Eigenstructure Assignment for Linear Systems

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The use of feedback (full state, output, and constrained output) is considered as a means of shaping the transient response of linear time invariant systems. The underlying importance of the eigenstructure (eigenvalues/eigenvectors) is highlighted. Also, the important results and techniques are presented along with a brief literature review. An extensive flight control example is presented which should give direction to the application of eigenstructure assignment in diverse areas.

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I. INTRODUCTION

One of the most fundamental topics of control theory is that of *feedback*. The usual purposes of feedback in practice (and attested to in most classical textbooks in control theory) are

(1) to improve or ensure the stability characteristics of the system
(2) to reduce the sensitivity of the system to modeling errors
(3) to improve the system's capability to reject disturbances and to attenuate noise
(4) to alter the transient response of the system.

The primary concern of this paper is the fourth aspect: the use of feedback in the alteration of transient response. We consider three types of feedback: full state feedback, output feedback, and constrained output feedback. Of these three types somewhat more attention is paid to the last two methods with emphasis on the usage of the techniques rather than a rigorous theorem–proof motif.

The format of this paper is as follows: the first section considers transient responses per se for a linear time invariant system. The second section contains some preliminaries related to linear control theory and problem statements. The next three sections deal with the various types of feedback (full state, output, constrained output). Included in these sections are some basic references and theorems concerning the problem statements. Each section contains the calculation of the appropriate feedback matrices. The final two sections follow with a specific choice of desirable eigenvectors for a given application and an extensive example.

II. TRANSIENT RESPONSES

The details of this section are generally well known but are repeated here for completeness. These details form the basis for an understanding of what is to follow.

We begin by considering a system described by n first-order, linear differential equations

$$\dot{\boldsymbol{x}}(t) = G\boldsymbol{x}(t); \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{1}$$

(Under feedback, an unforced closed loop system will be in this general form.) As is well known [14, 19], the solution of (1) is given by

$$\boldsymbol{x}(t) = \exp(Gt)\boldsymbol{x}_0. \tag{2}$$

Although (2) is a perfectly valid representation of the solution of (1), a more revealing one can be written in terms of the eigenvalues/eigenvectors of G. To this end, we note that if G is $(n \times n)$, then

$$G\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i, \qquad i = 1, 2, \ldots, n \tag{3}$$

where $\{\lambda_i\}_{i=1}^{n}$ are the n eigenvalues and $\{v_i\}_{i=1}^{n}$ are the corresponding eigenvectors. We assume that the eigenvalues of G are distinct.

Using the eigenvectors, we define the *modal matrix* M as

$$M = [v_1 \quad v_2 \quad \dots \quad v_n]. \tag{4}$$

Using this matrix (which is nonsingular due to the linear independence of the set $\{v_i\}_{i=1}^{n}$), we transform coordinates according to

$$x = Mz \tag{5}$$

and rewrite (1) as

$$\dot{z}(t) = M^{-1} GMz(t); \qquad z_0 = M^{-1} x_0. \tag{6}$$

Since M is the modal matrix for G and since G has distinct eigenvalues, we have

$$M^{-1} GM = \Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \dots, \lambda_n] \tag{7}$$

where λ_i is the ith eigenvalue of G. Hence

$$\exp(\Lambda t) = \mathrm{diag}\{\exp(\lambda_1 t), \exp(\lambda_2 t), \dots, \exp(\lambda_n t)\}, \tag{8}$$

$$z(t) = \exp(\Lambda t)\, z_0. \tag{9}$$

Returning to the original coordinate system, our solution may be written as

$$x(t) = M \exp(\Lambda t)\, M^{-1} x_0. \tag{10}$$

To resolve (10) more keenly, we calculate

$$M \exp(\Lambda t) = [v_1 \quad v_2 \quad \dots \quad v_n] \; \mathrm{diag}\{\exp(\lambda_1 t), \exp(\lambda_2 t), \dots, \exp(\lambda_n t)\} \tag{11}$$

or

$$M \exp(\Lambda t) = [\exp(\lambda_1 t)\, v_1; \exp(\lambda_2 t)\, v_2; \dots; \exp(\lambda_n t)\, v_n]. \tag{12}$$

Turning our attention to M^{-1}, we denote M^{-1} as L with rows l_i. Thus, using (11), (10) may be written as

$$x(t) = [\exp(\lambda_1 t)\, v_1; \exp(\lambda_2 t)\, v_2; \exp(\lambda_n t)\, v_n] \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} x_0 \tag{13}$$

or

$$x(t) = \sum_{i=1}^{n} \exp(\lambda_i t)\, v_i\, l_i x_0. \tag{14}$$

Denoting $l_i x_0 = \alpha_i$ and $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n]^{\mathrm{T}}$,

$$x(t) = \sum_{i=1}^{n} \alpha_i \exp(\lambda_i t)\, v_i \tag{15}$$

$$\alpha = M^{-1} x_0. \tag{16}$$

Equations (15) and (16) are merely the quantitative statements of the *modal expansion theorem* [14, 19]. We see that *every* solution representing a free response of (1) depends on three quantities:

(i) eigenvalues, which determine the *decay/growth rate* of the response
(ii) eigenvectors, which determine the *shape* of the response
(iii) initial condition, which determines the *degree* to which each mode will participate in the free response.

Thus it is immediately apparent that if feedback is to be used to alter the system transient response, eigenvector selection must be considered as well as pole (eigenvalue) placement.

III. PRELIMINARIES AND PROBLEM STATEMENTS

In general, we wish to consider a linear time invariant system described by the equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{17}$$

$$y(t) = Cx(t) \tag{18}$$

where (i) $x \in R^n$, $u \in R^m$, $y \in R^r$ are called the state, control, and output, respectively; (ii) A, B, C are real constant matrices of appropriate dimensions; and (iii) rank $B = m \neq 0$, rank $C = r \neq 0$.

From a theoretical (as well as practical) standpoint the concepts of controllability and observability are crucial to linear systems described by (17) and (18). For a thorough exposition of these concepts, the reader is referred to [2, 11, 15]. For our purposes, we simply state that

(1) the system described by (17) and (18) is controllable if and only if

$$\mathrm{rank}\, W = \mathrm{rank}[B \quad AB \quad \dots \quad A^{n-1}B] = n \tag{19}$$

(2) the system described by (17) and (18) is observable if and only if

$$\mathrm{rank}\, H = \mathrm{rank}[C^{\mathrm{T}} \; A^{\mathrm{T}}\, C^{\mathrm{T}} \dots (A^{\mathrm{T}})^{n-1}\, C^{\mathrm{T}}] = n. \tag{20}$$

Given this system and the controllability and observability "definitions," we have the following problem statements:

Problem 1. Full State Feedback

Given a self-conjugate[1] set of scalars $\{\lambda_i^d\}$, $i = 1, 2, \dots, n$ and a corresponding self-conjugate set of n-vectors $\{v_i^d\}$, $i = 1, 2, \dots, n$, find a real $(m \times n)$ matrix K such that the eigenvalues of $A + BK$ are precisely those of the self-conjugate set of scalars $\{\lambda_i^d\}$ with corresponding eigenvectors the self-conjugate set $\{v_i^d\}$.

Problem 2. Output Feedback

Given a self-conjugate set of scalars $\{\lambda_i^d\}$, $i = 1, 2, \dots, r$, and a corresponding self-conjugate set of n-vectors $\{v_i^d\}$, $i = 1, 2, \dots, r$, find a real $(m \times r)$ matrix F such that r of the eigenvalues of $A + BFC$ are precisely those of the self-conjugate set $\{\lambda_i^d\}$, with corresponding eigenvectors the self-conjugate set $\{v_i^d\}$.

[1] A set is said to be self-conjugate if, when a complex quantity is a member of the set, its complex conjugate is also a member.

Problem 3. Constrained Output Feedback

Given a self-conjugate set of scalars $\{\lambda_i^d\}$, $i=1, 2, \ldots, r$, and a corresponding self-conjugate set of n-vectors $\{\boldsymbol{v}_i^d\}$, $i=1, 2, \ldots, r$, find, *if possible,* a real $(m\times r)$ matrix F, some of whose elements are fixed as 0, such that r of the eigenvalues of $A + BFC$ are those of the self-conjugate set of scalars $\{\lambda_i^d\}$ with corresponding eigenvectors the self-conjugate set $\{\boldsymbol{v}_i^d\}$.

Hence the sets $\{\lambda_i^d\}$, $\{\boldsymbol{v}_i^d\}$ correspond to the desired closed loop eigenvalues and eigenvectors, respectively, while the matrices K and F are constant feedback gain matrices that yield control laws of the form

$$\boldsymbol{u}(t) = K\boldsymbol{x}(t)$$

or

$$\boldsymbol{u}(t) = F\boldsymbol{y}(t).$$

Note that under these feedback laws, the closed loop system is described by equations of the form (1).

IV. FULL STATE FEEDBACK

The exact relationship between feedback and system transient response has been a topic of interest for many decades. In the context of "modern" control theory and the state space formulation, great strides have taken place in the last 20 years. Recalling that the transient response depends most critically on the eigenstructure (eigenvalues/eigenvectors) of the system, any results concerning the alteration of transient response *must* change the eigenstructure of the system.

In terms of Problem 1, a *partial* resolution was rendered by Wonham [36] in 1967 with the following:

THEOREM. *A system described by* (17) *is controllable if and only if for every self-conjugate set of scalars* $\{\lambda_i^d\}$, $i=1, 2, \ldots, n$, *there exists a real* $(m\times n)$ *matrix* K *such that* $(A + BK)$ *has* $\{\lambda_i^d\}$, $i=1, 2, \ldots, n$, *as its eigenvalues.*

According to Kalman et al. [16], this theorem was first obtained for the single input case ($m=1$) by Bertram in 1959 using root locus methods. In 1961, Bass independently formulated and proved (but did not publish) the same result in the context of linear algebra. Also considering the single input case were Rosenbrock [28] and Rissanen [27]. Rissanen, using a direct analytical approach, stated and proved the result in an obscure paper. Popov [23] rendered an independent deduction in 1964 and actually considered the multi-input case. Other contributions concerning *eigenvalue* placement in multi-input systems include those of Langenhop [20], Simon and Mitter [29], and Brunovsky [1]. Wonham was the first to extend the controllability result from single input to multi-input systems. Since 1967, when Wonham's paper appeared, there have been literally hundreds of papers written concerning pole placement and its applications.

A few comments concerning pole placement using full state feedback are in order here. The primary impetus of most of the papers mentioned concerns the *stabilization* of systems as opposed to the shaping of transient response. Also, in the multi-input case, the calculated feedback gain matrix K for a given set of desired eigenvalues is not unique. This is a blessing which will be exploited later. Finally, relocation of the eigenvalues requires, potentially, a fully populated K matrix, i.e., feedback of every state variable to each input variable.

In 1976, Moore [21] was the first to identify the freedom offered by state feedback beyond specification of the closed loop eigenvalues for the case in which the closed loop eigenvalues are distinct. In [21], Moore considers systems described by (17) and derives necessary and sufficient conditions for the existence of a K which yields prescribed eigenvalues *and* eigenvectors. Moore's result includes a procedure for computing K.

