Nonlinear Control
Master in Aerospace Engineering 2023-2024
Author: Wonhee LEE, Rajashree SRIKANTH

Nonlinear Dynamic Inversion (NDI)

1 Introduction

The goal of this BE is to control the motion of a helicopter along the vertical plane using non-linear control techniques, namely an NDI controller. Since the dynamic model is non-linear in behaviour, controller design for such a system is not straightforward. On the other hand, well-established techniques exist exist if the model were linear.

The preliminary step towards controller design for this system is to find a method that effectively linearizes the input-output behaviour, and incorporate a controller that can work on this newly-linearized system. This is performed using the *Non-linear Dynamic Inversion (NDI)* technique. Following this, a PD controller for accelerations γ_x and γ_z is developed. Subsequently, P and PI controllers are added in cascade for the control of velocities (v_x and v_z) and positions (x and z) respectively.

2 Modelling

The lon-linear dynamic model describing the motion of helicopter along the $x - z$ plane is as follows:

Kinematic	**Forces equation**	**Torques equation**
$\dot{x} = v_x$	$m\dot{v_x} = Lsin\theta$	$\dot{\theta} = q$
$\dot{z} = v_z$	$m\dot{v_z} = Lcos\theta - mg$	$J\dot{q} = -fq + T$

Where, L, T are thrust force and torque, and are the taken as inputs. This system is modelled in Simulink, and the open-loop dynamics of the system is represented in the plots below (see Figure 2). It can be observed that the vertical acceleration diverges over time in response to a step input of thrust and torque loads, thus demonstrating the requirement of a controller to ensure convergence.

