Understanding anomalies from a math perspective

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1 Introduction

[1]The view we are going to adopt for defining an anomoly is:

Definition 1 (Anomaly) *a d-dimensional anomaly is controlled by a special type of $(d+1)$-dimensional topological theory.*

The aim of the notes is to be able to suffieniently describe anomalies in a QFT. Before we do this, we define a QFT as a functor from the category of cobordisms, which is naivelt speaking equivalent to space-time manifolds, to the category of Hilbert spaces:

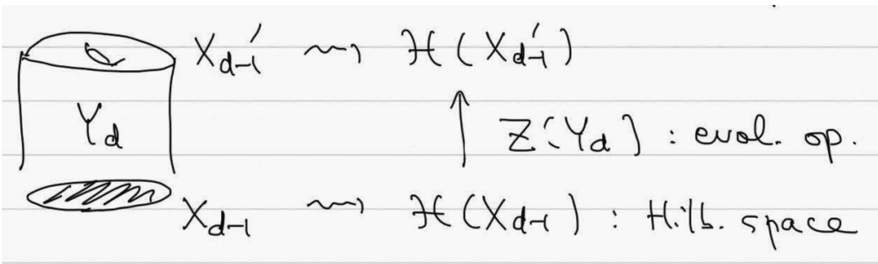


Figure 1: Functorial view of QFTs

In particular, we use the Atiyah-Segal axioms, which can be found in in my TQFT notes, where the null-space is mapped under the functor to the space of complex numers: QFT : $\varnothing \to \mathscr{H}(\varnothing)$, and the map then corresponds to the partition function, $Z : \mathbb{C} \to \mathbb{C} \implies Z \in \mathbb{C}$. Also, we can always add more structure to the manifolds which will affect the theroy, like a metric, orientation, spin structures, and other background fields.

1.1 Characterizing an anomaly

A unitary theory additionally contraints the partition function to be a phase, $Z \in U(1)$. From this perspective, an anomaly can be viewd as a *controllable* phase ambiguity in the partition function. We can characterize this phase ambiguity by considering $Z_Q(Y_d) \in V(Y_d)$, where the partition function is a vector ina 1-dimensional vector space without a canonical basis.[2]

[1]We review a TASI talk given by Yuji Tachikawa.

[2]Note that here we are trying to describe a general anomaly, and therefore the ambiguity is actually a complex number and not a phase ambiguity. The phase ambiguity refers to ambiguities in a unitary theory.

2 Connections to homology

In this section we will use the following definition for an anomaly:

Theorem 2 (Anomaly) *if a system couples to a background gauge field for symmetry G, but the partitioin function Z has a controllable phase ambiguity, this phase ambiguity is called an anomaly. Usually the anomaly is controlled by a theory in one higher dimension.*

To make things more intuitive we start off with a simple example[3]. We study a (0+1)D QFT, which is basically a quantum mechanical theory evolving in time, with an internal G symmetry which acts on the Hilbert space. Let $\mathscr{H}$ be finite dimensional and for simplicity let $H = 0$. We know that physically two states $v, cv \in \mathscr{H}$ where $c \in U(1)$ are equivalent. With the above in mind G symmetry acts on $\mathscr{H}$ in the following way: $G \ni g \implies \rho(g) : \mathscr{H} \to \mathscr{H}$ where ρ is some representation of G.

Theorem 3 (Layman's definiton of projective representation) *a projective representation of G is a collection of operators $\rho(g) \in GL(\mathscr{H})$[4], with $g \in G$ satisfying the homomorphism property upto a constant:*

$$\rho(g)\rho(h) = c(g,h)\rho(g \cdot h) \tag{1}$$

for some constant $c(g,h) \in U(1)$.

We consider projective representation of the group G with the projective phase $c \in U(1)$ precisely because additional phases do the affect the physical states. We can now look at partition functions of this system, with $t \in \mathbb{S}^1$ and consider in one case just inserting at some point just $g \cdot h$ and in the other inserting g and h at seperate times. This gives rise the following partition functions:

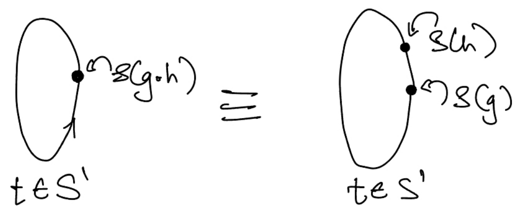


Figure 2: $Z_1 = \mathrm{Tr}(\rho(g \cdot h))$ $\qquad Z_2 = \mathrm{Tr}(\rho(g)\rho(h)) = c(g,h)Z_1$

where clearly there is an anomalous factor of $c(g,h)$. To study this further we look at the restrictions on the projective phases, by considering the simplification of $\rho(g) \cdot \rho(h) \cdot \rho(k)$ one can show that for consistency we require:

$$c(h,k)c(gh,k)^{-1}c(g,hk)c(g,h)^{-1} = 1 \tag{2}$$

to study the classification of these projective phases, we need to further understand equivalences under redefinitons of the representations:

[3]Note that the discussion here is moderately vague, and you should try to understand more deeply some of the comments in this section which are highlighted in red.

[4]recap: GL, the general linear group of degree n is the set of n×n invertible matrices, together with the operation of ordinary matrix multiplication.

$$\tilde{\rho}(g) = b(g)\rho(g) \text{ s.t. } b(g) \in G \implies \tilde{c}(g,h) = c(g,h)\Big[\frac{b(g\cdot h)}{b(g)b(h)}\Big] \tag{3}$$

Hence, the group of distinct projective phases is obtained by all phases satisfying Eqn. 2 quotionted out by the equivalence relation in Eqn. 3. This set can be shown to be homeomorphic to the second homology group $H^2(G,U(1))$.

