Exercise 1.2

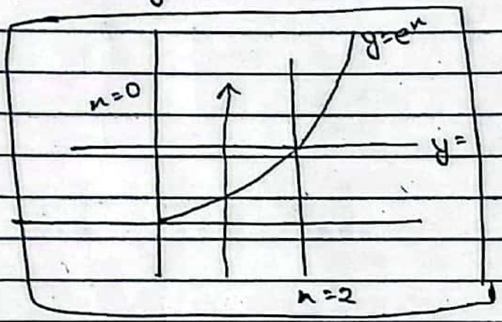
A.

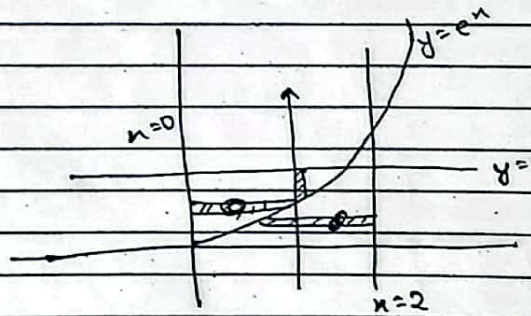
i. $\int_0^2 \int_1^{e^x} dy\, dx$

$\Rightarrow$ Range R:

$1 \le y \le e^x$ and $0 \le x \le 2$

$y = 1$, $y = e^x$ $x = 0$, $x = 2$.





Changing the order.

$\int_0^1 \int_0^{\ln y} dx\, dy$

$= \int_0^1 [x]_0^{\ln y}\, dy$

$= \int_0^1 \ln y\, dy$

$= [y \ln y - y]_0^1$

$= [(0-1) - 0]$

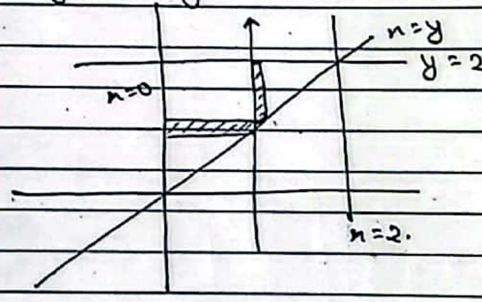
$= -1$

iii) $\int_0^2 \int_x^2 y^2 \sin xy \, dy \, dx$

⇒ Range R:

$x \le y \le 2$ $0 \le x \le 2$

$y = x, \; y = 2$ $x = 0, \; x = 2.$



Changing the order,

$$\int_0^2 \int_0^y y^2 \sin xy \, dx \, dy$$

$$= \int_0^2 y^2 \left[\frac{-\cos xy}{y}\right]_0^y dy$$

$$= -\int_0^2 y \left[\cos y^2 - 1\right] dy$$

$$= -\left[\frac{1}{2}\int_0^2 2y \cos y^2 \, dy - \int_0^2 y \, dy\right]$$

$$= -\frac{1}{2} \times \left[\sin y^2\right]_0^2 + \left[\frac{y^2}{2}\right]_0^2$$

$$= -\frac{1}{2} \sin 4 + \frac{1}{2} \times 4$$

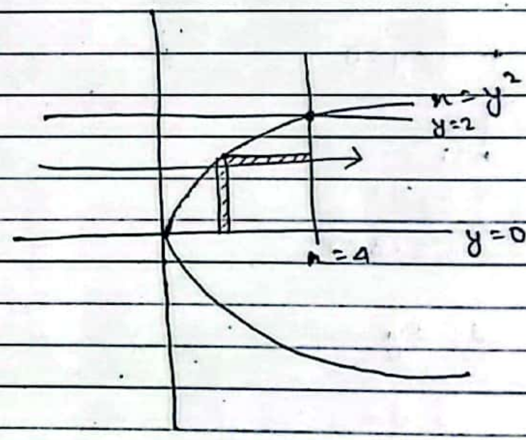
$$= 2 - \frac{1}{2} \sin 4$$

v) $\int_0^2 \int_{y^2}^4 y \cos x^2 \, dx \, dy$

⇒ Range R:

$0 \le y \le 2$ and $y^2 \le x \le 4.$

$y = 0, \; y = 2$ $x = y^2, \; x = 4.$



Changing the order

$$\int_0^4 \int_0^{\sqrt{x}} y \cos x^2 \, dy \, dx$$

$$= \int_0^4 \left[\frac{y^2}{2}\right]_0^{\sqrt{x}} \cos x^2 \, dx$$

$$= \frac{1}{4}\int_0^4 \cos x^2 \times 2x \, dx$$

$$= \frac{1}{4}\left[\sin x^2\right]_0^4$$

$$= \frac{1}{4} \sin 16$$

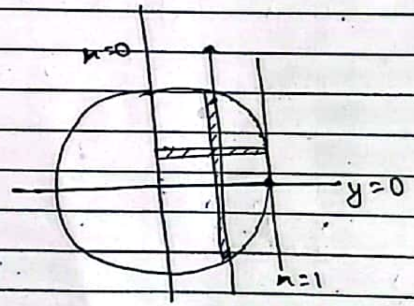
vi) $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$

$\Rightarrow$ Range R:

$0 \le x \le 1$ and $0 \le y \sqrt{1-x^2}$

$x=0, \; x=1$ $y=0, \; y=\sqrt{1-x^2}$

$x^2+y^2=1$



Changing the order,

$\int_{-1}^{1} \int_0^1 y^2 \, dx \, dy$

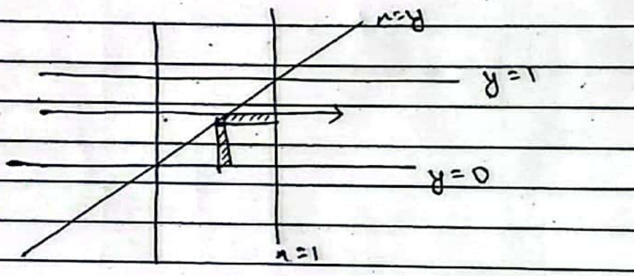
$= \int_{-1}^{1}$

vii) $\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$

⇒ Range R:

$0 \le y \le 1$ and $y \le x \le 1$

$y = 0,\ y = 1$ $\quad x = y,\ x = 1$



Changing the order,

$$\int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx$$

$$= \int_0^1 x^2 \left[\frac{e^{xy}}{x}\right]_0^x dx$$

$$= \int_0^1 x \left[e^{x^2} - 1\right] dx$$

$$= \frac{1}{2}\int_0^1 2x\, e^{x^2}\, dx - \int_0^1 x\, dx$$

$$= \frac{1}{2}\left[e^{x^2}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

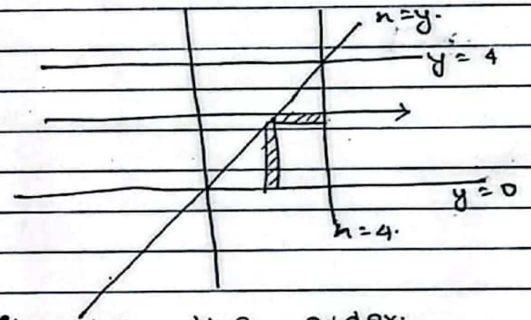
$$= \frac{1}{2}[e-1] - \frac{1}{2} = \frac{e}{2} - \frac{1}{2} - \frac{1}{2} = \frac{e-2}{2}$$

viii) $\int_0^4 \int_y^4 \frac{x\, dx\, dy}{x^2 + y^2}$

⇒ Range R:

$0 \le y \le 4$, $\quad y \le x \le 4$

$y = 0,\ y = 4$ $\quad x = y,\ x = 4.$



Changing the order.

$$\int_0^4 \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_0^4 x \times \frac{1}{x}\left[\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^4 \tan^{-1} 1 - 0\, dx$$

$$= \frac{\pi}{4} \times 4$$

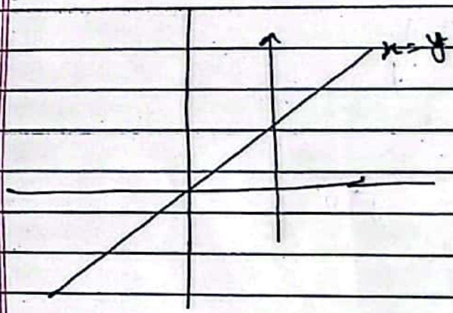
$$= \pi$$

$$\int_0^{\infty}\int_x^{\infty} \frac{e^y}{y}\, dy\, dx.$$

⇒ Range R:

$x \le y \le \infty$ $\qquad 0 \le x \le \infty$

$y = x, \quad y = \infty$ $\qquad x = 0, \quad x = \infty$

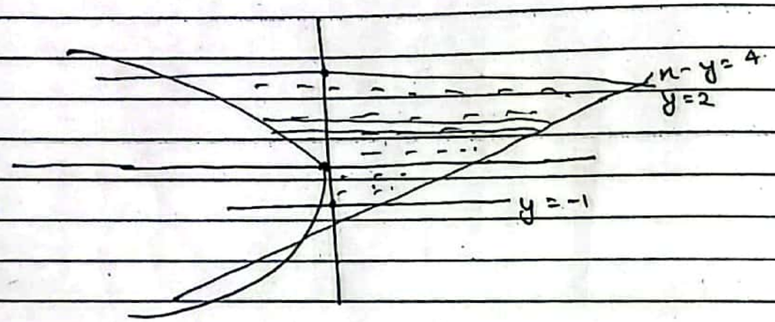


$$\int_0^{\infty}\int_y^{\infty} \frac{e^y}{y}\, dx\, dy$$

Sketch the region bounded by the curve.

$y^2 = -x \quad , \quad x - y = 4 \quad , \quad y = -1 \quad , \quad y = 2.$

Find its area by using double integral



$$\int_{-1}^{2}\int_{-y^2}^{y+4} 1 \; dx\,dy$$

$$= \int_{-1}^{2} [x]_{-y^2}^{y+4} \, dy$$

$$= \int_{-1}^{2} [y + 4 + y^2] \, dy$$

$$= \left[\frac{y^2}{2} + 4y + \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{4}{2} + 8 + \frac{8}{3}\right) - \left(\frac{1}{2} - 4 - \frac{1}{3}\right)$$

$$= \frac{33}{2}$$

6. Exercise 1.2

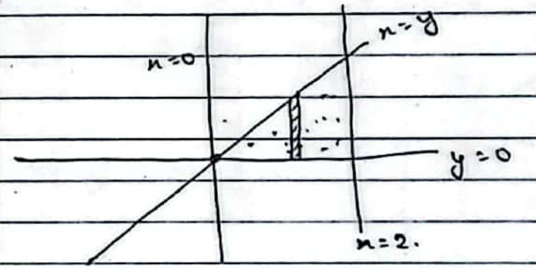
B.