To present Moore's result, we define

$$S_\lambda \triangleq [\lambda I - A \,\vdots\, B] \tag{21}$$

and a compatibly partitioned matrix

$$R_\lambda = \begin{bmatrix} N_\lambda \\ M_\lambda \end{bmatrix} \tag{22}$$

where the columns of R_λ form a basis for the nullspace of S_λ. For rank $B = m$, one can show that the columns of N_λ are linearly independent; also $N_{\lambda^*} = N_\lambda^*$ where the asterisk denotes a complex conjugate. With this background, the following is true [17]:

THEOREM. *Let* $\{\lambda_i\}_{i=1}^n$ *be a self-conjugate set of distinct complex numbers. There exists a real* $(m\times n)$ *matrix* K *such that*

$$(A + BK)\,\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i, \qquad i=1, 2, \ldots, n \tag{23}$$

if and only if, for each i,

(i) $\{\boldsymbol{v}_i\}_{i=1}^n$ are a linearly independent set in C^n, the space of complex n-vectors
(ii) $\boldsymbol{v}_i = \boldsymbol{v}_j^*$ when $\lambda_i = \lambda_j^*$
(iii) $\boldsymbol{v}_i \in \text{span}\,\{N_{\lambda_i}\}$.

Also, if K exists and rank $B = m$, then K is unique. For completeness, we present a version of the proof found in [21]:

PROOF: NECESSITY. The necessity of (i) and (ii) follows from matrix theory. Proceeding from (23), we obtain

$$(\lambda_i I - A)\boldsymbol{v}_i = BK\,\boldsymbol{v}_i \tag{24}$$

or

$$[\lambda_i I - A \,\vdots\, B]\begin{bmatrix} \boldsymbol{v}_i \\ -K\boldsymbol{v}_i \end{bmatrix} = \mathbf{0}. \tag{25}$$

Since the columns of R_{λ_i} form a basis for the nullspace of S_{λ_i}, it follows that $\boldsymbol{v}_i \in \text{span}\{N_{\lambda_i}\}$.

SUFFICIENCY. Assume that the set $\{v_i\}_{i=1}^{n}$ satisfy (i), (ii), and (iii). From (iii), there exists a vector z_i (real or complex) such that

$$v_i = N_{\lambda_i} z_i . \tag{26}$$

From our definitions of S_λ, R_λ, we have

$$(\lambda_i I - A) N_{\lambda_i} + BM_{\lambda_i} = 0 . \tag{27}$$

Hence,

$$(\lambda_i I - A) N_{\lambda_i} z_i + BM_{\lambda_i} z_i = \mathbf{0} \tag{28}$$

or, using (26),

$$(\lambda_i I - A) v_i + BM_{\lambda_i} z_i = \mathbf{0} . \tag{29}$$

If a K can be chosen so that

$$-M_{\lambda_i} z_i = K v_i \tag{30}$$

then

$$[\lambda_i I - (A + BK)] \, v_i = \mathbf{0} . \tag{31}$$

Moore proceeds to show this can be done in the following way: if such a K exists, it satisfies

$$K [v_1 \quad v_2 \quad \ldots \quad v_n] = [-M_{\lambda_1} z_1 \quad -M_{\lambda_2} z_2 \quad \ldots \quad -M_{\lambda_n} z_n] . \tag{32}$$

If each λ_i is real, then v_i, z_i are real and the matrix of eigenvectors is nonsingular. Hence,

$$K = [-M_{\lambda_1} z_1 \quad -M_{\lambda_2} z_2 \quad \cdots \quad -M_{\lambda_n} z_n] \, [v_1 \quad v_2 \quad \ldots \quad v_n]^{-1} . \tag{33}$$

If the desired eigenvalues are complex, a slight alteration of the above equation is required. Assume that $\lambda_1 = \lambda_2^*$. From (ii), $v_1 = v_2^*$, which implies $z_1 = z_2^*$. Thus, assuming all other eigenvalues are real, K must satisfy

$$K[v_{1R} + jv_{1I} \mid v_{1R} - jv_{1I} \mid v_3 \mid \ldots \mid v_n] = [w_{1R} + jw_{1I} \mid w_{1R} - jw_{1I} \mid w_3 \mid \ldots \mid w_n] . \tag{34}$$

where $w_i = -M_{\lambda_i} z_i$. Multiplication of both sides of the equation by the nonsingular matrix

$$\left[\begin{array}{cc|c} 1/2 & -j1/2 & \\ 1/2 & j1/2 & 0 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right]$$

will not alter the calculation of K and yields

$$K [v_{1R} \quad v_{1I} \quad v_3 \quad \ldots \quad v_n] = [w_{1R} \quad w_{1I} \quad w_3 \quad \ldots \quad w_n] . \tag{36}$$

Since the set $\{v_i\}_{i=1}^{n}$ is independent, the matrix

$$[v_{1R} \quad v_{1I} \quad v_3 \quad \ldots \quad v_n] \tag{37}$$

is nonsingular and, again, K may be calculated. The obvious modification for additional complex eigenvalues can be easily made.

Finally, since a fixed eigenstructure uniquely defines $A + BK$, it is straightforward that K is unique whenever rank $[B] = m$.

Essentially, with this result, Moore characterized the class of all closed loop eigenvector sets which can be attained with a given set of distinct eigenvalues and shows that one can, in addition to prespecifying the closed loop eigenvalues, select one of the allowable sets of eigenvectors. The assumption of distinct eigenvalues was dispensed with in a later paper by Klein and Moore [18]. However, two aspects which remained were the necessities of full state feedback and of the calculation of a basis for the nullspace of S_λ.

Other early researchers in the area of eigenstructure assignment include Porter and D'Azzo [24, 25]. The later paper [25] provided an algorithm for finding the vectors which span the nullspace of S_{λ_i}. Still other works which require full state feedback include Fahmy and O'Reilly [9], Dayawansa and Mukundan [4], Fallside and Seraji [10], and Chidambara et al. [3]. In each of these papers, it is difficult to assign a meaningful interpretation to the arbitrariness arising in the algorithms or it happens that a certain amount of freedom is sacrificed a priori by assigning a structure to the feedback matrix K. Also, if the initially found K matrix fails to yield a satisfactory solution, the design procedures give no guidance as to the means of achieving a system with improved response.

Finally, mention must be made of the thesis by Srinathkumar [33]. For controllable systems, Srinathkumar showed that (1) n eigenvalues and a maximum of $n \times m$ eigenvector entries can be arbitrarily specified, and (2) no more than m entries of any one eigenvector can be chosen arbitrarily.

In view of these results and their timing, we note the independent interests of many researchers in the mid-1970's. Of particular interest is the overlap of Moore's work and Srinathkumar's results. The assumption of controllability was not required in the theorem presented earlier and found in [21], but the theorem still "worked" as long as the uncontrollable eigenvalues were members of the desired closed loop set of eigenvalues. An example in [21] showed that even though there is no hope of relocating an uncontrollable eigenvalue, there is some flexibility in altering the eigenvector associated with the uncontrollable eigenvalue. Srinathkumar, with his controllability assumption, is unburdened when choosing eigenvalues and exhibits exactly the number of arbitrary choices in the eigenvector entries. In the next section this point will be investigated more closely.

V. OUTPUT FEEDBACK

From a practical standpoint, the necessity of full state feedback is undesirable. This is obvious when considering large order systems and the cost of measuring and feeding back *each* state variable. A more attractive procedure would be one which is based upon feeding back only the measured variables, i.e., *output feedback*. Hence, we have the motivation for Problem 2.

Among the first to respond to this question in terms of pole placement was Davison [5]. He showed that if the system is controllable and if rank$[C] = r$, then a linear feedback control law of the form

$$u(t) = Fy(t) \tag{38}$$

can always be found so that r eigenvalues of the closed loop system matrix $A + BFC$ are arbitrarily close (but not necessarily equal) to the r preassigned values. Later results by Davison and Chatterjee [6] and Sridhar and Lindorff [32] indicated that if the system is controllable and observable and if rank $[B] = m$ and rank $[C] = r$, then $\max(m,r)$ eigenvalues are assignable almost arbitrarily (as before [5]). In a later paper Kimura [17] showed that if the system is controllable and observable, and if $n \leqq r + m - 1$, then an almost arbitrary set of distinct closed loop poles is assignable by output feedback. It should be noted that in practice $n \geqq r + m - 1$. A noteworthy aspect of this paper is its reliance on the closed loop eigenvectors of the system, unlike previous works which relied on the characteristics equation. In a slightly different vein, Munro and Vardulakis [22] and Porter [26] presented necessary and sufficient conditions for arbitrary assignment of all of the system eigenvalues using only constant output feedback; results which are interesting from a theoretical standpoint. Still other contributions include [7, 8, 13, 35]. In each of the aforementioned works, pole assignment was of primary interest.

When one passes to the broadened question of affecting eigenvectors as well as eigenvalues, the paper of Srinathkumar [34] must be considered a benchmark. Later papers of Shapiro and Chung [30, 31] provided additional aspects which are illuminating. In [34], we find the following:

THEOREM. *Given the controllable and observable system described by* (17) *and* (18) *and the assumptions that the matrices B and C are of full rank, then max(m, r) closed loop eigenvalues can be assigned and max(m, r) eigenvectors (or reciprocal vectors by duality) can be partially assigned with min(m, r) entries in each vector arbitrarily chosen using gain output feedback, i.e., with a control law* $u = Fy$.

The proof is given in [34]. Here we present an exposition of the implications of the theorem and some aspects that have been highlighted by Harvey and Stein [12] and by Shapiro and Chung [30, 31].

A. Eigenvector Assignability

Recall the system equations (17) and (18) and the relevant assumptions concerning A, B, and C and the problem statements concerning $\{\lambda_i^d\}$, $\{v_i^d\}$. The purpose of this section is to consider the problems of (1) characterizing eigenvectors which can be assigned as closed loop eigenvectors, and (2) determining the best possible set of assigned closed loop eigenvectors in case a desired y_i^d is not assignable (since arbitrary eigenvector assignment is not, in general, possible, as we shall see).

1. Total Specification of v_i^d

We begin by considering the closed loop system

$$\dot{x}(t) = (A + BFC)\,x(t).$$

Assume we are given $\{\lambda_i\}_{i=1}^{r}$ as the desired closed loop eigenvalues and we assume v_i is the closed loop eigenvector corresponding to λ_i. Then we have for an eigenvalue/eigenvector pair, λ_i and v_i,

$$(A + BFC)v_i = \lambda_i v_i \tag{39}$$

or

$$v_i = (\lambda_i I - A)^{-1} BFCv_i. \tag{40}$$

It is tacitly assumed that none of the desired eigenvalues match the existing eigenvalues of A so that the inverse of $(\lambda_i I - A)$ exists; if this is not the case, a minor alteration is needed.

Analyzing (40), we define the m-vector m_i as

$$m_i = FCv_i. \tag{41}$$

Then (40) becomes

$$v_i = (\lambda_i I - A)^{-1} Bm_i. \tag{42}$$

The implication of (42) is of great importance: the eigenvector v_i *must be in the subspace spanned by the columns of* $(\lambda_i I - A)^{-1} B$. This subspace is of dimension m which is equal to rank B which is equal to the number of independent control variables. Therefore, the number of control variables available determines how large (dimension) the subspace is in which achievable eigenvectors must reside. The orientation of the subspace is determined by the open loop parameters described by A, B and the desired closed loop eigenvalue λ_i. We conclude that if we choose an eigenvector v_i which lies precisely in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$, it will be achieved exactly.