3 Digression into the math

Let us define the following map:

$$C^d(G,A) : G^d \longrightarrow A; \qquad d : C^d \longrightarrow C^{d+1} \tag{4}$$

where the first is a set of maps C^d, and the second is the exterior derivative defined on these maps. The exterior derivative is defined in the following way:

$$\begin{aligned} df(g_1,g_2,...,g_{d+1}) =& g_1 f(g_2,...,g_{d+1}) - f(g_1\cdot g_2, g_3,...,g_{d+1}) \\ & f(g_1,g_2\cdot g_3,g_4,...,g_{d+1}) - f(g_1,g_2,g_3\cdot g_4,...,g_{d+1}) \\ & ... \\ & ... \\ & ... \\ & (-1)^d f(g_1,g_2,...,g_d\cdot g_{d+1}) + (-1)^{d+1} f(g_1,...,g_d) \end{aligned}$$

from this definition it is easy to prove the nilpotent identity of exterior derivative algebra, $d^2 = 0$[5] Now, we go on to show that from this construction Eqn. 2 boils down to just being $dc = 1$, consider the case of $d = 2$:

$$\begin{aligned} d \ : \ C^2 \longrightarrow C^3 \qquad & G \circlearrowright U(1) \text{ trivially} \\ (dc)(g,h,k) = g_1 c(h,k)\cdot c(g\cdot h,k)^{-1} & \cdot c(g,h\cdot k)\cdot c(g,h)^{-1} \\ \implies (dc)(g,h,k) = 1 \ \forall \ g,h,k \in G & \end{aligned}$$

Differential forms which satisfy the property $dc = 0$ are called closed forms. The expression in Eqn. 3 becomes db:

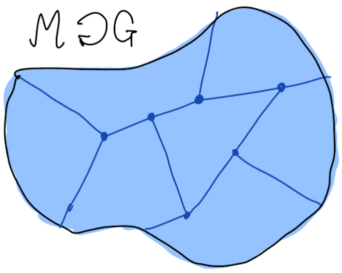
$$\begin{aligned} b : G \longrightarrow A \qquad & \implies b \in C^1(G,A) \\ (db)(g,h) = b(g_2)\cdot b(g_1\cdot g_2)^{-1} & \cdot b(g_1) \end{aligned}$$

which boils down to saying we want to look at equivalence classes where two forms differing by a total derivative are identifies in the same equivalence class. This is exactly the definition of a homology group which we refer to in the previous section.

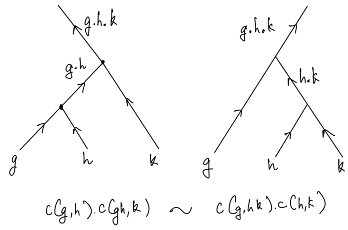
[5]Note that the equations above are written where the group action on A is written additively, but in our specific example the group G acts on $A = U(1)$ is written multiplicatively(or additively in the phases). So, $d^2 = 0$ becomes $d^2 = 1$ in our case

4 Intuitive picture of anomaly inflow

In this section we look at an intuitive and pictorial way of representing theories with finite groups acting on a manifold, and how anomalies on the boundary can be related to and viewed as boundaries of higher dimensional theories. This discussion follows from the 1990 paper by Witten et al. [1][6]. Consider a $(1+1)$D system with G symmetry, where G is finite. Naturally the space-tiem manifold is endowed with a G-bundle like structure.[7] We want to assign this G-bundle a partition function $Z^{2d} \in U(1)$. But how do we understand the action of finite G on this $(1+1)$D manifold. We use the following picture:



(a) General picture



(b) Assigning phase factor

where ... The action of continuous groups on a space-time manifold is naturally understood, this is not the case with finit groups. The idea is to realize the action of $g \in G$ as some sort of domain wall, so when we cross this domain wall we are acted on by this element g. In this picture, every G-bundle can now be viewed as being endowed with a complicated network of domain walls. The next question is how to assign a partition function to such a structure? We assign to each intersction points or nodes of domain walls the appropriate projective phase factor, as in the Fig. ??(b). Now, we can define the partion function as the product over all such projective phases

$$Z^{2d} = \prod_{\text{nodes}} c(g,h) \tag{5}$$

We want this partition function be 'gauge-invariant'. Here 'gauge-invariance' really means to be invariant under different domain wall network realizations of the same G gauge background, like in Fig. ??(a):

(a) Anomaly in the (2+1)D bulk

(b) Associated anomaly on the (1+1)D boundary

[6]the gauge can be promoted to be dynamical, and then you can sum over all possible gauge configurations. This was the initial point of the paper, where they tried to realize gauge field theory for finite groups.

[7]Where does this enter the discussion? Need to marry gauge theory with the algtop part of the discussion.

and we can see the partition function attributed to both these figures would be the same under the rules we discussed previously. We can now proceed to see how the $(0+1)$D anomaly we saw in Fig. 2. Looking at Fig. ?? above, we see that the anomaly in the $(0+1)$D theory is very simply cancelled by the appropriate cancellation in the bulk $(1+1)$D theory.

5 Category theory

References

1. this thesis by Bartlett [2] on a categorial and functorial view of quantum field theory is a good introduction to various ideas related to bordisms and tqfts.
2. these notes by Balsells(link) is also a good reference and is suited for someone who has absolutely no mathematical background for physics. Might be something to look into for journal clubs/group talks.
3. I also have some notes of my own in my notes on tqft, which I might merge with these notes. If I haven't here's the link.

6 Intuitive explanantion of Dijkgraaf-Witten theory

[8]A large class of TQFTS come from topological gauge theories. These are broadly categorized into two knids:

1. topological gauge theories with a finite gauge group G, called Dijkgraaf-Witten theories which is a state-sum TQFT.
2. Chern-Simons theories which have a continuous gauge group, for example $G = U(1)_k$ which correspond to CS theories with level k.

Remember, to define a TQFT we require the prescription of a partition function for spacetime evolution, and we also require the asignment of vector spaces to spatial manifolds. We first consider the first of these.

6.1 Mathematical preliminaries

We first formalize how we treat the spatial manifold in Dijkgraaf-Witten theory. We define the necessary ingrideients to contruct a triangulation, or for our purposes, a simplicial complex structure, on a manifold.