1. $\int_0^2 \int_0^x y \, dy \, dx$

$\Rightarrow$ Region R:

$0 \le y \le x$ and $0 \le x \le 2.$

$y = 0, \; y = x$ $\qquad$ $x = 0, \; x = 2$



To change polar coordinate

$x = r\cos\theta, \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta.$

For limit of r: $\qquad$, limit of θ:

$r = 0 \qquad x = 2$ $\qquad\qquad$ $\theta = 0 \; , \; y = x$

$r\cos\theta = 2$ $\qquad$ $r\sin\theta = r\cos\theta$

$r = 2\sec\theta$ $\qquad$ $\tan\theta = 1$

$0 \le r \le 2\sec\theta.$ $\qquad$ $\theta = \pi/4$

$0 \le \theta \le \pi/4.$

$$\int_0^{\pi/4}\int_0^{2\sec\theta} r\sin\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{2\sec\theta} r^2 \sin\theta\ dr\ d\theta$$

$$= \int_0^{\pi/4} \frac{\sin\theta}{3}\left[r^3\right]_0^{2\sec\theta}\ d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \sin\theta\ 8\sec^3\theta\ d\theta$$

$$= \frac{8}{3}\int_0^{\pi/4} \tan\theta\ \sec^2\theta\ d\theta$$

$$= \frac{8}{3}\int_0 \left[\frac{\tan^2\theta}{2}\right]_0^{\pi/4}$$

$$= \frac{4}{3}(1-0)$$

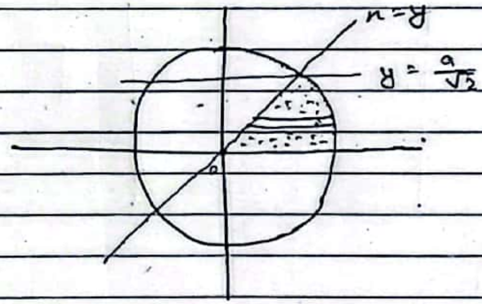
$$= 4/3$$

3) $$\int_a^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} x\ dx\ dy.$$

$\Rightarrow$ Range R:

$a \le y \le a/\sqrt{2}.$ and, $y \le x \le \sqrt{a^2-y^2}$

$y = a,\quad y = \frac{a}{\sqrt{2}}$. $x = y,\quad x^2+y^2=a^2$



$x = r\cos\theta$, $y = r\sin\theta.$

$dx\,dy = r\,dr\,d\theta.$

limit of r, limit of θ.

$r=0,\quad r=a$ $\qquad \theta = 0,\quad \theta = \pi/4.$

$$\int_0^{\pi/4}\int_0^a r\cos\theta\ r\ dr\ d\theta$$

$$= \int_0^{\pi/4} \cos\theta\left[\frac{r^3}{3}\right]_0^a d\theta = \frac{a^3}{3}\int_0^{\pi/4}\cos\theta\ d\theta$$

$$= \frac{a^3}{3}\left[\sin\theta\right]_0^{\pi/4} = \frac{a^3}{3}\times\frac{1}{\sqrt{2}} = \frac{a^3}{3\sqrt{2}}$$

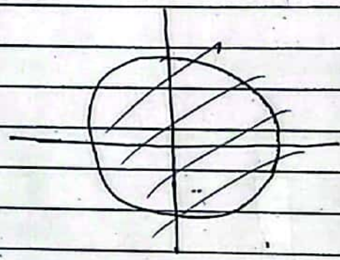
$$\int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{m+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

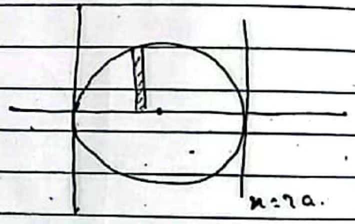
8th Bhadra

Q. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx.$

$0 \le y \le \sqrt{2ax-x^2}$ $\qquad 0 \le x \le 2a$

$y=0, \quad y^2+(x-a)^2=a^2 \qquad x=0, \; x=2a.$





$x=2a.$

Limit of r,

$r=0$ to $y^2 = 2ax - x^2$

$r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$

Limit of θ,

$\theta = 0$ to $\theta = \pi/2$

Then.

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 16a^4 \cos^4\theta \, d\theta.$$

$$= 4a^2 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= 4a^2 \int_0^{\pi/2} \cos^4\theta \,(\sin^0\theta)\, d\theta$$

$$= 4a^2 \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{2\times\Gamma\frac{6}{2}}$$

$$= 2a^2 \times \frac{\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2}\times\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= 2a^2 \times \frac{3}{2}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= \frac{3\pi a^4}{4}$$

$\iint_R dA \to$ area

$\iint_R f(x)\, dA \to$ volume

Exercise 1.3

Imp

2) The base ...., find the volume

→ Solution Here,

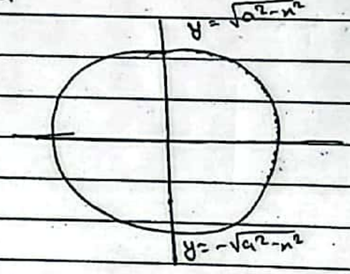
R: $x^2+y^2=a^2$

is the circle having centre (0,0) and radius a

$y=\pm\sqrt{a^2-x^2}$

Also,

$$z=\frac{x^2+y^2}{a}$$



Required volume is,

$$V=\iint_R z\, dA$$

$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{x^2+y^2}{a}\, dy\, dx$$

Limit of r, $r=0$ to $r=a$

Limit of θ, $\theta=0$ to $\theta=2\pi$

$$\int_0^a\int_0^{2\pi} \frac{r^2}{a}\, r\, d\theta\, dr$$

$$=\frac{1}{a}\int_0^a r^3\times 2\pi\, dr$$

$$=\frac{2\pi}{a}\int_0^a r^3\, dr$$

$$=\frac{2\pi}{a}\left[\frac{r^4}{4}\right]_0^a$$

$$=\frac{\pi a^4}{2a}$$

$$=\frac{\pi a^3}{2}$$

4) Solution

$y=4-x^2$ — (i)

$y=3x$ — (ii)

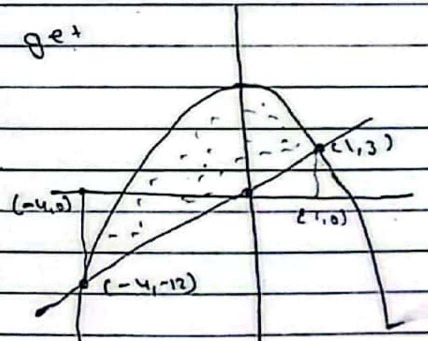
Solving (i) & (ii), we get

$x^2+3x-4=0$

$x^2+4x-x-4=0$

$x(x+4)-1(x+4)=0$

$\therefore x=1,-4$



$$\text{volume}=\int_{-4}^{1}\int_{3x}^{4-x^2}(x+4)\, dy\, dx$$

$$=\int_{-4}^{1}(x+4)(4-x^2-3x)\, dx$$

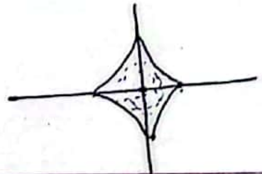
$$=\int_{-4}^{1}(4x-x^3-3x^2+16-4x^2-12x)\, dx$$

$$=\int (-x^3-\cancel{4x^2}-8x+16)\, dx$$

$$= \left[-\frac{x^4}{4} - \frac{7x^3}{3} - \frac{8x^2}{2} + 16x\right]_{-4}^{1}$$

$$= \left[\frac{+1}{4} + \frac{7}{3} + \frac{8}{2} - 16 - \left(\frac{\cancel{128}\ 216}{4} - \frac{7\times 64}{3} + \frac{6\times 16}{2} - 64\right)\right]$$

$$= \frac{625}{12}.$$



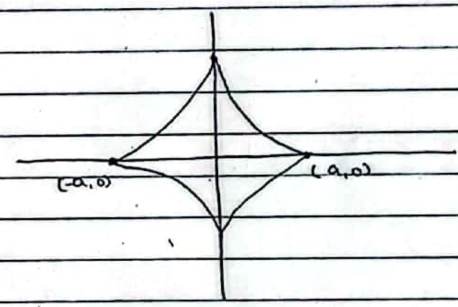
7.6. Find the area of asteroid.

$x^{2/3} + y^{2/3} = a^{2/3}.$

⇒ Solution

Given curve is:

$$x^{2/3} + y^{2/3} = a^{2/3}$$



Now

$$y^{2/3} = a^{2/3} - x^{2/3}$$

$$y = (a^{2/3} - x^{2/3})^{3/2}$$

$$\text{Required area} = 4\int_0^a \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy\, dx$$

$$= 4\int_0^a (a^{2/3} - x^{2/3})^{3/2}\, dx$$

Put

$x = a\sin^3\theta$ when, $x = 0,\ \theta = 0$

$\frac{dx}{d\theta} = a\, 3\sin^2\theta\cos\theta.$ $\quad x = a,\ \theta = \frac{\pi}{2}.$

$$= 4\int_0^{\pi/2} (a^{2/3} - a^{2/3}\sin^2\theta)^{3/2} \cdot 3a\sin^2\theta\cos\theta\, d\theta$$

$= 4a\int_0^{\pi/2} (1-\sin^2\theta)^{3/2}\ 3a\sin^2\theta\cdot\cos\theta\, d\theta$

$= 12a^2\int_0^{\pi/2} \cos^3\theta\cdot \sin^2\theta\cos\theta\, d\theta$

$= 12a^2\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta$

$= 12a^2 \dfrac{\Gamma\frac{3}{2}\ \Gamma\frac{5}{2}}{2\ \Gamma\frac{8}{2}}$

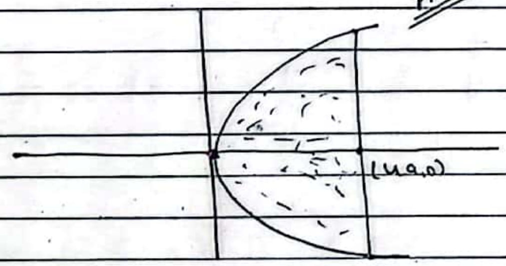
$= 6a^2 \times \dfrac{\frac{1}{2}\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{\Gamma 4}$

$= \dfrac{\cancel{6}^{3} a^2\times\cancel{3}\times\pi}{8\times\cancel{3}\times\cancel{2}\times 1}$

$= \dfrac{3\pi a^2}{8}$

8.a) Find the area bounded by $y^2 = 4ax$ and its latus rectum.

Ans latus rectum passes from $(a, 2a)$ and $(a, -2a)$



limit of x limit of y

$0 \le x \le 4a$ $0 \le y \le 2a\sqrt{ax}$

Required area is: $2\int_0^{4a}\int_0^{2\sqrt{ax}} dy\, dx$

$2\int_0^{4a} 2\sqrt{ax}\, dx$

$= 4\sqrt{a}\int_0^{4a} \sqrt{x}\, dx$

$4\sqrt{a}\left[\dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^{4a}$

$4\sqrt{a}\times\dfrac{2}{3}\left[x^{3/2}\right]_0^{4a}$

$\dfrac{8\sqrt{a}}{3}\left[(4a)^{3/2} - 0\right]$

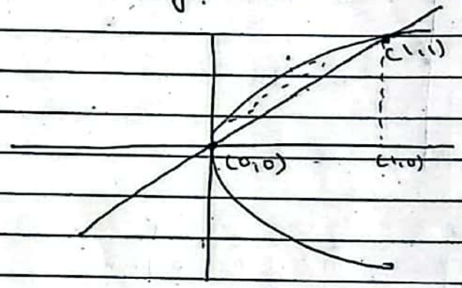
$\dfrac{8\sqrt{a}}{3}\times 2^{2\times\frac{3}{2}}\times a^{3/2}$

$\dfrac{8}{3}\times a^{\frac{1}{2}+\frac{3}{2}}\times 2^3 =$

8.c) Area of region bounded by $y^2 = x$ and st. line $y = x$.