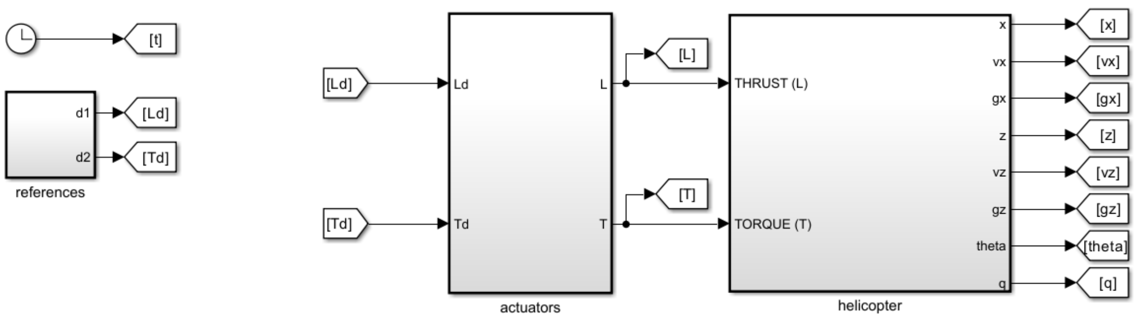


Figure 1. Model of the open-loop system

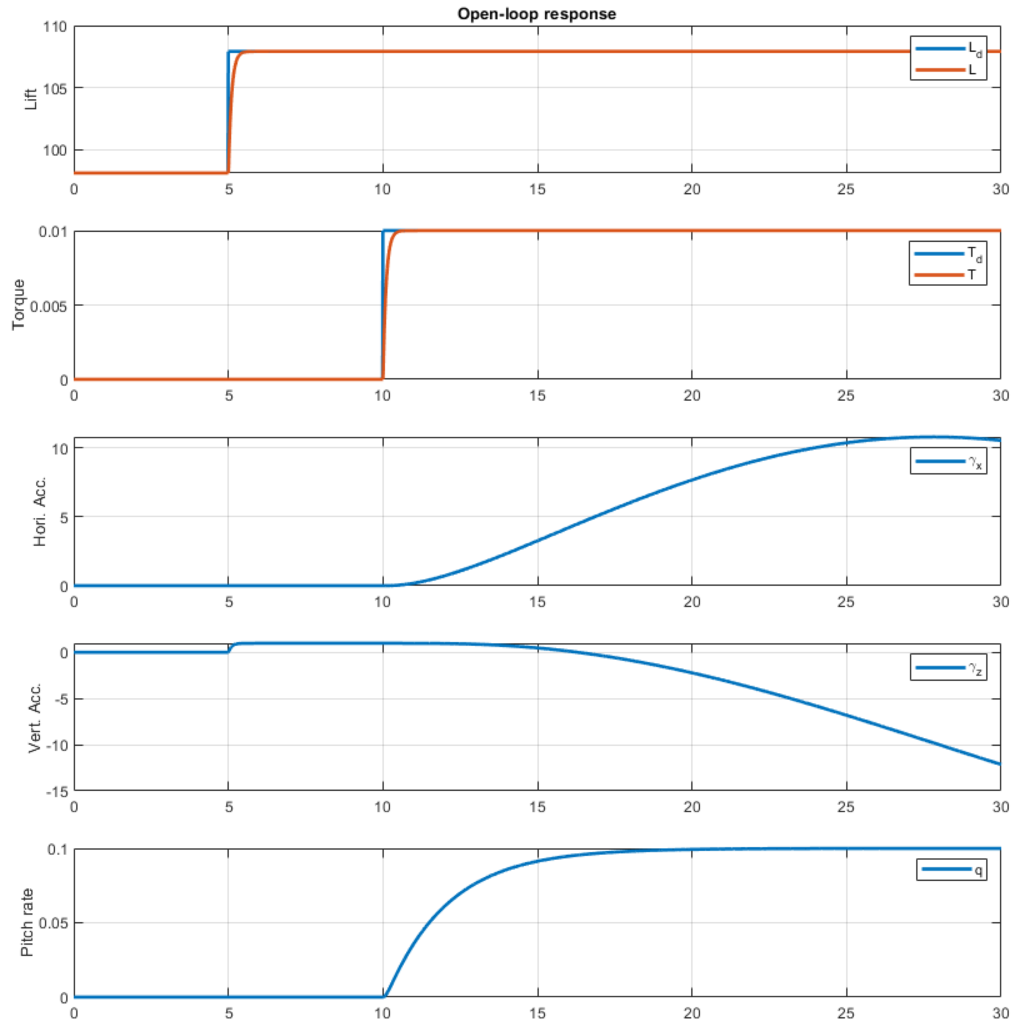


Figure 2. Simulated response of the open-loop system

3 Feedback Linearization

In this section, the input-output behaviour is linearized using Non-linear Dynamic Inversion. The general representation of model to be used for NDI is:

$$\mathcal{Y} = \Delta_0 + \Delta(x)u \tag{1}$$

and the associated NDI control law can be expressed of the form:

$$u = -\Delta^{-1}(\Delta_0 + v) \tag{2}$$

where v is the controlled input. For controlling the accelerations γ_x, γ_z, this is can be written as:

$$\begin{aligned} \begin{bmatrix} \gamma_x \\ \gamma_z \end{bmatrix} &= \begin{bmatrix} \frac{L}{m} sin\theta \\ \frac{L}{m} cos\theta - g \end{bmatrix} \\ \mathcal{Y} = \begin{bmatrix} \gamma_x \\ \gamma_z \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 \\ -g \end{bmatrix}}_{\Delta_0} + \underbrace{\begin{bmatrix} \frac{1}{m} sin\theta & 0 \\ \frac{1}{m} cos\theta & 0 \end{bmatrix}}_{\Delta(x)} \underbrace{\begin{pmatrix} L \\ T \end{pmatrix}}_{u} \end{aligned} \tag{3}$$