Definition 4 (Standard n-simplex) *is the convex hull of $(n+1)$ points in $\mathbb{R}^{n+1}$ given by:*

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1 \ \ and \ \ t_i \geqslant 0 \ \ \forall i\} \tag{6}$$

Definition 5 (Δ-complex structure) *on a topological space X is teh set of maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ such that:*

[8]Notes based on Meng Chang's lectures in TASI 2023, and these notes, Hatcher's book on Algebraic topology. Another great reference for understanding representative cocycles is Ref. [**Tiwari2018**].

1. $\sigma_\alpha|_{\dot{\Delta}^{n(\alpha)}}$ *is injective, and each point in* X *is in the image od exactly one such* $\sigma_\alpha|_{\dot{\Delta}^{n(\alpha)}}$.

2. *every restriction of* σ_α *to a face of* $\Delta^{n(\alpha)}$ *is a new map* $\sigma_\beta : \Delta^{n(\alpha)-1} \to X$.

3. *a set* $A \subset X$ *is ope iff* $\sigma^{-1}(A)$ *is open in* $\Delta^{n(\alpha)}$ *for all* α.

Definition 6 (Simplicial complex) *is a* Δ*-complex of which the simplicces are uniquely determined by its vertices. Or in other words, there cannot be multiple simplices with the same set of vertices.*

Definition 7 (n-chain) *Let* $\Delta_n(X) = \bigoplus_{\alpha:n(\alpha)=n} \mathbb{Z}\sigma_\alpha$. *This is a free abelian group, with elemenets of the form* $\sum_\alpha n_\alpha \sigma_\alpha$ *and are called n-chains.*

Definition 8 (Boundary operator) *is a map* $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ *given by:*

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} \tag{7}$$

where the hat indicates the removal of vertex v_i.

Lemma: the composition

$$\partial_{n-1}\partial_n : \Delta_n(X) \to \Delta_{n-2}(X) \tag{8}$$

is zero.

Definition 9 (Chain complex) *let* $C_0, C_1, \ldots$ *be abelian groups and* $\partial_n : C_n \to C_{n-1}$ *be homomorphisms such that* $\partial^2 = 0 \;\; \forall n \geqslant 0$, *where* $\partial_0 : C_0 \to 0$. *Then*

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \tag{9}$$

is called a chain complex and the groups C_n *are called chain groups.*

Definition 10 (Homology group) *is nth homology group of a chain complex is defined as*

$$H_n = ker\partial_n / im\partial_{n+1} \tag{10}$$

The definitions that follow are the most important, and a bit tricky:

Definition 11 (Cochain group and coboundary operator) *The cochain group is the dual of the chain group, ie.* $C^n := C_n^* = Hom(C_n, A)$ *for* A *an abelian group. The coboundary operator is the dual map* $\delta = \partial^* : C^{n-1} \to C^n$. *It is easy to show* $\delta^2 = 0$.

Definition 12 (Cohomology group) *For*

$$\ldots \leftrightarrow C^{n+1} \xleftarrow{\delta} C^n \xleftarrow{\delta} C^{n-1} \leftarrow \ldots \leftarrow C^1 \xleftarrow{\delta} C^0 \xleftarrow{\delta} 0 \tag{11}$$

a cochain complex, we call $H^n(C^n; A) = ker\delta / im\delta$ *the n-th cohomology group of* C^n *with coefficients in* A.

Definition 13 (i-cocycle) *Let G be a finite, discrete topological group and V a multiplicative abelian group. Let $\phi : C^i(G) \to V$ be a morphism, where $C^i(G) := \underbrace{G \times \ldots \times G}_{i \text{ times}}$ for $i \geqslant 1$, then d^i denotes an operator*

$$d^i\phi : C^{i+1}(G) \to V$$

such that $d^i\phi\left(g_1,\ldots,g_{i+1}\right) = \phi\left(g_1,\ldots,g_i\right)^{(-1)^{i+1}}\phi\left(g_2,\ldots,g_{i+1}\right)\Pi_{j=1}^{i}\phi\left(g_1,\ldots,g_jg_{j+1},\ldots,g_{i+1}\right)^{(-1)^j}$. The map ϕ is an i-cocycle if $d^i\phi = 1$.

For our calculations, G will usually be the symmetry group of the SPT and $V = U(1)$. Importantly, note here that the map ϕ takes $i - G$ arguments and not $(i+1)$ like in the usual definitions of a group cocycle. This point is unwarrentedly ignored in most papers on SPTs, and there is an abuse of terminology where such maps are called group cocycles. Next, we move on to defininting the Dijkgraaf-Witten invariant.

Definition 14 (Color) *of a manifold M is a map:*

$$\phi : \{\text{oriented edges of } M\} \to G, \tag{12}$$

satisfying two conditions:

1. *flatness condition: for any 2-simplex F we have $\phi(\partial F) = 1$, where the notation denotes the product of the group elements along the boundary of F.*
2. *for any oriented edge E we have $\phi(-E) = \phi^{-1}(E)$, where -E denotes the reversed edge.*

Observation: color $\iff$ gauge configuration. Let M be a manifold with boundary ∂M. Then, we denote set of all colors of manifold M and ∂M by $\mathrm{Col}(M)$ and $\mathrm{Col}(\partial M)$ respectively. If $\tau \in \mathrm{Col}(\partial M)$, then the set of colors on M compatible with τ is called $\mathrm{Col}(M,\tau)$.

Definition 15 (Weigth) *Fix a 3-cocycle $\alpha : C^3(G) \to U(1)$, and let $\phi \in Col(M)$, then for $\sigma = [v_0v_1v_2v_3]$ we denote*

$$\phi([v_1,v_2]) = g, \;\; \phi([v_2,v_3]) = h, \;\; \phi([v_3,v_4]) = k \tag{13}$$

The weight of σ with respect to color ϕ is $W(\sigma,\phi) = \alpha(g,h,k) \in U(1)$.