$\Rightarrow$ Given curve is

$$y^2 = x \quad \text{---} (i)$$
$$y = x \quad \text{---} (ii)$$



limit of x: $0 \le x \le 1$

limit of y: $x \le y \le \sqrt{x}$

$$\therefore \int_0^1 \int_x^{\sqrt{x}} dy\, dx$$

$$\int_0^1 [y]_x^{\sqrt{x}}\, dx$$

$$\int_0^1 (\sqrt{x} - x)\, dx$$

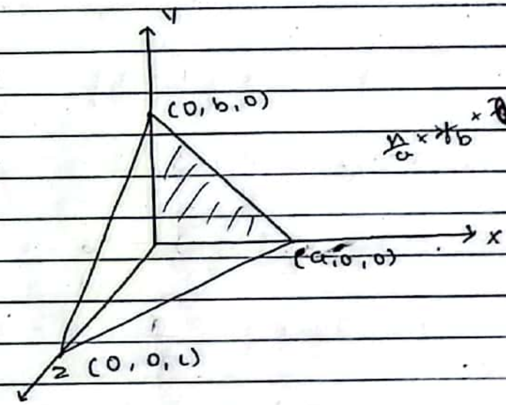
$$\left[x^{3/2} \times \frac{2}{3} - \frac{x^2}{2}\right]_0^1 = \cancel{\frac{2}{3} - \frac{1}{2} - \left(\frac{2}{3} \cdot 0\right)}$$

$$\cancel{\frac{2}{3} - \frac{1}{2}} \quad \cancel{\frac{4-3}{6} = \frac{1}{6}}$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}.$$

g) Find the volume bounded by the coordinate planes and plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

$\Rightarrow$



$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Here,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b}$$

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

Now,

Required volume $= \iint_R z\, dx\, dy$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} z\, dy\, dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= c\int_0^a \left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right) - \frac{xb}{a}\left(1-\frac{x}{a}\right) - \frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= c\int_0^a \left[\frac{b}{a}(a-x) - \frac{xb(a-x)}{a^2} - \frac{b^{\cancel{2}}(a-x)^2}{2a^2\cancel{b}}\right] dx$$

$$= \frac{c\times b}{a}\int_0^a a-x - \frac{x}{a}(a-x) - \frac{1}{2a}(a-x)^2 \, dx.$$

$$= \frac{bc}{a}\int_0^a a-x - \frac{ax}{a} + \frac{x^2}{a} - \frac{1}{2a}(a^2 - 2ax + x^2)\, dx$$

$$= \frac{bc}{a}\int_0^a \left(a - x - x + \frac{x^2}{a} - \frac{a}{2} + \frac{2ax}{2a} - \frac{x^2}{2a}\right) dx$$

$$= \frac{bc}{a}\int_0^a \left(a - 2x + \frac{x^2}{a} - \frac{a}{2} + x - \frac{x^2}{2a}\right) dx$$

$$= \frac{bc}{a}\left[ax - \frac{2x^2}{2} + \frac{x^3}{3a} - \frac{ax}{2} + \frac{x^2}{2} - \frac{x^3}{6a}\right]_0^a$$

$$= \frac{bc}{a}\left[\cancel{a^2} - \cancel{a^2} + \frac{a^3}{3a} - \frac{\cancel{a^2}}{2} + \frac{\cancel{a^2}}{2} - \frac{a^3}{6a} - 0\right]$$

$$= \frac{bc}{a}\times\left(\frac{a^3}{3a} - \frac{a^3}{6a}\right)$$

$$= \frac{bc}{a}\times\frac{6a^3 - 3a^3}{18a}$$

$$= \frac{bc}{\cancel{a}}\times\frac{3a^3}{18\cancel{a}} = \frac{abc}{6}$$ Ans

Q) Evaluate $\int_0^2\int_0^{\sqrt{4-y^2}} \cos(x^2+y^2)\, dx\, dy.$

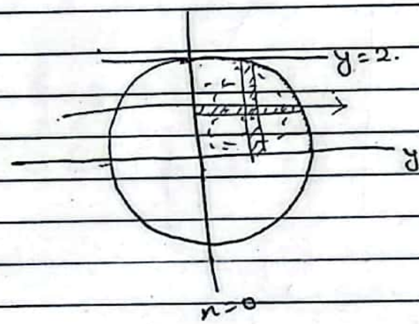
Limit of x: $0 \le x \le \sqrt{4-y^2}$

Limit of y: $0 \le y \le 2.$

$x = 0,$

&

$x^2 + y^2 = 2^2$



Changing the order,

$0 \le x \le 2$ $\quad$ $0 \le y \le \sqrt{4-x^2}.$

Changing into polar.

limit of θ: $0 \le \theta \le \pi/2$ $\quad$ limit of r: $0 \le r \le 2.$

$$\therefore \frac{1}{2}\int_0^{\pi/2}\int_0^2 \cos r^2 \, 2r \, dr \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} [\sin r^2]_0^2 \, d\theta \qquad = \frac{\sin 4}{2}\times\frac{\pi}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin 4 \, d\theta \qquad = \frac{\pi \sin 4}{4}$$

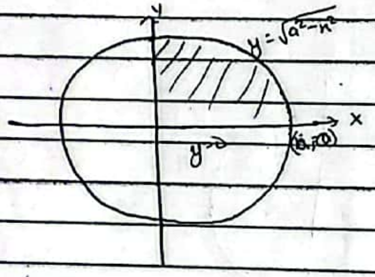
10) Find volume of $x^2+y^2+z^2=a^2$

⇒

Solution

$$x^2+y^2+z^2=a^2$$

$$z=\sqrt{a^2-x^2-y^2}$$



Required volume is

$$8\iint_R z \, dA$$

$$=8\int_0^a\int_0^{\sqrt{a^2-x^2}} z \, dy \, dx$$

$$=8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} \, dy \, dx$$

To change into polar coordinate

$x=r\cos\theta$, $y=r\sin\theta$

$dxdy = r\,dr\,d\theta$

$$=-\frac{18}{2}\int_0^{\pi/2}\int_0^a \sqrt{a^2-r^2} \cdot -2r\,dr\,d\theta$$

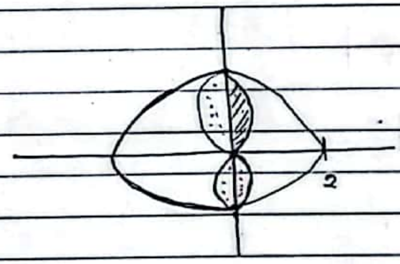
$$=-4\int_0^{\pi/2}\left[(a^2-r^2)^{3/2}\times\frac{2}{3}\right]_0^a d\theta$$

$$=-\frac{8}{3}\int_0^{\pi/2}[0-a^3]\,d\theta$$

$$=\frac{8a^3}{3}\frac{\pi}{2}$$

$$=\frac{4\pi a^3}{3}$$ Ans

11) $r=a(1-\cos\theta)$, $r=a(1+\cos\theta)$



Required Area is $4\iint_R r\,dr\,d\theta$

$$=4\int_0^{\pi/2}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$=\frac{4}{2}\int_0^{\pi/2}\left[r^2\right]_0^{a(1-\cos\theta)} d\theta$$

$$=2\int_0^{\pi/2}\left[a^2(1-\cos\theta)^2\right]d\theta$$

$$= 2a^2 \int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta$$

$$= 2a^2 \int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 2a^2 \left[\int_0^{\pi/2} d\theta - 2\int_0^{\pi/2} \cos\theta + \frac{1}{2}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta \right]$$

$$= 2a^2 \left[\frac{\pi}{2} - 2[\sin\theta]_0^{\pi/2} + \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \right]$$

$$= 2a^2 \left[\frac{\pi}{2} - 2 + \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}\times 0 - (0+0)\right]\right]$$

$$= 2a^2 \left[\frac{\pi}{2} - 2 + \frac{1}{2}\times\frac{\pi}{2} \right]$$

$$= 2a^2 \left[\frac{\pi}{2} - 2 + \frac{\pi}{4} \right]$$

$$= 2a^2 \left[\frac{2\pi+\pi}{4} - 2 \right]$$

$$= 2a^2 \left[\frac{3\pi}{4} - 2 \right]$$

$$= a^2 \left[\frac{3\pi}{2} - 4 \right]$$

Ans

Exercise 1.4

i) $\int_2^3 \int_1^2 \int_0^1 (x+y+z)\, dx\, dy\, dz$.

$$= \int_2^3 \int_1^2 \left(\frac{x^2}{2} + yx + zx\right)_0^1 dy\, dz$$

$$= \int_2^3 \int_1^2 \left(\frac{1}{2} + y + z\right) dy\, dz$$

$$= \int_2^3 \left[\frac{1}{2}y + \frac{y^2}{2} + zy\right]_1^2 dz$$

$$= \int_2^3 \left[\frac{2}{2} + \frac{4}{2} + 2z - \frac{1}{2} - \frac{1}{2} - z\right] dz$$

$$= \int_2^3 [3 + z - 1]\, dz$$

$$= \int_2^3 (2+z)\, dz$$

$$= \left[2z + \frac{z^2}{2}\right]_2^3$$

$$= \left[6 + \frac{9}{2} - 4 - \frac{4}{2}\right]$$

$$= \cancel{\left[\frac{9}{2}\right]} \; \frac{9}{2}$$

2.i) $\int_{-1}^{1}\int_{0}^{2}\int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

$= \int_{-1}^{1}\int_{0}^{2} \left[xy + \frac{y^2}{2} + zy \right]_{x-z}^{x+z} dx\, dz$

$= \int_{-1}^{1}\int_{0}^{2} \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z) - x(x-z) + \frac{(x-z)^2}{2} + z(x-z) \right] dx\, dz.$

$= \int_{-1}^{1}\int_{0}^{2} \left[\cancel{x^2} + xz + \frac{(x+z)^2}{2} + xz + z^2 - \cancel{x^2} + \cancel{xz} + \frac{(x-z)^2}{2} - \cancel{xz} + z^2 \right] dx\, dz$

$= \int_{-1}^{1}\int_{0}^{2} \left[2xz + 2z^2 + \frac{1}{2}(\cancel{x^2} - 2xz + \cancel{z^2} - \cancel{x^2} - 2xz - \cancel{z^2}) \right]$

$= \int_{-1}^{1}\int_{0}^{2} \left[\cancel{2xz} + 2z^2 - \cancel{2xz} \right] dx\, dz$

$= \int_{-1}^{1}\int_{0}^{2} 2z^2\, dx\, dz$

$= \int_{-1}^{1} 2z^2 \times 2\, dz$

$= 4\left[\frac{z^3}{3} \right]_{-1}^{1}$

$= \frac{4}{3}[1+1] = \frac{8}{3}$

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3) Find volume of hemisphere $x^2+y^2+z^2=a^2$ by triple integral