Equation 3 already contains the input coefficient L, thus, has a characteristic order $\rho_k = 0$. However, Δ_0 is non-invertible, hence, it is required to perform state-augmentation to overcome this issue. This is represented in a Simulink block called *state augmentation*. The equation 3 is differentiated twice, in order to obtain both input terms T and L in the output model equation. Eventually, this yields:

$$\mathcal{Y} = \begin{bmatrix} \ddot{\gamma}_x \\ \ddot{\gamma}_z \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{L}{m}\frac{f}{J}cos\theta q - \frac{L}{m}sin\theta q^2 + 2\frac{\dot{L}}{m}cos\theta q \\ \frac{L}{m}\frac{f}{J}cos\theta - \frac{L}{m}cos\theta q^2 - 2\frac{\dot{L}}{m}sin\theta q \end{bmatrix}}_{\Delta_0} + \underbrace{\begin{bmatrix} \frac{1}{m}sin\theta & \frac{L}{Jm}cos\theta \\ \frac{1}{m}cos\theta & -\frac{L}{Jm}sin\theta \end{bmatrix}}_{\Delta(x)} \begin{pmatrix} \ddot{L} \\ T \end{pmatrix} \tag{4}$$

The new matrix Δ_0 obtained through two state augmentations is non-singular, hence, invertible. The NDI control law can now be applied to the system represented by the Equation 4. This control law, according to Equation 2 is:

$$u = \begin{bmatrix} v_L \\ v_T \end{bmatrix} = \Delta^{-1}\Delta_0 + \Delta^{-1}\begin{pmatrix} \omega_x \\ \omega_z \end{pmatrix} \tag{5}$$

This is implemented as a MATLAB function as shown below:

```
function mu = fcn(m, f, J, omega, theta, q, L, L_dot)

delta_inv_delta_zero  = [-L*q^2*cos(2*theta)
                          q*(2*J*L_dot/L - f + J*q*sin(2*theta))];

delta_inv = m*[       sin(theta),          cos(theta)
               (J/L)*cos(theta),  -(J/L)*sin(theta)];

mu = -delta_inv_delta_zero + delta_inv*omega;
end
```

The code snippet provides a simplified expression, which is implemented in a block in Simulink as *feedback linearization*. Figure 3 depicts this and *state augmentation* blocks to form the NDI system.

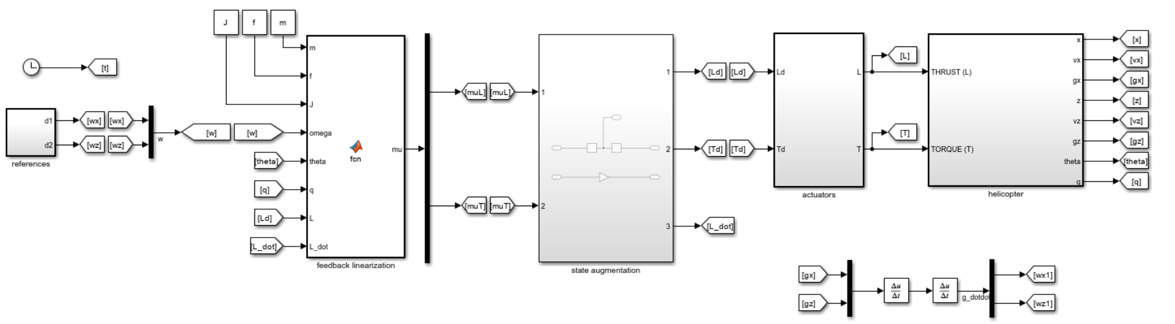


Figure 3. Model of the feedback linearized system

A simulation of the new system is run, and the time response curves obtained are as shown in Figure 4. For a non-linear system represented as S, NDI effectively obtains an expression S^{-1} such that $\mathcal{Y} = \mathcal{Y}^*$. Thus, it is expected that $\ddot{\gamma}_x = \ddot{\gamma}_x{}^*$ and $\ddot{\gamma}_z = \ddot{\gamma}_z{}^*$, which can be observed in the first two plots, proving the correctness of the NDI model thus formulated, and hence, establishes a linear system between overall system input and output that works on state feedback.