2. Best Possible v_i^d

In general, however, a desired eigenvector v_i^d will not reside in the prescribed subspace and hence cannot be achieved. Instead a "best possible" choice for an *achievable* eigenvector is made. This best possible eigenvector is the projection of v_i^d onto the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$ as depicted in Fig. 1.

Analytically, we compute v_{i_A}. Begin by defining

$$L_i \triangleq (\lambda_i I - A)^{-1} B. \tag{43}$$

An achievable eigenvector, as we see, must reside in the required subspace and hence

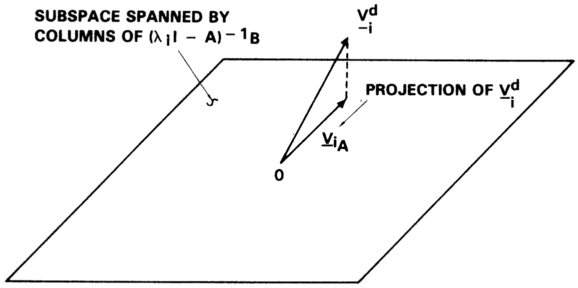

Fig. 1. Geometric interpretation of v_i^d, v_{i_A} where v_i^d is the desired eigenvector and v_{i_A} is the achievable eigenvector. Underlined symbols correspond to boldface symbols in the text.

$$\boldsymbol{v}_{i_A} = L_i z_i; \qquad z_i \in R^m. \tag{44}$$

To find the value of z_i corresponding to the projection of $\boldsymbol{v}_i^d$ onto the "achievability subspace," we choose z_i which minimizes

$$J = \|\boldsymbol{v}_i^d - \boldsymbol{v}_{i_A}\|^2 = \|\boldsymbol{v}_i^d - L_i z_i\|^2. \tag{45}$$

Now

$$dJ/dz_i = 2 L_i^T (L_i z_i - \boldsymbol{v}_i^d). \tag{46}$$

Hence $dJ/dz_i = \mathbf{0}$ implies

$$z_i = (L_i^T \quad L_i)^{-1} \quad L_i^T \boldsymbol{v}_i^d \tag{47}$$

$$\boldsymbol{v}_{i_A} = L_i (L_i^T \quad L_i)^{-1} \quad L_i^T \boldsymbol{v}_i^d. \tag{48}$$

Due to possible ill conditioning, care must be taken when inverting $L_i^T L_i$. A Cholesky decomposition or a singular value decomposition is advised at this stage [12].

We re-emphasize the importance of the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$. From the previous analysis, the following comments are illuminating:

(1) If a desired eigenvector $\boldsymbol{v}_i^d$ is nearly orthogonal to the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$, little hope can be had that the desired eigenvector will translate to better system responses.

(2) If a "larger" subspace in the sense of dimension is required, more independent control variables are needed. This follows from the fact that the dimension of the subspace generated by $(\lambda_i I - A)^{-1}B$ is m. Thus, to increase m, the dimension and rank of B must be increased.

(3) A single input system (B is simply a column vector) allows no hope of selecting an eigenvector although an element of an eigenvector can be specified.

In summary, a designer usually finds his/her problem between the worst case where rank$[B] = 1$, in which case there is no hope of specifying the eigenvector, and the best case where rank$[B] = n$, in which case the entire eigenvector can be completely specified.

3. *Partial Specification of $\boldsymbol{v}_i^d$*

In many practical situations, complete specification of $\boldsymbol{v}_i^d$ is neither required nor known but rather the designer is interested only in certain elements of the eigenvector. To handle such a situation, we again consider our eigenvector $\boldsymbol{v}_i^d$ and assume that it has the following structure:

$$\boldsymbol{v}_i^d = \begin{bmatrix} v_{i1} \\ x \\ x \\ x \\ x \\ v_{ij} \\ x \\ x \\ v_{in} \end{bmatrix}$$

where v_{ij} are designer specified components and x is an unspecified component.

Proceeding, we define, as in [17], reordering operator $\{\cdot\}^{R_i}$ as follows:

$$\{\boldsymbol{v}_i^d\}^{R_i} = \begin{bmatrix} \boldsymbol{l}_i \\ \boldsymbol{d}_i \end{bmatrix}$$

where $\boldsymbol{l}_i$ is a vector of specified components of $\boldsymbol{v}_i^d$ and $\boldsymbol{d}_i$ is a vector of unspecified components of $\boldsymbol{v}_i^d$.

We also reorder the rows of the matrix $(\lambda_i I - A)^{-1}B$ to conform with the reordered components of $\boldsymbol{v}_i^d$, i.e.,

$$\{(\lambda_i I - A)^{-1} B\}^{R_i} = \begin{bmatrix} \tilde{L}_i \\ D_i \end{bmatrix}. \tag{49}$$

If we proceed in precisely the same manner as in Section VA-2 (to obtain z_i) with $\boldsymbol{l}_i$ replacing $\boldsymbol{v}_i^d$ and $\tilde{L}_i$ replacing L_i, we obtain

$$z_i = (\tilde{L}_i^T \tilde{L}_i)^{-1} \tilde{L}_i^T \boldsymbol{l}_i \tag{50}$$

$$\boldsymbol{v}_{i_A} = L_i (\tilde{L}_i^T \tilde{L}_i)^{-1} \tilde{L}_i^T \boldsymbol{l}_i. \tag{51}$$

A tacit assumption for these formulas is that the dimension of the desired vector is less than the row dimension and rank of $\tilde{L}_i$. If this is not the case, the formulas are trivially modified. However, it should be noted that our assumption is valid for most practical situations.

Thus we remain mindful of the natures of a desired eigenvector $\boldsymbol{v}_i^d$, of an eigenvector $\boldsymbol{v}_i$, and of an assignable eigenvector $\boldsymbol{v}_{i_A}$. Also, implicit in the previous presentation is the assumption that the desired eigenvalue is real. For complex eigenvalues/eigenvectors the presentation is formally correct with minor arithmetic adjustments to accommodate complex quantities [21].

4. *Feedback Gain Computation*

We now turn our attention to computing the feedback gain matrix F. We assume in what follows that when the term *eigenvector* is used, we are considering an *assignable* eigenvector, i.e., desired eigenvectors have been projected to the appropriate subspaces and the achievable closed loop eigenvectors are obtained.

Feedback Gain Calculation (Unconstrained). Assuming feedback of the form $\boldsymbol{u}(t) = F\boldsymbol{y}(t)$, the closed loop system is

$$\dot{\boldsymbol{x}}(t) = (A + BFC)\boldsymbol{x}(t).$$

For advantages to be realized later,[2] it is necessary to transform the input matrix B to the following form:

$$B \rightarrow \begin{bmatrix} I_m \\ \text{----} \\ 0 \end{bmatrix}. \tag{52}$$

There is no loss of generality in doing this since B is of full rank and there exists a transformation (nonunique) which does this.

In order to obtain this transformation T, consider the following

$$T = [B \,\vdots\, P] \tag{53}$$

where P is *any* matrix such that rank$[T] = n$. Using T, we consider the change of coordinates,

$$\boldsymbol{x} = T\tilde{\boldsymbol{x}}.$$

Thus the open loop system (17) and (18) is transformed to

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{A}\tilde{\boldsymbol{x}}(t) + \tilde{B}\boldsymbol{u}(t)$$

$$\boldsymbol{y}(t) = \tilde{C}\tilde{\boldsymbol{x}}(t) \tag{54}$$

where

$$\tilde{A} = T^{-1} A\, T$$

$$\tilde{B} = T^{-1} B = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

$$\tilde{C} = C\, T. \tag{55}$$

Under this transformation, the eigenvalues of the original system are identical to the eigenvalues of the transformed system and the eigenvectors of the two systems are related by

$$T^{-1}\, \boldsymbol{v}_i = \tilde{\boldsymbol{v}}_i.$$

Note. In what follows, it is assumed that all matrices and eigenvectors have been transformed to obtain the necessary structure of the B matrix. For notational convenience, we suppress the $(\tilde{\cdot})$ notation.

Having concluded with these preliminaries, we continue our analysis. If λ_i is a closed loop eigenvalue and $\boldsymbol{v}_i$ its associated eigenvector, then the following must hold:

$$(A + BFC)\boldsymbol{v}_i = \lambda_i\, \boldsymbol{v}_i, \qquad i = 1, 2, \ldots, r. \tag{56}$$

[2]The advantages will become obvious when we consider the case of constrained output feedback.

Rewriting the closed loop eigenvalue/eigenvector equation, we obtain

$$(\lambda_i I - A)\, \boldsymbol{v}_i = BFC\boldsymbol{v}_i. \tag{57}$$

We partition (57) conformally, mindful of the special structure of the B matrix:

$$\left[\begin{array}{c|c} \lambda_i I_m - A_{11} & -A_{12} \\ \hline -A_{21} & \lambda_i I_{n-m} - A_{22} \end{array}\right] \left[\begin{array}{c} z_i \\ \hline \boldsymbol{w}_i \end{array}\right] = \left[\begin{array}{c} I_m \\ \hline 0 \end{array}\right] FC \left[\begin{array}{c} z_i \\ \hline \boldsymbol{w}_i \end{array}\right]. \tag{58}$$

where

$$\boldsymbol{v}_i = \begin{bmatrix} z_i \\ \boldsymbol{w}_i \end{bmatrix},$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

are partitioned appropriately.

Upon considering the first matrix equation from the partitioned form, we obtain

$$[\lambda_i I_m - A_{11} \quad -A_{12}] \begin{bmatrix} z_i \\ \boldsymbol{w}_i \end{bmatrix} = FC \begin{bmatrix} z_i \\ \boldsymbol{w}_i \end{bmatrix}. \tag{59}$$

Multiplying through yields

$$(\lambda_i I_m - A_{11})\, z_i - A_{12}\boldsymbol{w}_i = FC\boldsymbol{v}_i \tag{60}$$

or

$$\lambda_i I_m z_i - (A_{11} z_i + A_{12}\boldsymbol{w}_i) = FC\boldsymbol{v}_i \tag{61}$$

or

$$\lambda_i z_i - A_1 \boldsymbol{v}_i = FC\boldsymbol{v}_i \tag{62}$$

where $A_1 = [A_{11} \quad A_{12}]$. We rewrite (62) as

$$(A_1 + FC)\boldsymbol{v}_i = \lambda_i z_i. \tag{63}$$

This last equation holds for each desired eigenvalue/achievable eigenvector pair, i.e.,

$$(A_1 + FC)\boldsymbol{v}_1 = \lambda_1 z_1$$

$$(A_1 + FC)\boldsymbol{v}_2 = \lambda_2 z_2$$

$$(A_1 + FC)\boldsymbol{v}_r = \lambda_r z_r \tag{64}$$

or, in condensed form,

$$(A_1 + FC)\, V = Z \tag{65}$$

where

$$V = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \ldots \quad \boldsymbol{v}_r], \qquad V \text{ is } (n \times r) \tag{66}$$

$$Z = [\lambda_1 z_1 \quad \lambda_2 z_2 \quad \ldots \quad \lambda_r z_r], \qquad Z \text{ is } (m \times r). \tag{67}$$

It is from (65) that we calculate F. We note that, in general, V and Z are complex. To alleviate the need for complex arithmetic we use the transformation presented by Moore (35), as presented previously, to transform V and Z to real matrices. Hence, without loss of generality, we assume Z, V real and obtain

$$F = (Z - A_1\, V)\,(C\, V)^{-1}. \tag{68}$$

The matrix F will exist provided the matrix CV is nonsingular. From a mathematical standpoint, the inverse of CV is guaranteed provided the nullspace of C and the subspace spanned by the columns of V intersect only at the origin. From a physical standpoint, CV will be singular (or extremely ill conditioned) when measurements taken (reflected by the C matrix) have little or no impact on the achievable eigenvectors (reflected by the V matrix). The singularity or nonsingularity of CV, therefore, provides an excellent check as to the reasonableness of our eigenvectors in relation to the outputs being measured and fed back.