$\Rightarrow$ Solution

Given,

$x^2+y^2+z^2=a^2$

For the limit of z,

$z = \pm\sqrt{a^2-x^2-y^2}$ For hemisphere, $0 \le z \le \sqrt{a^2-x^2-y^2}$

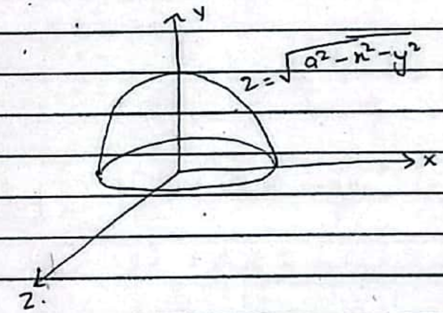
For the limit of y, $z=0$

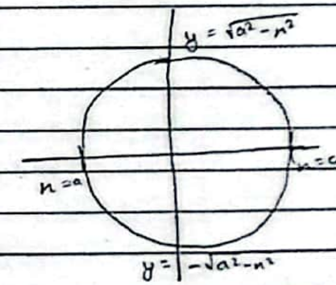
$\therefore x^2+y^2=a^2$

$y = \pm\sqrt{a^2-x^2}$, $-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

For the limit of x, $y=0, z=0$ $-a \le x \le a$

$x = \pm a$





Now, volume is

$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$

$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$

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Changing into polar,

$x = r\cos\theta, \quad y = r\sin\theta.$

$dy\,dx = r\,dr\,d\theta$

limit of θ limit of r

$0 \le \theta \le 2\pi$ $0 \le r \le a.$

$$\frac{-1}{2}\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\cdot -2r\,dr\,d\theta$$

$$= \frac{-1}{2}\int_0^{2\pi}\left[\frac{(a^2-r^2)^{3/2}}{3}\times 2\right]_0^a d\theta$$

$$= \frac{-1}{3}\int_0^{2\pi}(0-a^3)\,d\theta$$

$$= \frac{a^3}{3}\times 2\pi = \frac{2\pi a^3}{3}.$$

4) $\iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}$; $V: 0 \le x \le 1,\ 0 \le y \le (1-x)$

$0 \le z \le (1-x-y)$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[(\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1)^{-2}-(x+y+1)^{-2}\right]dy\,dx.$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[2^{-2}-(x+y+1)^{-2}\right]dy\,dx.$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left(\frac{1}{4}-(x+y+1)^{-2}\right)dy\,dx.$$

$$= \frac{-1}{2}\int_0^1\left(\frac{1}{4}[y]_0^{1-x}-\left[\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x}\right)dx.$$

$$= \frac{-1}{2}\int_0^1\left(\frac{1}{4}(1-x)+\left[(\cancel{x}+1-\cancel{x}+1)^{-1}-(x+1)^{-1}\right]\right)dx$$

$$= \frac{-1}{2}\int_0^1\left(\frac{1}{4}(1-x)+2^{-1}-(x+1)^{-1}\right)dx.$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1}{4}(1-x)+\frac{1}{2}-(x+1)^{-1}\right]dx.$$

$$= -\frac{1}{2}\left[\frac{1}{4}\left(x-\frac{x^2}{2}\right)_0^1+\frac{1}{2}[x]_0^1-[\ln(x+1)]_0^1\right]$$

$$= \frac{-1}{2}\left[\frac{1}{4}\left(1-\frac{1}{2}\right)+\frac{1}{2}-\ln(2)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4}\times\frac{1}{2}+\frac{1}{2}-\ln(2)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{8}+\frac{1}{2}-\ln(2)\right]$$

$$= -\frac{1}{2}\left[\frac{1+4}{8}-\ln(2)\right]$$

$$= -\frac{1}{2}\left[\frac{5}{8}-\ln(2)\right]$$

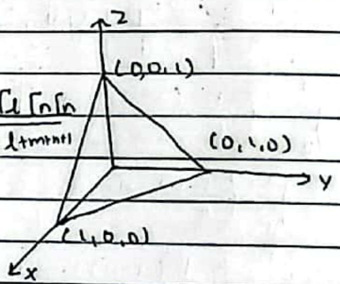
$$= \frac{\ln(2)}{2}-\frac{5}{16}$$

Integration by Using Dirichlet Theorem

Dirichlet Theorem: If V is the region bounded by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x+y+z \leq 1$ (i.e fi octant) as shown in figure.

Then,

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dz\, dy\, dx = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma (l+m+n+1)}$$



5) Evaluate:

$$\iiint_V x^2\, dx\, dy\, dz$$ over the region V

bounded by $x=0$, $y=0$, $z=0$ and.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

⇒ Solution

Given,

$$\iiint_V x^2\, dx\, dy\, dz \quad \text{——(i)}$$

and $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{——(ii)}$$

To apply Dirichlet theorem,

$\frac{x}{a} = u$ $\quad$ $\frac{y}{b} = v$ $\quad$ $\frac{z}{c} = w$

$dx = a\,du$ $\quad$ $dy = b\,dv$ $\quad$ $dz = c\,dw$.

Then,

$$\iiint_V a^2u^2\; a\,du\; b\,dv\; c\,dw$$

$$= a^3bc \iiint_V u^{3-1}\, v^{1-1}\, w^{1-1}\, du\, dv\, dw$$

$$= a^3bc\, \frac{\Gamma 3\, \Gamma 1\, \Gamma 1}{\Gamma (3+1+1+1)}$$

$$= \frac{a^3bc \times 2}{\Gamma 6}$$

$$= \frac{a^3bc \times 2}{5!}$$

$$= \frac{a^3bc \times \cancel{2}}{5 \times 4 \times 3 \times \cancel{2}}$$

$$= \frac{a^3bc}{60}$$ □

8. Find the volume of ellipsoid $\frac{n^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by triple integral.

⇒

$$\frac{n^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{z^2}{c^2} = 1 - \frac{n^2}{a^2} - \frac{y^2}{b^2}$$

$$z = \pm c \sqrt{1 - \frac{n^2}{a^2} - \frac{y^2}{b^2}}$$

For limit of y, $z = 0$

$$\frac{n^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{n^2}{a^2}$$

$$y = \pm b\sqrt{1 - \frac{n^2}{a^2}}$$

For the limit of n, $z = 0$, $y = 0$.

$$\frac{n^2}{a^2} = 1$$

$$n = \pm a.$$

$$\int_{-a}^{a} \int_{-b\sqrt{1-\frac{n^2}{a^2}}}^{b\sqrt{1-\frac{n^2}{a^2}}} \int_{c\sqrt{1-\frac{n^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{n^2}{a^2}}}$$

$4u_1'' - u'v_2 + u v_3$

Example 18

Find the Fourier series of $f(x) = x - x^2$ in $-1 < x < 1$.

⇒ Given,

$f(x) = x - x^2 \quad , \quad -1 < x < 1$

We know that Fourier series of $f(x)$ is $(-L, L)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n x\pi}{L} + b_n \sin\frac{n x\pi}{L}\right) \quad -①$$

Given, Period of $f(x)$ is $1-(-1) = 2$

$2L = 2$

$\therefore L = 1$

We know,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2\cdot 1}\int_{-1}^{1}(x - x^2)\,dx$$

$$= \frac{1}{2}\cdot -\int_{-1}^{1} x^2\,dx$$

$$= \frac{-2}{2}\int_{0}^{1} x^2\,dx$$

$$= -1\left[\frac{x^3}{3}\right]_0^1$$

$$= -\frac{1}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{1}\int_{-1}^{1}(x - x^2)\cos n\pi x\,dx$$

$$= \int_{-1}^{1}(x\cos n\pi x - x^2\cos n\pi x)\,dx$$

$$= -\int_{-1}^{1} x^2\cos n\pi x\,dx$$

$$= -2\int_{0}^{1} x^2\cos n\pi x\,dx$$

$$= -2\left[x^2\frac{\sin n\pi x}{n\pi} - 2x\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= -2\left[x^2\frac{\sin n\pi x}{n\pi} + 2x\frac{\cos n\pi x}{n^2\pi^2} - \frac{2\sin n\pi x}{n^3\pi^3}\right]_0^1$$

$$= -2\left[\cancel{1\,\frac{\sin n\pi x}{\pi}}\right.$$

$$= -2\left[1\,\frac{\sin n\pi}{n\pi} + \frac{2\cos n\pi}{n^2\pi^2} - \frac{2\sin n\pi}{n^3\pi^3} - (0+0-0)\right]$$

$$= -2\left[\cancel{\frac{\sin n\pi}{n\pi}}^{0} + \frac{2\cos n\pi}{n^2\pi^2} - \cancel{\frac{2\sin n\pi}{n^3\pi^3}}^{0}\right]$$

$$= -2\left[\frac{2(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \sin n\pi x\ dx$$

$$= \frac{1}{1}\int_{-1}^{1} (x-x^2)\ \sin n\pi x\ dx$$

$$= \int_{-1}^{1} x \sin n\pi x\ dx \quad + 0$$

$$= 2\int_{0}^{1} x \sin n\pi x\ dx$$

$$= 2\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[\frac{-\cos n\pi}{n\pi} - \left(\frac{-\sin n\pi}{n^2\pi^2}\right) - 0\right]$$

$$= 2\left[-\frac{(-1)^n}{n\pi} - 0\right]$$

$$= \frac{2\,(-1)^{n+1}}{n\pi}$$

Now,

Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\cancel{= \frac{-1}{3} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi x + 2(-1\right.}$$

$$= \frac{-1}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi x + \frac{2(-1)^{n+1}}{n\pi}\sin n\pi x\right]$$

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Fourier Series for Even and Odd Function
(Fourier half range cosine and sine series)

We know that Fourier series of $f(x)$ in $(-L, L)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \text{--- (1)}$$

where,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

(1) If $f(x)$ is even function,

$\Rightarrow a_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$\Rightarrow a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$\Rightarrow b_n = 0$

Then, series (1) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \quad \text{--- (2)}$$

where $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$\int e^{ax} \sin bx$

$\frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$.

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The series (2) is know as fourier cosine series in half range $(0, L)$.

(2) If : $F(x)$ is odd function,

$a_0 = 0$

$a_n = 0$

$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$.

$$\therefore \boxed{F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}} \quad \text{—— (3)}$$

where,

$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$.

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Fourier Series of Even / odd Function in $[-\pi, \pi]$

We know that, fourier series of $f(x)$ in $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{—— (1)}$$

where,

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \, dx$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$.

(1) If $f(x)$ is even function,

$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x) \, dx$.

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

$b_n = 0$

Then series is:

$F(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx)$

Then ~~series~~ where,

$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x) \, dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$.

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(2) If $F(x)$ is odd function

$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} F(x)\sin nx\,dx.$

The series becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

x — x — x — x — x — x — x — x — x

Example 00

Find the Fourier cosine and sine series of the function $f(x)=x$ in the interval $(0, L)$.

