Understanding anomalies from a math perspective

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1 Introduction

[1]The view we are going to adopt for defining an anomoly is:

Definition 1 (Anomaly) *a d-dimensional anomaly is controlled by a special type of $(d+1)$-dimensional topological theory.*

The aim of the notes is to be able to suffieniently describe anomalies in a QFT. Before we do this, we define a QFT as a functor from the category of cobordisms, which is naivelt speaking equivalent to space-time manifolds, to the category of Hilbert spaces:

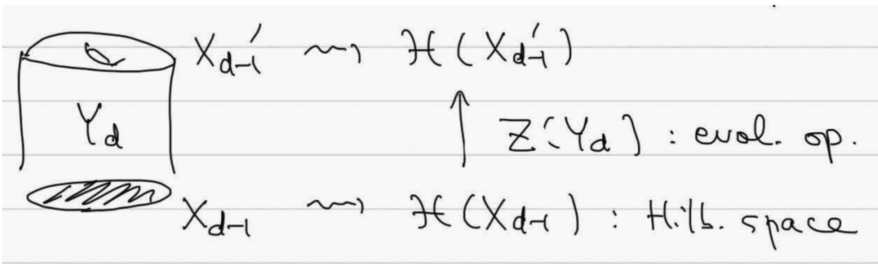


Figure 1: Functorial view of QFTs

In particular, we use the Atiyah-Segal axioms, which can be found in in my TQFT notes, where the null-space is mapped under the functor to the space of complex numers: QFT : $\varnothing \to \mathscr{H}(\varnothing)$, and the map then corresponds to the partition function, $Z : \mathbb{C} \to \mathbb{C} \implies Z \in \mathbb{C}$. Also, we can always add more structure to the manifolds which will affect the theroy, like a metric, orientation, spin structures, and other background fields.

1.1 Characterizing an anomaly

A unitary theory additionally contraints the partition function to be a phase, $Z \in U(1)$. From this perspective, an anomaly can be viewd as a *controllable* phase ambiguity in the partition function. We can characterize this phase ambiguity by considering $Z_Q(Y_d) \in V(Y_d)$, where the partition function is a vector ina 1-dimensional vector space without a canonical basis.[2]

[1]We review a TASI talk given by Yuji Tachikawa.

[2]Note that here we are trying to describe a general anomaly, and therefore the ambiguity is actually a complex number and not a phase ambiguity. The phase ambiguity refers to ambiguities in a unitary theory.

2 Connections to homology

In this section we will use the following definition for an anomaly:

Theorem 2 (Anomaly) *if a system couples to a background gauge field for symmetry G, but the partitioin function Z has a controllable phase ambiguity, this phase ambiguity is called an anomaly. Usually the anomaly is controlled by a theory in one higher dimension.*

To make things more intuitive we start off with a simple example[3]. We study a (0+1)D QFT, which is basically a quantum mechanical theory evolving in time, with an internal G symmetry which acts on the Hilbert space. Let $\mathscr{H}$ be finite dimensional and for simplicity let $H = 0$. We know that physically two states $v, cv \in \mathscr{H}$ where $c \in U(1)$ are equivalent. With the above in mind G symmetry acts on $\mathscr{H}$ in the following way: $G \ni g \implies \rho(g) : \mathscr{H} \to \mathscr{H}$ where ρ is some representation of G.

Theorem 3 (Layman's definiton of projective representation) *a projective representation of G is a collection of operators $\rho(g) \in GL(\mathscr{H})$[4], with $g \in G$ satisfying the homomorphism property upto a constant:*

$$\rho(g)\rho(h) = c(g,h)\rho(g \cdot h) \tag{1}$$

for some constant $c(g,h) \in U(1)$.

We consider projective representation of the group G with the projective phase $c \in U(1)$ precisely because additional phases do the affect the physical states. We can now look at partition functions of this system, with $t \in \mathbb{S}^1$ and consider in one case just inserting at some point just $g \cdot h$ and in the other inserting g and h at seperate times. This gives rise the following partition functions:

