HW1: Modeling the Flying-Chardonnay

This exercise is part of a series of activities leading to the implementation of an interactive MIMO control law for the `Flying-Chardonnay`, an automatic drink delivery device. Our study starts with physics-based modeling of our system and implementing its simulation counterpart in MATLAB.

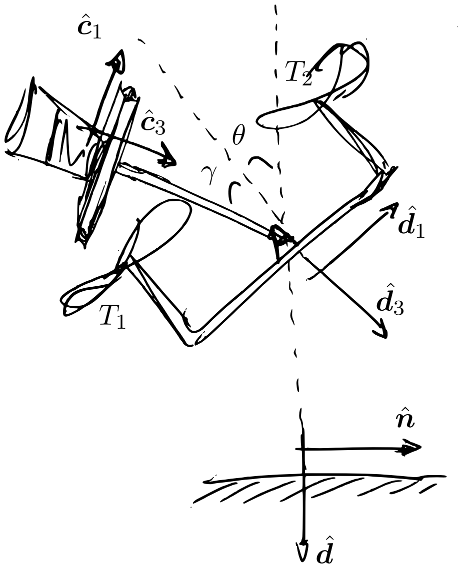

Figure 1: The Flying-Chardonnay reference frames and relevant parameters.

The relevant reference frames and coordinates are given in Fig. 1. The coordinate systems $(\hat{\boldsymbol{n}}, \hat{\boldsymbol{d}})$, $(\hat{\boldsymbol{d}}_1, \hat{\boldsymbol{d}}_3)$ and $(\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_3)$ are fixed to the inertial, drone and cup platform frames, respectively. The propellers thrust are denoted by T_1 and T_2 and numbered according to Fig. 1.

The center of mass of the drone (without the platform) is assumed to be located in the base of the inverted pendulum. The distance of each propeller to the center of mass is denoted l_d while the distance of the cup to the center of mass is l. Their masses are denoted by m_d and m_c, respectively. We assume the mass of the platform is negligible in comparison to the weight of the cup m_c.

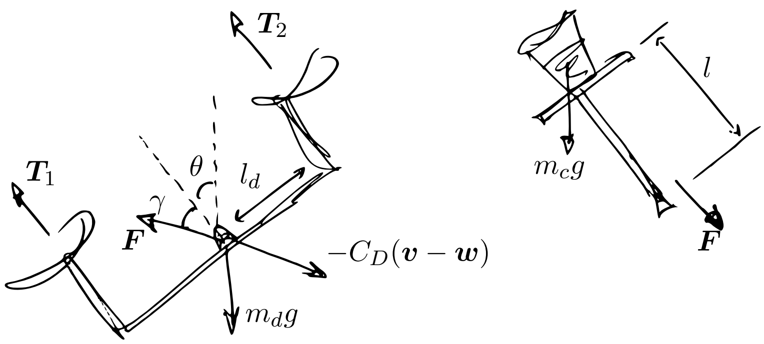

Figure 2: Free body diagrams.

In view of the above assumptions, application of Newton's Second Law to each sub-element (see Fig. 2) yields

$$\left(\sum_{\text{drone}} \boldsymbol{F}\right)_i = m_d {}^i\boldsymbol{a}_i^d \tag{1}$$

$$\left(\sum_{\text{cup}} \boldsymbol{F}\right)_i = m_c {}^i\boldsymbol{a}_i^c \tag{2}$$

where the resultant forces are described w.r.t. the inertial coordinate system $(\hat{\boldsymbol{n}}, \hat{\boldsymbol{e}}, \hat{\boldsymbol{d}})$ by

$$\left(\sum_{\text{drone}} \boldsymbol{F}\right)_i = m_d g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (T_1 + T_2) D_i^d \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - C_D \begin{pmatrix} v_n - w_n \\ 0 \\ v_d - w_d \end{pmatrix} + F D_i^d \begin{pmatrix} -\sin\gamma \\ 0 \\ -\cos\gamma \end{pmatrix} \tag{3}$$

and

$$\left(\sum_{\text{cup}} \boldsymbol{F}\right)_i = m_c g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - F D_i^d \begin{pmatrix} -\sin\gamma \\ 0 \\ -\cos\gamma \end{pmatrix} \tag{4}$$

where

$$D_d^i = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{5}$$

On the other hand, the inertial accelerations described in the inertial coordinates system is given accordingly by

$${}^i\boldsymbol{a}_i^d = \begin{pmatrix} \dot{v}_n \\ 0 \\ \dot{v}_d \end{pmatrix} \tag{6}$$

and

$${}^i\boldsymbol{a}_i^c = {}^i\boldsymbol{a}_i^d + {}^d\boldsymbol{a}_i^c + {}^i\boldsymbol{\alpha}_i^d \times \boldsymbol{p}_i^{c/d} + 2{}^i\boldsymbol{\omega}_i^d \times {}^d\boldsymbol{v}_i^c + {}^i\boldsymbol{\omega}_i^d \times ({}^i\boldsymbol{\omega}_i^d \times \boldsymbol{p}_i^{c/d}) \tag{7}$$