⇒ Solution

$$F(x) = x, \quad 0 < x < L$$

→ First we find fourier cosine series

$$a_0 = \frac{1}{L}\int_0^L F(x)\,dx$$

$$= \frac{1}{L}\int_0^L x\,dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{1}{L}\times\frac{L^2}{2}$$

$$= \frac{L}{2}$$

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$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[x\cdot\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}} + \frac{\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right]_0^L$$

$$= \frac{2}{L}\left[\frac{x\cdot L}{n\pi}\sin\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{2}{L}\left[\frac{L^2\cdot 0}{n\pi} + \frac{L^2}{n^2\pi^2}\cos n\pi - \left(0 + \frac{L^2}{n^2\pi^2}\right)\right]$$

$$= \frac{2}{L}\left[\frac{L^2(-1)^n}{n^2\pi^2} - \frac{L^2}{n^2\pi^2}\right]$$

$$= \frac{2}{K}\left[\frac{L^2[(-1)^n - 1]}{n^2\pi^2}\right]$$

$$= \frac{2L}{n^2\pi^2}[(-1)^n - 1]$$

∴ Fourier cosine series is

$$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi x}{L}$$

Fourier sine series is.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L x \cdot \sin\frac{n\pi x}{L}\, dx.$$

$$= \frac{2}{L}\left[x \cdot \left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) + \frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}} \right]_0^L$$

$$= \cancel{\frac{2}{L}\left[\frac{xL}{n\pi}\right.}$$

$$= \frac{2}{L}\left[\frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L} - \frac{xL}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{2}{L}\left[0 - \frac{L^2}{n\pi}(-1)^n - (0-0)\right]$$

$$= \frac{2}{L}\left(-\frac{L^2(-1)^n}{n\pi}\right)$$

$$= \frac{2}{\cancel{L}} \times \frac{L^{\cancel{2}}(-1)^{n+1}}{n\pi}$$

$$= \frac{2L(-1)^{n+1}}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{L}.$$

Laplace Transformation

Definition

Let $t > 0$ and $f(t)$ be a continuous function of t. Then the integral $\int_0^\infty e^{-st} f(t)\,dt$ is defined as the laplace transform of $f(t)$. It is denoted by $F(s)$.

$$\therefore F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Transformation of Elementary functions

1) $L(1) = \frac{1}{s} \quad (s > 0)$

2) $L(e^{at}) = \frac{1}{s-a} \quad (s > a)$

3) $L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$ where, $n = 0, 1, 2, \ldots$

4) $L(\sin at) = \frac{a}{s^2+a^2} \quad (s > 0)$

5) $L(\cos at) = \frac{s}{s^2+a^2} \quad (s > 0)$

6) $L(\sinh at) = \frac{a}{s^2-a^2} \quad (s > |a|)$

7) $L(\cosh at) = \frac{s}{s^2-a^2} \quad (s > |a|)$

$F(s) = \int e \ldots = L\{F(t)\}$

$L^{-1} F(s) = F(t)$

Imp

Linearity of Laplace transformation

Show that Laplace operator is a linear operator.

⇒

Proof:

Let $F(t)$ and $g(t)$ are two functions whose laplace transformation exist. Let a and b are constant.

Then we have,

$$L(aF(t) + b\,g(t)) = a\,L(F(t)) + b\,L(g(t))$$

By definition of Laplace transformation,

$$L[aF(t) + b\,g(t)] = \int_0^\infty e^{-st}(aF(t) + b\,g(t))\,dt$$

$$= \int_0^\infty a\,e^{-st} F(t)\,dt + \int_0^\infty b\,e^{-st} g(t)\,dt$$

$$= a\int_0^\infty e^{-st} F(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= a\,L\{F(t)\} + b\,L\{g(t)\}$$

$$\therefore L\{aF(t) + b\,g(t)\} = a\,L\{F(t)\} + b\,L\{g(t)\}$$

Hence, Laplace operator is a linear operator.

Imp

First shifting Theorem [Multiplication of a function by e^{at}]

Statement : If $L\{F(t)\} = F(s)$. Then,

$$\therefore L(e^{at}F(t)) = F(s-a)$$

OR, $$L\{e^{at}F(t)\} = [F(s)]_{s \to s-a}$$

⇒ Proof:

By definition of Laplace transformation.

$$L\{e^{at}F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\,dt$$

$$= \cancel{\int_0^\infty e^{-(a-s)}}$$

$$= \int_0^\infty e^{-(s-a)t} F(t)\,dt$$

$$\therefore L\{e^{at}F(t)\} = F(s-a)$$

Enercise 3.1

1)

c) $F(t) = e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$.

⇒ Now,

$$L(F(t)) = L\{e^{2t}\} + 4\,L(t^3) - 2\,L(\sin 3t) + 3\,L(\cos 3t)$$

$$= \frac{1}{s-2} + 4 \times \frac{\Gamma 4}{s^4} - 2 \times \frac{3}{s^2+3^2} + 3 \times \frac{s}{s^2+3^2}$$

$$F(s) = \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+3^2} + \frac{3s}{s^2+9} \text{ Ans}$$

$$L^{-1}\{F(s)\} = L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\left(\frac{24}{s^4}\right) - L^{-1}\left(\frac{6}{s^2+3^2}\right) + 3\,L^{-1}\left(\frac{s}{s^2+3^2}\right)$$

$$= e^{2t} + L^{-1}\left(\frac{1}{s^{3+1}} \cdot \frac{\Gamma 3+1}{} \times \frac{24}{3!}\right) - L^{-1}\left(\frac{3}{s^2+3^2} \times \frac{6}{3}\right) + 3\cos 3t$$

$$= e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t.$$

ISMT

09/02. Tuesday

Laplace Transform of the Integral of the Function.

If $L\{F(t)\} = F(s)$. Then $L\left[\int_0^t F(u)\,du\right] = \frac{1}{s} F(s)$

Multiplication of Function by t^n

If $L\{F(t)\} = F(s)$. Then $L\left[t^n F(t)\right] = (-1)^n \frac{d^n}{ds^n} F(s)$

Division of a Function by t

If $L\{F(t)\} = F(s)$. Then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Laplace transformation of Derivative of a Function

If $L\{F(t)\} = F(s)$, Then $L\{F'(t)\} = s\,L\{F(t)\} - F(0)$

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Q) Laplace transformation of.

a) $t \cdot \sinh at$.

$\Rightarrow$ $L\{t \cdot \sinh at\} = (-1)^1 \frac{d}{ds}[F(s)]$

$= \cancel{(-1)^1}$

$F(s) = L\{\sinh at\}$

$F(s) = \frac{a}{s^2-a^2}$

$\therefore L\{t \cdot \sinh at\} = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$

$= -a \frac{d}{ds}(s^2-a^2)^{-1}$

$= a \times 2s \times (s^2-a^2)^{-2}$.

$= \frac{2as}{(s^2-a^2)^2}$.

b) $t e^{-3t} \cos 2t$.

$L\{t e^{-3t} \cos 2t\} = (-1)^1 \frac{d F(s)}{ds}$

$= \cancel{(-1)^1}$

$F(s) = \left.\frac{s}{s^2+4}\right|_{s \to s-a(-3)}$

$= \cancel{\frac{s+a}{s^2+4}} \quad \frac{s+3}{(s+3)^2+4}$.

$\frac{d}{dx}\frac{u}{v} = \frac{\cancel{udv - vdu}}{} \frac{vdu - udv}{v^2}$.

$+5 \quad -5 \quad +3 \quad -3$

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$\frac{d F(s)}{ds} = \frac{d}{ds}\left(\frac{s+3}{(s+3)^2+4}\right)$

$= \frac{(s+3)^2+4 - (s+3)\, 2(s+3)}{[(s+3)^2+4]^2}$

$= \frac{(s+3)^2+4 - 2(s+3)^2}{[(s+3)^2+4]^2}$

$\frac{d F(s)}{ds} = \frac{4-(s+3)^2}{[(s+3)^2+4]^2}$

$\therefore L\{t e^{-3t} \cos 2t\} = \frac{(s+3)^2-4}{[(s+3)^2+4]^2}$.

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Q) $t e^{-t} \cosh t$

$$L\{t e^{-t} \cosh t\} = (-1)^1 \frac{d F(s)}{ds} \quad -(1)$$

$$F(s) = L\{e^{-t} \cosh t\}$$
$$= \left.\frac{s}{s^2-1}\right|_{s \to s+1}$$
$$= \frac{(s+1)}{(s+1)^2-1}$$

Now,

$$\frac{dF(s)}{ds} = \frac{\{(s+1)^2-1\} \cancel{2(s+1)} - (s+1) \times 2(s+1)}{[(s+1)^2-1]^2}$$
$$= \frac{\cancel{2(s+1)}\{(s+1)^2-1\} - 2(s+1)^2}{[(s+1)^2-1]^2}$$
$$= \frac{(s+1)^2-1-2(s+1)^2}{[(s+1)^2-1]^2}$$

$$\therefore L\{t e^{-t} \cosh t\} = \frac{(s+1)^2+1}{[(s+1)^2-1]^2}.$$

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4-a $\frac{\sin at}{t}$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty F(s)\, ds$$
$$= \int_s^\infty L\{\sin at\}\, ds$$
$$= \int_s^\infty \frac{a}{s^2+a^2}\, ds$$
$$= a \times \frac{1}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$
$$= \cot^{-1}\left(\frac{s}{a}\right).$$

$\int_0^t f(u)\,du = \frac{F(s)}{s}$

$L\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

09/03 wednesday

$L\{\frac{F(t)}{t}\} = \int_0^\infty F(s)\,ds$

S. 1

Q) $\int_0^t \frac{\sin t}{e^t}\,dt$

$= \int_0^t e^{-t} \sin t\,dt$

$= \frac{F(s)}{s}$

Here,

$F(s) = L\{e^{-t}\sin t\}$

$= \left[\frac{1}{s^2+1}\right]_{s\to s+1}$

$= \frac{1}{(s+1)^2+1}$

$\therefore \int_0^t \frac{\sin t}{e^t}\,dt = \frac{1}{s} \times \frac{1}{(s+1)^2+1}$

c) $\int_0^t \frac{\sin t}{t}\,dt$

$\Rightarrow \int_s^\infty F(s)\,ds.$

$F(s) = L\{\sin t\}$

$= \frac{1}{s^2+1}$

c) $\int_0^t \frac{\sin t}{t}\,dt.$

$\Rightarrow$ Solution

$L\{\frac{\sin t}{t}\} = \int_s^\infty \frac{1}{s^2+1}\,ds$

$= \left[\tan^{-1} s\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$

$L\{\int_0^t \frac{\sin t}{t}\,dt\} = \frac{1}{s}\cot^{-1} s$

d) $\int_0^t \frac{1-\cos t}{t}\,dt.$

$\Rightarrow$ Solution

$L\{\frac{1-\cos t}{t}\} = L\{\frac{1}{t}\} - L\{\frac{\cos t}{t}\}$

$= \int_s^\infty \frac{1}{s}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+1}\,ds$

$= [\ln s]_s^\infty - \frac{1}{2}[\ln(s^2+1)]_s^\infty$

$= \left[\ln \frac{s}{(s^2+1)^{1/2}}\right]_s^\infty$

$= \ln\left[\frac{1}{(1+\frac{1}{s^2})^{1/2}}\right]_s^\infty$

$= \ln\left(\frac{1}{1}\right) - \ln\left(\frac{s}{(s^2+1)^{1/2}}\right)$

$= 0 - \ln \frac{s}{(1+s^2)^{1/2}}$

$= \ln \frac{(1+s^2)^{1/2}}{(s^2)^{1/2}}$

$= \frac{1}{2}\ln\frac{s^2+1}{s^2}$

$L(e^{at}) = \frac{1}{s-a}$

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e) $\int_0^t \frac{1-e^{-t}}{t}\, dt$

$\Rightarrow L\left\{ \frac{1}{t} - \frac{e^{-t}}{t} \right\}$

$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1} \right) ds$

$= \left[\ln s - \ln(s+1) \right]_s^\infty$

$= \left[\ln \frac{s}{s+1} \right]_s^\infty$

$= \left[\ln \frac{1}{1+\frac{1}{s}} \right]_s^\infty$

$= \left[\ln 1 - \ln \frac{s}{s+1} \right]$

$= \ln\left(\frac{s+1}{s} \right)$

$\therefore \int_0^t \frac{1-e^{-t}}{t}\, dt = \frac{1}{s} \ln\left(\frac{s+1}{s} \right)$

F) $\int_0^t t^2 e^{-t}\, dt$

$\Rightarrow L\{ e^{-t} t^2 \} = \left[\frac{\lceil 3}{s^3} \right]_{s \to s+1}$

$= \frac{2}{(s+1)^3}$

$\int_0^t t^2 e^{-t}\, dt = \frac{1}{s} \frac{2}{(s+1)^3}$

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b) $\int_0^t \frac{\cos at - \cos bt}{t}\, dt$

$\Rightarrow$

$L\left\{ \frac{\cos at - \cos bt}{t} \right\} = \frac{1}{2} \int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2} \right) ds$

$= \frac{1}{2} \left[\ln \frac{s^2+a^2}{s^2+b^2} \right]_s^\infty$

$= \frac{1}{2} \left[\ln \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}} \right]_s^\infty$

$= \frac{1}{2} \left[0 - \ln \frac{s^2+a^2}{s^2+b^2} \right]$

$= \frac{1}{2} \ln\left(\frac{s^2+b^2}{s^2+a^2} \right)$

$\therefore \int_0^t \frac{\cos at - \cos bt}{t}\, dt = \frac{1}{2s} \ln\left(\frac{s^2+b^2}{s^2+a^2} \right)$

4)

a. $L\left\{ \frac{\sin at}{t} \right\} = \int_s^\infty \frac{a}{s^2+a^2}\, ds$

$= a \times \frac{1}{a} \left[\tan^{-1} \frac{s}{a} \right]_s^\infty$

$= a \left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a} \right) \right]$

$= a \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a} \right) \right]$

$= a \cot^{-1} \frac{s}{a}$

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c) $\frac{2as}{(s^2-a^2)^2}$.