Definition 16 (Dijkgraaf-Witten invariant) *Given a simplicail complex structure on a manifold M, with $\{\sigma_i\}_{i=1}^n$ the set of 3-simplices of M and N_0 the number of 0-simplices of M, and $\tau \in Col(\partial M)$, then the Dijkgraaf-Witten inariant is given by:*

$$Z(M,\tau) = \frac{1}{|G|^{N_0}} \sum_{\phi \in Col(M,\tau)} \prod_{i=1}^{n} W(\sigma_i,\phi)^{\varepsilon_i} \tag{14}$$

where ε_i is the relative orientation of σ_i with respect to teh orientation of M.

6.2 Dijkgraaf-Witten partition function

Here, we build up the theory up in a down-up approach, instead of the top-down approach considered in the McGrrevy notes. We consider first a topological manifold in, say $(2+1)$D, and call it M_3. Then, we triangulate this manifold and give the triangulation a branching structure. Now, a gauge configuration defined on this manifold is given by a set of assignments $[ij] \mapsto \{g_{ij}\}$ of group elements to every 1-simplex on the manifold. Note that the order matters, and $g_{ij} = g_{ji}^{-1}$,We further impose that for a configuration to be valid, it needs to be flat:

$$g_{ij}g_{jk}g_{ki} = 1 \;\; \forall \;\; [ijk] \implies \delta g[ijk] = 1 \tag{15}$$

or, in other words, it needs to satisfy the 2-cocycle condition. We can now define partition functions for this type of theory. First, we talk about the untwisted Dijkgraaf-Witten theories:

$$Z[M_3] = \frac{1}{|G|^{N_0}} \sum_{\{g\}}{}' 1 \tag{16}$$

where N_0 is the unmber of 0-simplices, and the primed summation is over gauge configurations satisfying the cocycle conodition mentioned before. The prefactor acts as a normalization which quotients out the equivalent gauge transformations. We can also consider twisted Dijkgraaf-Witten theories:

$$Z[M_3] = \frac{1}{|G|^{N_0}} \sum_{\{g\}}{}' \prod_{[ijkl]} \omega^{s([ijkl])}(g_{ij}, g_{jk}, g_{kl}) \tag{17}$$

where $[ijkl]$ is a tetrahedrom with a branching structure, $s([ijkl])$ is the relative orientation of the tetrahedran and $\omega \in H^3(G, U(1))$. This partition function is well defined, and does not depend on the choice of triangulation[9]. We note that each tetrahedron with flat gauge fields only has 3 independent gauge labels. The theory is called twisted because twists encode the existence of topological fluxes through the spacetime manifold M_3. This can be understood intuitively in terms of holonomies around a topological defect resulting in non-trivial values in the gauge group. For example, a $U(1)$ defect leads to an additional phase given by $\mathrm{Hol}_\gamma = \exp\left(iq \S a\right)$.

6.3 Dijkgraaf-Witten theory Hilbert space assignments

We now consider how we can prescribe the assignment of a vector space to a spatial manifold. The simplest way to work out this vector space is by quantizing the theory on the spatial manifold. To do this, we consider a spacetime evolution of the form $M_3 = \Sigma \times [0,1]$ and try to work out the Hamiltonian for the system by figuring out the correspondence:

$$Z[M_3, \{g_b\}] \leftrightarrow \langle \{g'\} | \exp\{-\beta H\} | \{g\} \rangle \tag{18}$$

Note that M_3 has a boundary, and so the partition function now depends on the gauge configurations on the boundary(boundary conditions) which we denote by $\{g_b\}$. In this particular setting, we note that the spatial manifold is the same at all times, ie. $\Sigma_0 = \Sigma_1$, and wlog let the triangulation of the spatial manifolds be the same. Also, we define an enlarged Hilbert space for the spatialmanifold

[9]This is clearly true in the first case because we are counting the number of non-trivial G-bundles on M_3, which is a topological invariant.

as $\mathbb{H} = \text{Span}(\{g\})$ which is the span of flat gauge configurations on the spatial manifold.[10] To be precise, we have the following transition amplitude:

$$Z[M_3, \{g_b\}] = \delta_{g'_{ij}, h_i g_{ij} h_j^{-1}} = \prod_i A_i; \quad \langle \{g'\} | A_i | \{g\} \rangle = \frac{1}{|G|} \sum_{h \in G} \prod_j \delta_{g'_{ij}, h g_{ij}}$$
$$\implies \exp\{-\beta H\} \sim \prod_i A_i \implies H = -\sum_i A_i$$

where $h_i \in G$ and in the first line the delta function triggers for equivalent gauge configurations, and the operators A_i basically simplify the equivalence by locally resolving the equivalence condition at each site i. Note that this Hamiltonian describes dynamics in the enlarged Hilbert space, however when we zoom into the ground state space of this Hamiltonian we will see the TQFT Hilbert space.

7 Bordism invariants

Definition 17 (Bordant manifolds) *two d-dimensional manifolds X_d and Y_d are said to be **bordant** if there exists a $(d+1)$-dimensional manifold M_{d+1} such that:*

$$\partial M_{d+1} = X_d \sqcup \bar{Y}_d \tag{19}$$

the propotypical example is that of a pair of pants which provides a bordism between S^1 and $S^1 \sqcup S^1$. We say that the partition function is bordism invariant if:

$$Z[X_d] = Z[Y_d] \tag{20}$$

it is invariant under bordisms. It is beleived that Bordism invariants classify all invertible gapped phases. d-mainfolds with bordisms form an abelian group, which is denoted by Ω_d, witht he following properties:

1. the addition operation is just the disjoint union.
2. the identity elemets is just the empty set, denoted $\varnothing$.
3. and the inverse is orientation reversal. Why? Because the manifold $X_d \sqcup \bar{X}_d$ is always bordant to the empty set, ie. $X_d \sqcup \bar{X}_d \sim \varnothing$.