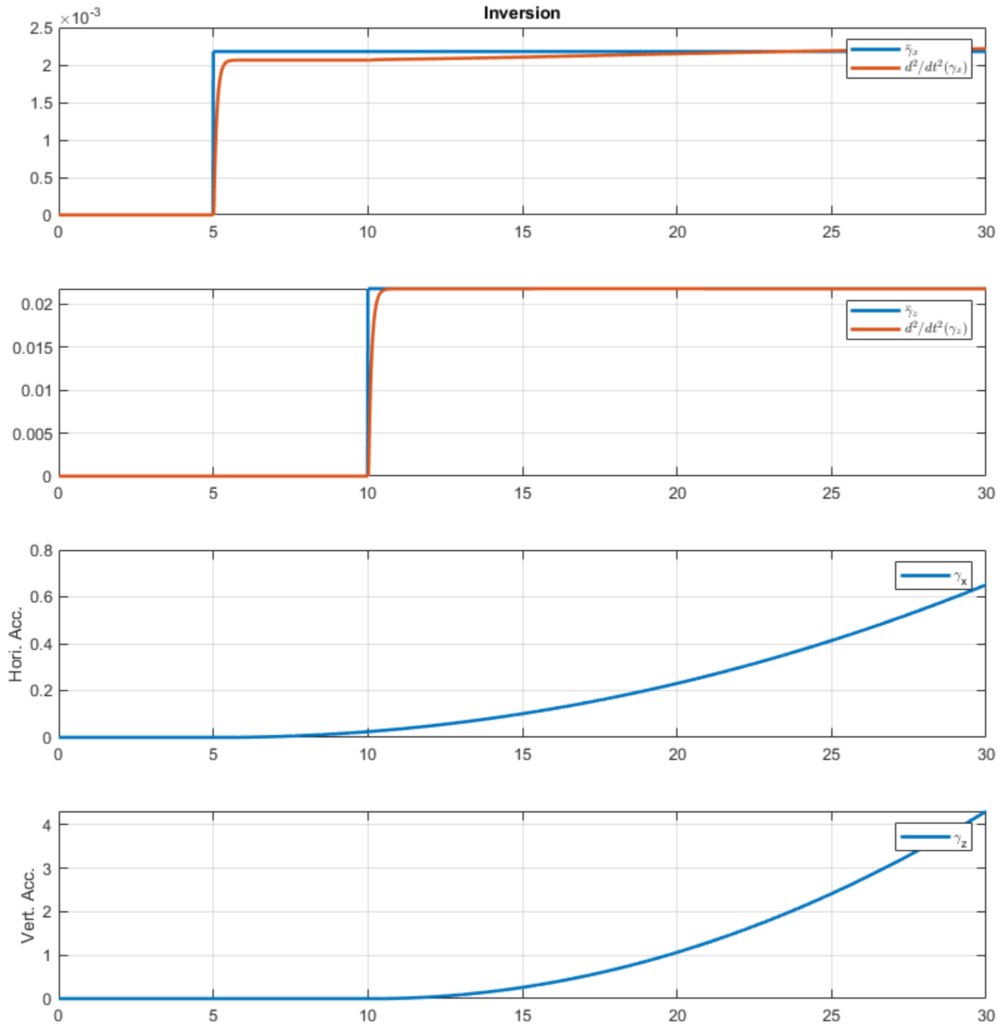


Figure 4. Simulated response of the feedback linearized system

4 Acceleration Control

The design requirements for the acceleration controller is given in Table I.

TABLE I. Acceleration controller design requirements

T_{rep} [s]	ζ [-]
1	**0.7**

Consider the second order acceleration error dynamics,

$$\ddot{e}_\gamma + 2\zeta\omega\dot{e}_\gamma + \omega^2 e_\gamma = 0 \tag{6}$$

$$\ddot{\gamma} = \omega^2 e_\gamma + 2\zeta\omega\dot{e}_\gamma + \ddot{\gamma}^* \tag{7}$$

where $e_\gamma = \gamma^* - \gamma$, and since $\ddot{\gamma}^* = 0$, Equation 7 is the control law that achieves the convergence $\gamma \to \gamma^*$ at the steady state.

ω for the second order system is approximated by Equation 8.

$$\omega \approx \frac{3}{\zeta T_{rep}} \approx 4.29 \tag{8}$$

Referring to Equation 7 and Equation 8, the PD controller gains are calculated and given in Table II.