$L^{-1}\left\{\frac{2as}{(s^2-a^2)^2}\right\}$

3. d) e^{2t} $\sin t \cdot \sin 2t \cdot \sin 3t$

$= e^{2t} \times \frac{1}{2} \times 2\sin t \sin 3t \quad \sin 2t$

$= \frac{e^{2t}}{2} \left[\cos(2t) - \cos 4t\right] \sin 2t$

$= \frac{e^{2t}}{4} \left[2\cos 2t \cdot \sin 2t - 2\cos 4t \sin 2t\right]$

$= \frac{e^{2t}}{4} \left[\sin 4t - 2\sin 2t \cos 4t\right]$

$= \frac{e^{2t}}{4} \left[\sin 4t - [\sin(6t) + \sin(2t-4t)\right]$

$= \frac{e^{2t}}{4}\left[\sin 4t - \sin 6t + \sin 2t\right]$

$\frac{1}{4} L\left\{e^{2t} \sin 4t - e^{2t} \sin 6t + e^{2t} \sin 2t\right\}$

$= \frac{1}{4}\left[\frac{4}{s^2+16} - \frac{6}{s^2+36} + \frac{2}{s^2+4}\right]_{s \to s-2}$

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$\frac{1}{2}\left[\frac{2}{s^2-4s+20} - \frac{3}{s^2-4s+40} + \frac{1}{s^2-4s+8}\right]$

$= \frac{1}{2}\left[\frac{2}{(s-2)^2+16} - \frac{3}{(s-2)^2+36} + \frac{1}{(s-2)^2+4}\right]$

$= \frac{1}{2}\left[\frac{4}{(s-2)^2+4^2} \times \frac{2}{4} - \frac{6}{(s-2)^2+36} \times \frac{3}{6} + \frac{2}{(s-2)^2+2^2} \times \frac{1}{2}\right]$

$= \frac{1}{4}\left[\frac{4}{(s-2)^2+4^2} - \frac{6}{(s-2)^2+36} + \frac{2}{(s-2)^2+2^2}\right]$

$= \frac{1}{4}\left[\frac{4}{s+4^2} - \frac{6}{s+6^2} + \frac{2}{(s)^2+2^2}\right]_{s \to s-2.}$

$= \frac{1}{4}\left[\sin 4t - \sin 6t + \sin 2t\right]_{s \to s-2.}$

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Q) $te^{2t}\cosh t$.

$L\{te^{2t}\cosh t\}$

$L\{\cosh t\} = \frac{s}{s^2-1}$

$L\{t\cosh t\} = (-1)\frac{d}{dt}\left(\frac{s}{s^2-1}\right)$

$= -1\left[\frac{(s^2-1)-s(2s)}{(s^2-1)^2}\right]$

$= -1\left[\frac{s^2-1-2s^2}{(s^2-1)^2}\right]$

$= \frac{s^2+1}{(s^2-1)^2}$.

$L\{e^{2t}\frac{s^2+1}{(s^2-1)^2}\} = \frac{(s-2)^2+1}{\{(s-2)^2-1\}^2}$

$= \frac{s^2-4s+5}{(s^2-4s+3)^2}$

Q) $\cosh t \sin at$

$= \frac{e^{x}+e^{-x}}{2}\sin at$.

$= \frac{1}{2}\left[e^{t}\sin at + e^{-at}\sin at\right]$

$= e^{t}\frac{a}{s^2+a^2} \quad + e^{-t}\frac{a}{s^2+a^2}$

$= \frac{1}{2}\left[\frac{a}{(s-1)^2+a^2} + \frac{a}{(s+1)^2+a^2}\right]$

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Q) $t^2\cos at$

$L\{t^2\cos at\}$

$= (-1)^2\frac{d^2 L(\cos at)}{ds^2}$

Now,

$L(\cos at) = \frac{s}{s^2+a^2}$.

$\frac{d\,L(\cos at)}{ds} = \frac{(s^2+a^2)-s(2s)}{(s^2+a^2)^2}$.

$\frac{d\,L(\cos at)}{ds} = \frac{a^2-s^2}{(s^2+a^2)^2}$

or, $\frac{d^2 L(\cos at)}{ds^2} = \frac{(s^2+a^2)^2(-2s)-(a^2-s^2)2(s^2+a^2)2s}{(s^2+a^2)^4}$

$= \frac{-2s(s^2+a^2)^2 - 4s(s^2+a^2)(a^2-s^2)}{(s^2+a^2)^4}$

$= \frac{-2s(s^2+a^2)^2+4s(s^4-a^4)}{(s^2+a^2)^4}$

$= \frac{-2s(s^4+2a^2s^2+a^4)+4s^5-4sa^4}{(s^2+a^2)^4}$

$= \frac{+2s^5-4a^2s^3-2a^4s+4s^5-4sa^4}{(s^2+a^2)^4}$

$= \frac{2s^5-4a^2s^3-6sa^4}{(s^2+a^2)^4}$.

$= \frac{+2s^3(s^2+a^2)-6sa^2(s^2+a^2)}{(s^2+a^2)^4}$

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$\frac{\sin^2 t}{t}$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty L\{\sin^2 t\}\, ds.$$

$$= \int_s^\infty L\left\{\frac{1-\cos 2t}{2}\right\} ds$$

$$= \frac{1}{2}\int_s^\infty L\{1-\cos 2t\}\, ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds.$$

$$\cancel{= \frac{1}{2}\int_s^\infty \frac{s^2+4-2s}{s(s^2+4)}}$$

$$\cancel{= \frac{1}{2}\left[[\ln s]_s^\infty - 2\left[\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty\right]}$$

$$\cancel{= \frac{1}{2}\left[\ln\right.}$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln s - \ln\left(\sqrt{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left(\ln\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln\frac{1}{\sqrt{1+\frac{4}{s^2}}} - \ln\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\left[0 - \ln\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\ln\frac{\sqrt{s^2+4}}{s}$$

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$$\int_0^t \left(\cosh t \cdot t^2 \cdot \cos^2 at\right) dt$$

$$L\{\cos^2 at\} = L\left\{\frac{1+\cos 2at}{2}\right\}$$

$$= L\left\{\frac{1}{2} + \frac{\cos 2at}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4a^2}\right).$$

$$L\{t^2 \cos^2 at\} = \cancel{L\{}t^2.$$

$$= (-1)^2 \frac{1}{2}\frac{d^2}{ds^2}\left\{\frac{1}{s} + \frac{s}{s^2+4a^2}\right\}$$

$$= \frac{1}{2}$$

$$\frac{d}{ds}\left\{\frac{1}{s} + \frac{s}{s^2+4a^2}\right\} = \frac{-1}{s^2} + \frac{(s^2+4a^2) - s(2s)}{(s^2+4a^2)^2}$$

$$= -\frac{1}{s^2} + \frac{4a^2-s^2}{(s^2+4a^2)^2}.$$

$$\frac{d^2}{ds} F(s) = \frac{d}{ds}\left(\frac{-1}{s^2} + \frac{4a^2-s^2}{(s^2+4a^2)^2}\right)$$

$$= \frac{2}{s^3} + \frac{(s^2+4a^2)^2(-2s) - (4a^2-s^2)2(s^2+4a^2) \times 2s}{(s^2+4a^2)^4}$$

$$= \frac{2}{s^3} + \frac{\cancel{(s^2+4a^2)}\{(s^2+4a^2)(-2s) - 4s(4a^2-s^2)\}}{(s^2+4a^2)^{\cancel{4}\,3}}$$

$$= \frac{2}{s^3} + \frac{-2s^3 - 8a^2 s - 16a^2 s + 4s^3}{(s^2+4a^2)^3}$$

$$= \frac{2}{s^3} + \frac{2s^3 - 24a^2 s}{(s^2+4a^2)^3}.$$

$$L\left\{ \frac{e^{t}+e^{-t}}{2} \left(\frac{2}{s^3} + \frac{2s^3-24a^2s}{(s^2+4a^2)^3} \right) \right\}$$

$$= \frac{1}{2s}\left\{ \frac{2}{(s-1)^3} + \frac{2(s-1)^3 - 24a^2(s-1)}{\{(s-1)^2+4a^2\}^3} + \frac{2}{(s+1)^3} + \frac{2(s+1)^3 - 2a^2(s+1)}{\{(s+1)^2+4a^2\}^3} \right\}$$

$$\iint_R \vec{F}\cdot\vec{N}\, dx\, dy$$

$$\iint (\vec{F}\cdot\hat{n})\, ds.$$

9. $\vec{F} = 3x\hat{i} + xz\hat{j} + z^2\hat{k}$

$x^2+y^2 = 2.$

S: $z = 4 - x^2 - y^2.$

$\vec{r} = x\hat{i} + y\hat{j} + (4-x^2-y^2)\hat{k}$

$\vec{r_x} = 1\hat{i} + 0\hat{j} + (-2x)\hat{k}$

$\vec{r_y} = 0\hat{i} + \hat{j} + (-2y)\hat{k}$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$$

$= 2x\hat{i} - \hat{j}(-2y) + \hat{k}(1)$

$= 2x\hat{i} + 2y\hat{j} + \hat{k}.$

$\vec{F}\cdot\vec{N} = (3x,\ x(4-x^2-y^2) + (4-x^2-y^2)^2).$
$(2x,\ 2y,\ 1)$

$= 6x^2 + 2xy(4-x^2-y^2) + (4-x^2-y^2)^2$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(6x^2 + 8xy - 2x^2y - 2xy^3 + 16 + x^4 + y^4 - 8x^2 + 2x^2y^2 - 8y^2 \right) dy\, dx$$

2081/09/16
Tuesday

Convolution of Laplace Transform

Let $F(t)$ and $g(t)$ be two function defined $t > 0$. Then the convolution of two function $F(t)$ and $g(t)$ is denoted by $F*g$ and given as.

$$(F*g)(t) = \int_0^t F(u)\, g(t-u)\, du$$

OR,

$$= \int_0^t F(t-u)\, g(u)\, du.$$

Find the convolution of $t * e^t$

$\Rightarrow$

Let, $F(t) = t$, $g(t) = e^t$

$F(t-u) = t-u$ $g(u) = e^u$

Now,

$$(F*g)(t) = \int_0^t (t-u)\, e^{-u}\, du$$

$$= \cancel{\int_0^t t \cdot e^{-u}\, du - \int_0^t u\, du}$$

$$= \cancel{t \cdot e^{-t} [u]_0^t - \left[\frac{u^2}{2}\right]_0^t}$$

$$= \cancel{t \cdot e^{-t} [t] - \frac{t^2}{2}}$$

$$\cancel{t^2 e^{-t} - \frac{t^2}{2}}$$

$$= \int_0^t t e^{u}\, du - \int_0^t u e^{u}\, du$$

$$= t[e^{u}]_0^t - [u e^{u} - e^{u}]_0^t$$

$$= t[e^{t} - e^0] - [u e^{-u} - e^{-u}]_0^t$$

$$= t[e^{t} - 1] - [u e^{u} - e^{-u}]_0^t$$

$$= t[e^{t} - 1] - [t e^{t} - e^{t} - 0 + 1]$$

$$\cancel{= t e^{-t} - t + t e^{-t} + e^{-t} - 1}$$

$$= t e^{t} - t - t e^{t} + e^{t} - 1$$

$$= e^{t} - t - 1$$ Ans

Convolution Theorem

Statement:- If $L(f(t)) = F(s)$ and
$L(f(t)) = G(s)$ Then;
$L(f*g) = F(s) \cdot G(s)$.