Before proceeding to the constrained output case, a few comments are in order:

(1) The F calculated in (68) will place precisely r *desired eigenvalues* and will assign the associated eigenvectors as close as possible to the desired eigenvectors in a least squares sense.

(2) If control over a larger number of eigenvalues is desired, the rank of C *must* be increased, i.e., additional independent sensors must be added to the system.

(3) If improved eigenvector assignability is required, the rank of B *must* be increased, i.e., additional control variables must be allowed to participate in the control process.

(4) The solution is *truly* output feedback in that only measured quantities are fed back but the structure of the matrix F is fixed as every output is fed to each input.

VI. CONSTRAINED OUTPUT FEEDBACK

From the analysis of the previous section, we note that every output is fed back to each input. In this section, we wish to consider the possibility of *not* feeding back certain outputs to certain inputs, i.e., we wish to impose constraints on the feedback matrix F in the form of fixed zeros within the matrix to reflect physically desirable feedback combinations. This feature of constrained feedback lends practicality and flexibility to our procedure. Also, it will provide the designer, as we shall see, the spectrum of tradeoffs between dynamic performance and the structural complexity of the controller (reliability). The only references known to the authors concerning this problem are [30, 31, 37].

To begin, recall (65),

$$(A_1 + FC)\,V = Z$$

and that the equation is of this form due to the special structure of B. (Had we not imposed this structure on B, the matrix B would be premultiplying FC.) Therefore

$$A_1\,V + FCV = Z \tag{69}$$

or

$$FCV = Z - A_1\,V. \tag{70}$$

Proceeding, we let

$$\Omega \triangleq C\,V \tag{71}$$

$$\Psi \triangleq Z - A_1\,V. \tag{72}$$

Then, to re-emphasize B's structure, (70) may be rewritten as

$$I_m\,F\,\Omega = \Psi.$$

We rewrite this equation in terms of the Kronecker product and a *row* stacking operator,

$$[I_m \otimes \Omega^{\mathrm{T}}]\,S(F) = S(\Psi) \tag{73}$$

where S represents the row stacking operator. Letting $\boldsymbol{f}_i$ denote the ith row of F and $\boldsymbol{\psi}_i$ denote the ith row of $\boldsymbol{\Psi}$, we obtain

$$\begin{bmatrix} \Omega^{\mathrm{T}} & 0_m & \cdots & 0_m \\ 0_m & \Omega^{\mathrm{T}} & \cdots & 0_m \\ \vdots & & & \\ \vdots & & & 0_m \\ & & & \Omega^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_1^{\mathrm{T}} \\ \boldsymbol{f}_2^{\mathrm{T}} \\ \vdots \\ \vdots \\ \boldsymbol{f}_m^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_1^{\mathrm{T}} \\ \boldsymbol{\psi}_2^{\mathrm{T}} \\ \vdots \\ \vdots \\ \boldsymbol{\psi}_m^{\mathrm{T}} \end{bmatrix} \tag{74}$$

The matrix Ω^{T} is repeated along the main diagonal while 0_m matrices reside off the diagonal.

The structure of (74) is due solely to the transformation of the B matrix. If the transformation were not made, the coefficient matrix of the feedback gains would possibly have been full. The obvious advantage of this structure is that each row of feedback gains ($\boldsymbol{f}_i$) can be computed independently of all other rows, i.e.,

$$\boldsymbol{f}_i = \boldsymbol{\psi}_i\,\Omega^{-1}. \tag{75}$$

Let us focus on this ith row equation. Expanding we have

$$[\Omega^{\mathrm{T}}] \begin{bmatrix} f_{i1} \\ f_{i2} \\ f_{ij} \\ f_{ir} \end{bmatrix} = \boldsymbol{\psi}_i^{\mathrm{T}}. \tag{76}$$

If we were to constrain f_{ij} to be zero, then we delete f_{ij} from $\boldsymbol{f}_i^{\mathrm{T}}$ and delete the jth column of Ω^{T}. We now solve the reduced problem,

$$\tilde{\Omega}^{\mathrm{T}}\,\tilde{\boldsymbol{f}}_i^{\mathrm{T}} = \boldsymbol{\psi}_i^{\mathrm{T}} \tag{77}$$

where $\tilde{\Omega}^{\mathrm{T}}$ is the matrix Ω^{T} with its jth column deleted and $\tilde{\boldsymbol{f}}_i^{\mathrm{T}}$ is the vector $\boldsymbol{f}_i^{\mathrm{T}}$ with its jth element deleted. Our reduced problem is overdetermined in that we now have more equations than unknowns. Using a pseudoinverse, our "solution" for $\boldsymbol{f}_i^{\mathrm{T}}$ (the remaining active gains in ith row) is

$$\tilde{\boldsymbol{f}}_i^{\mathrm{T}} = (\tilde{\Omega}^{\mathrm{T}})^{\dagger}\,\boldsymbol{\psi}_i^{\mathrm{T}} \tag{78}$$ [3]

or

$$\tilde{\boldsymbol{f}}_i = \boldsymbol{\psi}_i\,\tilde{\Omega}^{\dagger}. \tag{79}$$

[3] $(\ )^{\dagger}$ indicates the appropriate pseudoinverse of $(\ )$.

If more than one gain in a row of F is to be set to zero, then f_i^{T} and $\tilde{\Omega}^{\mathrm{T}}$ must be modified appropriately.

The potential offered by this procedure is great and will be highlighted by examples to follow. What will be seen is that a designer can now look at a spectrum of possibilities in synthesizing a feedback controller. For example, in considering a given problem, one might obtain a certain dynamic performance given unconstrained output feedback. Upon further consideration, the designer may be interested in determining performance if, for instance, certain states are not fed to certain inputs. By suppressing certain gains to zero, the designer reduces controller complexity and increases reliability. This gives the designer the ability to examine a spectrum of tradeoffs between performance and complexity/reliability.

It should be noted that the tradeoff in performance comes from the fact that once constraints are put on the feedback structure, it may not be possible to assign the closed loop eigenvalues to the *exact* desired locations (see [30, 31]). However, if the eigenvalues are "close," the dynamic behavior may be acceptable to the designer.

VII. CHOICE OF DESIRED EIGENVECTORS

Before presenting an example, a few words are in order concerning the choice of v_i^d, the set of desired closed loop eigenvectors. From the previous sections, we know where the eigenvectors must reside, from a mathematical standpoint, in order to be achieved. However, a designer's desires or requirements are usually not formulated in subspaces; typically, in any practical situation, a designer knows what is "good" or "not good" from a performance standpoint. We present here generically good eigenvectors for the handling of aircraft. For practitioners in the control design of other physical systems, analogous choices can be made.

If one is considering the linearized perturbed longitudinal equations of an aircraft and the state vector is

$$\boldsymbol{x} = \begin{bmatrix} \alpha \\ q \\ \theta \\ u \\ \delta_e \end{bmatrix} \begin{array}{l} \text{angle of attack} \\ \text{pitch rate} \\ \text{pitch angle} \\ \text{forward velocity (perturbed)} \\ \text{elevator deflection} \end{array}$$

then a good choice of closed loop eigenvectors might be

$$\begin{array}{cccc} \begin{bmatrix} 1 \\ x \\ x \\ 0 \\ x \end{bmatrix} & \begin{bmatrix} x \\ 1 \\ x \\ 0 \\ x \end{bmatrix} \quad \text{and} & \begin{bmatrix} 0 \\ x \\ 1 \\ x \\ x \end{bmatrix} & \begin{bmatrix} 0 \\ x \\ x \\ 1 \\ x \end{bmatrix}. \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \end{array}$$

The vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are called short period vectors and are chosen such that the variation in forward velocity is zero and the angle of attack and pitch rate are coupled together. The vectors $\boldsymbol{v}_3$, $\boldsymbol{v}_4$ are the phugoid vectors which couple pitch angle and (perturbed) forward velocity while holding angle of attack constant. This choice of eigenvectors coupled with the flight dynamics of the problem, i.e., during the short period mode the pitch angle is small, renders the subvector made up of the first four components of the short period vectors "almost orthogonal" to the first four components of the phugoid vectors. This yields a good degree of decoupling between these modes.

A relevant comment at this point is that it will be impossible to fully realize this "decoupling" if only an elevator deflection command is used as an input. (This is due to the rank of the B matrix being 1.) However, if canards (or vectored thrusts) are added to the aircraft, more authority can be exercised over these modes.

If we consider the linearized perturbed lateral axis equations with the state vector given by

$$\boldsymbol{x} = \begin{bmatrix} r \\ \beta \\ p \\ \varphi \\ \delta_r \\ \delta_a \end{bmatrix} \begin{array}{l} \text{yaw rate} \\ \text{sideslip angle} \\ \text{roll rate} \\ \text{bank angle} \\ \text{rudder deflection} \\ \text{aileron deflection} \end{array}$$

then, for a generic lateral stability augmentation system, a desirable choice of eigenvectors would be

$$\begin{array}{ccc} \begin{bmatrix} 1 \\ x \\ 0 \\ 0 \\ x \\ x \end{bmatrix}, & \begin{bmatrix} x \\ 1 \\ 0 \\ 0 \\ x \\ x \end{bmatrix} \quad \text{and} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ x \\ x \\ x \end{bmatrix}. \\ \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{array}$$

The vectors $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ are the dutch roll vectors in which the yaw rate and sideslip are coupled while roll rate and bank angle are suppressed. The vector $\boldsymbol{w}_3$ is the roll subsidence vector where roll rate (and hence, bank angle) are emphasized while yaw rate and sideslip are set to zero. The effect of these choices is again to obtain an orthogonality of the subvector composed of the first four components of the dutch roll vectors with respect to the appropriate subvector from roll subsidence.

If the lateral axis equations are considered again with the roll attitude loop closed, then a desirable set of eigenvectors is

$$\underset{w_1}{\begin{bmatrix}1\\x\\0\\0\\x\\x\end{bmatrix}} \quad \underset{w_2}{\begin{bmatrix}x\\1\\0\\0\\x\\x\end{bmatrix}} \quad \text{and} \quad \underset{w_3}{\begin{bmatrix}0\\0\\1\\x\\x\\x\end{bmatrix}} \quad \underset{w_4}{\begin{bmatrix}0\\0\\x\\1\\x\\x\end{bmatrix}}$$

In this case, the appropriate subvectors are orthogonal and a decoupling of the roll mode (w_3, w_4) from the dutch roll (w_1, w_2) will be realized if these eigenvectors are achieved.

