Multiple-Input Multiple-Output Control

Master of Science in Aerospace Engineering

HW7 - Transmission Zeros, Zero-Pole Cancellation and Sigma Plots

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December 8, 2023

2023/2024 – 3rd Semester

1 Exercise 1

We are given a transfer function $G(s)$ that establishes a relationship between the frequency response of a building $p(s)$ and the wind velocity $w(s)$,

$$G(s) = \begin{bmatrix} \frac{-0.25}{s^2+0.2s+0.25} & \frac{0.25}{s^2+0.2s+0.25} \\ \frac{-1}{s+1} & \frac{-1}{s+1} \end{bmatrix} \tag{1}$$

with $p(s)$ denoting the displacement of the apex of the tower. The vectors $p(s)$ and $w(s)$ are described in the north-east coordinates.

1.1 Wind Frequency for the Highest Building Vibration Amplitude

The wind frequency that leads to the highest amplitude can be estimated by obtaining the σ plot of the given transfer function $G(s)$. This plot is obtained using the MATLAB function $sigma(G)$. The code snippet used is shown below:

```
%% Transfer function
    G1 = tf([-0.25],[1 0.2 0.25]);
    G2 = tf([0.25],[1 0.2 0.25]);
    G3 = tf([-1],[1 1]);
    G4 = tf([-1],[1 1]);
    G = [G1 G2; G3 G4];

%% Obtaining Sigma Plot
[sigma_max, omega_max] = sigma(G);
[sigma_max,i] = max(sigma_max(1,:));
omega_max = omega_max(i)
figure;
sigma(G)
grid on;
```

The σ-plot hence obtained is shown in figure 1. The largest gain value from this plot corresponds to the highest building vibration amplitude. From the plot, we can see that this maximum gain is **11.1 dB** and occurs when frequency $\omega_{max} = 0.479\, rad/s$.

1.2 Wind Direction $\hat{w} \in \mathbb{C}^2$ Corresponding to Highest Vibration Amplitude

Computation of $\hat{w}$ requires frequency domain analysis of the system in the complex domain. For any system, the output $\boldsymbol{y(t)}$ can be expressed as:

$$\boldsymbol{y(t)} = G(j\omega)\boldsymbol{u(t)} \tag{2}$$

In case of MIMO systems, the vectors $\boldsymbol{y(t)}, \boldsymbol{u(t)}$ can be written in terms of complex amplitudes $\hat{y}, \hat{u}$ as:

$$\hat{y}e^{jwt} = F(jw)\hat{u}e^{jwt} \tag{3}$$

Due to the property of sinusoidal signals having the same input and output frequencies for LTI systems, the time-dependant terms in the above equation can be cancelled out on both sides. Eventually, we are left with:

$$\hat{y} = F(jw)\hat{u} \tag{4}$$

To compute the wind direction for the highest building amplitude, we substitute the highest frequency $\omega = \omega_{max} = 0.479\, rad/s$ in equation 4. Following this, we decompose the resulting transfer

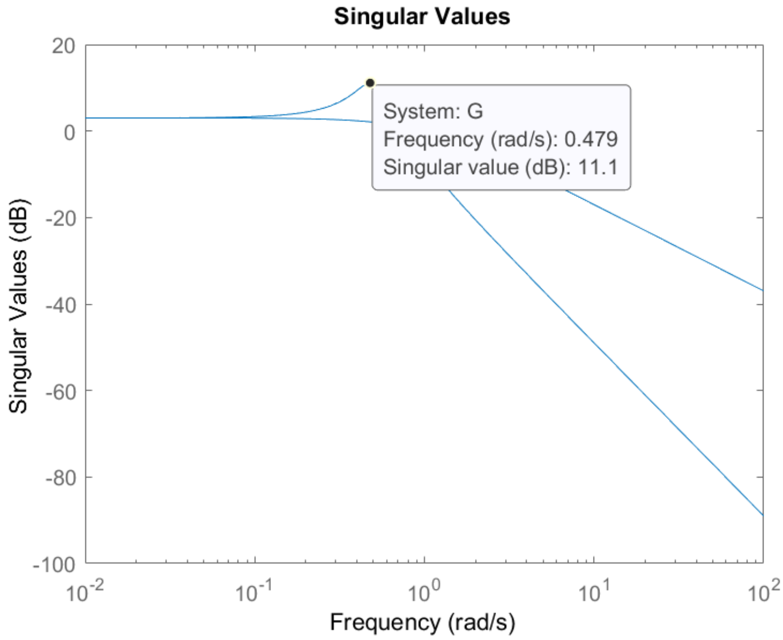

Figure 1: Sigma plot for transfer function $G(s)$

function $(G(j\omega))$ into its corresponding SVD form as:

$$\hat{p} = \underbrace{U\Sigma V^*}_{G(j\omega)} \hat{u} = U \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} V^* \hat{w}$$

