

2D Linear Elasticity

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In the two dimensional elasticity problem we need to find the displacement $u = u_x \hat{e}_x + u_y \hat{e}_y$ at a point $(x, y) \in \Omega$, the domain. Extrapolation to two dimensions of the simplistic one dimensional strain $= \Delta l / l$, leads to a 2nd order strain tensor.

$$\begin{aligned}\epsilon &= \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix} \\ \epsilon_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right) \quad i, j \text{ are placeholders for } x, y \\ \implies \epsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \epsilon_{xy} &= \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) / 2 \\ \epsilon_{yx} &= \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) / 2 \\ \epsilon_{yy} &= \frac{\partial u_y}{\partial y}\end{aligned}$$

The constitutive relation is linear and is defined by a 4th order tensor C . So, the 2nd order stress tensor is given by

$$\sigma_{ij} = \sum_{k=x,y} \sum_{l=x,y} C_{ijkl} \epsilon_{kl}$$

where again i and j are placeholders for x or y . Let \hat{n} be the outward normal on a point on the boundary of the domain Ω . Now, the components of the traction vector is given by

$$t_i = \sum_{j=x,y} \sigma_{ij} n_j \quad i = x, y$$

The essential boundary condition would be to specify u , whereas the natural boundary condition would be to specify the traction t .

To make our lives easier we consider the engineering strain $[\epsilon_{xx} \ \epsilon_{yy} \ \gamma_{xy}]^T$ where $\gamma_{xy} = 2\epsilon_{xy}$ is engineering shear strain. Now, we can write the constitutive relation using a matrix C .

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = C \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Two of the usual choices for C are

$$\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

where E is the Young's modulus and ν is the Poisson's ratio.

Now that we have clearly defined the stress and strain we can write the governing equation

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y &= 0 \end{aligned}$$

where $b_x \hat{e}_x + b_y \hat{e}_y$ is the given body force at the point (x, y) .