Chapter 5

2-D Transformations

Contents

Homogeneous Coordinates

- There are three types of co-ordinate systems

1. *Cartesian Co-ordinate System*
   - *Left Handed Cartesian Co-ordinate System*(Clockwise)
   - *Right Handed Cartesian Co-ordinate System* (Anti Clockwise)
2. *Polar Co-ordinate System*
3. *Homogeneous Co-ordinate System*

We can always change from one co-ordinate system to another.

Homogeneous Coordinates

- A point (x, y) can be re-written in **homogeneous coordinates** as (x_h, y_h, h)
- The **homogeneous parameter** h is a non-zero value such that:

$$x = \frac{x_h}{h} \qquad y = \frac{y_h}{h}$$

- We can then write any point (x, y) as (hx, hy, h)
- We can conveniently choose $h = 1$ so that (x, y) becomes $(x, y, 1)$

Homogeneous Coordinates

Advantages:

1. Mathematicians use homogeneous coordinates as they allow scaling factors to be removed from equations.
2. All transformations can be represented as 3*3 matrices making homogeneity in representation.
3. Homogeneous representation allows us to use matrix multiplication to calculate transformations extremely efficient!
4. Entire object transformation reduces to single matrix multiplication operation.
5. Combined transformation are easier to built and understand.

Contents

1. Homogeneous coordinates
2. **Matrices**
3. Transformations
4. Geometric Transformations
5. Inverse Transformations
6. Coordinate Transformations
7. Composite transformations

Matrices

- **Definition**: A ***matrix*** is an n X m array of scalars, arranged conceptually in n rows and m columns, where n and m are positive integers. We use A, B, and C to denote matrices.
- If n = m, we say the matrix is a **square matrix**.
- We often refer to a matrix with the notation

  A = [a(i,j)], where a(i,j) denotes the scalar in the ith row and the jth column
- Note that the text uses the typical mathematical notation where the i and j are subscripts. We'll use this alternative form as it is easier to type and it is more familiar to computer scientists.

Matrices

- **Scalar-matrix multiplication:**

  $\alpha A = [\alpha \; a(i,j)]$

- **Matrix-matrix addition:** A and B are both n X m

  $C = A + B = [a(i,j) + b(i,j)]$

- **Matrix-matrix multiplication:** A is n X r and B is r X m

  $$C = AB = [c(i,j)] \quad \text{where} \quad c(i,j) = \sum_{k=1}^{r} a(i,k)\, b(k,j)$$

Matrices

- **Transpose:** A is n X m. Its transpose, A^T, is the m X n matrix with the rows and columns reversed.
- **Inverse:** Assume A is a square matrix, i.e. n X n. The identity matrix, In has 1s down the diagonal and 0s elsewhere The inverse A^{-1} does not always exist. If it does, then

$$A^{-1} A = A A^{-1} = I$$

Given a matrix A and another matrix B, we can check whether or not B is the inverse of A by computing AB and BA and seeing that AB = BA = I

Matrices

– Each point P(x,y) in the homogenous matrix form is represented as
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{3x1}$$

– Recall matrix multiplication takes place:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3x3} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3x1} = \begin{bmatrix} a*x+b*y+c*z \\ d*x+e*y+f*z \\ g*x+h*y+i*z \end{bmatrix}_{3x1}$$

Matrices

- Matrix multiplication does NOT *commute*:

$$\mathbf{M}\,\mathbf{N} \neq \mathbf{N}\ \mathbf{M}$$

- Matrix composition works *right-to-left.*
  - Compose:

$$\mathbf{M} = \mathbf{A}\ \mathbf{B}\ \mathbf{C}$$

  - Then apply it to a column matrix v:

$$\mathbf{v}' = \mathbf{M}\ \mathbf{v}$$

$$\mathbf{v}' = (\mathbf{A}\ \mathbf{B}\ \mathbf{C})\,\mathbf{v}$$

$$\mathbf{v}' = \mathbf{A}\ (\mathbf{B}\ (\mathbf{C}\ \mathbf{v}))$$

  - It first applies **C** to **v,** then applies **B** to the result, then applies **A** to the result of that.

Contents

Transformations

- A ***transformation*** is a function that maps a point (or vector) into another point (or vector).
- An ***affine transformation*** is a transformation that maps lines to lines.
- Why are affine transformations "nice"?
  - We can define a polygon using only points and the line segments joining the points.
  - To move the polygon, if we use affine transformations, we only must map the points defining the polygon as the edges will be mapped to edges!
- We can model many objects with polygons---and should---for the above reason in many cases.

Transformations

- Any affine transformation can be obtained by applying, in sequence, transformations of the form
  - Translate
  - Scale
  - Rotate
  - Reflection

- So, to move an object all we have to do is determine the sequence of transformations we want using the 4 types of affine transformations above.

Transformations

- **Geometric Transformations:** In Geometric transformation an object itself is moved relative to a stationary coordinate system or background. The mathematical statement of this view point is described by geometric transformation applied to each point of the object.

- **Coordinate Transformation:** The object is held stationary while coordinate system is moved relative to the object. These can easily be described in terms of the opposite operation performed by Geometric transformation.

Transformations

- What does the transformation do?

- What matrix can be used to transform the original points to the new points?

- Recall--- moving an object is the same as changing a frame so we know we need a 3 X 3 matrix

- It is important to remember the form of these matrices!!!

Contents

Geometric Transformations

- In Geometric transformation an object itself is moved relative to a stationary coordinate system or background. The mathematical statement of this view point is described by geometric transformation applied to each point of the object. Various Geometric Transformations are:
  - **Translation**
  - Scaling
  - Rotation
  - Reflection
  - Shearing