Where $\hat{w} = \hat{u}$ the input to the system, and $\hat{p} = \hat{y}$, the system output. This computation is performed in MATLAB, and the code snippet is provided below:

```
Gjw = evalfr(G, omega_max*1j);
[U,S,V] = svd(Gjw);
U
S
V = V'
```

The SVD form was obtained as:

$$G(j\omega) = \underbrace{\begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}}_{V^*} \tag{5}$$

with $e^{j\theta_1} = -0.2041+0.9789j$, $e^{j\theta_2} = 0.9017+0.4324j$, $\sigma_1 = 3.6084$, $\sigma_2 = 1.2752$. $20log_{10}\sigma_1 = 11.1dB$, which agrees with the result that we obtained from the σ-plot in figure 1.

Equation 4 can be written as:

$$\hat{p} = \underbrace{\begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}}_{V^*} \hat{w} \tag{6}$$

which, when simplified, becomes

$$\hat{p} = \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \end{bmatrix} = \begin{bmatrix} 0.7071\, \sigma_1\, e^{\theta_1 j}\, (\hat{u}_1 - \hat{u}_2) \\ 0.7071\, \sigma_2\, e^{\theta_2 j}\, (\hat{w}_1 + \hat{w}_2) \end{bmatrix} \tag{7}$$

Since we are only interested in obtaining for the case of the highest amplitude, we only want omit the influence of the smaller of the two gains (σ_2), which is present only in the second row of equation 7. Thus, we want the second row to effectively go to zero, which occurs when

$$\hat{w_1} = -\hat{w_2}$$

Thus, the wind direction $\hat{w} = c \cdot [1 \ -1]^T$, where c is some constant. Thus, the unit wind vector that results in maximum building amplitude is:

$$\hat{\boldsymbol{w}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8}$$

1.3 Resulting Building Vibration $\hat{y} \in \mathbb{C}^2$

Substituting the computed $\hat{w} = \frac{1}{\sqrt{2}}[1 \ -1]^T$ in equation 4 yields the desired result $\hat{y} = \hat{p}$.

$$\hat{p} = \begin{bmatrix} -0.736 + 3.5322j \\ 0 \end{bmatrix} \tag{9}$$

Interestingly, the magnitude

$$\frac{|\hat{y}|}{|\hat{u}|} = 3.6081 = \sigma_1$$

which is the computed maximum gain/amplitude of the system.

1.4 Minimum h Required to Record Full Building Apex Trajectory

Given $w(t) = [2\,cos(10\pi t)\ 3\,sin(10\pi t)]^T$, we want to compute the minimum h that can enable the camera to fully capture the displacement of the building apex.

For this, the first step is to obtain the wind direction in terms of the complex amplitude $\hat{w}$. $w(t)$ can be re-written as:

$$w(t) = \begin{bmatrix} 2cos(10\pi t) \\ 3sin(10\pi t) \end{bmatrix} = \begin{bmatrix} 2sin(10\pi t + \frac{\pi}{2}) \\ 3sin(10\pi t) \end{bmatrix}$$

Since we are only interested in the complex amplitude, $\hat{w}$ can be written as;

$$\hat{w} = \begin{bmatrix} 2\,e^{j\frac{\pi}{2}} \\ 3 \end{bmatrix}$$

With this, the displacement of the building apex $\hat{p}$ for the corresponding input wind can be computed easily using relation 4. The MATLAB code is as follows:

```
    w = [2*exp(1j*pi/2); 3];
    omega_2 = 10*pi;
    Gjw_2 = evalfr(G, omega_2*1j);
    y_2 = Gjw_2*w;

%obtaining magnitude of displacement
    vib_mag = norm(y_2)
    h_min = vib_mag*sqrt(3)
```

The resulting $\hat{p}$ is found to be

$$\hat{p} = \begin{bmatrix} -0.0008 + 0.0005j \\ -0.0666 + 0.0934j \end{bmatrix} m$$

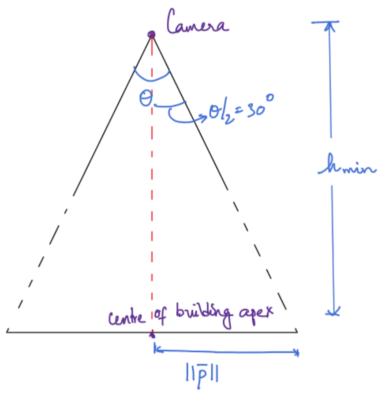

Figure 2: Computation of h

The magnitude of the apex displacement is:

$$|\hat{p}| = 0.1147m$$

Finally, from figure 2, h can found using the simple trigonometric relation:

$$h = \frac{|\hat{p}|}{tan(\frac{\theta}{2})} = 0.1987\,m$$