$\Rightarrow$

Proof: By definition of Laplace ~~theorem~~ transformation,

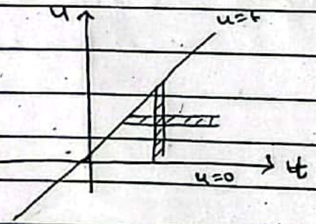
$$L(f*g) = \int_0^\infty e^{-st} (f*g)\, dt$$

$$= \int_0^\infty e^{-st} \left[\int_0^t f(u)\, g(t-u)\, du \right] dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt.$$

$$= \int_0^\infty \int_0^t e^{-s(t-u)} e^{-su} f(u)\, g(t-u)\, du\, dt$$

$0 \le u \le t$ $\qquad$ $0 \le t \le \infty$



changing the order,
$u \le t \le \infty$ $\qquad$ $0 \le u \le \infty$

$$= \int_0^\infty \int_u^\infty e^{-s(t-u)} e^{-su} f(u)\, g(t-u)\, dt\, du.$$

$$= \int_0^\infty e^{-su} f(u)\, du \cdot \int_u^\infty e^{-s(t-u)} g(t-u)\, dt.$$

Put $t-u = p$ $\qquad$ when $t=u$, $p=0$
$1 = \frac{dp}{dt}$ $\qquad$ when $t \to \infty$, $p \to \infty$.

$\therefore dt = dp$

$$= \int_0^\infty e^{-su} f(u)\, du \cdot \int_0^\infty e^{-sp} g\,p\, dp$$

$$= F(s) \times G(s)$$

$\therefore L(f*g) = F(s) \times G(s)$

Exercise 3-2

1. Find the inverse laplace transform.

a) $\frac{3s+2}{s^2+2s+2}$

$\Rightarrow$ Given,

$F(s) = \frac{3s+2}{s^2+2s+2}$

$= \frac{3s+2}{s^2+2s+1+1}$

$= \frac{3s+2}{(s+1)^2+1}$

$= \frac{3s+3-1}{(s+1)^2+1}$

$F(s) = \frac{3(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$

$L^{-1}(F(s)) = 3L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$

$= \cancel{3 \sin}$

$= 3\cos t\, e^{-t} - \sin t\, e^{-t}$

$= (3\cos t - \sin t)e^{-t}$

b) $\frac{2s+3}{s^2+4}$

$\Rightarrow$ $F(s) = \frac{2s+3}{s^2+2^2} = \frac{2s}{s^2+2^2} + \frac{3}{s^2+2^2}$

$L^{-1}(F(s)) = 2L^{-1}\left(\frac{s}{s^2+2^2}\right) + \frac{3}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$

$= 2 \times \cos 2t + \frac{3 \sin 2t}{2}$

c) $\frac{1}{s^2-9}$

$\Rightarrow$ $F(s) = \frac{1}{s^2-3^2} = \frac{1}{3} \times \frac{3}{s^2-3^2}$

$L^{-1}(F(s)) = \frac{1}{3}L^{-1}\left(\frac{3}{s^2-3^2}\right) = \frac{1}{3}\sinh 3t$

d) $\frac{s-1}{(s-1)^2+4}$

$\Rightarrow$ $F(s) = \frac{s-1}{(s-1)^2+2^2}$

$L^{-1}(F(s)) = L^{-1}\left(\frac{s-1}{(s-1)^2+2^2}\right)$

$= e^{t} L^{-1}\left(\frac{s}{(s)^2+2^2}\right)$

$= e^{t} \times \cancel{\sin 2t} \cos 2t$

e) $\frac{1}{s^2+2s+1+4}$

$F(s) = \frac{1}{(s+1)^2+2^2} = \frac{1}{2} \times \frac{2}{(s+1)^2+2^2}$

$L^{-1}(F(s)) = \frac{1}{2}L^{-1}\left(\frac{2}{(s+1)^2+2^2}\right)$

$= \frac{1}{2} \times e^{-t} L^{-1}\left(\frac{2}{s^2+2^2}\right)$

$= \frac{e^{-t}}{2} \times \sin 2t$

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f. $\frac{s-4}{s^2-4}$.

$F(s) = \frac{s}{s^2-2^2} - \frac{4}{2} \times \frac{2}{s^2-2^2}$.

$L^{-1}(F(s)) = \cosh 2t - 2 \times \sinh 2t$.

g. $\frac{2s^3}{s^4-1} = \frac{2s^3}{(s^2-1)(s^2+1)} = \frac{As+B}{s^2-1} + \frac{Cs+D}{s^2+1}$

$\frac{(s^2+1)(As+B) + (Cs+D)(s^2-1)}{(s^2-1)(s^2+1)}$

$\Rightarrow 2s^3 = (s^2+1)(As+B) + (Cs+D)\cdot(s^2-1)$

Here,

$2 = A+C$ — (i)

$0 = B+D$ — (ii)

$0 = A-C$ — (iii)

$0 = B-D$ — (iv)

$\therefore B=0, \quad D=0$.

$2=A+C$

$0=A-C$

$2A=2$

$\therefore A=1$

$2=1+C$

$\therefore C=1$

$\frac{2s^3}{(s^2-1)(s^2+1)} = \frac{s}{s^2-1} + \frac{s}{s^2+1}$

$\therefore \cancel{2s^3} =$

$\therefore L^{-1}\left(\frac{2s^3}{s^4-1}\right) = L^{-1}\left(\frac{s}{s^2-1}\right) + L^{-1}\left(\frac{s}{s^2+1}\right)$

$\therefore L^{-1}\left(\frac{2s^3}{s^4-1}\right) = \cosh t + \cos t$

2. e) $\frac{s}{(s+1)^2(s^2+1)} = \frac{s}{(s^2+2s+1)(s^2+1)}$

$\frac{s}{(s+1)^2(s^2+1)} = \frac{As+B}{s^2+2s+1} + \frac{Cs+D}{s^2+1}$ — (1)

$\frac{s}{(s+1)^2(s^2+1)} = \frac{(As+B)(s^2+1) + (s^2+2s+1)(Cs+D)}{(s^2+2s+1)(s^2+1)}$

$s = (As+B)(s^2+1) + (s^2+2s+1)(Cs+D)$

$0 = A+C$ — (i)

$0 = B+D+2C$ — (ii)

$1 = A+2D+C$ — (iii)

$0 = B+D$ — (iv)

$1 = 2D \Rightarrow D = 1/2, \quad B = -1/2$.

$C=0, \quad A=0$.

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$$= \frac{-1}{2(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}$$

$$\therefore L^{-1}\left(\frac{s}{(s+1)^2(s^2+1)}\right) = L^{-1}\left(\frac{-1}{2(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}\right)$$

$$= \frac{-1}{2} L^{-1}\left(\frac{1}{(s+1)^2}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{-1}{2} L^{-1}\left(\frac{1}{(s+1)^2}\right) + \frac{1}{2}\sin t$$

$$= \frac{-1}{2} e^{-t}\, t + \frac{1}{2}\sin t$$

~~3(a)~~

4-b) $\frac{1}{(s^2+a^2)^2}$

$$\Rightarrow \frac{1}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} \times \frac{1}{s^2+a^2}$$

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{1}{s^2+a^2} \times \frac{1}{s^2+a^2}\right) \quad \text{--- ①}$$

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$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$$

By convolution theorem.

$$L^{-1}(F(s)\cdot G(s)) = f * g.$$

$$L^{-1}\left(\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at * \frac{1}{a}\sin at$$

$$= \frac{1}{a^2}\sin at * \sin at$$

$$= \frac{1}{a^2}\int_0^t \sin au \cdot \sin a(t-u)\, du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au \,\sin(at-au)\, du$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at) - \cos at]\, du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} \quad \cos at\cdot u\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right]$$

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4e) $\log\left(\frac{s^2-a^2}{s^2}\right)$

$F(s) = \log(s^2-a^2) - \log s^2$

$F'(s) = \frac{2s}{s^2-a^2} - \frac{2s}{s^2}$

$F'(s) = \frac{2s}{s^2-a^2} - \frac{2}{s}$

we know,

$L(t\,f(t)) = -\frac{d}{ds}F(s) = -F'(s)$

$= \frac{2}{s} - \frac{2s}{s^2-a^2}$

$t\,f(t) = L^{-1}\left(\frac{2}{s} - \frac{2s}{s^2-a^2}\right)$

$= 2(1 - \cosh at)$

$\therefore f(t) = \frac{2(1-\cosh at)}{t}$

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4.f) $\frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

$= \frac{1}{2}\left[\ln(s^2+b^2) - \ln(s^2+a^2)\right]$

$L(t\,f(t)) = -\frac{d}{ds}F(s)$

where,

$F(s) = \frac{1}{2}\left[\ln(s^2+b^2) - \ln(s^2+a^2)\right]$

$\frac{dF(s)}{ds} = \frac{1}{2}\left[\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right]$

$= \left[\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}\right]$

$L(t\,f(t)) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$t\,f(t) = \cos at - \cos bt$

$\therefore f(t) = \frac{\cos at - \cos bt}{t}$

h) $\ln\left(\frac{s(s+1)}{s^2+4}\right).$

$= \ln(s^2+s) - \ln(s^2+4)$

$L(t\,f(t)) = -\frac{d}{ds}F(s)$

$\frac{dF(s)}{ds} = \frac{2s+1}{s^2+s} - \frac{2s}{s^2+4}$

$= \frac{2s}{s^2+s} + \frac{1}{s^2+s} - \frac{2s}{s^2+4}$

$= \frac{2}{s+1} + \frac{1}{s^2+s} - \frac{2s}{s^2+4}.$

Let,

$\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{(s+1)}$

or, $1 = A(s+1) + Bs$

$\therefore A+B=0$

$A=1$

$\therefore B=-1.$

$\therefore \frac{dF(s)}{ds} = \frac{2}{s+1} + \frac{1}{s} - \frac{1}{s+1} - \frac{2s}{s^2+4}.$

$= \frac{1}{s+1} + \frac{1}{s} - \frac{2s}{s^2+4}.$

Q) $\cot^{-1}\frac{s}{2}$

$t\,f(t) = L^{-1}\left(\frac{2s}{s^2+4} - \frac{1}{s+1} - \frac{1}{s}\right).$

$\therefore \boxed{f(t) = \frac{2\cdot\cos 2t - e^{-t} - 1}{t}}$

Q) $\tan^{-1}\frac{2}{s^2}.$

$F(s) = \tan^{-1}\frac{2}{s^2}.$

$\frac{dF(s)}{ds} = \frac{1}{1+\frac{4}{s^4}} \times 2\cdot(-2)\,s^{-3}$

$= \frac{s^4}{s^4+4} \times \frac{(-4)}{s^3}$

$= \frac{-4s}{s^4+4}$

$L(t\,f(t)) = \frac{4s}{s^4+4}.$

$t\,f(t) = L^{-1}\left(\frac{4s}{s^4+4}\right)$ —— ①

$s^4+4 = (s^2+2)^2 - 4s^2$

$s^2+4 \quad = (s^2+2s+2)\,(s^2-2s+2)$

$\therefore \frac{4s}{s^4+4} = \frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}.$

$L\left(\frac{F(t)}{t}\right) = \int_s^\infty F(s)\, ds$

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$t\, f(t) = L^{-1}\left(\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right)$

$= e^{t}\sin t - e^{-t}\sin t$

$= 2\sin t \left(\frac{e^{t} - e^{-t}}{2}\right)$

$\therefore \boxed{f(t) = \frac{2\sin t \sinh t}{t}}$

3.d. $\frac{s^2}{s^4+4a^4}$.

