Loss function for Poisson 2D inverse problem of finding a constant unknown forcing

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Consider the triangular element T as seen in figure 1. Given a point $O(x, y)$ inside the triangle $V_1V_2V_3$, we define ξ and η, the barycentric coordinates.

$$\xi = \frac{\text{area}(OV_2V_3)}{\text{area}(V_1V_2V_3)}$$
$$\eta = \frac{\text{area}(OV_1V_3)}{\text{area}(V_1V_2V_3)}$$

Let the value of the trial function $u(x, y)$ at the vertices V_1, V_2 and V_3 be u_1, u_2 and u_3 respectively. Similarly, let the value of the test function $v(x, y)$ at the vertices V_1, V_2 and V_3 be v_1, v_2 and v_3 respectively. Now, the points inside or on the triangle are easily parameterised by ξ and η in the following manner.

$$x = x_1\xi + x_2\eta + x_3(1 - \xi - \eta) \tag{1}$$
$$y = y_1\xi + y_2\eta + y_3(1 - \xi - \eta) \tag{2}$$

The finite dimensional weak formulation for the Poisson's problem in two dimensions for the triangular element T is to find $\hat{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{\mathsf{T}}$ such that

$$\int_T \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} dxdy = \int_T fvdxdy \tag{3}$$

for all $\hat{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{\mathsf{T}}$, where,

$$u(x(\xi, \eta), y(\xi, \eta)) = N\hat{u}$$
$$v(x(\xi, \eta), y(\xi, \eta)) = N\hat{v}$$
$$\text{and, } N = \begin{bmatrix} \xi & \eta & 1 - \xi - \eta \end{bmatrix}$$

Following standard procedure, to find algebraic equations for the element, we start from

$$\begin{bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{bmatrix} = J \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} \qquad J = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$
$$\implies \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = J^{-1} \begin{bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = B\hat{u} \tag{4}$$

$$\begin{bmatrix} \partial v/\partial x \\ \partial v/\partial y \end{bmatrix} = B\hat{v} \tag{5}$$

$$\tag{6}$$

where,

$$B = \frac{1}{\det J} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

So, the LHS of equation 3 is the following.

$$\hat{v}^\mathsf{T} K_e \hat{u}$$

where,

$$K_e = \frac{\det J}{2} B^\mathsf{T} B$$

whereas, the RHS of equation 3 is the following.

$$\begin{aligned} &\int_T f\hat{v}^\mathsf{T} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \det J d\xi d\eta \\ =& f\hat{v}^\mathsf{T} \begin{bmatrix} \det J/6 \\ \det J/6 \\ \det J/6 \end{bmatrix} \end{aligned}$$

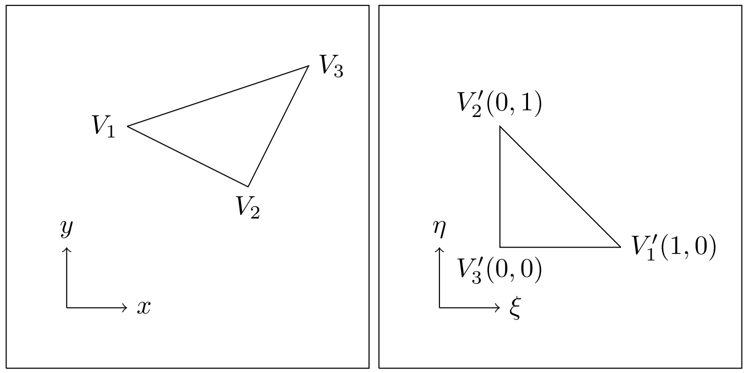

Figure 1: The triangular element T in the computaional domain is shown on the left while its reference element T' is shown on the right.

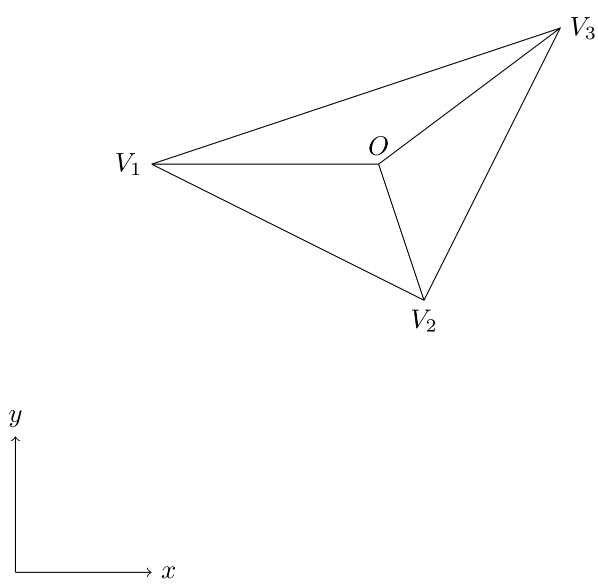

Figure 2: To define parameters ξ and η we form three triangles OV_2V_3, OV_3V_1, and OV_1V_2 which put together form the triangle $V_1V_2V_3$.