# Multiple-Input Multiple-Output Control

## Master of Science in Aerospace Engineering

---

**HW5 - Controllable and Unobservable Spaces**

---

**Authors:**

Rajashree Srikanth rajashree.srikanth@student.isae-supaero.fr
João Fernandes Barbara joao.fernandes-barbara@student.isae-supaero.fr

**Tutor:**
Leandro Lustosa

December 8, 2023

**2023/2024 – 3rd Semester**

# 1 Exercise 1

The purpose of this exercise is to analyse and comment on the stability and related parameters of the Flying Chardonnay during hovering. From HW3,the linearised state space model was represented as:

$$\boldsymbol{\Delta\dot{x}} = A\boldsymbol{\Delta x} + B\boldsymbol{\Delta x} + E\boldsymbol{\Delta w}$$

and the value of the matrices during hover was computed as:

$$A = \begin{bmatrix} -0.010 & 0 & -10 & 0 & 10 & 0 \\ 0 & -0.005 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.01 & 0 & 0 & 0 & 20 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ -0.50 & -0.50 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.0050 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.01 & 0 \end{bmatrix}$$

## 1.1 Stability of the System

The stability of the system depends on the state matrix $A$. The poles of the system ($p$) can be computed using the $eig(A)$ function on MATLAB.

$$p = \begin{bmatrix} -0.005 \\ -4.4746 \\ 4.4696 \\ 5.89025e-08 \\ -5.8902e-08 \\ -0.005 \end{bmatrix}$$

The system has 3 stable poles ($\lambda_1 = \lambda_6 = -0.005, \lambda_2 = -4.4746$). However, since we have a positive pole of significant magnitude $\lambda_3 = (4.4696)$, we can conclude that the system is unstable.

## 1.2 Controllability and Stabilisabilty

The controllability of the system can be determined by computing the rank of the controllability matrix, which can be obtained in MATLAB using the function *ctrb(A,B)*. The rank is found to be 6. Since the controllability matrix is a rank-full matrix, and is equal to the size of the state matrix $A = dim_{max}(A) = 6$, we can conclude that the system is fully controllable.

A system is stabilisable if and only if its uncontrollable states are stable. Since the system is fully controllable, it is stabilisable as well.

### 1.3 Actuation of the System

The rank of $B$ matrix $= 2$, but the rank of $A = 6$. Hence, the system is under-actuated.

## 2 Exercise 2

We are given the following linear system:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{1}$$

### 2.1 Checking if (0,1) is on the Controllable Subspace

The controllability matrix $K$ can be computed as:

$$K = [B\ |AB] \tag{2}$$

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The matrix $K$ was thus computed to be:

$$K = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

The rank of the controllability matrix, $rank(K) = 1$, but the rank of matrix $A$ $(rank(A)) = 2$. Since $rank(K) < rank(A)$, the system is not fully controllable.

To check whether $x = (0, 1)$ is within the controllable subspace, we use the concept of span of the controllability matrix.

$span\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right)$ can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 \\ c_1 - c_2 \end{bmatrix} \tag{3}$$

From equation 3, we can see that

$$x_1 = x_2 \tag{4}$$

This straight line equation indicates the set of points on the x-y plane that can be controlled by the system. On substituting the given $x = [x_1\ x_2] = [0\ 1]$ in equation 4, we can see that the equation is not satisfied. Thus we can conclude that $x = [x_1\ x_2] = [0\ 1]$ does not lie in the controllable subspace.

### 2.2 The smallest achievable distance x(t) can have with respect to (0, 1) from origin

The figure 1 depicts the trajectory of the controllable subspace as derived in the previous subsection. We need to compute the point $\boldsymbol{x(t)}$ (along the blue line) that is closest to the point (0,1). The shortest distance between (0,1) and the line can be found using simple straight-line relation, and is computed to be $h = \frac{1}{\sqrt{2}}$.