Here,

$s^4+4a^4 = (s^2)^2 + (2a^2)^2$

$= (s^2+2a^2)^2 - 2\cdot s^2\cdot 2a^2$

$= (s^2+2a^2)^2 - (2as)^2$

$= (s^2-2as+2a^2)(s^2+2as+2a^2)$

$= \{(s-a)^2+a^2\}\{(s+a)^2+a^2\}$

we know,

$\cancel{t\, F(t) = L^{-1}\left\{-\frac{d}{ds}F(s)\right\}}$

$\frac{s^2}{s^4+4a^4} = \frac{s}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$

$L^{-1}\left[\frac{s^2}{s^4+4a^4}\right] = L^{-1}\left[\frac{1}{4a}\left[\frac{s}{s^2-2as+2a^2} - \frac{s}{s^2+4as+2a^2}\right]\right]$

$= \frac{1}{4a} L^{-1}\left[\frac{s}{(s-a)^2+a^2} - \frac{s}{(s+a)^2+a^2}\right]$

$s^2+4a^2+4as.$

$L\left[\int_0^t F(t)\, dt\right] = \frac{1}{s} F(s)$

e^{+at} e^{-at}

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$= \frac{1}{4a}\left[e^{at}\cdot \cos at - e^{-at}\cos at\right] =$

$= \frac{1}{2a} \cos at \cdot \sin at$

3.b) $\frac{1}{s^4+4a^4}$.

$s^4+4a^4 = (s^2+2a^2)^2 - 2\cdot s^2\cdot 2a^2$

$= (s^2+2a^2)^2 - (2as)^2$

$s^4+4a^4 = (s^2-2as+2a^2)(s^2+2as+2a^2)$

$\therefore \frac{1}{s^4+4a^4} = \frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}$

$= \frac{1}{4a}\times\frac{1}{s}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$

$\frac{1}{s^4+4a^4} = \frac{1}{4a}\times\frac{1}{s}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$

$\therefore L^{-1}\left(\frac{1}{s^4+4a^4}\right) = \frac{1}{4a} L^{-1}\left(\frac{1}{s}\left(\frac{1}{(s-a)^2+a^2}\right) - \frac{1}{s}\left(\frac{1}{(s+a)^2+a^2}\right)\right)$

$= \frac{1}{4a}\int_0^t \left(\frac{1}{a} e^{at}\sin at\, dt - \frac{1}{4a}\int_0^t \frac{1}{a} e^{-at} \sin at\, dt\right.$

$= \frac{1}{4a^2}\int_0^t \left(e^{at}\cdot \sin at - e^{-at}\sin at\right) dt$

2081/10/02
Wednesday

Legendre Differential Equation & solution

The differential equation of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{— (I)}$$

where n is the real number. Then the Equation (I) is the legendre differential Equation.

$x=0$ is the ordinary point of diff. eq^n (I). so $y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$ — (II)

be the solution of (I)

$$y' = \sum_{m=1}^{\infty} a_m m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} a_m m(m-1) x^{m-2}$$

Put y, y', y'' in (I)

~~$\therefore (1-x^2) a_m +$~~

$$\therefore (1-x^2)\sum_{m=2}^{\infty} a_m m(m-1)x^{m-2} - 2x\sum_{m=1}^{\infty} a_m m x^{m-1}$$

$$+ n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

or, $\sum_{m=2}^{\infty} a_m m(m-1)x^{m-2} - \sum_{m=2}^{\infty} a_m (m-1) x^m$

$$-2\sum_{m=1}^{\infty} a_m m x^m + \sum_{m=0}^{\infty} n(n+1) a_m x^m = 0$$

To make same power of x i.e x^m in all series, replace m by $m+2$ in first series,

or $\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m - \sum_{m=2}^{\infty} a_m m(m-1)x^m - \sum_{m=1}^{\infty} 2a_m m x^m + \sum_{m=0}^{\infty} n(n+1)a_m x^m = 0$

Equating the coeff. of x^m, we get.

$a_{m+2}(m+2)(m+1) - a_m \cdot m(m-1) - 2a_m m + n(n+1)a_m = 0$

or, $a_{m+2}(m+2)(m+1) - a_m[m(m-1) + 2m - n(n+1)] = 0$

or, $a_{m+2}(m+2)(m+1) = a_m[m^2 - m + 2m - n^2 - n]$

or, $a_{m+2} = \frac{(m^2-n^2)+(m-n)}{(m+2)(m+1)} a_m$

$\therefore a_{m+2} = \frac{(m-n)(m+n+1)}{(m+2)(m+1)} a_m$

$\therefore a_{m+2} = -\frac{(n-m)(m+n+1)}{(m+2)(m+1)} a_m \quad m \geq 2.$

which is the recursive formula for determining a_m for $m \geq 2$. In terms of a_0, a_1 by taking $m = 0, 1, 2, 3, 4, \ldots$

When $m = 0$,

$a_2 = \frac{-n(n+1)}{2 \cdot 1} \cdot a_0$

When $m = 1$,

$a_3 = -\frac{(n-1)(n+2)}{3 \cdot 2} \cdot a_1$

When $m = 2$,

$a_4 = -\frac{(n-2)(n+3)}{4 \cdot 3} a_2$

$= + \frac{(n-2)(n+3)}{4 \cdot 3} \cdot \frac{n(n+1)}{2 \cdot 1} a_0$

$= \frac{n(n+1)(n-2)(n+3)}{4 \cdot 3 \cdot 2 \cdot 1} a_0$

$= \frac{n(n+1)(n-2)(n+3)}{4!} a_0$

When $m = 3$,

$a_5 = + \frac{(n-3)(n+4)}{5 \cdot 4} \cdot \frac{(n-1)(n+2)}{3 \cdot 2} \cdot a_1$

$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$

Put $a_2, a_3, a_4, a_5 \ldots$ in (2).

$$y = a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 x^2 - \frac{(n-1)(n+2)}{3!} a_1 x^3$$
$$+ \frac{n(n+1)(n-2)(n+3)}{4!} a_0 x^4 +$$
$$+ \frac{(n-3)(n+4)(n-1)(n+2)}{5!} a_1 x^5 + \ldots$$

$$\therefore y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 - \ldots\right]$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}\right.$$

which is in the form:

$$y = a_0 y_1(x) + a_1 y_2(x)$$

Rodrigue's Formula for Legendre Polynomial

The legendre polynomial is denoted by $P_n(x)$ By Rodrigues formula polynomial is given by:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

First few terms of $P_n(x)$ are:

$P_0(x) = 1$

$P_1(x) = x$

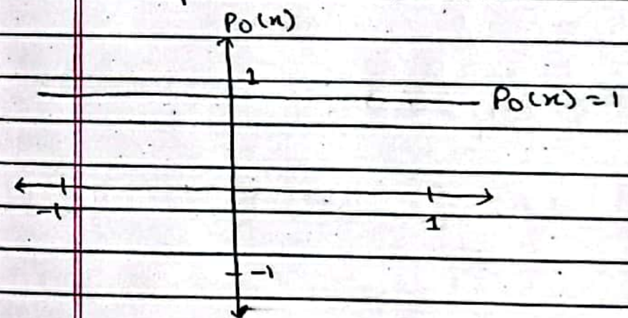
$P_2(x) = \frac{1}{2}\left[3x^2 - 1\right]$

$P_3(x) = \frac{1}{2}\left[5x^3 - 3x\right]$

$P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$

$P_5(x) = \frac{1}{8}\left[63x^5 - 70x^3 + 15x\right]$

Graph of $P_0(x)$

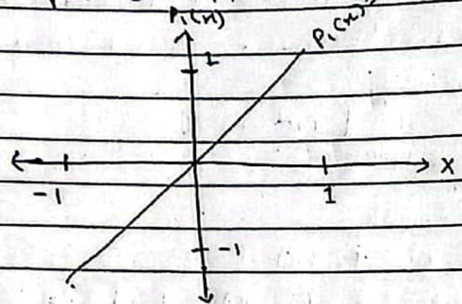


$(1-x^2)\,y''$

$x = \pm 1$

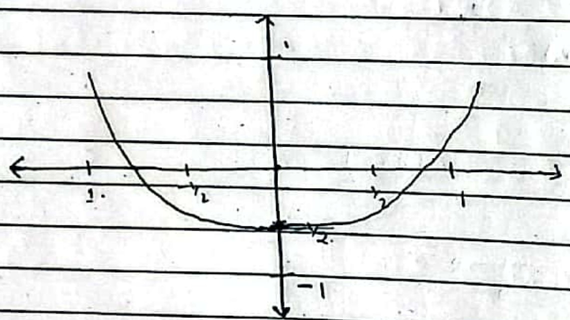
$-1 < x < 1$

Graph of $P_1(x)$



Graph of $P_2(x)$

$$P_2(x) = \frac{1}{2}[3x^2 - 1]$$

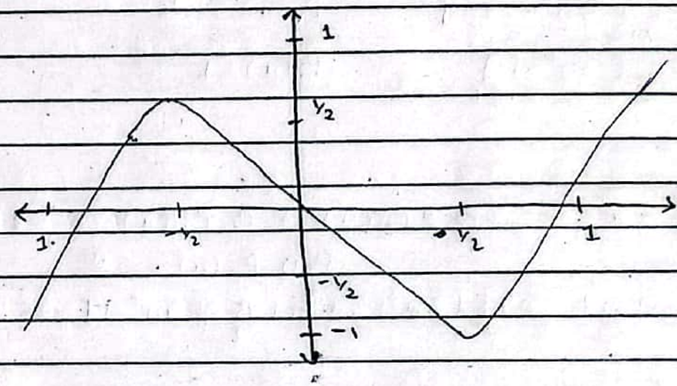


$0 = 3x^2 - 1$

$x = \pm\sqrt{\frac{1}{3}}$

Graph of $P_3(x)$

$$P_3(x) = \frac{1}{2}[5x^3 - 3x]$$



Q) Graph of $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$