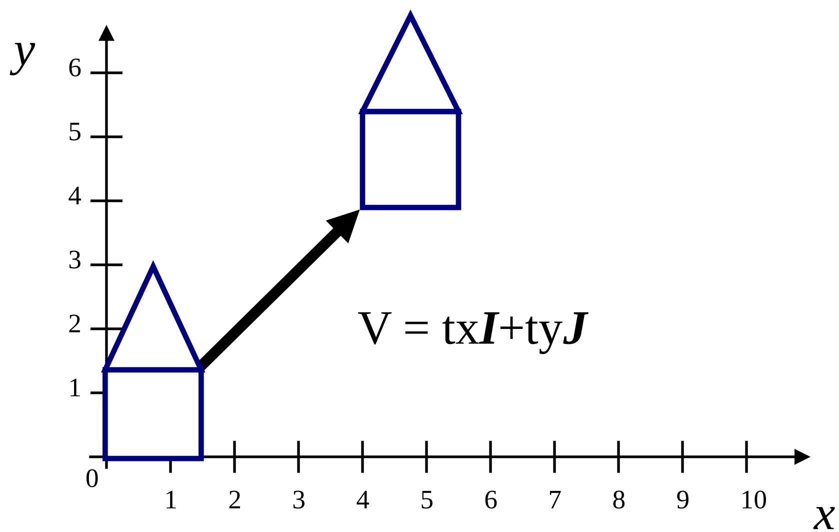
Geometric Transformations

- –**Translation**
- –Scaling
- –Rotation
- –Reflection
- –Shearing

Geometric Translation

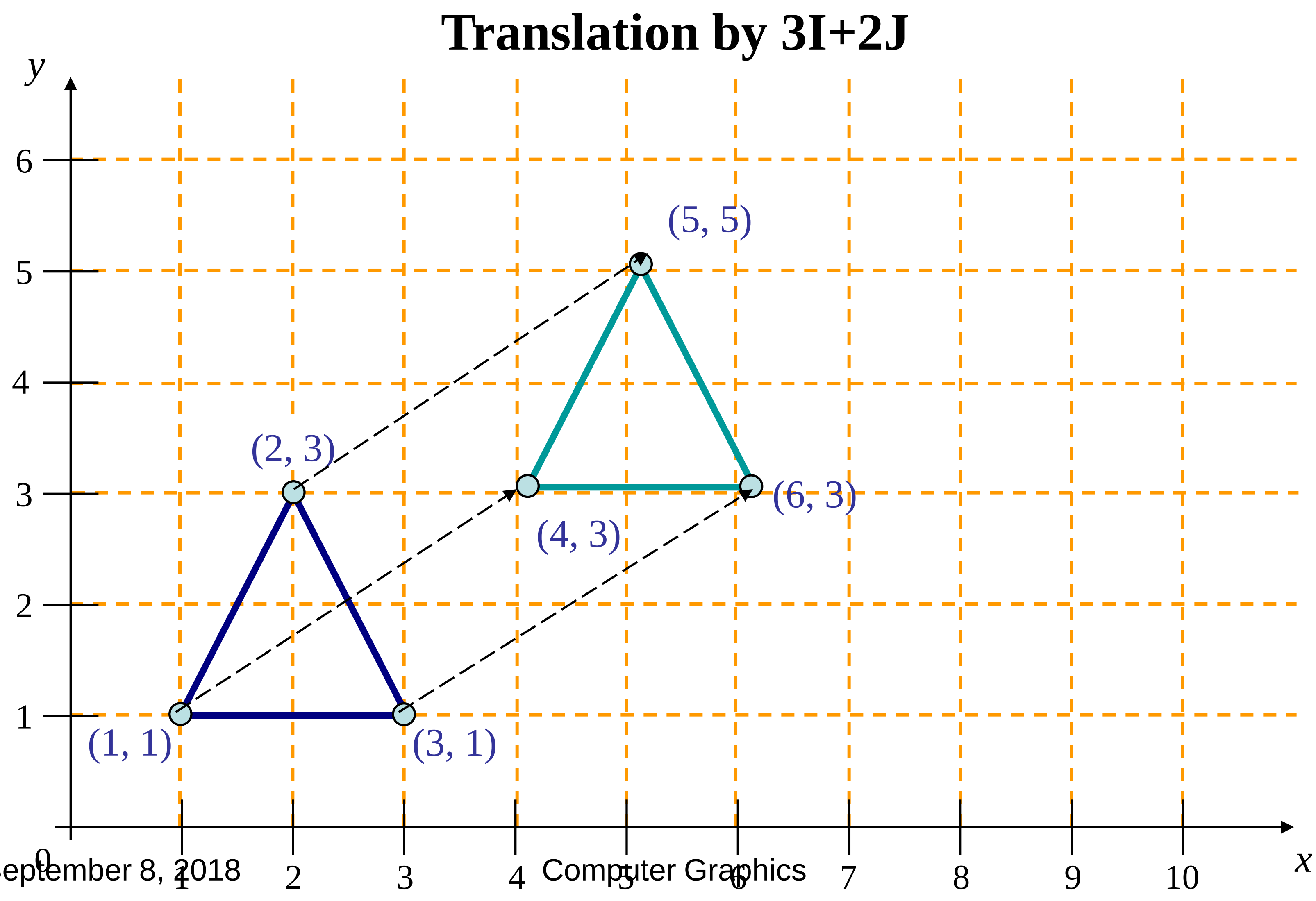
- Is defined as the displacement of any object by a given distance and direction from its original position. In simple words it moves an object from one position to another.

$x' = x + tx \qquad\qquad y' = y + ty$



Note: House shifts position relative to origin

Geometric Translation Example



Geometric Translation

- To make operations easier, 2-D points are written as homogenous coordinate column vectors
- The translation of a point P*(x,y)* by *(tx, ty)* can be written in matrix form as:

$$P' = T_v(P) \quad where\ v = tx\vec{I} + ty\vec{J}$$

$$T_v = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Geometric Translation

– Representing the point as a homogeneous column vector we perform the calculation as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1*x+0*y+tx*1 \\ 0*x+1*y+ty*1 \\ 0*x+0*y+1*1 \end{bmatrix} = \begin{bmatrix} x+tx \\ y+ty \\ 1 \end{bmatrix}$$

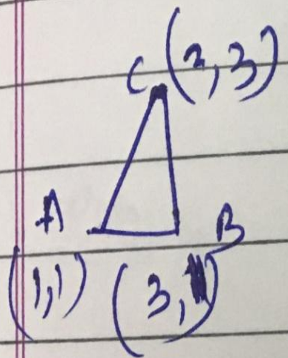
$$on \quad comparing$$

$$x' = x + tx$$

$$y' = y + ty$$

for full object

$$\begin{bmatrix}1 & 0 & tx\\ 0 & 1 & ty\\ 0 & 0 & 1\end{bmatrix} * \begin{bmatrix}A(x) & B(x) & C(x)\\ A(y) & B(y) & C(y)\\ 1 & 1 & 1\end{bmatrix}$$



Translate $3i + 2j$.

$$\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} * \begin{bmatrix}1 & 3 & 2\\ 1 & 1 & 3\\ 1 & 1 & 1\end{bmatrix}$$

Geometric Transformations

–Translation

–**Scaling**

–Rotation

–Reflection

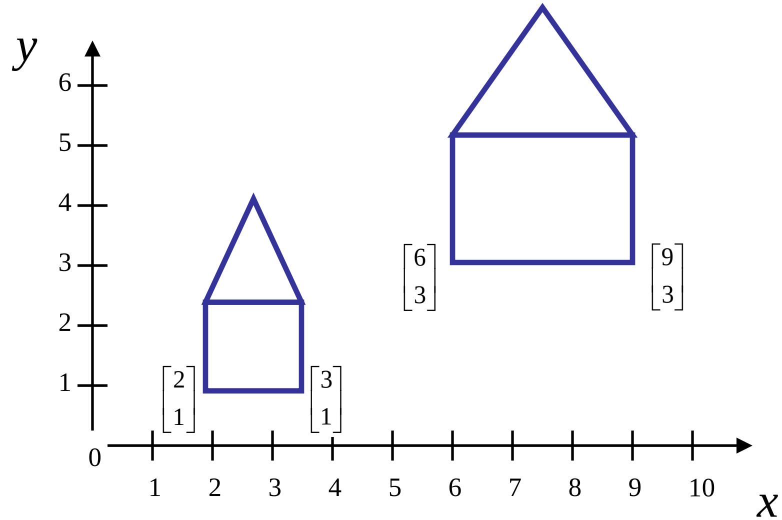
–Shearing

Geometric Scaling

- Scaling is the process of expanding or compressing the dimensions of an object determined by the scaling factor.
- Scalar multiplies all coordinates

$x' = Sx \times x \qquad\qquad y' = Sy \times y$

- **WATCH OUT:**
  - Objects grow and move!



Note: House shifts position relative to origin

Geometric Scaling

– The scaling of a point P*(x,y)* by scaling factors *Sx* and *Sy* about origin can be written in matrix form as:

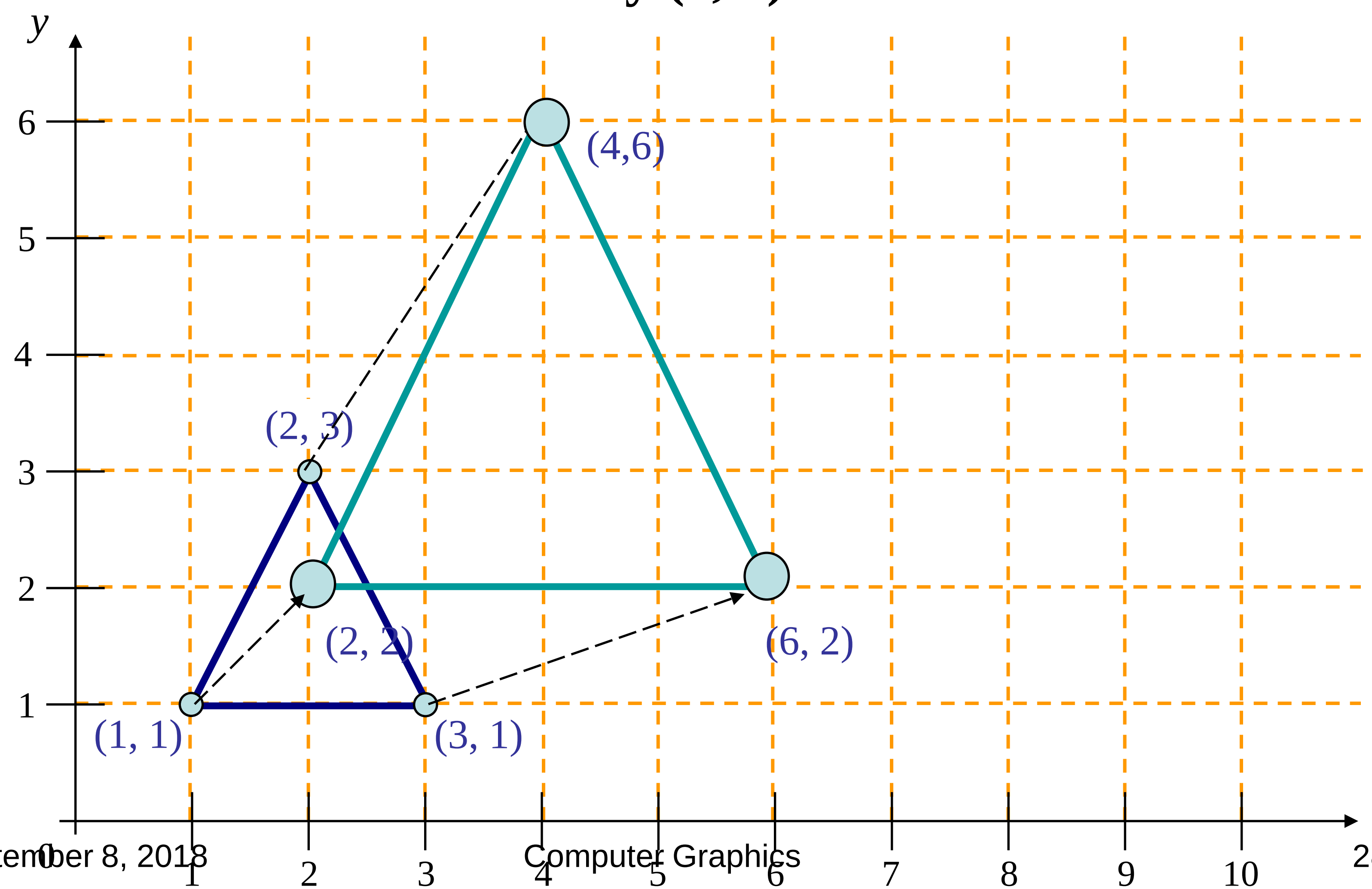
$$P' = S_{sx,\, sy}(P) \quad where$$

$$S_{sx,\, sy} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x \times x \\ s_y \times y \\ 1 \end{bmatrix}$$