We notice that in each case, the "control" deflections elevator (δ_e), rudder (δ_r), and aileron (δ_a) were unspecified. This is done to allow them to take on values necessary to achieve the desired subvectors. As long as the deflections stay within physically reasonable limits the values of δ_e, δ_r, and δ_a are appropriate.

For problems in areas other than flight control, it is obvious that a designer can peform analogous tasks. Decoupling or partial decoupling of modes in the system can be achieved by judicious choices of elements in the desired eigenvector. We emphasize that once a choice of eigenvector is made, a best possible (in the least square sense) eigenvector is obtained from the analysis of the previous section.

VIII. NUMERICAL EXAMPLE

We consider the lateral axis model of the L-1011 at cruise flight condition. The model will include actuator dynamics and a washout (high-pass) filter on the yaw rate. The system equations, evaluated at the cruise flight condition, are composed of the following components. The state vector is given by

$$x = \begin{bmatrix}\delta_r\\ \delta_a\\ \varphi\\ r\\ p\\ \beta\\ x_7\end{bmatrix} \begin{matrix}\text{rudder deflection (rad)}\\ \text{aileron deflection (rad)}\\ \text{bank angle (rad)}\\ \text{yaw rate (rad/s)}\\ \text{roll rate (rad/s)}\\ \text{sideslip angle (rad)}\\ \text{washout filter state.}\end{matrix}$$

The A, B, and C matrices are given, respectively, as

$$A = \begin{bmatrix} -20 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -25 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ -0.744 & -0.032 & 0 & -0.154 & -0.0042 & 1.54 & 0\\ 0.337 & -1.12 & 0 & .249 & -1.0 & -5.2 & 0\\ 0.02 & 0 & 0.0386 & -0.996 & -0.000295 & -0.117 & 0\\ 0 & 0 & 0 & 0.5 & 0 & 0 & -0.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 20 & 0\\ 0 & 25\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system's inputs are

$$u = \begin{bmatrix}\delta_{r_c}\\ \delta_{a_c}\end{bmatrix} \begin{matrix}\text{rudder command (rad)}\\ \text{aileron command (rad).}\end{matrix}$$

The system outputs are

$$y = \begin{bmatrix}r_{wo}\\ p\\ \beta\\ \varphi\end{bmatrix} \begin{matrix}\text{washed out yaw rate}\\ \text{roll rate}\\ \text{sideslip angle}\\ \text{bank angle.}\end{matrix}$$

For this system the open loop eigenvalues (i.e., the eigenvalues of the A matrix) are

$$\begin{aligned} \lambda_1 &= -20.0 && \text{rudder mode}\\ \lambda_2 &= -25.0 && \text{aileron mode}\\ \lambda_{3,4} &= -0.0884 \pm j1.272 && \text{dutch roll mode}\\ \lambda_5 &= -1.085 && \text{roll subsidence mode}\\ \lambda_6 &= -0.00911 && \text{spiral mode}\\ \lambda_7 &= -0.5 && \text{washout filter mode.} \end{aligned}$$

We wish to design a closed loop controller to provide for the function of a lateral stability augmentation system as well as closure of the roll attitude loop.

As an illustration, we choose the desired closed loop dutch roll and roll modes to be

$$\lambda_{3,4}^d = -1.5 \pm j1.5 \quad \text{dutch roll}$$

$$\lambda_{5,6}^d = -2.0 \pm j1.5 \quad \text{roll mode.}$$

Since rank$[C] = 4$, we will be able to modify four closed loop eigenvalues.

Referring to the discussion of the previous section, we choose the following as desired closed loop eigenvectors:

$$\begin{array}{c} \delta_r \\ \delta_a \\ \varphi \\ r \\ p \\ \beta \\ x_7 \end{array} \begin{bmatrix} x \\ x \\ 0 \\ 1 \\ 0 \\ x \\ x \end{bmatrix} \begin{bmatrix} x \\ x \\ 0 \\ x \\ 0 \\ 1 \\ x \end{bmatrix} \qquad \begin{bmatrix} x \\ x \\ 1 \\ 0 \\ x \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x \\ x \\ x \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

dutch roll roll mode

(Recall x corresponds to an unspecified component.)

We now consider four possible controller configurations. These configurations are the following:

$$F(1) = \begin{bmatrix} X & X & X & X \\ X & X & X & X \end{bmatrix}$$

i.e., every output is fed to every input (truly output feedback controller).

$$F(2) = \begin{bmatrix} X & 0 & X & 0 \\ X & X & X & X \end{bmatrix}$$

i.e., no feedback from roll rate and bank angle to rudder.

$$F(3) = \begin{bmatrix} X & 0 & X & 0 \\ 0 & X & X & X \end{bmatrix}$$

i.e., no feedback from roll rate and bank angle to rudder, no feedback from washed out yaw rate to aileron.

$$F(4) = \begin{bmatrix} X & 0 & X & 0 \\ 0 & X & 0 & X \end{bmatrix}$$

i.e., no feedback from roll rate and bank angle to rudder, no feedback from washed out yaw rate and sideslip to aileron.

With each controller, we present

(1) calculated gains
(2) closed loop eigenvalues
(3) transient responses for
  (a) initial sideslip angle
  (b) initial bank angle.

The computed feedback gains for each configuration are

Controller I:

$$F(1) = \begin{bmatrix} -3.35 & 0.159 & 4.88 & 0.379 \\ -1.42 & -2.38 & 6.36 & -3.8 \end{bmatrix}.$$

Controller II:

$$F(2) = \begin{bmatrix} -3.34 & 0 & 4.87 & 0 \\ -1.42 & -2.38 & 6.36 & -3.8 \end{bmatrix}.$$

Controller III:

$$F(3) = \begin{bmatrix} -3.34 & 0 & 4.87 & 0 \\ 0 & -2.4 & 3.51 & -0.389 \end{bmatrix}.$$

Controller IV:

$$F(4) = \begin{bmatrix} -3.34 & 0 & 4.87 & 0 \\ 0 & -2.42 & 0 & -3.98 \end{bmatrix}.$$

The closed loop eigenvalues for each controller are given in Table I.

The open loop transient response as well as the closed loop transient response for each controller to an initial sideslip angle of 1° (all other initial conditions are zero) is shown in Figs. 2–6, respectively.

The same order of presentation is preserved in Figs. 7–11 which depict the transient responses to an initial bank angle of 1° (all other initial conditions are zero).

From these, the following results emerge.

(1) For the modes of primary interest, the eigenvalues were placed exactly for unconstrained output feedback $F(1)$; but we also note that sufficient accuracy was obtained for all configurations of the feedback controller, i.e.,

$$\text{dutch roll} \quad \begin{cases} 2.118 \leqq \omega \leqq 2.223 \\ 0.6395 \leqq \zeta \leqq 0.708 \end{cases}$$

$$\text{roll mode} \quad \begin{cases} 2.114 \leqq \omega \leqq 2.278 \\ 0.8946 \leqq \zeta \leqq 0.9213. \end{cases}$$

Hence gain suppression (in a rational manner) seems to have negligible effect on closed loop eigenvalues. This is also visible when considering sideslip and yaw rate time histories in response to an initial β (0) (Figs. 3–6) or when considering roll rate and bank angle time histories in response to an initial (0) (Figs. 8–11). This shows that the dutch roll mode is almost invariant (when viewing β and r in the first set of figures) relative to controller configuration as is roll mode (when viewing ϕ and p in the second set of figures).

(2) Returning to the figures, one can see that the eigenvector orientation is affected more by gain suppression than the eigenvalues were. This is done by investigating the time histories of roll rate and bank angle in response to an initial sideslip and the time histories of yaw rate and sideslip in response to an initial bank angle. For example, increased roll rate is visible as certain gains are eliminated from the F matrix (Figs. 3–6) and the same type of observation can be made concerning yaw rate when considering Figs. 8–11.

(3) A spectrum of results is presented which allows the designer to consider tradeoffs between performance and structural controller complexity.

IX. SUMMARY

The concept of eigenvalue/eigenvector assignment using feedback can be a very powerful tool in the design of control systems. Depending on a given physical situation, the designer can decide among a number of options which contain inherent tradeoffs.

TABLE I
Closed loop eigenvalues for feedback matrices $F(1)$, $F(2)$, $F(3)$, $F(4)$ and open loop eigenvalues.

CONTROLLER	RUDDER MODE	AILERON MODE	DUTCH ROLL MODE	ROLL MODE	WASHOUT FILTER POLE
$F(1)=\begin{bmatrix} -3.35 & .159 & 4.88 & .379 \\ -1.42 & -2.38 & 6.36 & -3.8 \end{bmatrix}$	-17.05	-22.01	$-1.502 \pm j1.497$ $\omega = 2.12$ $\zeta = .708$	$-2.001 \pm j.9995$ $\omega = 2.33$ $\zeta = .8946$	-.6989
$F(2)=\begin{bmatrix} -3.34 & 0 & 4.87 & 0 \\ -1.42 & -2.38 & 6.36 & -3.8 \end{bmatrix}$	-17.12	-22.02	$-1.496 \pm j1.500$ $\omega = 2.11$ $\zeta = .706$	$-1.971 \pm j.9838$ $\omega = 2.07$ $\zeta = .8947$	-.6946
$F(3)=\begin{bmatrix} -3.34 & 0 & 4.87 & 0 \\ 0 & -2.40 & 3.51 & -3.89 \end{bmatrix}$	-17.17	-22.02	$-1.521 \pm j1.622$ $\omega = 2.2236$ $\zeta = .6838$	$-1.918 \pm j.8896$ $\omega = 2.114$ $\zeta = .9072$	-.6991
$F(4)=\begin{bmatrix} -3.34 & 0 & 4.87 & 0 \\ 0 & -2.4\underline{2} & 0 & -3.98 \end{bmatrix}$	-17.17	-21.99	$-1.378 \pm j1.657$ $\omega = 2.1557$ $\zeta = .6395$	$-2.098 \pm j.8856$ $\omega = 2.2768$ $\zeta = .9213$	-.6579
OPEN LOOP $F = \underline{0}$	-20.0	-25.0	$-.0889 \pm j1.269$	Roll Subsid -1.085 Spiral -.009165	-.50

Note: Underlined symbols correspond to boldface symbols in the text.

In honesty, as users of the above techniques, an in-depth knowledge of the system to be controlled is required. The reader must note that once a designer enters the realm of output feedback in eigenstructure assignment, stability *is not mathematically guaranteed.* Hence care must be exercised in its use.

In conclusion, the techniques presented here may prove useful in the feedback control of linear systems.