TABLE II. Acceleration controller gains

Gain	Expression	Value
$k_{p,\,\gamma_x}$	ω^2	18.37
$k_{d,\,\gamma_x}$	$2\zeta\omega$	6.00
$k_{p,\,\gamma_z}$	ω^2	18.37
$k_{d,\,\gamma_z}$	$2\zeta\omega$	6.00

The resulting model of the PD acceleration controller is shown in Figure 5.

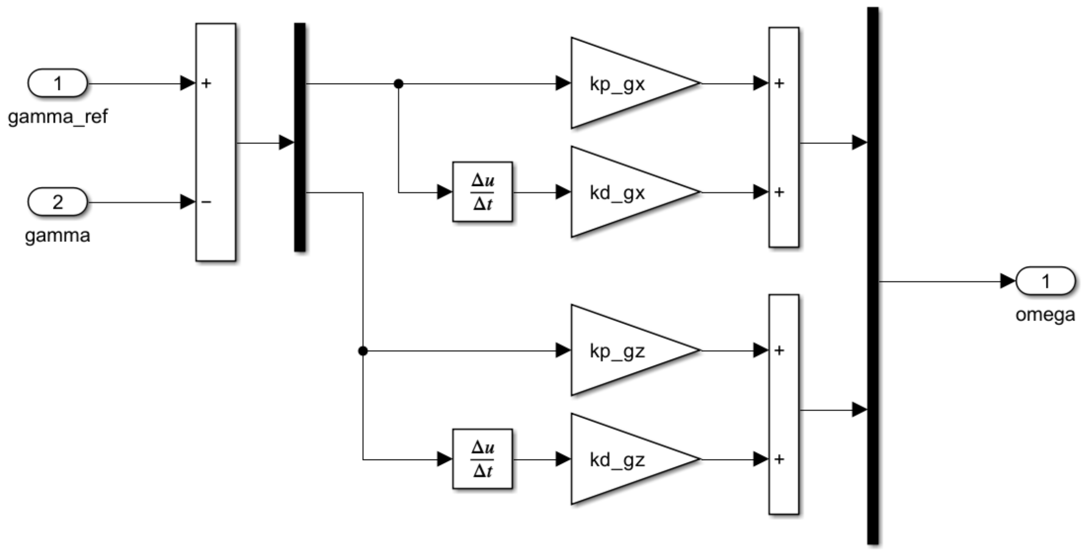


Figure 5. Model of the PD acceleration controller

The resulting model of the feedback linearized system with the PD acceleration controller is shown in Figure 6, and its simulated response is shown in Figure 7.

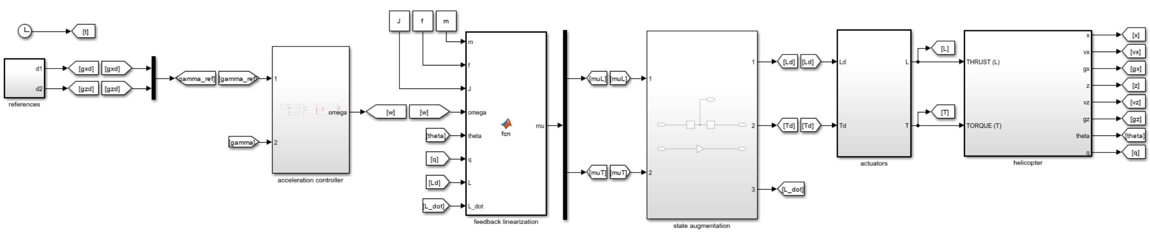


Figure 6. Model of the feedback linearized system with the PD acceleration controller