Geometric Scaling Example

Scale by (2, 2)



Geometric Transformations

- –Translation
- –Scaling
- –**Rotation**
- –Reflection
- –Shearing

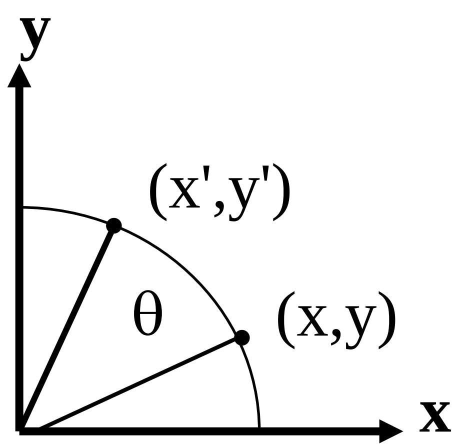
Geometric Rotation

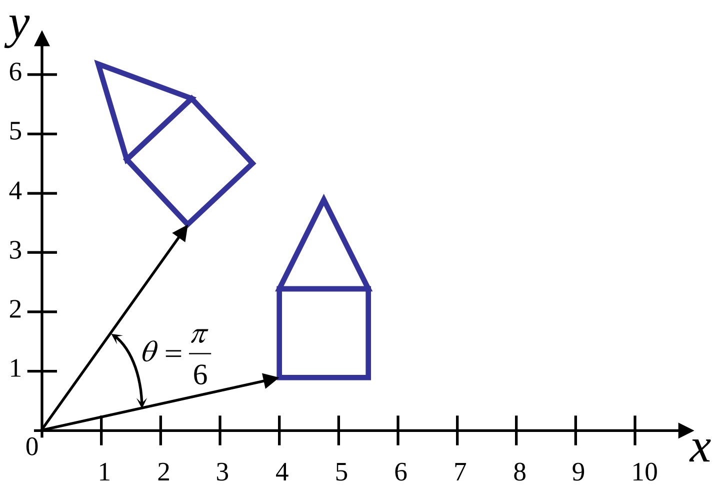
- The rotation of a point P (x,y) ***about origin***, by specified angle θ *(>0* counter clockwise) can be obtained as

$$x' = x \times \cos\theta - y \times \sin\theta$$

$$y' = x \times \sin\theta + y \times \cos\theta$$

- To rotate an object we have to rotate all coordinates





Geometric Rotation

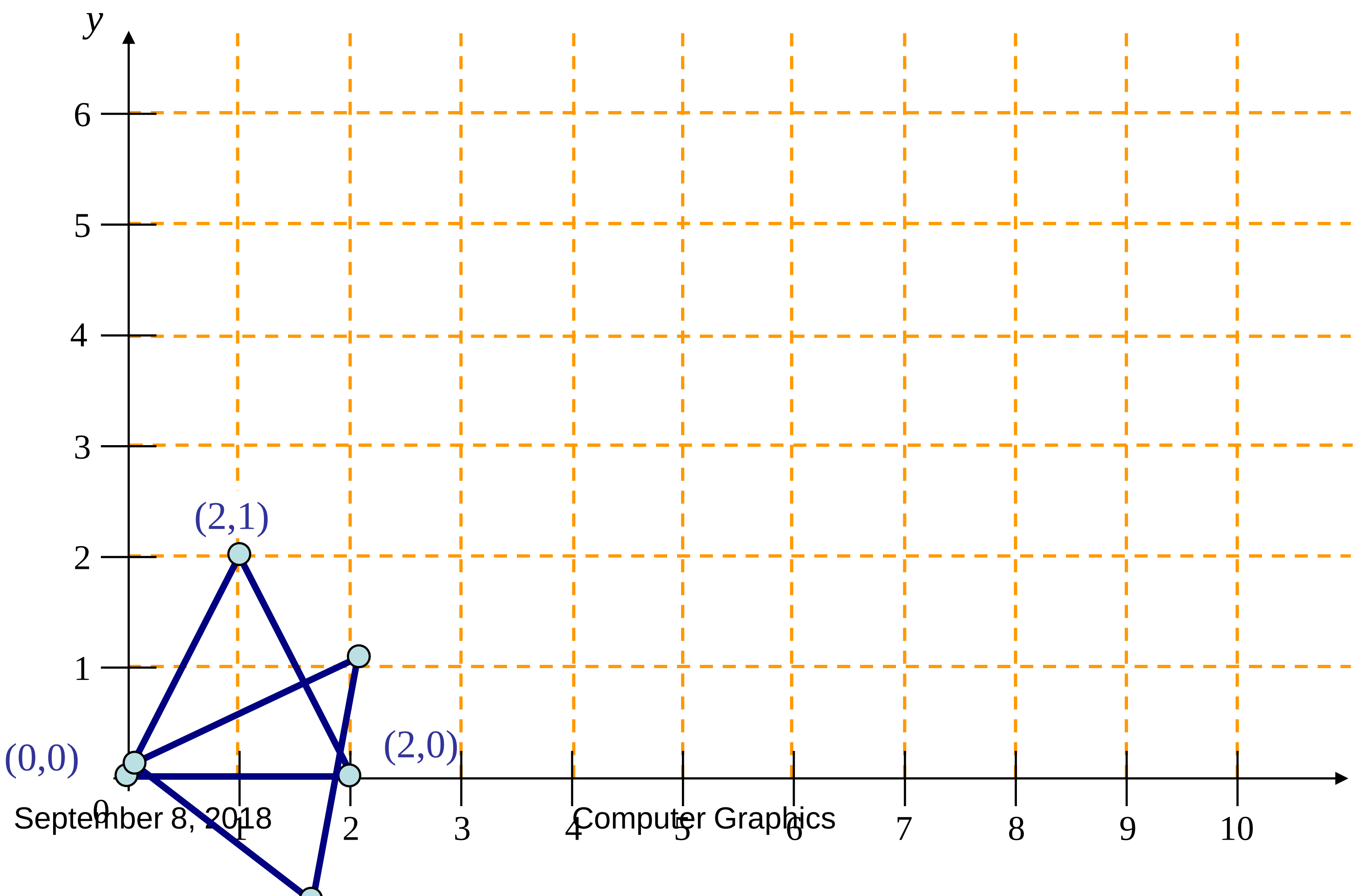
– The rotation of a point P(x,y) by an angle θ about origin can be written in matrix form as:

$$P' = R_\theta(P) \quad where$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta \times x - \sin\theta \times y \\ \sin\theta \times x + \cos\theta \times y \\ 1 \end{bmatrix}$$

Geometric Rotation Example



Geometric Transformations

–Translation

–Scaling

–Rotation

–**Reflection**

–Shearing

Geometric Reflection

– Mirror reflection is obtained about X-axis

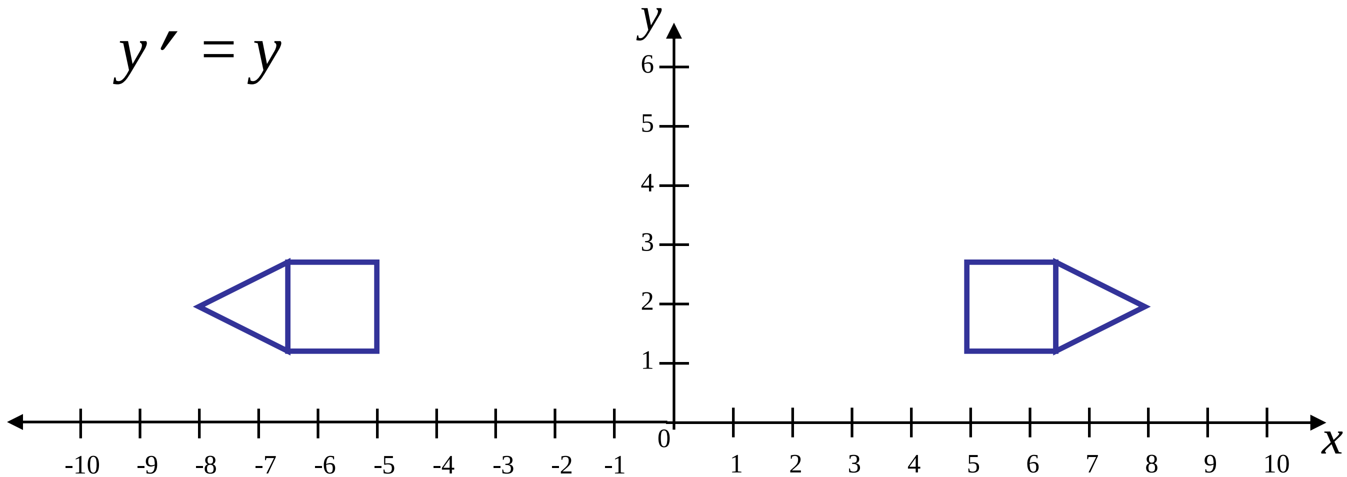
$x' = x$

$y' = -y$

– Mirror reflection is obtained about Y-axis

$x' = -x$

$y' = y$



Geometric Reflection

– The reflection of a point P*(x,y)* about X-axis can be written in matrix form as:

$$P' = M_x(P) \quad where$$

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ -y \\ 1 \end{bmatrix}$$

Geometric Reflection

– The reflection of a point P*(x,y)* about Y-axis can be written in matrix form as:

$$P' = M_y(P) \quad where$$

$$M_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -x \\ y \\ 1 \end{bmatrix}$$