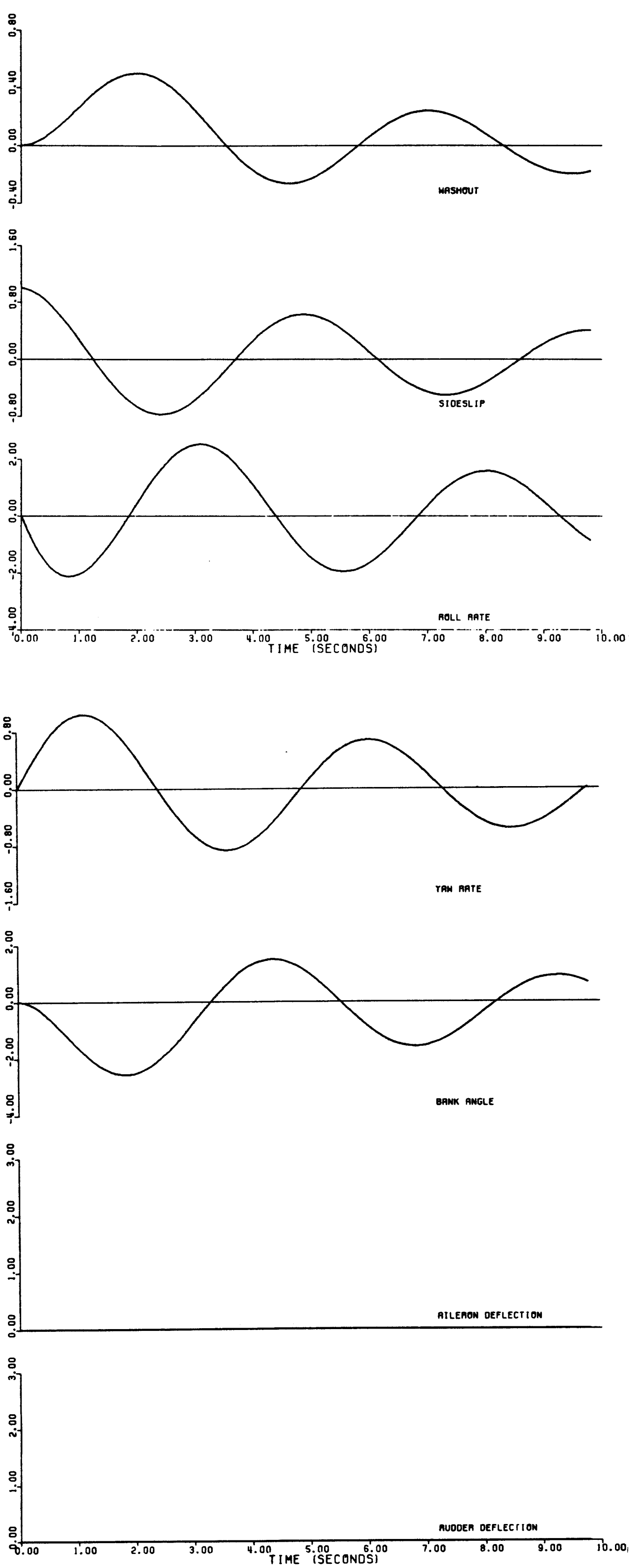

Fig. 2. Open loop responses, $\beta(0) = 1°$.

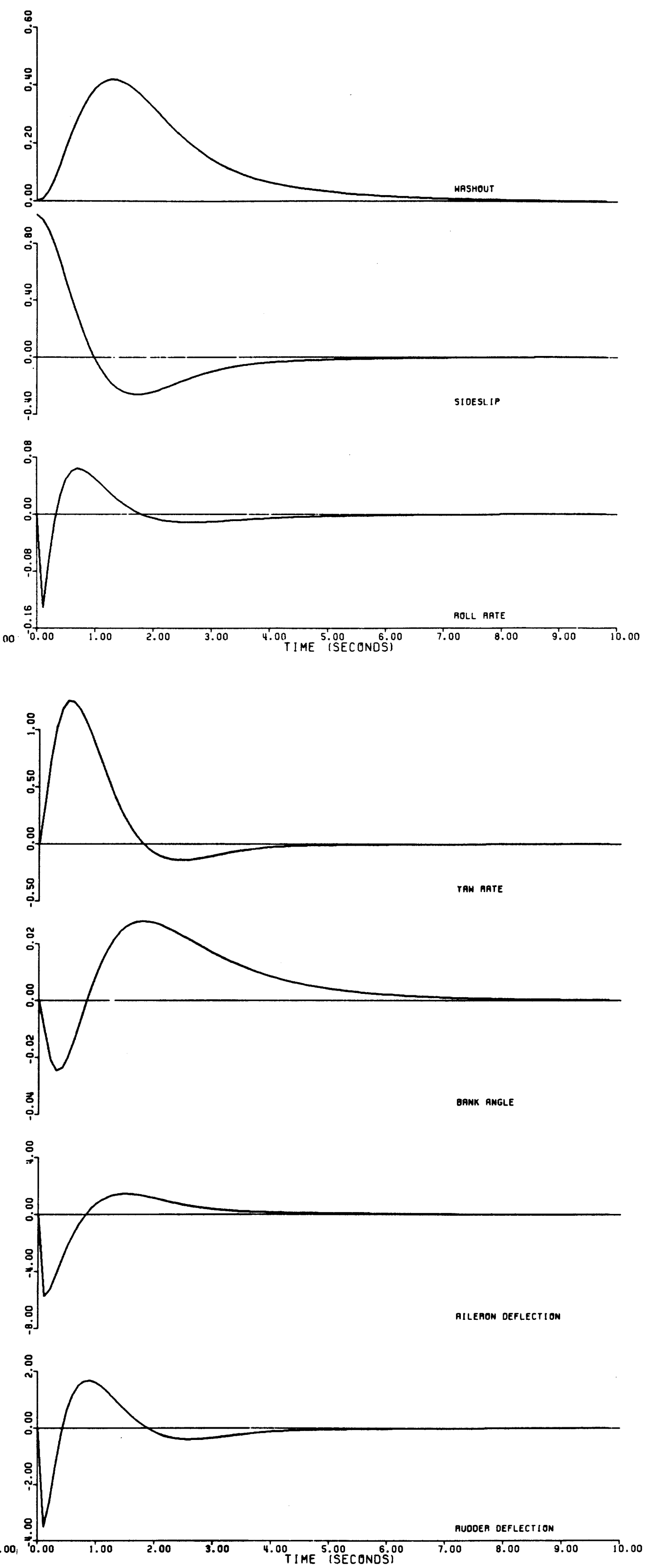

Fig. 3. Closed responses for $F(1)$ controller, $\beta(0) = 1°$.

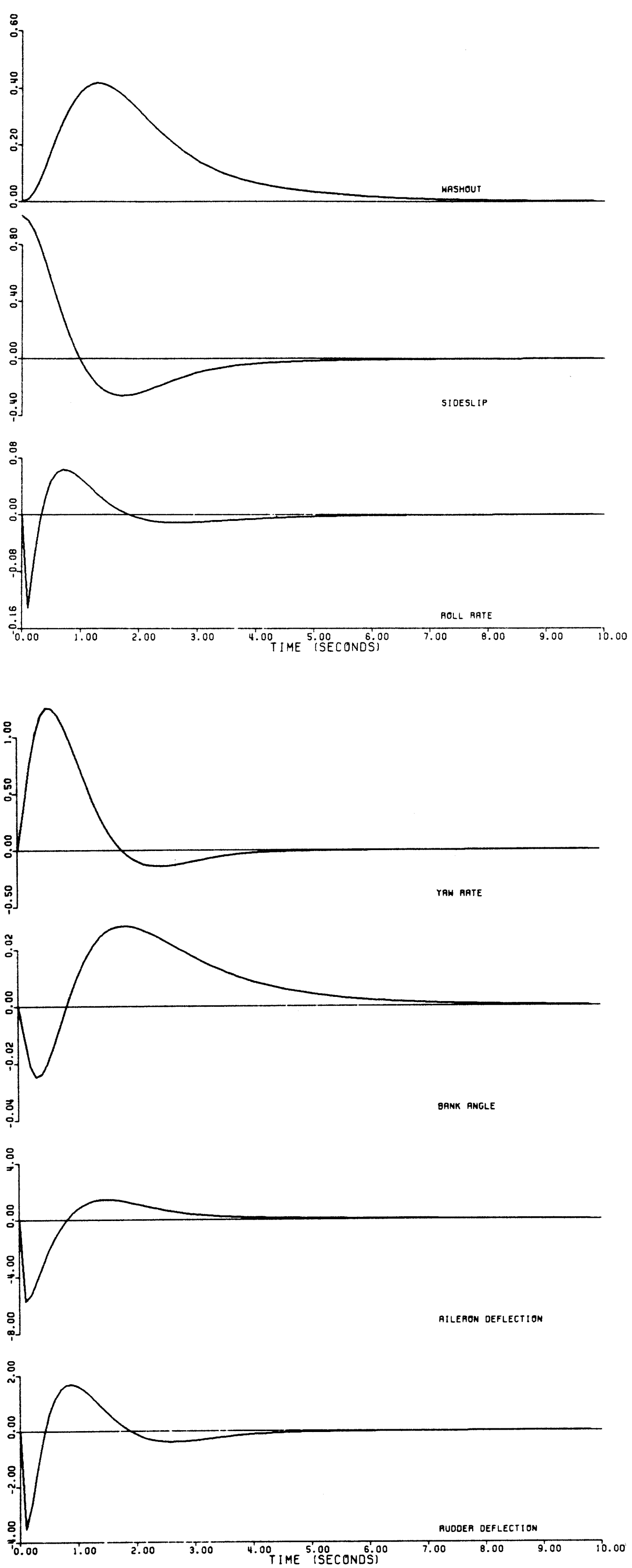

Fig. 4. Closed loop responses for $F(2)$, $\beta(0) = 1°$.

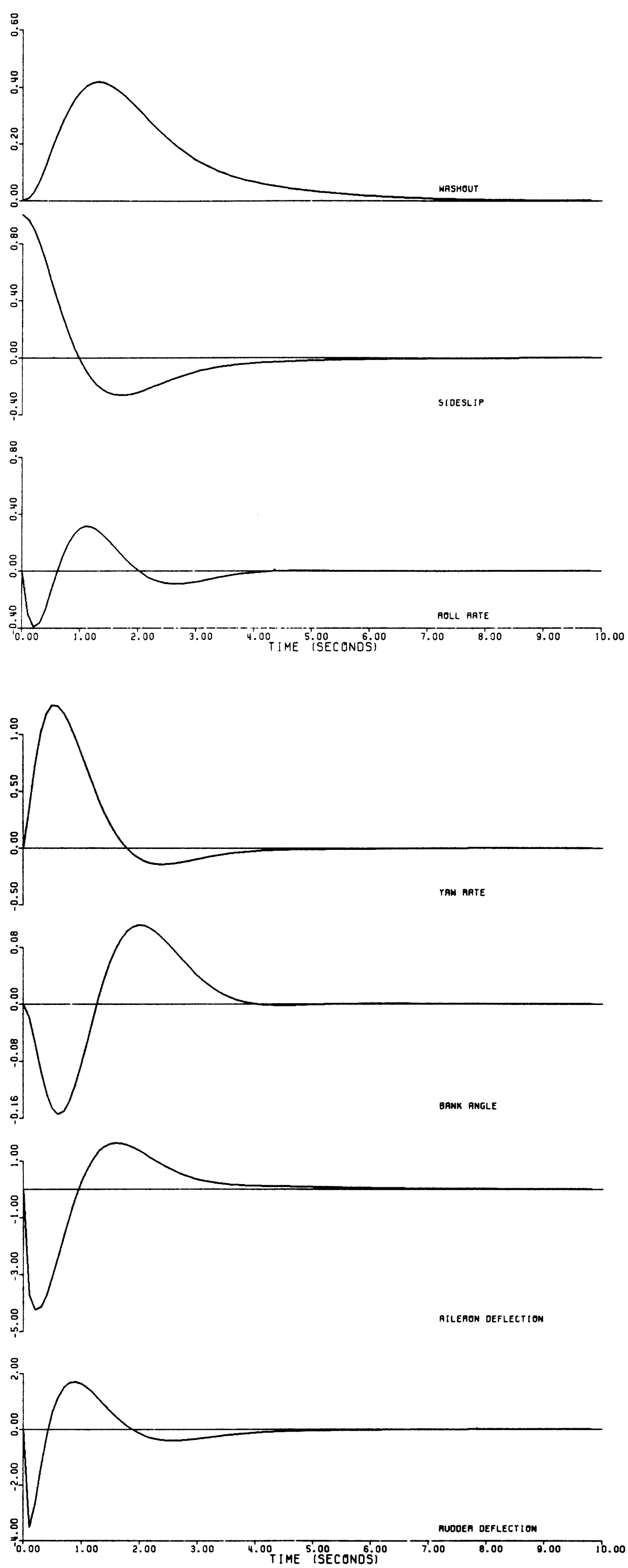

Fig. 5. Closed loop responses for $F(3)$, $\beta(0) = 1°$.

