{\theta}{-\theta}\left[e^{-\theta x}\right]_t^{\infty}$$

$$= -\left[e^{-\infty} - e^{-\theta t}\right]$$

$$= e^{-\theta t}.$$

$$\therefore LHS = \frac{P(X>s+t)}{P(X>t)} = \frac{e^{-\theta s - \theta t}}{e^{-\theta t}}$$

$$= e^{-\theta s} \quad \text{——— (a)}$$

Also,

For RHS;

$$P(X>s) = \int_s^{\infty} \theta e^{-\theta x} dx$$

$$= \frac{\theta}{-\theta}\left[e^{-\theta x}\right]_s^{\infty}$$

$$= -\left[e^{-\infty} - e^{-\theta s}\right]$$

$$= e^{-\theta s} \quad \text{——— (b)}$$

From (a) and (b)

LHS = RHS

$$\therefore P\left(X>s+t / X>t\right) = P(X>s).$$

which is the memoryless property of exponential distribution.

Q) The time between arrivals of taxi ~~and~~ at a busy intersection is exponentially distributed with a mean of 10 minutes.

① What is the prob. that you wait longer than 1 hour for a taxi.

② What is the prob. that you will wait only 15 minutes for a taxi?

③ Determine x such that the prob. that you will wait less than x minute is 0.1.

⇒ Solution

Let X = time to wait for a taxi (in minutes)

$\therefore X \sim Exp(\theta)$

$\therefore$ The pdf is:

$$f(x) = \theta e^{-x/\theta}.$$

Given,

$$\text{Mean}: E(x) = 10 \text{ minutes} = \frac{1}{\theta}$$

$$\therefore \theta = \frac{1}{10}.$$

$$\therefore f(x) = \frac{1}{10} e^{-\frac{x}{10}}$$

1) Probability that you wait longer than 1 hr = 60 min

$$\therefore P(X > 60) = 1 - P(X \leq 60)$$

$$= 1 - \int_0^{60} \frac{1}{10} e^{-\frac{x}{10}} \, dx$$

$$= 1 - \frac{1}{10} \left[e^{-x/10} \right]_0^{60} \times (-10)$$

$$= 1 + \left[e^{-6} - e^{0} \right] = 1 + e^{-6} - 1$$

$$= 0.00247$$

iii) $P(X<x) = 0.1$

or, $\int_0^x f(x)\,dx = 0.1$

or, $-\int_0^x -\frac{1}{10} e^{-\frac{x}{10}}\,dx = 0.1$

or, $-\left[e^{-\frac{x}{10}}\right]_0^x = 0.1$

or, $e^{-x/10} - 1 = -0.1$

or, $e^{-x/10} = 0.9$

or, $-\frac{x}{10} = \ln(0.9)$

$\therefore\ x = 1.05$ minutes.