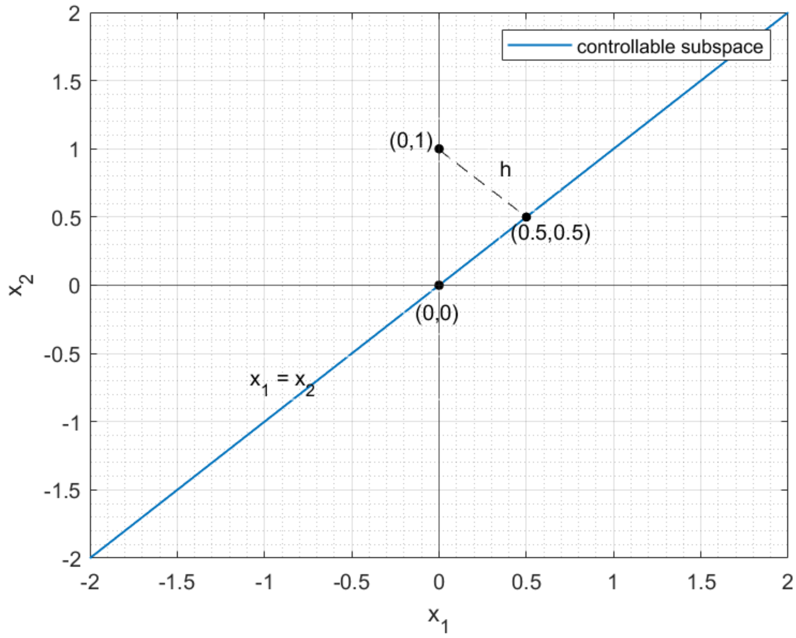


**Figure 1:** Trajectory of Controllable Subspace

# 3 Exercise 3

We are given a linearized dynamic model as follows:

$$\frac{d}{dt}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} = \begin{bmatrix}10 & -2\\ 2 & 5\end{bmatrix}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}0 & 3 & 2\\ 1 & 5 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} \tag{5}$$

## 3.1 Part 1

### 3.1.1 Controllability

The rank of the controllability matrix is found to be $2 = \text{rank }(A)$. Hence, the system is fully controllable.

### 3.1.2 Actuation

The rank of the matrix $B = 2$. However, we have more than 2 inputs, hence, we can independently control the system with more than 2 inputs. The additional input could also be used to control the trajectory. Thus, the system is over-actuated.

## 3.2 Part 2 - Jamming of Actuators

### 3.2.1 New Controllable Space

Due to jamming, the linearized dynamic system becomes

$$\frac{d}{dt}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} = \begin{bmatrix}10 & -2\\ 2 & 5\end{bmatrix}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}0 & 3 & 2\\ 1 & 5 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\\ u_3\end{bmatrix} \tag{6}$$

$$\frac{d}{dt}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} = \begin{bmatrix}10 & -2\\ 2 & 5\end{bmatrix}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} + 0\begin{bmatrix}0\\ 1\end{bmatrix} + 1\begin{bmatrix}3\\ 5\end{bmatrix} + u_3\begin{bmatrix}2\\ 1\end{bmatrix}$$

Thus, the new dynamics system is represented as:

$$\frac{d}{dt}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} = \begin{bmatrix}10 & -2\\ 2 & 5\end{bmatrix}\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} + u_3\begin{bmatrix}2\\ 1\end{bmatrix} + \begin{bmatrix}3\\ 5\end{bmatrix}$$

The new controllable subspace is along the vector $[2\ 1]^T$. The unit vector direction along which the new system is controllable is $\frac{1}{\sqrt{5}}[2\ 1]^T$. This is shown in figure 2.

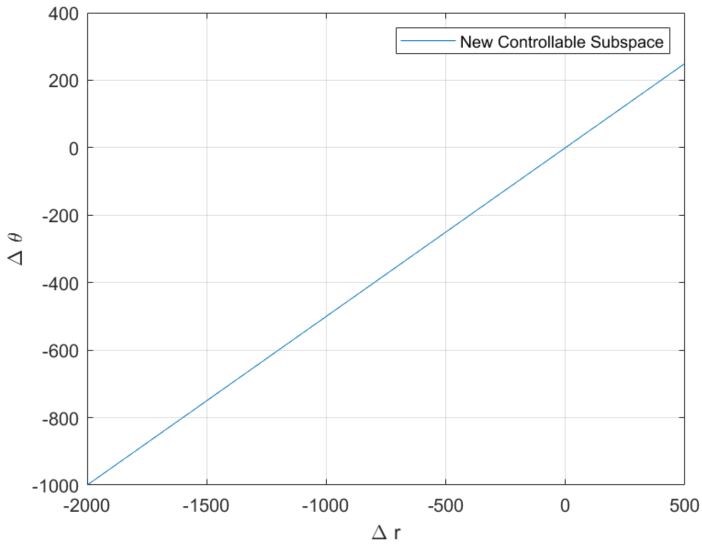


**Figure 2:** New Controllable Subspace