The following discussion seems to be a bit confusing in connection to out notes on teh functorial TQFTs. But I present here what wwas shown in the lectures.

If the partion function of a theory is invariant under bordisms, then it stands to reason that thre should exist the homomorphism:

$$Z : \Omega_d \to \mathbb{C} \tag{21}$$

And we can claim the follwing identity: $Z[X_d \sqcup Y_d] = Z[X_d]Z[Y_d]$, which we say is aximatic. Then it follows that $Z[X_d \sqcup \bar{X}_d] = Z[] = 1 \implies Z[X_d]Z[\bar{X}_d] = 1$.

For Bosons without symmetry, we think about manifolds with orientations, and the bordism group of d-dimesntional manifolds, denoted Ω_d^{SO}. However, for fermions without symmetry, we

[10]This enlarged Hilbert space is not reallyb teh TQF Hilbert space, but is a useful intermediate step.

think about the bordism group of d-dimnesional manifolds with spin structures, denoted Ω_d^{spin}. Now, under addition of some symmetry, say G, we need to also consider gauge fields associated to this symmetries which live on the manifolds. We say the partition function in the presence of the gauge field, $Z[X_d, A]$, is bordism invariant. For example:

1. for bosons in $(1+1)$d, and no symmetries we have $\Omega_2^{SO} = 1$. This is because any oriented 2-dimensional manifold can be filled in, or in other words is the boundary of a 3-dimnesional manifold and are therefore always bordant to teh empty set. This classification is confirmed by studies of models in 1+1d.

2. for fermions in $(1+1)$d however we have $\Omega_2^{\text{spin}} = \mathbb{Z}_2$, with the homomorphism: $Z : \mathbb{Z}_{\not{2}} \to U(1)$. The generator of the non-trivial class is teh torus with adjacent Ramond edges, where the map gives $Z[T^2, R, R] = -1$. This non-trivial class is the **Kitaev chain**. This classification is confirmed by studies of models in 1+1d.

3. for bosons with T symmetry in $(1+1)$d, we have the bordism group denoted as ${}_2^O = \mathbb{Z}_2$, with the generator being the real projective plane $\mathbb{R}P^2$, ie. $Z[\mathbb{R}P^2] = -1$. This non-trivial class has a well-known representative called the **Haldane chain**. This classification is confirmed by studies of models in 1+1d.

4. for fermions with T symmetry in $(1+1)$d, we have two possibilities: $T^2 = \pm 1$. This is associated to the fact that the universal cover of the rotation group in Euclidean space isa double cover: $1 \to \mathbb{Z}_2 \to \text{Spin}(d) \to SO(d) \to 1$. For the case $T^2 = +1$ we have the bordism group denoted as ${}_2^{pin-} = \mathbb{Z}_8$, with $Z[\mathbb{R}P^2] = e^{2\pi k i/8}$, where $k \in \mathbb{Z}_8$. This is the same class discussed by Bernevig, which we presented at IITM in 2019. This classification is confirmed by studies of models in 1+1d.

5. fermions in $(3+1)$d with $T^2 = (-1)^F$ which is associated to the *spin−* configuraation. The correspoinding Bordism group is $\Omega_4^{pin+} = \mathbb{Z}_{16}$, with the generator of the classication given by $Z[\mathbb{R}P^4] = e^{2\pi i k/16}$ with $k \in \mathbb{Z}_{16}$. For free femion model, we have a $\mathbb{Z}$ classification, which is conjectured to go down to $\mathbb{Z} \to \mathbb{Z}_{16}$ with the addition of interactions.

8 TQFTs and the Atiyah-Segal axioms

References

I came across notes, and TASI lectures, by David Simmons-Duffin during a discussion with Akash where we were talking about our respective views of what quatization means.

1. The view I presented was that: nature gives us spacetime manifold M(or equivalently some notion of cobordisms) and symmetries G on your system, we note that we usually mean some kind of on-site symmetry here(and expending upon this notion of symmetry is crucial for our understanding of crytalline symmetris.). Given these inputs we can construct a principal G-bundle over M, using which we can describe the transformation of various $\mathbb{K}$-valued fields by constructing the associated $\mathbb{K}$-fiber bundles. Beyond this we then use canonical quantization, the kind talked about in Schottenburg's book, we promote canonical Automorphisms to Aut($\mathbb{P}$), and then we quantize the symmetries which is fairly more complicated and involves using the universal cover of G rather than G itself. Then I talked about some notion of a functorial view of TQFTs which I still hadn't understood myself yet. The whole argument was therefore very incomplete, I would like to complete it sometime in the future.

2. Akash talked about what he read in Simmons-Duffin's notes, which I give an excerpt of here, since it tries to give a simple view of the Atiyah-Segal axioms. A more complete discussion can be found in my notes on CFTs, in the section "Conformal bootstrap". His answer in terms of descibing a QFT was: a QFT is described by the specification of two objects:

   (a) a Hilbert space $\mathscr{H}$

   (b) and an algebra of quantum observables(self-adjoint operators): $\mathscr{A}_{QFT}$

   both of which are intimately related to "operators", which are intrinsically encoded with information about the symmetries of the system by way of defining how they transform. The operators are objects on which the algebra $\mathscr{A}_{QFT}$ is defined, and furthermore we can construct the Hilbert space of the theory by postulation the existence of a fully symmetric vacuum state $|0\rangle$ and saying $\mathscr{H} = \text{Span}(\prod_i \theta_i(x)|0\rangle)$.