Geometric Reflection

– The reflection of a point P*(x,y)* about origin can be written in matrix form as:

$$P' = M_{xy}(P) \quad where$$

$$M_{xy} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ 1 \end{bmatrix}$$

Geometric Transformations

- –Translation
- –Scaling
- –Rotation
- –Reflection
- –**Shearing**

Geometric Shearing

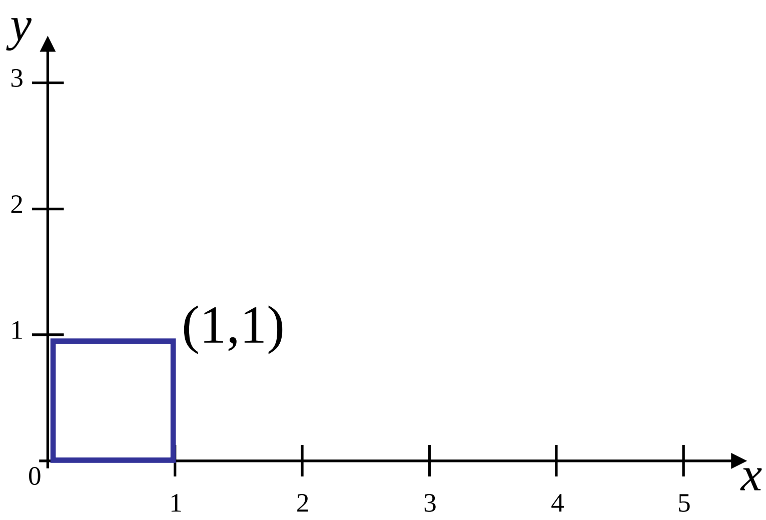
- It us defined as tilting in a given direction
- Shearing about y-axis

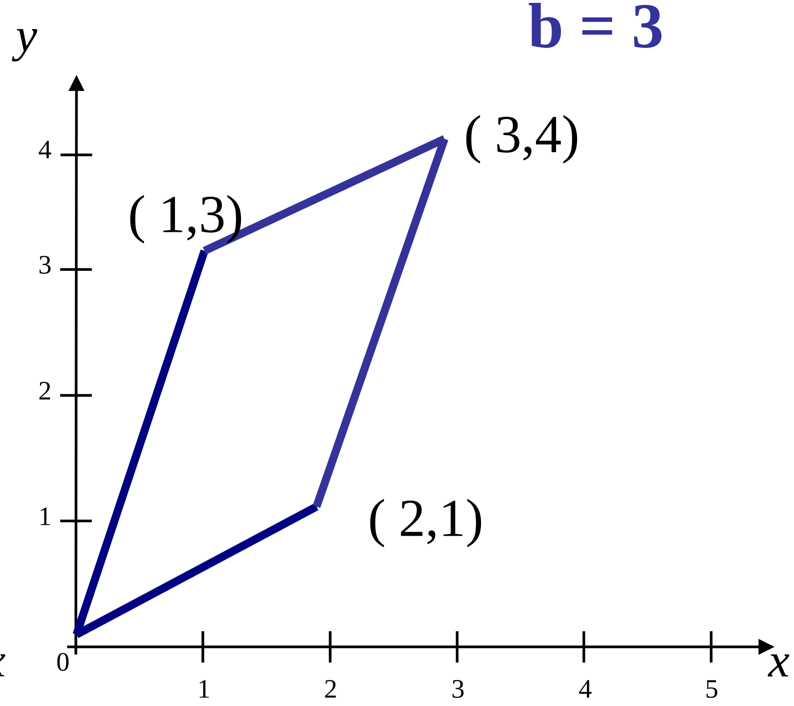
$x' = x + ay$

$y' = y + bx$

a = 2

b = 3





Geometric Shearing

– The shearing of a point P*(x,y)* in general can be written in matrix form as:

$$P' = Sh_{a,b}(P) \quad where$$

$$Sh_{a,b} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

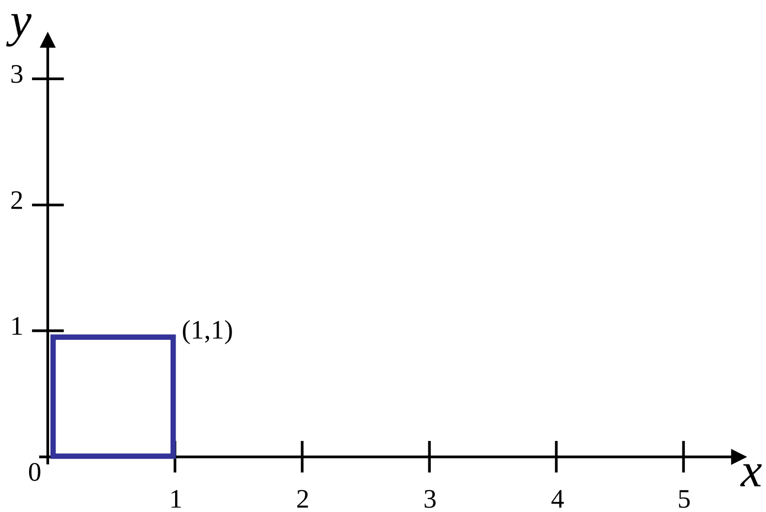
$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + ay \\ y + bx \\ 1 \end{bmatrix}$$

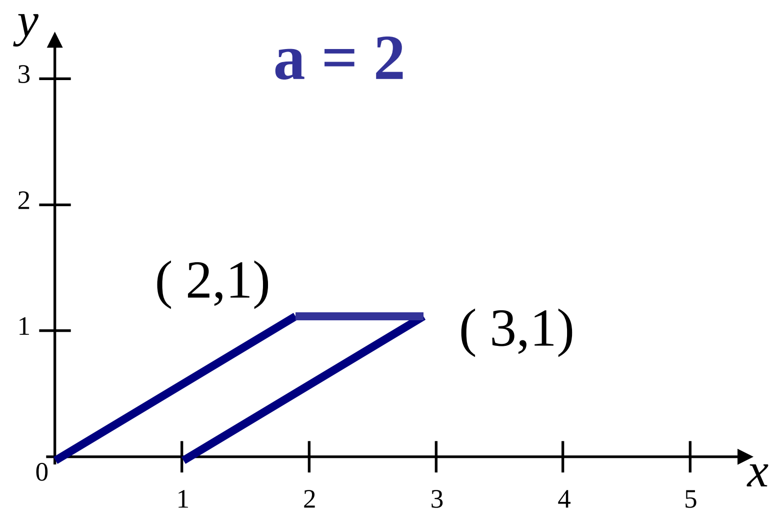
Geometric Shearing

– If b = 0 becomes Shearing about X-axis

$x' = x + ay$

$y' = y$





Geometric Shearing

– The shearing of a point P*(x,y)* about X-axis can be written in matrix form as:

$$P' = Sh_{a,0}(P) \quad where$$

$$Sh_{a,0} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

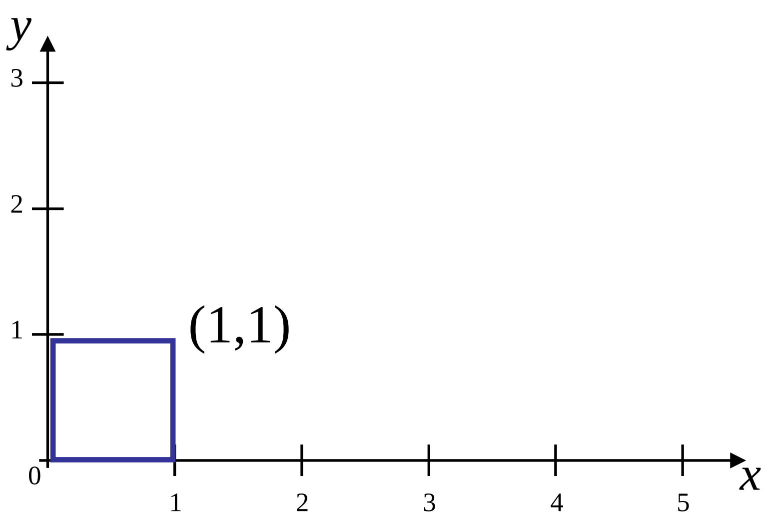
$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + ay \\ y \\ 1 \end{bmatrix}$$