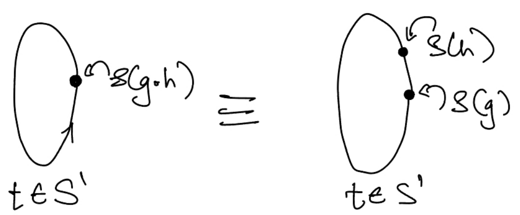


Figure 2: $Z_1 = \mathrm{Tr}(\rho(g \cdot h)) \qquad Z_2 = \mathrm{Tr}(\rho(g)\rho(h)) = c(g,h)Z_1$

where clearly there is an anomalous factor of $c(g,h)$. To study this further we look at the restrictions on the projective phases, by considering the simplification of $\rho(g) \cdot \rho(h) \cdot \rho(k)$ one can show that for consistency we require:

$$c(h,k)c(gh,k)^{-1}c(g,hk)c(g,h)^{-1} = 1 \tag{2}$$

to study the classification of these projective phases, we need to further understand equivalences under redefinitons of the representations:

[3]Note that the discussion here is moderately vague, and you should try to understand more deeply some of the comments in this section which are highlighted in red.

[4]recap: GL, the general linear group of degree n is the set of n×n invertible matrices, together with the operation of ordinary matrix multiplication.

$$\tilde{\rho}(g) = b(g)\rho(g) \text{ s.t. } b(g) \in G \implies \tilde{c}(g,h) = c(g,h)\Big[\frac{b(g\cdot h)}{b(g)b(h)}\Big] \tag{3}$$

Hence, the group of distinct projective phases is obtained by all phases satisfying Eqn. 2 quotionted out by the equivalence relation in Eqn. 3. This set can be shown to be homeomorphic to the second homology group $H^2(G,U(1))$.

3 Digression into the math

Let us define the following map:

$$C^d(G,A) : G^d \longrightarrow A; \qquad d : C^d \longrightarrow C^{d+1} \tag{4}$$

where the first is a set of maps C^d, and the second is the exterior derivative defined on these maps. The exterior derivative is defined in the following way:

$$\begin{aligned} df(g_1,g_2,...,g_{d+1}) =& g_1 f(g_2,...,g_{d+1}) - f(g_1\cdot g_2, g_3,...,g_{d+1}) \\ & f(g_1,g_2\cdot g_3,g_4,...,g_{d+1}) - f(g_1,g_2,g_3\cdot g_4,...,g_{d+1}) \\ & ... \\ & ... \\ & ... \\ & (-1)^d f(g_1,g_2,...,g_d\cdot g_{d+1}) + (-1)^{d+1} f(g_1,...,g_d) \end{aligned}$$

from this definition it is easy to prove the nilpotent identity of exterior derivative algebra, $d^2 = 0$[5] Now, we go on to show that from this construction Eqn. 2 boils down to just being $dc = 1$, consider the case of $d = 2$:

$$\begin{aligned} d\ :\ C^2 \longrightarrow C^3 \qquad & G \circlearrowright U(1) \text{ trivially} \\ (dc)(g,h,k) = g_1 c(h,k)\cdot c(g\cdot h,k)^{-1}&\cdot c(g,h\cdot k)\cdot c(g,h)^{-1} \\ \implies (dc)(g,h,k) = 1\ \forall\ & g,h,k \in G \end{aligned}$$

Differential forms which satisfy the property $dc = 0$ are called closed forms. The expression in Eqn. 3 becomes db:

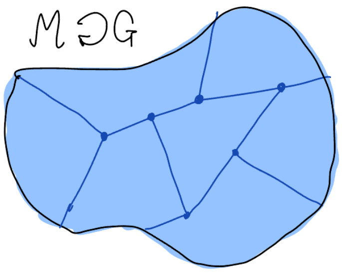
$$\begin{aligned} b : G \longrightarrow A \qquad & \implies b \in C^1(G,A) \\ (db)(g,h) = b(g_2)\cdot b(&g_1\cdot g_2)^{-1}\cdot b(g_1) \end{aligned}$$

which boils down to saying we want to look at equivalence classes where two forms differing by a total derivative are identifies in the same equivalence class. This is exactly the definition of a homology group which we refer to in the previous section.