where each of the individual terms above (e.g., centripetal acceleration, tangential acceleration, Coriolis acceleration) are given by

$${}^d\boldsymbol{a}_i^c = D_i^c \begin{pmatrix} -l\ddot{\gamma} \\ 0 \\ l\dot{\gamma}^2 \end{pmatrix} = l \begin{pmatrix} -\ddot{\gamma}\cos(\gamma+\theta) + \dot{\gamma}^2\sin(\gamma+\theta) \\ 0 \\ \ddot{\gamma}\sin(\gamma+\theta) + \dot{\gamma}^2\cos(\gamma+\theta) \end{pmatrix} \tag{8}$$

$${}^i\boldsymbol{\alpha}_i^d = \begin{pmatrix} 0 \\ \ddot{\theta} \\ 0 \end{pmatrix} \tag{9}$$

$$\boldsymbol{p}_i^{c/d} = D_i^c \begin{pmatrix} 0 \\ 0 \\ -l \end{pmatrix} = \begin{pmatrix} -l\sin(\gamma+\theta) \\ 0 \\ -l\cos(\gamma+\theta) \end{pmatrix} \tag{10}$$

$${}^i\boldsymbol{\omega}_i^d = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} \tag{11}$$

$${}^d\boldsymbol{v}_i^c = D_i^c \begin{pmatrix} -l\dot{\gamma} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -l\dot{\gamma}\cos(\gamma+\theta) \\ 0 \\ l\dot{\gamma}\sin(\gamma+\theta) \end{pmatrix} \tag{12}$$

$$ {}^{i}\boldsymbol{\alpha}_i^d \times \boldsymbol{p}_i^{c/d} = \begin{pmatrix} -l\ddot{\theta}\cos(\gamma+\theta) \\ 0 \\ l\ddot{\theta}\sin(\gamma+\theta) \end{pmatrix} \tag{13} $$

$$ 2{}^{i}\boldsymbol{\omega}_i^d \times {}^{d}\boldsymbol{v}_i^c = 2l\dot{\gamma}\dot{\theta} \begin{pmatrix} \sin(\gamma+\theta) \\ 0 \\ \cos(\gamma+\theta) \end{pmatrix} \tag{14} $$

$$ {}^{i}\boldsymbol{\omega}_i^d \times \boldsymbol{p}_i^{c/d} = \dot{\theta} l \begin{pmatrix} -\cos(\gamma+\theta) \\ 0 \\ \sin(\gamma+\theta) \end{pmatrix} \tag{15} $$

$$ {}^{i}\boldsymbol{\omega}_i^d \times ({}^{i}\boldsymbol{\omega}_i^d \times \boldsymbol{p}_i^{c/d}) = \dot{\theta}^2 l \begin{pmatrix} \sin(\gamma+\theta) \\ 0 \\ \cos(\gamma+\theta) \end{pmatrix} \tag{16} $$

As for the Newton's Second Law applied to rotation modes of the drone (we model the cup as a point particle), we have

$$ (T_2 - T_1)l_d = J\ddot{\theta} \tag{17} $$

Summarizing the above equations, we have the following equations of motion for the Flying-Chardonnay system:

$$ m_d\dot{v}_n = -(T_1+T_2)\sin\theta - C_D(v_n - w_n) - F\sin(\gamma+\theta) \tag{18} $$

$$ m_d\dot{v}_d = m_d g - (T_1+T_2)\cos\theta - C_D(v_d - w_d) - F\cos(\gamma+\theta) \tag{19} $$

$$ m_c\Big(\dot{v}_n - l(\ddot{\theta}+\ddot{\gamma})\cos(\gamma+\theta) + l(\dot{\theta}+\dot{\gamma})^2\sin(\gamma+\theta)\Big) = F\sin(\gamma+\theta) \tag{20} $$

$$ m_c\Big(\dot{v}_d + l(\ddot{\theta}+\ddot{\gamma})\sin(\gamma+\theta) + l(\dot{\gamma}+\dot{\theta})^2\cos(\gamma+\theta)\Big) = m_c g + F\cos(\gamma+\theta) \tag{21} $$

and

$$ (T_2 - T_1)l_d = J\ddot{\theta} \tag{22} $$

Exercise 1

1. **(20pts)** Implement a MATLAB function $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})$ of this system, where $\boldsymbol{u} = (T_1, T_2)$, $\boldsymbol{x} = (v_n, v_d, \theta, \dot{\theta}, \gamma, \dot{\gamma})$, and $\boldsymbol{w} = (w_n, w_d)$. I recommend using the following function signature:

```
function x_dot = drone_dynamics(x,u,w,drone)
% DRONE_DYNAMICS
% All values are in S.I. units!!
% x = [ pn; pd; vn; vd; the; thed; gam; gamd ]
% u = [ T1; T2 ]
% w = [ wn; wd ]
% drone = structure containing all drone physical parameters (e.g., mass, length)
end
```

What is the value of $\dot{\boldsymbol{x}}$ for $\boldsymbol{x} = (1\text{m/s}, 0.1\text{m/s}, 10^{\text{o}}, 10^{\text{o}}/\text{s}, 5^{\text{o}}, 5^{\text{o}}/\text{s})$, $\boldsymbol{u} = (4.8N, 5.3N)$ and $\boldsymbol{w} = (2\text{m/s}, -3\text{m/s})$, given the parameters in S.I. units below?