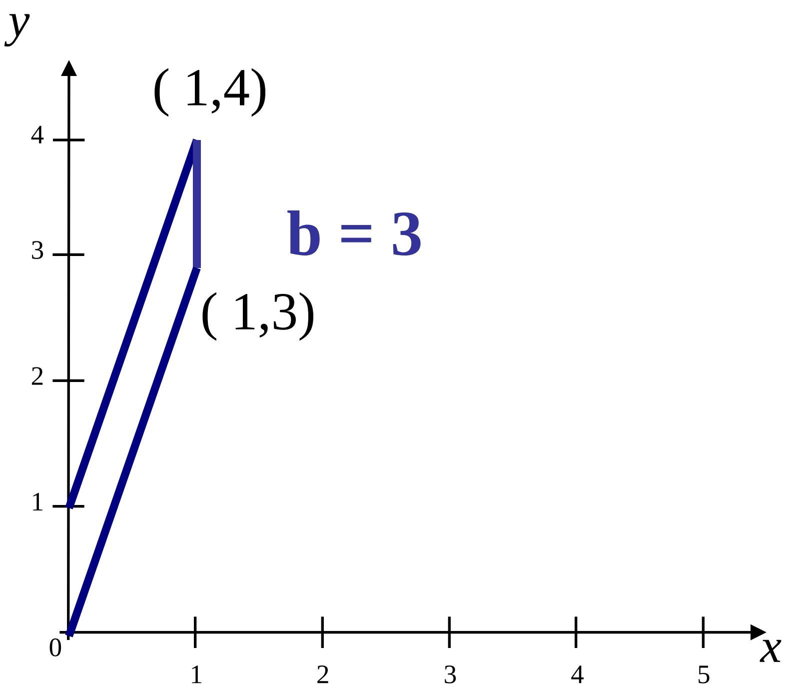
Geometric Shearing

– If a = 0 it becomes Shearing about y-axis

$x' = x$

$y' = y + bx$





Geometric Shearing

– The shearing of a point P*(x,y)* about Y-axis can be written in matrix form as:

$$P' = Sh_{0,b}(P) \quad where$$

$$Sh_{0,b} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$such\ that \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y + bx \\ 1 \end{bmatrix}$$

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Inverse Transformations

– **Inverse Translation**: Displacement in direction of –V

$$T_v^{-1} = T_{-v} = \begin{bmatrix} 1 & 0 & -tx \\ 0 & 1 & -ty \\ 0 & 0 & 1 \end{bmatrix}$$

– **Inverse Scaling**: Division by S_x and S_y

$$S_{sx,sy}^{-1} = S_{1/sx,1/sy} = \begin{bmatrix} 1/S_x & 0 & 0 \\ 0 & 1/S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Transformations

– **Inverse Rotation**: Rotation by an angle of $-\theta$

$$R_\theta^{-1} = R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– **Inverse Reflection**: Reflect once again

$$M_x^{-1} = M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Coordinate Transformations

– **Coordinate Transformation:** The object is held stationary while coordinate system is moved relative to the object. These can easily be described in terms of the opposite operation performed by Geometric transformation.

Coordinate Transformations

– **Coordinate Translation**: Displacement of the coordinate system origin in direction of –V

$$\overline{T}_{v} = T_{-v} = \begin{bmatrix} 1 & 0 & -tx \\ 0 & 1 & -ty \\ 0 & 0 & 1 \end{bmatrix}$$

– **Coordinate Scaling**: Scaling an object by S_x and S_y or reducing the scale of coordinate system.

$$\overline{S}_{sx,sy} = S_{1/sx,1/sy} = \begin{bmatrix} 1/S_x & 0 & 0 \\ 0 & 1/S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Coordinate Transformations

– **Coordinate Rotation**: Rotating Coordinate system by an angle of $-\theta$

$$\overline{R}_{\theta} = R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– **Coordinate Reflection**: Same as Geometric Reflection

$$\overline{M}_{x} = M_{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Composite Transformations

– A number of transformations can be combined into one matrix to make things easy
  - Allowed by the fact that we use homogenous coordinates
– Matrix composition works *right-to-left.*

Compose:

$$\mathbf{M} = \mathbf{A}\ \mathbf{B}\ \mathbf{C}$$

Then apply it to a point:

$$\mathbf{v}' = \mathbf{M}\ \mathbf{v}$$

$$\mathbf{v}' = (\mathbf{A}\ \mathbf{B}\ \mathbf{C})\ \mathbf{v}$$

$$\mathbf{v}' = \mathbf{A}\ (\mathbf{B}\ (\mathbf{C}\ \mathbf{v}))$$

It first applies **C** to **v,** then applies **B** to the result, then applies **A** to the result of that.

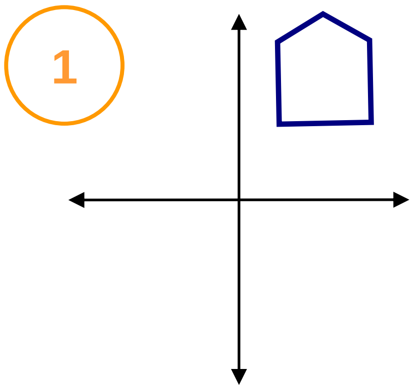
Composite Transformations

Rotation about Arbitrary Point (h,k)

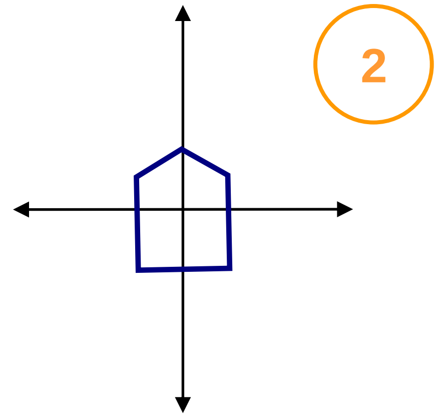
– Imagine rotating an object around a point (h,k) other than the origin
  - Translate point (h,k) to origin
  - Rotate around origin
  - Translate back to point

Composite Transformations

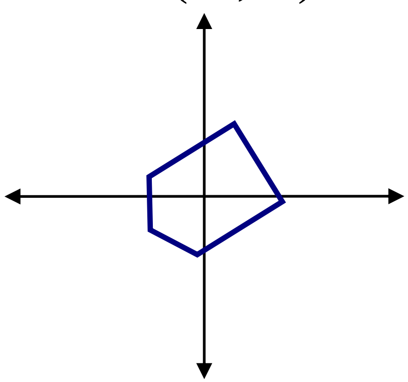
$House(H)$



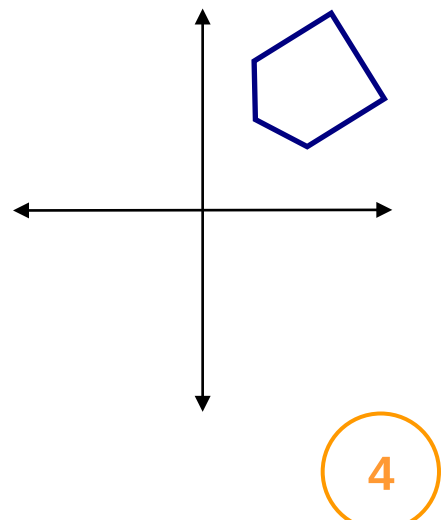
$T_{(-h,-k)}H$



$R_\theta T_{(-h,-k)}H$



$T_{(h,k)}R_\theta T_{(-h,-k)}H$



Composite Transformations

Let P is the object point whose rotation by an angle θ about the fixed point (h,k) is to be found. Then the composite transformation $R_{\theta,(h,k)}$ can be obtained by performing following sequence of transformations :

1. Translate (h,k) to origin and the new object point is found as

   $P^1 = T_V(P)$ where $V = -hI - kJ$

2. Rotate object about origin by angle θ and the new object point is

   $P^2 = R_\theta(P^1)$

3. Retranslate (h,k) back the final object point is

   $P^F = T^{-1}{}_V(P^2) = T_{-V}(P^2)$

The composite transformation can be obtained by back substituting

$$
\begin{aligned}
P^F &= T^{-1}{}_V(P^2) \\
&= T_{-V}\, R_\theta(P^1) \\
&= T_{-V}\, R_\theta T_V(P) \text{ where } V = -hI - kJ
\end{aligned}
$$

Thus we form the matrix to be $R_{\theta,(h,k)} = T_{-V} R_\theta T_V$

Composite Transformations

– The composite rotation transformation matrix is

$$R_{\theta,(h,k)} = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & -h\cos\theta + k\sin\theta + h \\ \sin\theta & \cos\theta & -h\sin\theta - k\cos\theta + k \\ 0 & 0 & 1 \end{bmatrix}$$