Consider the theory of the Ising model in 3 dimensions. We have the discrete model:

$$Z(K,h) = \sim_{\{s_i\}} e^{-S[s]}; \quad S[s] = -K\sum_{\langle ij\rangle} s_i s_j - h\sum_j s_j \tag{22}$$

where $s_i \in \{\pm 1\}$ and we define the theory on a lattice $i \in \mathbb{Z}^d$.which we can promote to a continuum field theory which falls in the same universality class as the Ising model. We promote the discrete spins to real-valued functions $s : \mathbb{Z}^d \to \{\pm\} \implies \phi : \mathbb{R}^3 \to \mathbb{R}$:

$$Z(K,h) = \int D\phi e^{-S[\phi]}; \quad S[\phi] = \int d^3x\left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + g\phi^4\right) \tag{23}$$

where we note that ϕ has mass dimension $1/2$, and m, g both have mass dimension 1. The phase diagram of this model looks like:

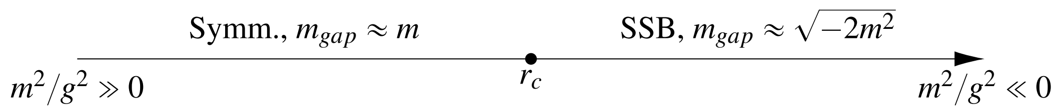


Figure 3: Phase diagram of Ising theory

the critical point, $m^2/g^2 = r_c$, is where the gap closes and is described by a CFT. And all the critical exponents which characterize the universality class are critical exponents associated to this point. Importantly, the CFT behaves differently in the UV and IR limits:

Free Boson — Ising CFT

Short distances(high energies/UV) ? Long distance(low energies/IR)

Figure 4: Behaviour of CFT($m^2/g^2 = r_c$) in different regimes.

we see here that there is an RG flow from one CFT to another[11], where we start in the UV CFT and perturb it so that it flows to an IR CFT[12]. This is one possible definition of a general QFT. In this definition we allow for the IR theory to be gapped, in which case $\text{CFT}_{\text{IR}} \equiv \text{TQFT}$, which is a CFT with all local correlation function zero.

This heirarchy among field theories, TQFT $\to$ CFT $\to$ general QFT, shows the fundamental importance of underlying structures.

8.1 Atiyah-Segal axioms

Motivated by the discussion of quantization of theories in my notes and Simmon-Duffin's notes, expecially the examples where we quantize the Ising model in 2D, we can come to a intuitive understanding of the Atiyah-Segal axioms, which I enumerate below with a discussion connecting to the Ising model. Consider a d-dimesnional QFT Q:

1. **Action of functor Q on objects**: for every $(d-1)$-dimesnional manofold N, Q assigns a *Hilbert space* $\mathscr{H}_N$. For theories with a path integral formulation, or Lagrangian theories, this assignment can be as simple as:

$$\mathscr{H}_N := \text{Span}\{|\phi_b\rangle \,|\, |\phi_b\rangle \in C(N,\mathbb{R})\} \quad \text{or} \quad \mathscr{H}_N = L^2(C(N,\mathbb{R}),\mathbb{C}) \tag{24}$$

where we consider a theory with a real-valued scalar field.

2. **Monoidal category**: the Hilbert space associated to disjoint unions is the tensor product of Hilbert space on each:

$$\mathscr{H}_{N\sqcup N'} := \mathscr{H}_N \otimes \mathscr{H}_{N'} \tag{25}$$

where we note that the Hilbert space associated to the empty set is given by:

$$\mathscr{H}_{\varnothing} = \mathbb{C} \tag{26}$$

3. **Action of functor Q on arrows**: for d-dimensional manifolds with in-boundary N and out-boundary N', Q assigns a transition probability:

$$Z : N \to N' \tag{27}$$

[11]zoom-in to very low energies, the masses are at infinity. Zoom-out far enough, and the masses should be irrelevant as well.

[12]this argument is precise for UV complete theories, which are theories who correlators are *unambiguously* finite for arbitrarily high energy cutoffs. This implies the theory is *physical* in some sense, ie. including higher energy dofs will not lead to divergencies.

which for Lagrangian theories corresponds to fixing the boundary configurations of fields at N and N', for example for a scalar theory we have:

$$Z_M = \int_{\phi|_N=\phi_N}^{\phi|_{N'}=\phi'_{N'}} D\phi e^{-S[\phi]} \tag{28}$$

there are some enlightening special cases which we note here:

(a) for a closed manifold, the in and out boundaries are empty, so the trnasition amplitude:

$$Z_M : \mathbb{C} \to \mathbb{C} \tag{29}$$

is just a complex number, which is the partition function associated to the manifold.

(b) M with in boundary $\varnothing$ and out-boundary N:

$$Z_M : \mathbb{C} \to \mathscr{H}_N \tag{30}$$

any such map is uniquely determined by its value for $1 \in \mathbb{C}$, which makes the map equivalent to preparing the state: $|\psi\rangle = Z_M(1) \in \mathscr{H}_N$.

(c) M with in-boundary N and out-boundary $\varnothing$:

$$Z_M : \mathscr{H}_N \to \mathbb{C} \tag{31}$$

which is equivalent to defining the dual state $\langle\psi \in \mathscr{H}_N^*$.

4. **Gluing rule/composition law**: if we have two arrows/spacetime manifolds M_1 with boundaries (N, N') and M_2 with boundaries (N', N''), then we can glue them together over the boundary N', giving us the new manifolds $M_1 \cup_{N'} M_2$[13]. This composition rule has an analogue under the functor Q, where the total transition amplitude of the new manifold is the composition:

$$Z_{M_1 \cup_{N'} M_2} = Z_{M_1} Z_{M_2} : \mathscr{N} \to \mathscr{N}'' \tag{32}$$

in Lagrangian theories this corresponds to integrating over field configurations over the intersecting boundary:

$$\int_{\phi|_N=\phi_b}^{\phi|_{N''}=\phi''_b} D\phi e^{-S[\phi]} = \int_{y\in N'} D\phi'_b(y) \int_{\phi_2|_{N'}=\phi'_b}^{\phi_2|_{N''}=\phi''_b} D\phi_2(x) e^{-S[\phi_2]} \int_{\phi_1|_N=\phi_b}^{\phi_1|_{N'}=\phi'_b} D\phi_1(x) e^{-S[\phi_1]} \tag{33}$$

This is the field-theory analog of gluing transition amplitudes in QM by inerting a complete set of states:

$$\langle\phi''_b|Z_{M_2} Z_{M_1}|\phi_b\rangle = \int_{y\in N'} D\phi'_b(y) \langle\phi''_b|Z_{M_2}|\phi'_b\rangle\langle\phi'_b|Z_{M_1}|\phi_b\rangle \tag{34}$$

[13] Note that we have to define orientations on these manifolds to define this gluing procedure rigorously.