$m_d = 1$	$m_c = 1$	$l = 1$	$l_d = 1$	$J = 1$	$C_D = 0.01$	$g = 10$

In your report, please add the value of $\dot{\boldsymbol{x}}$ and your MATLAB function code.

Solution Hints

Assuming $\alpha = \gamma + \theta$, the equations of motion can be rewritten as

$$\begin{bmatrix} m_d & 0 & 0 & 0 & \sin\alpha \\ 0 & m_d & 0 & 0 & \cos\alpha \\ m_c & 0 & -m_c l\cos\alpha & -m_c l\cos\alpha & -\sin\alpha \\ 0 & m_c & m_c l\sin\alpha & m_c l\sin\alpha & -\cos\alpha \\ 0 & 0 & J & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{v}_n \\ \dot{v}_d \\ \ddot{\theta} \\ \ddot{\gamma} \\ F \end{pmatrix} = \begin{pmatrix} -(T_1+T_2)\sin\theta - C_D(v_n - w_n) \\ m_d g - (T_1+T_2)\cos\theta - C_D(v_d - w_d) \\ -m_c l(\dot{\theta}+\dot{\gamma})^2\sin\alpha \\ m_c g - m_c l(\dot{\theta}+\dot{\gamma})^2\cos\alpha \\ (T_2 - T_1)l_d \end{pmatrix} \tag{23}$$

and therefore,

$$\begin{pmatrix} \dot{v}_n \\ \dot{v}_d \\ \ddot{\theta} \\ \ddot{\gamma} \\ F \end{pmatrix} = \begin{bmatrix} m_d & 0 & 0 & 0 & \sin\alpha \\ 0 & m_d & 0 & 0 & \cos\alpha \\ m_c & 0 & -m_c l\cos\alpha & -m_c l\cos\alpha & -\sin\alpha \\ 0 & m_c & m_c l\sin\alpha & m_c l\sin\alpha & -\cos\alpha \\ 0 & 0 & J & 0 & 0 \end{bmatrix}^{-1} \begin{pmatrix} -(T_1+T_2)\sin\theta - C_D(v_n - w_n) \\ m_d g - (T_1+T_2)\cos\theta - C_D(v_d - w_d) \\ -m_c l(\dot{\theta}+\dot{\gamma})^2\sin\alpha \\ m_c g - m_c l(\dot{\theta}+\dot{\gamma})^2\cos\alpha \\ (T_2 - T_1)l_d \end{pmatrix} \tag{24}$$

The above matrix is always invertible! You can see so by computing the determinant:

$$\det \begin{bmatrix} m_d & 0 & 0 & 0 & \sin\alpha \\ 0 & m_d & 0 & 0 & \cos\alpha \\ m_c & 0 & -m_c l\cos\alpha & -m_c l\cos\alpha & -\sin\alpha \\ 0 & m_c & m_c l\sin\alpha & m_c l\sin\alpha & -\cos\alpha \\ 0 & 0 & J & 0 & 0 \end{bmatrix} = J m_d m_c l(m_d + m_c) \tag{25}$$

Therefore, in state space, the equations of motion can be described as

$$\Pi(\boldsymbol{x}) \begin{pmatrix} \dot{\boldsymbol{x}} \\ F \end{pmatrix} = h(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) \tag{26}$$

with $\det \Pi(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x}$, where

$$\Pi(\boldsymbol{x}) = \begin{bmatrix} m_d & 0 & 0 & 0 & 0 & 0 & \sin(x_3+x_5) \\ 0 & m_d & 0 & 0 & 0 & 0 & \cos(x_3+x_5) \\ m_c & 0 & 0 & -m_c l\cos(x_3+x_5) & 0 & -m_c l\cos(x_3+x_5) & -\sin(x_3+x_5) \\ 0 & m_c & 0 & m_c l\sin(x_3+x_5) & 0 & m_c l\sin(x_3+x_5) & -\cos(x_3+x_5) \\ 0 & 0 & 0 & J & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{27}$$

and

$$h(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) = \begin{pmatrix} -(u_1+u_2)\sin x_3 - C_D(x_1 - w_1) \\ m_d g - (u_1+u_2)\cos x_3 - C_D(x_2 - w_2) \\ -m_c l(x_4+x_6)^2\sin(x_3+x_5) \\ m_c g - m_c l(x_4+x_6)^2\cos(x_3+x_5) \\ (u_2-u_1)l_d \\ x_4 \\ x_6 \end{pmatrix} \tag{28}$$