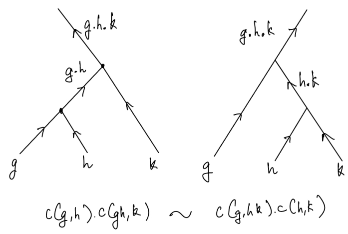
[5]Note that the equations above are written where the group action on A is written additively, but in our specific example the group G acts on $A = U(1)$ is written multiplicatively(or additively in the phases). So, $d^2 = 0$ becomes $d^2 = 1$ in our case

4 Intuitive picture of anomaly inflow

In this section we look at an intuitive and pictorial way of representing theories with finite groups acting on a manifold, and how anomalies on the boundary can be related to and viewed as boundaries of higher dimensional theories. This discussion follows from the 1990 paper by Witten et al. [1][6]. Consider a $(1+1)$D system with G symmetry, where G is finite. Naturally the space-tiem manifold is endowed with a G-bundle like structure.[7] We want to assign this G-bundle a partition function $\mathrm{Z}^{2d} \in U(1)$. But how do we understand the action of finite G on this $(1+1)$D manifold. We use the following picture:



(a) General picture



(b) Assigning phase factor

where ... The action of continuous groups on a space-time manifold is naturally understood, this is not the case with finit groups. The idea is to realize the action of $g \in G$ as some sort of domain wall, so when we cross this domain wall we are acted on by this element g. In this picture, every G-bundle can now be viewed as being endowed with a complicated network of domain walls. The next question is how to assign a partition function to such a structure? We assign to each intersction points or nodes of domain walls the appropriate projective phase factor, as in the Fig. ??(b). Now, we can define the partion function as the product over all such projective phases

$$\mathrm{Z}^{2d} = \prod_{\text{nodes}} c(g,h) \tag{5}$$

We want this partition function be 'gauge-invariant'. Here 'gauge-invariance' really means to be invariant under different domain wall network realizations of the same G gauge background, like in Fig. ??(a):

(a) Anomaly in the (2+1)D bulk

(b) Associated anomaly on the (1+1)D boundary

[6]the gauge can be promoted to be dynamical, and then you can sum over all possible gauge configurations. This was the initial point of the paper, where they tried to realize gauge field theory for finite groups.

[7]Where does this enter the discussion? Need to marry gauge theory with the algtop part of the discussion.

and we can see the partition function attributed to both these figures would be the same under the rules we discussed previously. We can now proceed to see how the $(0+1)$D anomaly we saw in Fig. 2. Looking at Fig. ?? above, we see that the anomaly in the $(0+1)$D theory is very simply cancelled by the appropriate cancellation in the bulk $(1+1)$D theory.

5 Category theory

References

1. this thesis by Bartlett [2] on a categorial and functorial view of quantum field theory is a good introduction to various ideas related to bordisms and tqfts.
2. these notes by Balsells(link) is also a good reference and is suited for someone who has absolutely no mathematical background for physics. Might be something to look into for journal clubs/group talks.
3. I also have some notes of my own in my notes on tqft, which I might merge with these notes. If I haven't here's the link.

6 Using DW theory as an instance

In this section we try to understand the functorial view of TQFTs by using Dijkgraaf-Witten theory as an instance of such a view. We use this perspective to understand the anomaly-inflow picture.

The simplest possible example of any such view is for a closed spacetime manifold, where we view a "flow" from an empty subset to the empty subset. The transition is understood in terms of the partition function for which we have an explicit expression in Dijkgraaf-Witten theory.

7 Bordism invariants

Definition 4 (Bordant manifolds) *two d-dimensional manifolds X_d and Y_d are said to be* ***bordant*** *if there exists a $(d+1)$-dimensional manifold M_{d+1} such that:*

$$\partial M_{d+1} = X_d \sqcup \bar{Y}_d \tag{6}$$

the propotypical example is that of a pair of pants which provides a bordism between S^1 and $S^1 \sqcup S^1$. We say that the partition function is bordism invariant if:

$$Z[X_d] = Z[Y_d] \tag{7}$$

it is invariant under bordisms. It is beleived that Bordism invariants classify all invertible gapped phases. d-mainfolds with bordisms form an abelian group, which is denoted by Ω_d, witht he following properties:

1. the addition operation is just the disjoint union.
2. the identity elemets is just the empty set, denoted $\varnothing$.
3. and the inverse is orientation reversal. Why? Because the manifold $X_d \sqcup \bar{X}_d$ is always bordant to the empty set, ie. $X_d \sqcup \bar{X}_d \sim \varnothing$.

The following discussion seems to be a bit confusing in connection to out notes on teh functorial TQFTs. But I present here what wwas shown in the lectures.

If the partion function of a theory is invariant under bordisms, then it stands to reason that thre should exist the homomorphism:

$$Z : \Omega_d \to \mathbb{C} \tag{8}$$

And we can claim the follwing identity: $Z[X_d \sqcup Y_d] = Z[X_d]Z[Y_d]$, which we say is aximatic. Then it follows that $Z[X_d \sqcup \bar{X}_d] = Z[] = 1 \implies Z[X_d]Z[\bar{X}_d] = 1$.