REMEMBER: Matrix multiplication is not commutative so order matters

Composite Transformations

Scaling about Arbitrary Point (h,k)

– Imagine scaling an object around a point (h,k) other than the origin
  - Translate point (h,k) to origin
  - Scale around origin
  - Translate back to point

Composite Transformations

Let P is the object point which is to be scaled by factors sx and sy about the fixed point (h,k). Then the composite transformation $S_{sx,sy,(h,k)}$ can be obtained by performing following sequence of transformations :

1. Translate (h,k) to origin and the new object point is found as

   $P^1 = T_V(P)$ where $V = -hI - kJ$

2. Scale object about origin and the new object point is

   $P^2 = S_{sx,sy}(P^1)$

3. Retranslate (h,k) back the final object point is

   $P^F = T^{-1}_V(P^2) = T_{-V}(P^2)$

The composite transformation can be obtained by back substituting

$P^F = T^{-1}_V(P^2)$

$= T_{-V}\, S_{sx,sy}(P^1)$

$= T_{-V}\, S_{sx,sy}.T_V(P)$ where $V = -hI - kJ$

Thus we form the matrix to be $S_{sx,sy,(h,k)} = T_{-V} S_{sx,sy} T_V$

Composite Transformations

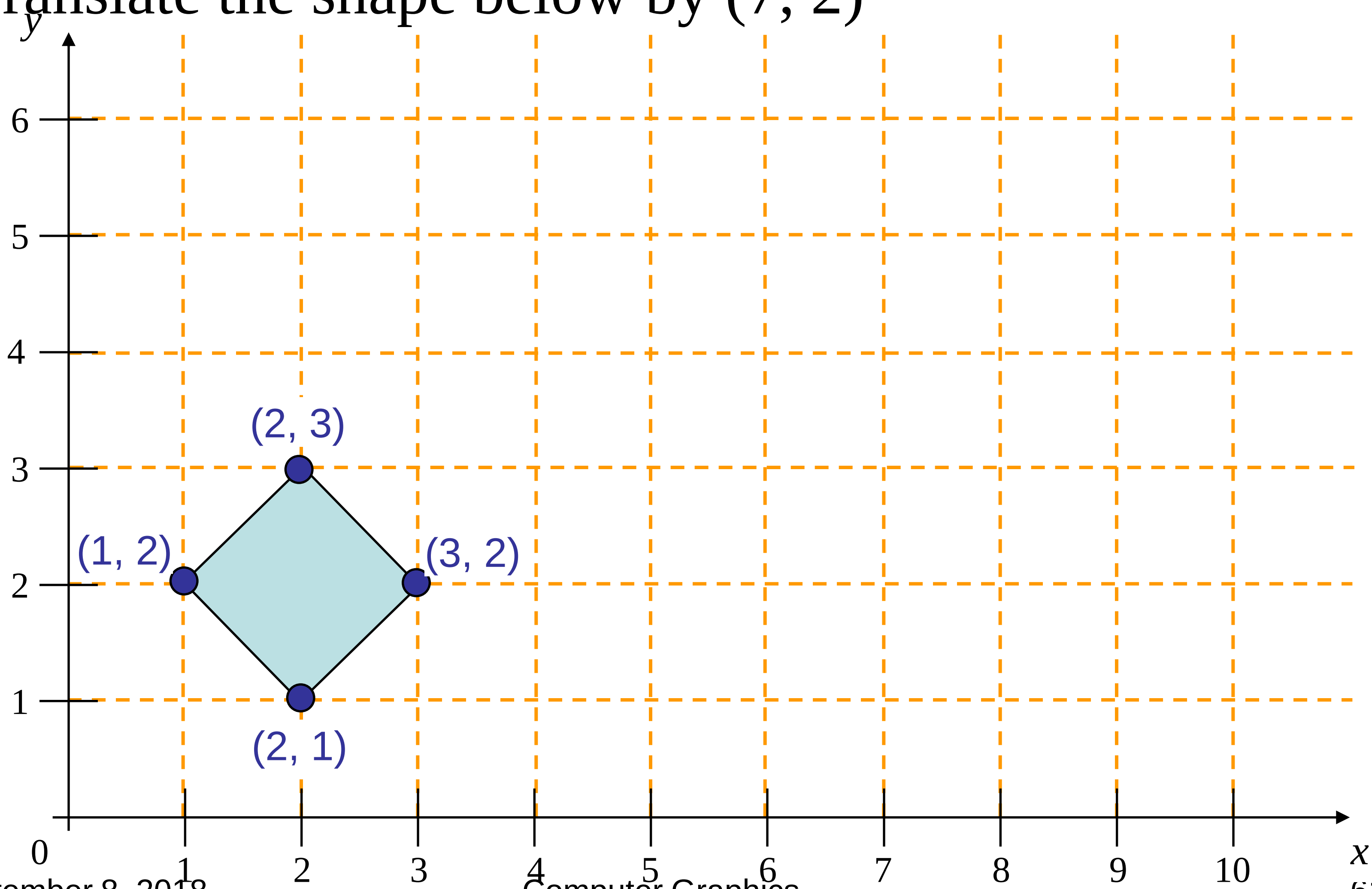
– The composite scaling transformation matrix is

$$S_{sx,sy,(h,k)} = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sx & 0 & -h.sx + h \\ 0 & sy & -k.sy + k \\ 0 & 0 & 1 \end{bmatrix}$$

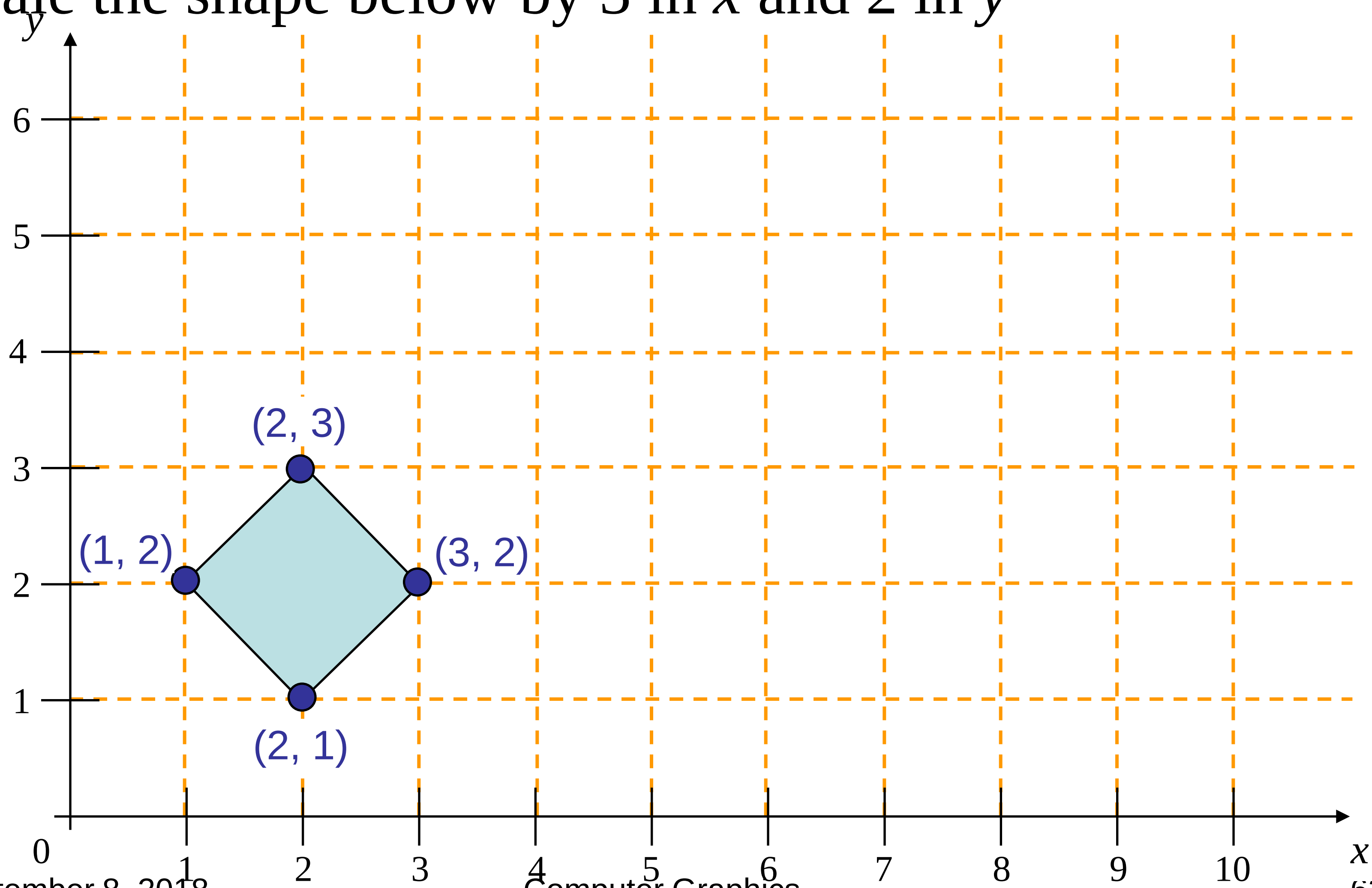
Exercises 1

Translate the shape below by (7, 2)