9 Functorial view of TQFTs and cobordisms

References

1. reference for introduction is the Bartlett thesis [**bartlett2005**]
2. more rigorous reference for the material on differential cohomology is from lectures by Moore.
3. cross-referencing notes on anomalies in TQFTs here.
4. Note: we should connect this to literature on Dijkgraff-Witten theories and how to obtain them from microscopic theories. As far as I can tell there is one good reference for this, and that is work by Juven Wang [**putrov2017**, **juven2015**]. Our view should be something like this:

$$\text{TQFTs and Cob-th} \iff \text{DW theory (effective theories?)} \iff \text{Micr. models} \tag{35}$$

Ref. [**juven2015**] has a good discussion on deriving DW theory as an effective theory for some bosonic models with discrete symmetry groups. And Ref. [**putrov2017**] is a more general discussion in trying to bring all the above together.

9.1 Introduction

The discussion in this section is based on the disussion in Barlett thesis [**bartlett2005**]. The view is based on the effort by mathematicians to understand topological properties of spcaes by defining extra structures on the underlying space. For example, in the Chern-Weil theory the aim is to compute homology groups associated to topological spaces, denoted M, by defining connections, A, on these spaces and then computing differential forms associated to the connection, say $f(A)$, which is then used to calculate $H_n(M)$. Remarkably, the results in this theory are invariant with respect to the choice of the connection $A \in \mathscr{A}(M)$. TQFTs go a bit further than this construction where we calculate a number associated to the connection, $\exp(iS[A])$, which does depend on the choice of A but we get rid of the dependence by summing over all field configurations:

$$Z = \int DA \exp(iS[A]) \tag{36}$$

where we can now view the partition function of the TQFT as the simplest topological invariant associated to the theory. In addition to this topological perspective, we alse have *dynamical* features in a TQFT, which is captured by the formalism in terms of calculating *transition probabilities*. In particular, we can understand TQFTs as functors from the category of cobordisms to the category of Hilber spaces:

$$TQFT_n : \mathbf{nCob} \to \mathbf{Hilb} \tag{37}$$

On a high level we can clearly see that a QFT is a description over some spacetime. In particular, we will push the idea that a QFT takes as input some evolution of spatial manifolds(Σ_t) in time and gives us rules to construct over them Hilbert spaces $\mathscr{A}(\Sigma_t)$[14] and rules(correlation functions) for

[14]clearly we need to specify what kind of fields we are interested in to define the Hilbert space, for example spinors. Given a field, we can construct principal G-bundles associated to the symmetries of these fields to describe them.

calculating the time evolution operators $U(M)$[15]. A number of results are simplified by such an approach, especially in QIS, see Ref. [**abramsky2007**] and notes on UMTCs for details.

9.1.1 Category of Cobordisms and Hilbert spaces

Before we move onto the functorial definition of a TQFT, we spend some time trying to understand the properties of the underlying categories. We can view a n-dimensional spacetime manifold M, as the evolution of a (n-1) dimensional spatial manifold, say Σ_1, to the final spatial manifold, say Σ_2, as in the figure. This process is known mathematically as a *cobordism*.

Figure 5: Spacetime manifold M can be viewed as evolution of spatial manifolds from Σ_1 to Σ_2.

9.1.2 Cobordisms

Failed to find a better reference, still writing from Bartlett thesis. Right now compiling notes for course on applied category theory in my math notes, will come back here later.

[15]Given the fields and an underlying principal G-bundle to describe transformation of these fields, an evolution specifies the 'Hamiltonian', or the specific way of evolution.

10 Differential Cohomology

10.0.1 Preliminary definition of a TQFT

As sone in the introduction, we mantain that a QFT is a functor from the category of cobordisms to the category of Hilbert spaces. The objects of the cobordism category, nCob, are spatial manifolds in $(n-1)$-dimensions, and they are mapped on to a Hilbert space: $\mathscr{N}_{(n-1)} \mapsto \mathscr{H}$.[16] A space-time manifold, M with $\partial M_{(n)} = \bar{N}^0_{(n-1)} \sqcup N^1_{(n-1)}$, can be seen to be a map between spatial manifolds, or a map in the category of Cobs. And under the action of the QFT functor, we obtain a transition amplitude:

$$Z[A] = \int_{A|_{N_0}=A_0}^{A|_{N_1}=A_1} dA \exp(-iS[A]) \tag{38}$$

which can be viewed as a linear map between Hilbert spaces: $Z : \mathscr{H}_0 \to \mathscr{H}_1$.[17] Z is often computed using a path-integral formalism. And so, it is fruitful to axiomatize the properties of path-integrals. Importantly, for TQFTs, we axiomatize to only mantain the properties of **locality** because it is computable, in the sense that it is minimal and therefore mathematically tractable, and interesting. In particular, to a closed spatial manifold, $N_{(n-1)}$, a TQFT assigns a complex vector space, $N_{(n-1)} \xrightarrow{\text{TQFT}} F(N_{(n-1)})$, which is linear(in most cases). And we want to throw out as much as we possibly can. The condition of locality imposes the further constraints:

1. locality condition 1: $F(N_{(n-1)} \sqcup N_{(n-1)'}) = F(N_{(n-1)}) \otimes F(N'_{(n-1)})$. This is motivated by physical intuition, the total Hilbert space of completely non-interacting systems is a tensor product.
   (a) furthermore, we say $F(N_{(n-1)})$ only depends on the diffeomorphism type of $N_{(n-1)}$.
   (b) implication from 1: $\varnothing \sqcup N \cong N \implies F(N) \otimes F() \cong F(N) \implies F(\varnothing) \cong \mathbb{C}$.
   (c) implication from 1: we now want to compare with topological invariant, and we know from homology theory: $H_k(N \sqcup N') = H_k(N) \oplus H_k(N')$, which is different from the law we say for QM. So, we call $F(N)$ **quantum invariants**.