For Bosons without symmetry, we think about manifolds with orientations, and the bordism group of d-dimesntional manifolds, denoted Ω_d^{SO}. However, for fermions without symmetry, we think about the bordism group of d-dimnesional manifolds with spin structures, denoted Ω_d^{spin}. Now, under addition of some symmetry, say G, we need to also consider gauge fields associated to this symmetries which live on the manifolds. We say the partition function in the presence of the gauge field, $Z[X_d, A]$, is bordism invariant. For example:

1. for bosons in $(1+1)$d, and no symmetries we have $\Omega_2^{SO} = 1$. This is because any oriented 2-dimensional manifold can be filled in, or in other words is the boundary of a 3-dimnesional manifold and are therefore always bordant to teh empty set. This classification is confirmed by studies of models in 1+1d.

2. for fermions in $(1+1)$d however we have $\Omega_2^{\text{spin}} = \mathbb{Z}_2$, with the homomorphism: $Z : \mathbb{Z}_{\not{\mathscr{E}}} \to U(1)$. The generator of the non-trivial class is teh torus with adjacent Ramond edges, where the map gives $Z[T^2, R, R] = -1$. This non-trivial class is the **Kitaev chain**. This classification is confirmed by studies of models in 1+1d.

3. for bosons with T symmetry in $(1+1)$d, we have the bordism group denoted as ${}^{O}_{2} = \mathbb{Z}_2$, with the generator being the real projective plane $\mathbb{R}P^2$, ie. $Z[\mathbb{R}P^2] = -1$. This non-trivial class has a well-known representative called the **Haldane chain**. This classification is confirmed by studies of models in 1+1d.

4. for fermions with T symmetry in $(1+1)$d, we have two possibilities: $T^2 = \pm 1$. This is associated to the fact that the universal cover of the rotation group in Euclidean space isa double cover: $1 \to \mathbb{Z}_2 \to \text{Spin}(d) \to SO(d) \to 1$. For the case $T^2 = +1$ we have the bordism group denoted as ${}^{pin-}_{2} = \mathbb{Z}_8$, with $Z[\mathbb{R}P^2] = e^{2\pi ki/8}$, where $k \in \mathbb{Z}_8$. This is the same class discussed by Bernevig, which we presented at IITM in 2019. This classification is confirmed by studies of models in 1+1d.

5. fermions in $(3+1)$d with $T^2 = (-1)^F$ which is associated to the *spin*$-$ configuraation. The correspoinding Bordism group is $\Omega_4^{pin+} = \mathbb{Z}_{16}$, with the generator of the classication given by $Z[\mathbb{R}P^4] = e^{2\pi ik/16}$ with $k \in \mathbb{Z}_{16}$. For free femion model, we have a $\mathbb{Z}$ classification, which is conjectured to go down to $\mathbb{Z} \to \mathbb{Z}_{16}$ with the addition of interactions.

8 Discussion notes

Discussion with James Weijun: On connections between hep-th and condmat

There were multiple points on contention/doubt when it came to differnt definitions between the fields, I will summarize some of the important questions below which we should answer in order if possible:

1. the important question was: why is the group cohomology classification discussed in Ref. [**Else2014**] not applicable to invertiple topological orders. It seems that the classification mathods are indeed based on similar approaches, but there exists an extra step when it comes to invertible models and it has to do with marrying gauge theory ideas with the algtop ideas using classifying spacces and cobordism based arguments.

2. second question was: what is the difference between non-local and non-on-site symmetries? Then we went on to discuss about the bosonic model in (2+1)D given in the ELse paper, and discussed in length also in a recent paper by Seiberg and Cheng, Ref. [**Seiberg2022**].

The answers to the above questions seem to be crucial to hte resolution of doubts in our discussion.

References

[1] Robbert Dijkgraaf and Edward Witten. "Topological gauge theories and group cohomology". In: *Communications in Mathematical Physics* 129.2 (Apr. 1990), pp. 393–429. ISSN: 1432-0916. DOI: 10.1007/BF02096988. URL: https://doi.org/10.1007/BF02096988.

[2] Bruce H. Bartlett. *Categorical Aspects of Topological Quantum Field Theories*. 2005. arXiv: math/0512103 [math.QA].