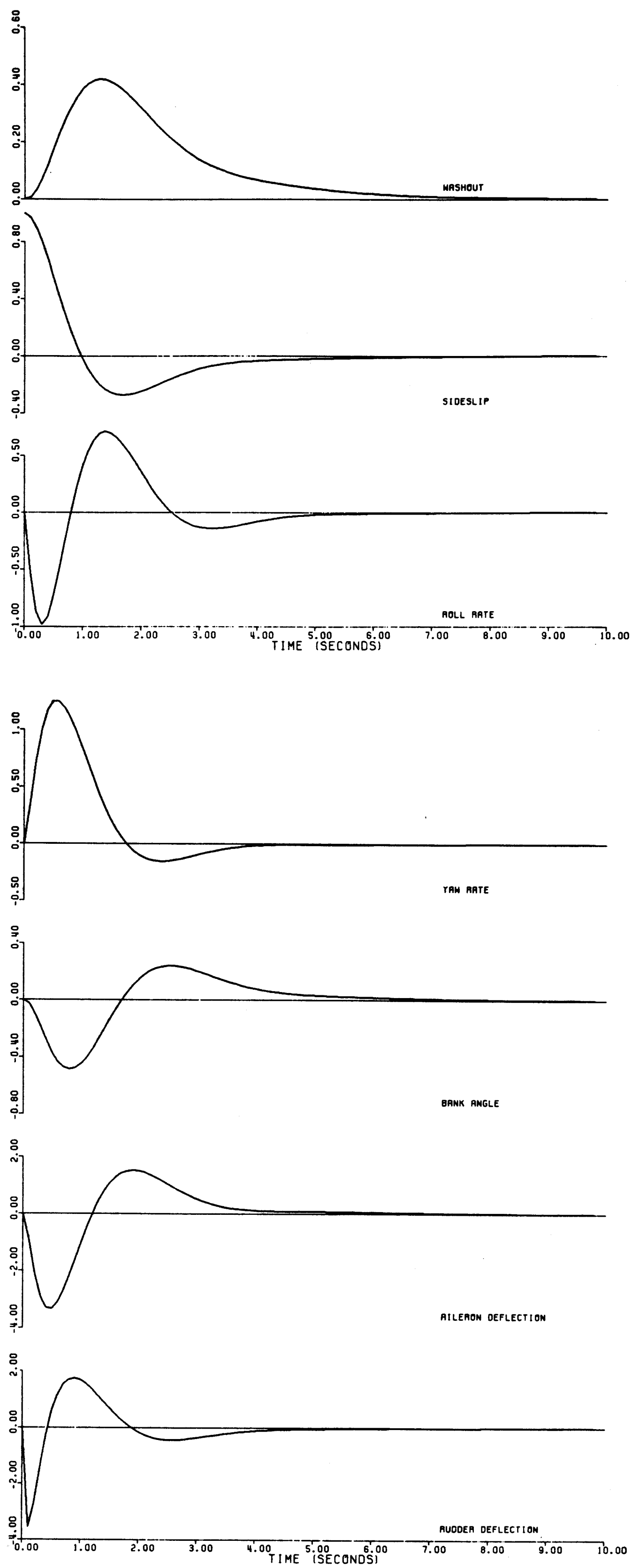

Fig. 6. Closed loop responses for $F(4)$, $\beta(0) = 1°$.

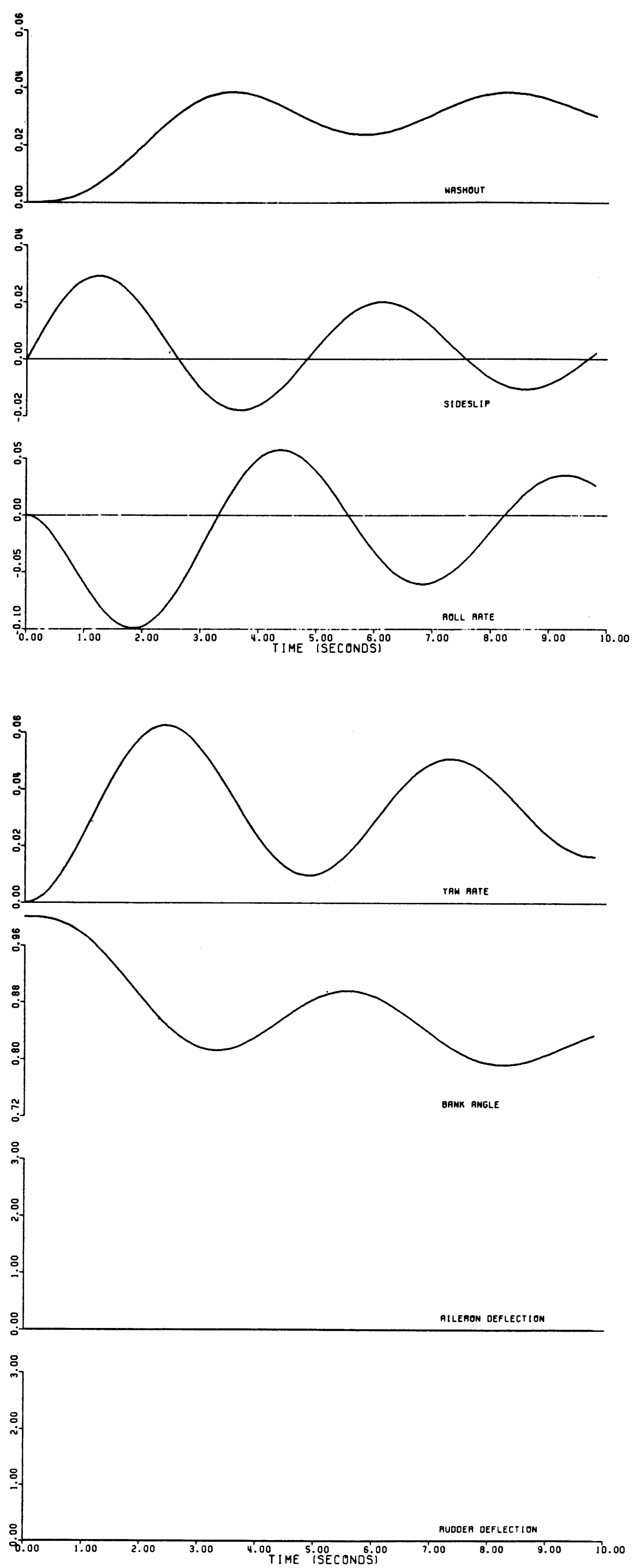

Fig. 7. Open loop responses to initial bank angle, $\phi(0) = 1°$.

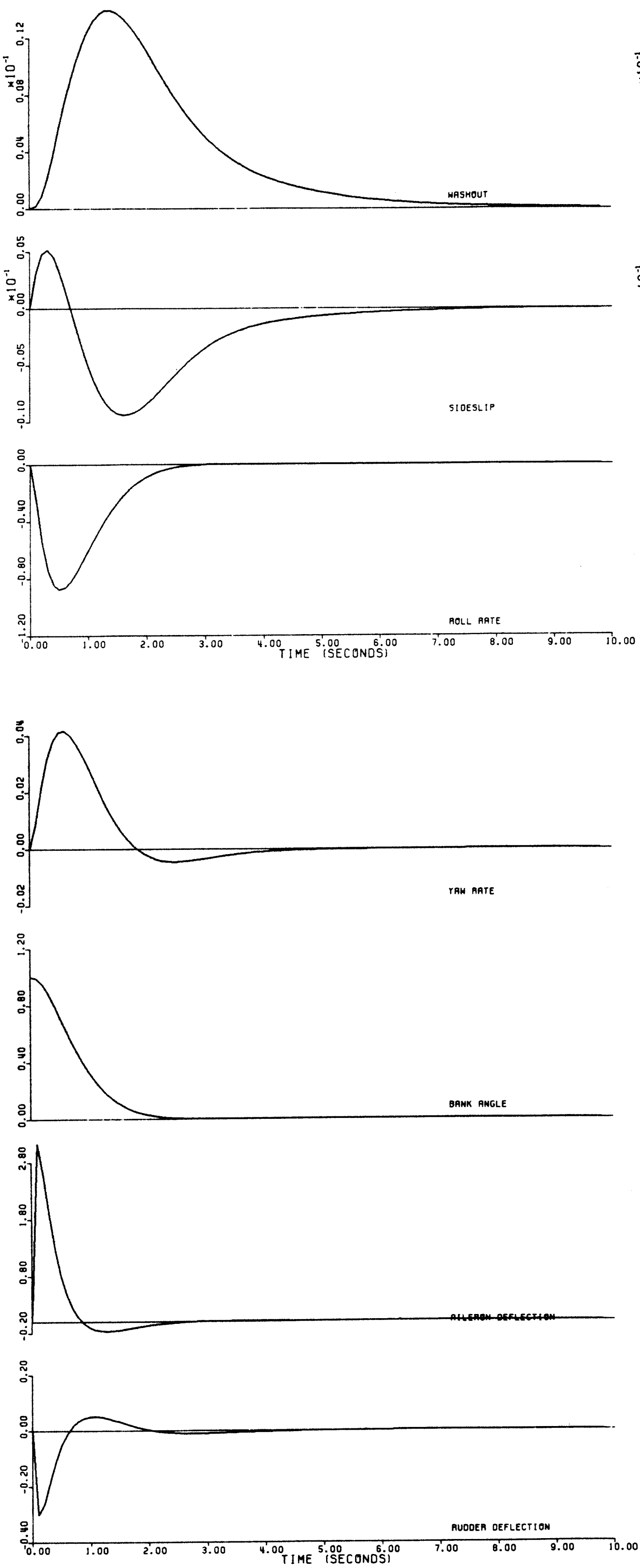

Fig. 8. Closed loop responses for $F(1)$, $\phi(0) = 1°$.