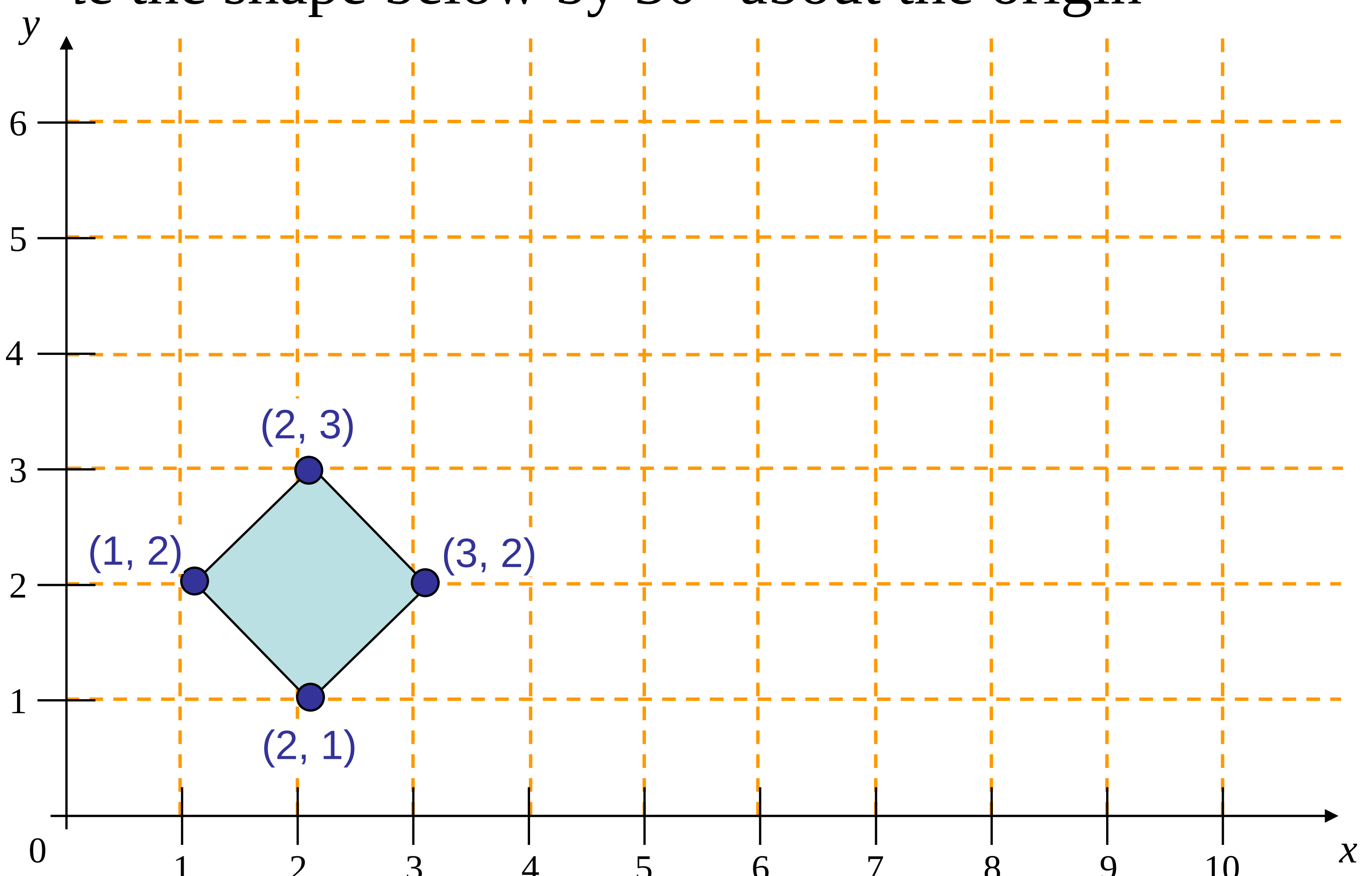
Exercises 2

Scale the shape below by 3 in *x* and 2 in *y*



Exercises 3

Rotate the shape below by 30° about the origin



Exercise 4

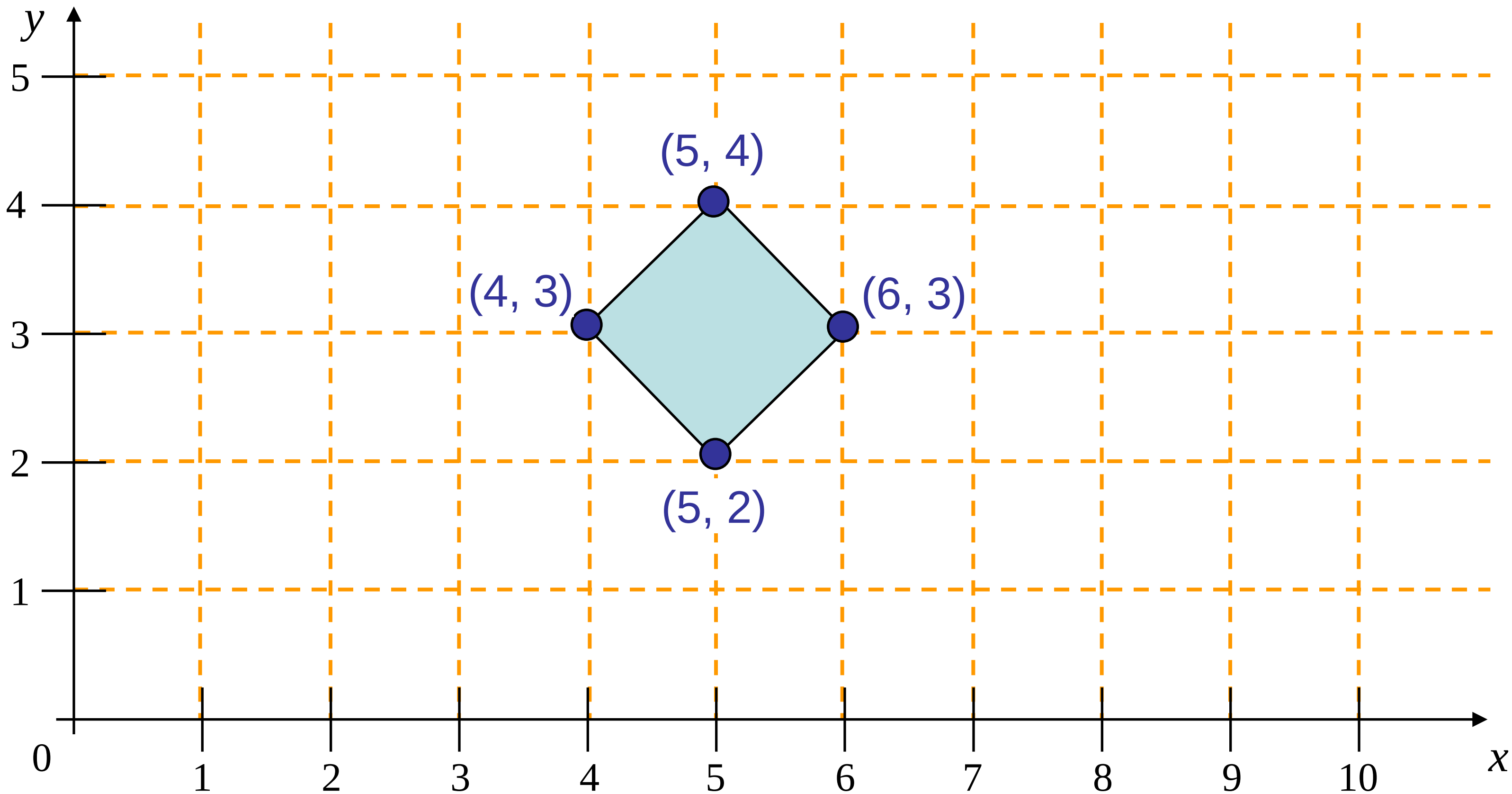
Write out the homogeneous matrices for the previous three transformations

Translation Scaling Rotation

$$\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

Exercises 5

Using matrix multiplication calculate the rotation of the shape below by 45° about its centre (5, 3)



Exercise 6

Rotate a triangle ABC A(0,0), B(1,1), C(5,2) by 45^0

1. About origin (0,0)
2. About P(-1,-1)

$$R_{45^0} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[ABC] = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_{45^0,(-1,-1)} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & -1 \\ \sqrt{2}/2 & \sqrt{2}/2 & \sqrt{2}-1 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 7

Magnify a triangle ABC A(0,0), B(1,1), C(5,2) twice keeping point C(5,2) as fixed.