Now, say we have fields which live on the spatial manifolds that take values in some target space: $\phi : N \to X$, then it is important to note that the space of maps is just a Banach space, with the notation: $X^N := \{\gamma \,|\, \gamma : N \to X\}$. But this doesn't qualify as a Hilbert space, we need to further define a hermitian form to get some space like"$L^2(X^N) := \{\Psi : X^N \to \mathbb{C} | \int d\phi |\Psi[\phi]|^2 = 1\}$". Then, given a map $\gamma \in X^N$, we have some functional $\Psi[\gamma] \in L^2(X^N)$, which is an element of the Hilbert space[18]. A cobordism $M_n : N^0 \to N^1$ with $\phi_i \in X^{N^0}$ and $\phi_f \in X^{N^1}$ has a path integral $K_{M_n}(\phi_f, \phi_i)$. The transition amplitude is then given by:

$$\Big(F(M_n)\Psi_i\Big)(\phi_f) = \int_{\phi_i \in X^{N^0}} d\phi_i K_{M_n}(\phi_f, \phi_i)\Psi_i(\phi_i) \tag{39}$$

[16]We emphasize that the Hilbert space is the space of physical states, so it is isomorphic to the projective Hilbert space or the space of density matrices, and **not** the Hilbert space of states as is referred to usually.

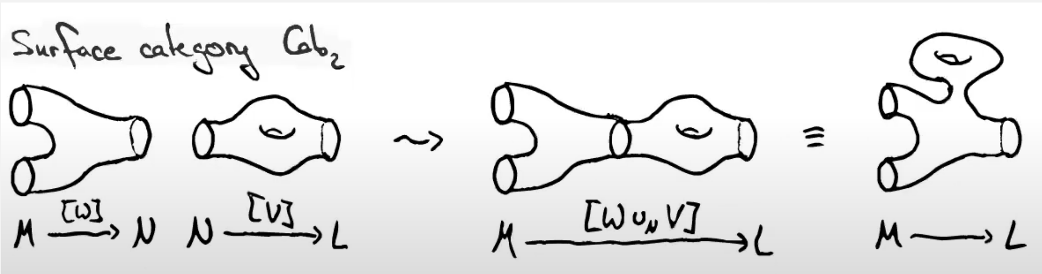
[17]The above definition needs more work, where do the fields come into the description? Actually, it is not always linear, but a majority of the known theories work with linear theories.

[18]This statement is, as far as I know, quite non-trivial. And we need some amount of functional analysis to write down the map here. See Schwarz functions.

One-dimensional Bordism category by Jan Steinebrunner

The one-dimensional bordism category is the category corresponding to intervals and circles.

Definition 18 (Bordism category) *For $d \geqslant 0$, the bordism category Cob_d has as objects closed oriented $(d-1)$-manifolds $M, N, \ldots$ and morphisms $M \to N$ are diffeomorphism classes $[w]$ of compact oriented d-manifolds w with $\phi : \partial w \cong M^- \sqcup N$. Composition is defined by gluing.*



The above figure demonstrates the Cob_2 category.

Definition 19 (Topologically enriched bordism category) *The topologically enriched bordism category C_d has as objects: $M \subset \mathbb{R}^\infty$ which are closed oriented $(d-1)$-dimensional submanifolds, and as morphisms: $(w,t) : M \to N$ where $t \in [0,)$ is the length and $w \subset [0,t] \times \mathbb{R}^\infty$ is a compact d-dimensional submanifold such that $\partial w = M^- \times \{0\} \sqcup N \times \{t\}$. $Hom_{C_d}(M,N)$ is topologised using the Whitney-C^∞-topology.*

$Hom(M,N)$ is an object, referring to homomorphisms, usually defined in abstract category theory and should not be confused with homology group. Also, note that in this class, two diffeomorphic bordisms $w_0, w_1 : M \to N$ are no longer identified, but they will always be connected by a path $[0,1] \to \mathrm{Hom}_{C_d}(M,N), \quad t \mapsto w_t$. This relaxation is made possiible because $\mathbb{R}^\infty$ is so big that we can always find an isotopy which maps the two embedded manifolds. In fact, $\mathrm{Hom}_{C_d}(M,N) = \sqcup_{[w]:M\to N} \mathrm{BDiff}^*(w/\partial w)$.

1-category $\mathbf{Cob}_d$	$(\infty,1)$**- category $\mathbf{Bord}_d$**
recovered as homotopy category hC_d	modelled by C_d
diffeomorphic boordisms identified	keeps track of higher structure
Atiyah TQFTs	variant features in Lurie's cobordism hypothesis

11 Discussion notes

Discussion with James Weijun: On connections between hep-th and condmat

There were multiple points on contention/doubt when it came to differnt definitions between the fields, I will summarize some of the important questions below which we should answer in order if possible:

1. the important question was: why is the group cohomology classification discussed in Ref. [**Else2014**] not applicable to invertiple topological orders. It seems that the classification mathods are indeed based on similar approaches, but there exists an extra step when it comes to invertible models and it has to do with marrying gauge theory ideas with the algtop ideas using classifying spacces and cobordism based arguments.

2. second question was: what is the difference between non-local and non-on-site symmetries? Then we went on to discuss about the bosonic model in (2+1)D given in the ELse paper, and discussed in length also in a recent paper by Seiberg and Cheng, Ref. [**Seiberg2022**].

The answers to the above questions seem to be crucial to hte resolution of doubts in our discussion.

References

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[2] Bruce H. Bartlett. *Categorical Aspects of Topological Quantum Field Theories*. 2005. arXiv: math/0512103 [math.QA].