Numerical Integration for Triangular Elements

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Given a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$, we need to find

$$\int_K f(x, y) dxdy \tag{1}$$

where K is the triangular element defined by the vertices $V_1(x_1, y_1)$, $V_2(x_2, y_2)$ and $V_3(x_3, y_3)$ as seen in figure 1.

1 Transformation of coordinates

As we have to consider points inside the triangular element we switch to a different coordinate system (barycentric?). As seen in figure 2, given a point O inside the triangle $V_1V_2V_3$, we define

$$\xi = \frac{\text{area}(OV_2V_3)}{\text{area}(V_1V_2V_3)}$$
$$\eta = \frac{\text{area}(OV_1V_3)}{\text{area}(V_1V_2V_3)}$$

Now, the points inside or on the triangle are easily parameterised by ξ and η in the following manner.

$$x = x_1\xi + x_2\eta + x_3(1 - \xi - \eta) \tag{2}$$
$$y = y_1\xi + y_2\eta + y_3(1 - \xi - \eta) \tag{3}$$

The tranformation from the parameters (ξ, η) to (x, y) is given by the following linear relation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \tag{4}$$

To transform the integral given in equation 1, in terms of ξ and η, we have the following considerations.

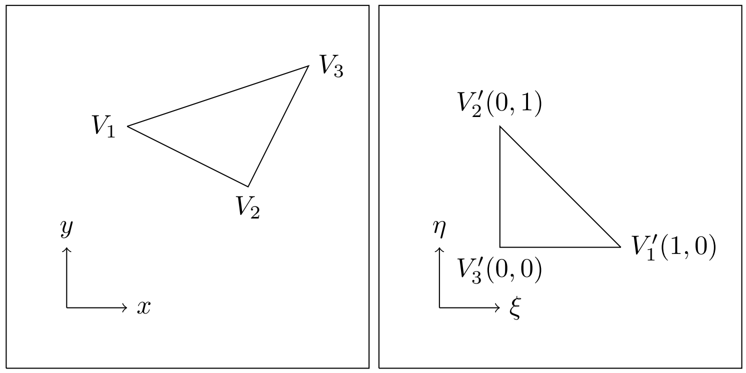

Figure 1: The triangular element K in the computaional domain is shown on the left while its reference element K' is shown on the right.

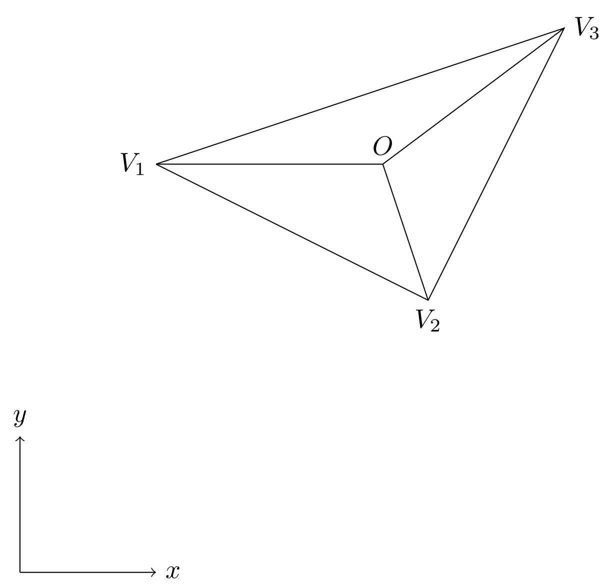

Figure 2: To define parameters ξ and η we form three triangles OV_2V_3, OV_3V_1, and OV_1V_2 which put together form the triangle $V_1V_2V_3$.

Limits We can see that $(\xi = 1, \eta = 0)$, $(\xi = 0, \eta = 1)$ and $(\xi = 0, \eta = 0)$ correspond to the vertices V_1, V_2 and V_3 respectively as shown in figure 1. Since, this mapping is linear we can conclude that the limits of the integration in equation 1 would be ξ going from 0 to 1 and η going from 0 to $1 - \xi$.

Integrand The integrand $f(x, y)$ will be written as $f(x(\xi, \eta), y(\xi, \eta)$ using equations 2 and 3.

Infinitesimal area element Small changes in (x, y) due to small change $(\delta\xi, \delta\eta)$ in (ξ, η) can be found by using equation 4.

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = J \begin{bmatrix} \delta \xi \\ \delta \eta \end{bmatrix}$$

where

$$J = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}$$

Therefore the infinitesimal area element $dxdy$ will have to be replaced by $\det(J) d\xi d\eta$. Fore more details regarding this refer to section 3.

2 Integration in the reference element

3 On scaling and determinants