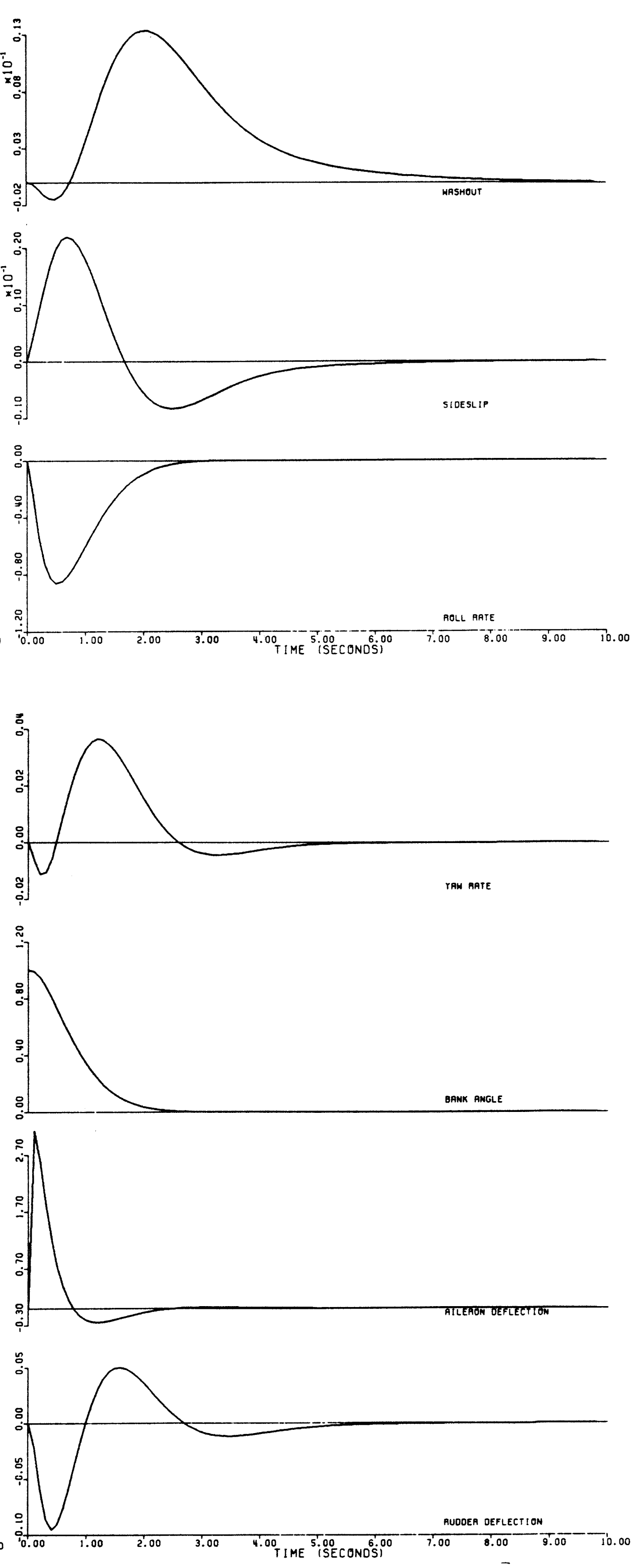

Fig. 9. Closed loop responses for $F(2)$, $\phi(0) = 1°$.

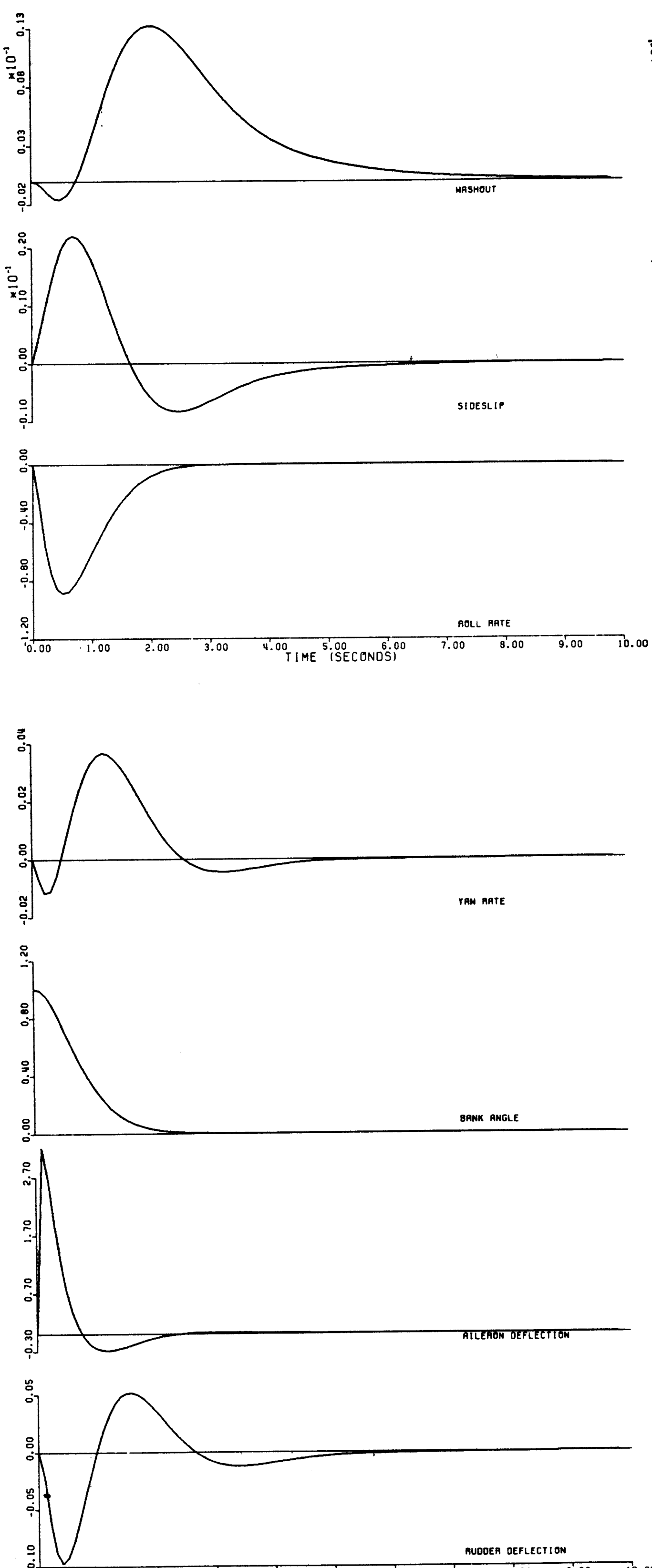

Fig. 10. Closed loop responses for $F(3)$, $\phi(0) = 1°$.

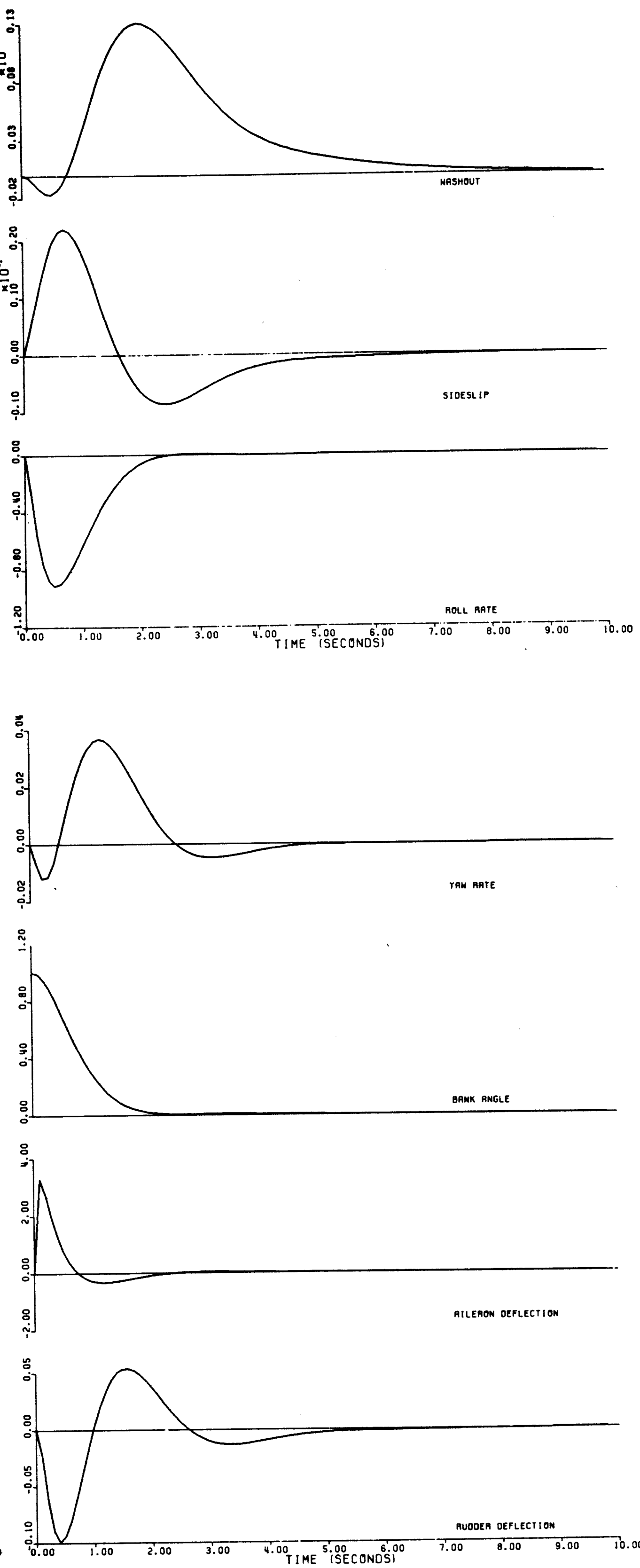

Fig. 11. Closed loop responses for $F(4)$, $\phi(0) = 1°$.

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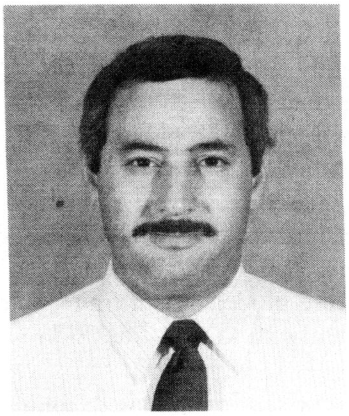

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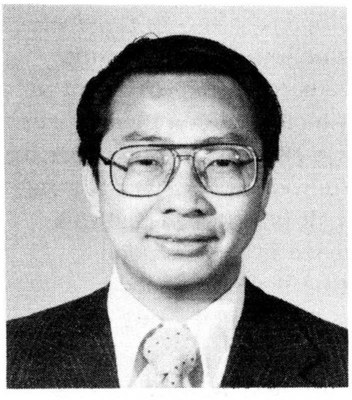
Dr. Shapiro is a member of Sigma Xi and an Associated Fellow of the American Institute of Aeronautics and Astronautics.



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