$$[ABC] = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

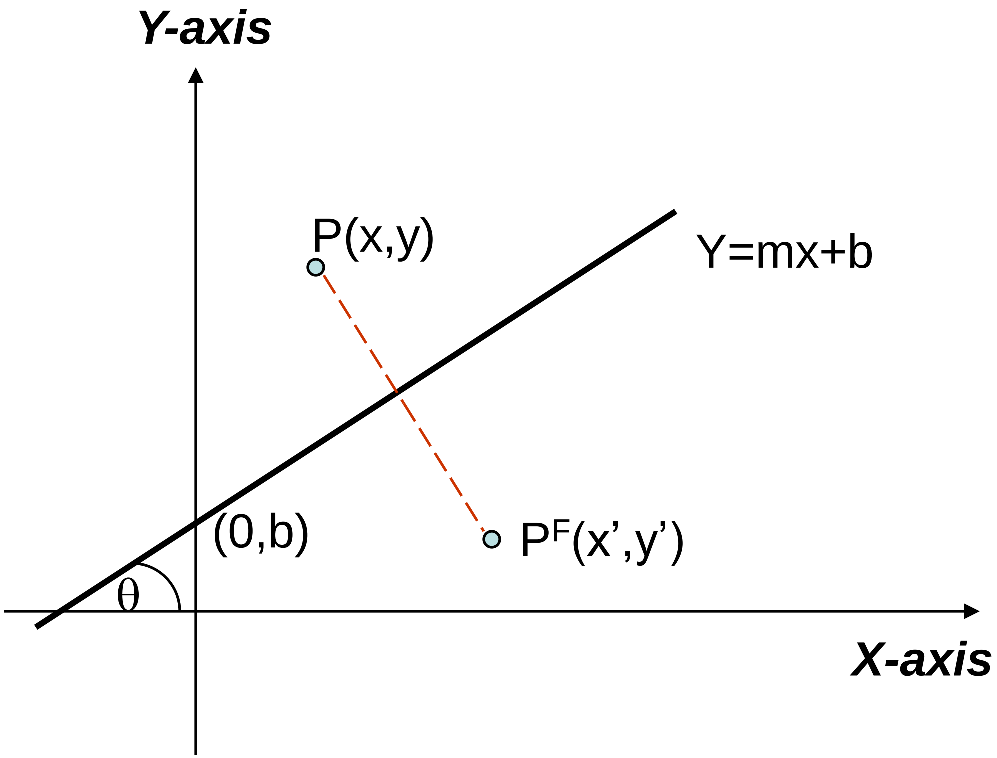
$$S_{2,2,\,(5,2)} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A'B'C'] = \begin{bmatrix} -5 & -3 & 5 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 8

Describe transformation M_L which reflects an object about a Line L: y=m*x+b.

Let P be the object point whose reflection is to taken about line L that makes an angle θ with +ve X-axis and has Y intercept as (0,b). The composite transformation M_L can be found by applying following transformations in sequence:



Exercise 8

1. Translate (0,b) to origin so that line passes through origin and P is transformed as

   $P^{I} = T_{V}(P)$ where $V = -hI - kJ$

2. Rotate by an angle of $-\theta$ so that line aligns with +ve X-axis

   $P^{II} = R_{-\theta}(P^{I})$

3. Now take mirror reflection about X-axis.

   $P^{III} = M_{x}(P^{II})$

4. Re-rotate line back by angle of θ

   $P^{IV} = R_{\theta}\,(P^{III})$

5. Retranslate (0,b) back.

   $P^{F} = T_{-V}(P^{IV})$

Exercise 8

The composite transformation can be obtained by back substituting

$$
\begin{aligned}
P^F &= T_{-V}(P^{IV}) \\
&= T_{-V}.R_\theta\,(P^{III}) \\
&= T_{-V}.R_\theta\,.\,M_x(P^{II}) \\
&= T_{-V}.R_\theta\,.\,M_x\,.\,R_{-\theta}(P^{I}) \\
&= T_{-V}.R_\theta\,.\,M_x\,.\,R_{-\theta}.\,T_V(P)
\end{aligned}
$$

Thus we form the matrix to be $M_L = T_{-V}.R_\theta\,.\,M_x\,.\,R_{-\theta}.\,T_V$

where $V = -0.I - b.J$

Exercise 8

$$M_L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta & -b\sin\theta \\ -\sin\theta & \cos\theta & -b\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta & -b\sin\theta \\ \sin\theta & -\cos\theta & b\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & -2b\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & b(\cos^2\theta - \sin^2\theta) + b \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 8

$$M_L = \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & -2b\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & b(\cos^2\theta - \sin^2\theta) + b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta & -b\sin 2\theta \\ \sin 2\theta & -\cos 2\theta & b\cos 2\theta + b \\ 0 & 0 & 1 \end{bmatrix}$$

$$\textit{putting} \quad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}, \quad \sin 2\theta = \frac{\tan^2\theta}{1+\tan^2\theta}, \quad \textit{and} \quad \tan\theta = m \quad (\textit{why}?)$$

$$= \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} & \frac{-2bm}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} & \frac{2b}{1+m^2} \\ 0 & 0 & 1 \end{bmatrix} \quad C.T.M.$$

Exercise 9

Reflect the diamond shaped polygon whose vertices are A(-1,0) B(0,-2) C(1,0) and D(0,2) about

1. Horizontal Line y=2
2. Vertical Line x = 2
3. Line L: y=x+2.

$$M_{y = x + 2} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{y = 2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{x = 2} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 10

Obtain reflection about Line y = x

$Method\ \ I$: Rotate by 45^0, take reflection about Y axis and Rerotate

$$R_{-45^0} M_y R_{45^0}$$

$Method\ \ II$: Rotate by - 45^0, take reflection about X axis and Rerotate

$$R_{45^0} M_x R_{-45^0}$$

$$Here\ R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{y\,=\,x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 11

Prove that

a. Two successive translations are additive /commutative.
b. Two successive rotations are additive /commutative.
c. Two successive Scaling are multiplicative /commutative.
d. Two successive reflections are nullified /Invertible.

Is Translation followed by Rotation equal to Rotation followed by translation ?

Exercise 11

a. Two Successive translations are additive/commutative.

Let two translations are described by translation vectors

$V = txI + tyJ$ and $V' = tx'I + ty'J$

We first formulate the translation by V followed by translation by V'.

$$T_{v'}.T_{v} = \begin{bmatrix} 1 & 0 & tx' \\ 0 & 1 & ty' \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & tx'+tx \\ 0 & 1 & ty'+ty \\ 0 & 0 & 1 \end{bmatrix}$$

$$= T_{v'+v}$$

Hence two successive translations are additve

Exercise 11

Also,

$$T_{v+v'} = T_v . T_{v'} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & tx' \\ 0 & 1 & ty' \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & tx + tx' \\ 0 & 1 & ty + ty' \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & tx' + tx \\ 0 & 1 & ty' + ty \\ 0 & 0 & 1 \end{bmatrix}$$

$$= T_{v'+v} = T_{v'} . T_v$$

$$Hence \quad two \quad successive \quad translations \quad are \quad commutative$$

Exercise 11

b. Two Successive scaling are multiplicative/commutative.

Let two scalings are described by scaling factors

sx, sy and sx' , sy'

We first formulate the scaling with sx and sy followed by scaling with sx' and sy'.

$$S_{sx',sy'}.S_{sx,sy} = \begin{bmatrix} sx' & 0 & 0 \\ 0 & sy' & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sx'.sx & 0 & 0 \\ 0 & sy'.sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= S_{sx'.sx,sy'.sy}$$

Hence two successive scalings are additve

Exercise 11

Also,

$$S_{sx.sx',sy.sy'} = S_{sx,sy}.S_{sx',sy'} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} sx' & 0 & 0 \\ 0 & sy' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sx.sx' & 0 & 0 \\ 0 & sy.sy' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sx'.sx & 0 & 0 \\ 0 & sy'.sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= S_{sx'.sx,sy'.sy} = S_{sx',sy'}.S_{sx,sy}$$

Hence two successive scalings are commutative

Exercise 11

c. Two Successive rotations are additive/commutative.

Let two rotations are described by angle θ_1 and θ_2

We first formulate the rotation by θ_1 followed by rotation by θ_2.

$$R_{\theta_2}.R_{\theta_1} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_2+\theta_1) & -\sin(\theta_2+\theta_1) & 0 \\ \sin(\theta_2+\theta_1) & \cos(\theta_2+\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R_{\theta_2+\theta_1}$$

Hence two successive rotations are additve

Exercise 11

Also,

$$R_{\theta_1}.R_{\theta_2} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_2+\theta_1) & -\sin(\theta_2+\theta_1) & 0 \\ \sin(\theta_2+\theta_1) & \cos(\theta_2+\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R_{\theta_2+\theta_1} = R_{\theta_2}.R_{\theta_1}$$

$$Hence \quad two \quad successive \quad rotations \quad are \quad commutative$$

Exercise 11

d. Two Successive reflections are nullified/Invertible.

$$\text{Let us consider reflection about X-axis}$$

$$M_x.M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I \quad (Identity \quad matrix)$$

$$Hence \quad two \quad successive \quad reflections \quad are \quad Invertible$$

Any Question !