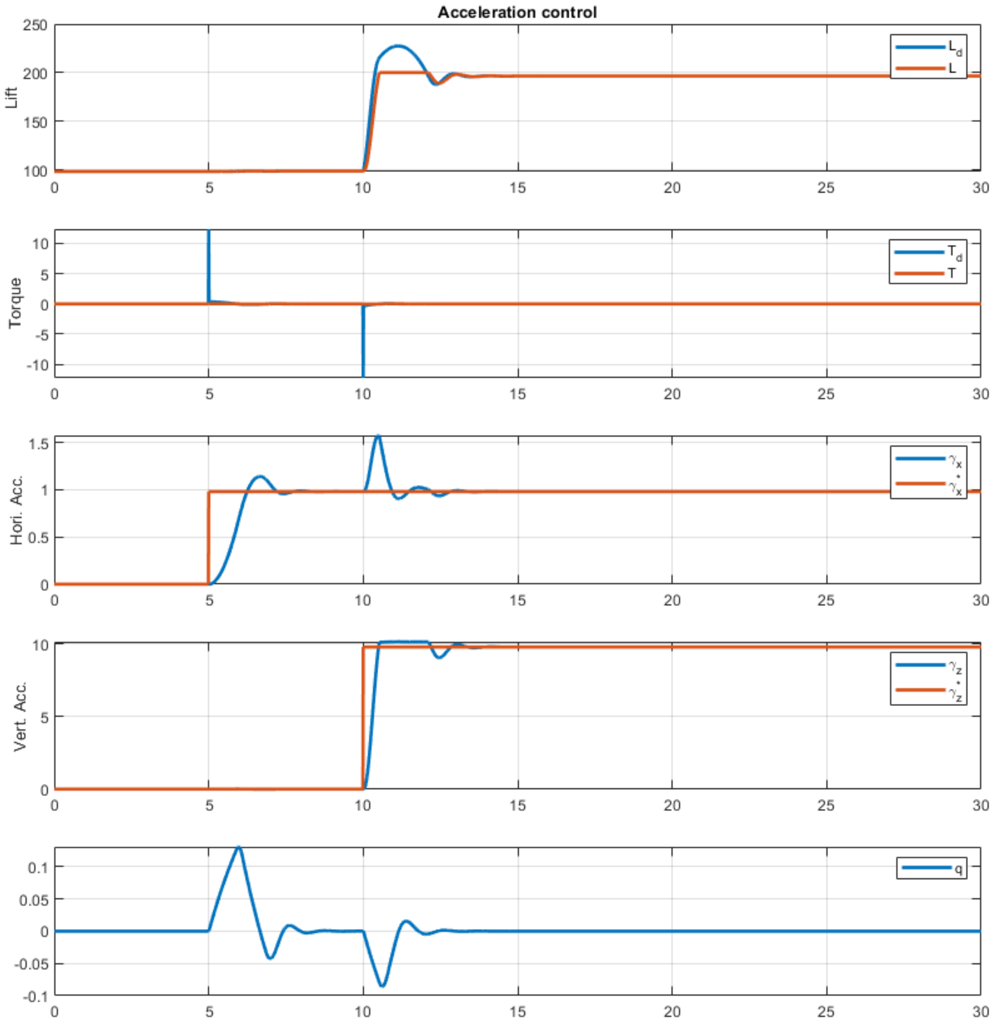


Figure 7. Simulated response of the feedback linearized system with the PD acceleration controller

5 Velocity Control

The design requirements for the acceleration controller is given in Table I.

TABLE III. Velocity controller design requirements

T_{rep} [s]
3

Consider the first order P controller shown in Figure 8.

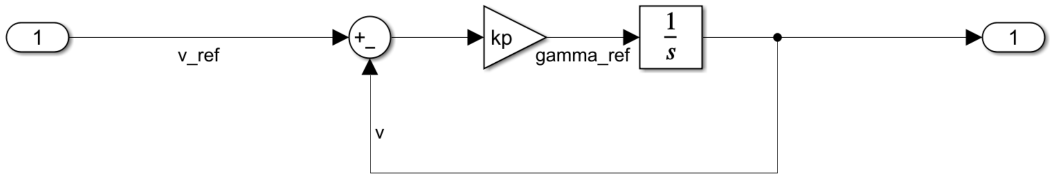


Figure 8. Model of a P controller

Its dynamics is described by

$$\frac{k_p}{s + k_p}$$

hence,

$$k_p = \frac{1}{\tau} \tag{9}$$

τ for the second order system is approximated by Equation 10.

$$\tau \approx \frac{T_{rep}}{3} \approx 1 \tag{10}$$

Referring to Equation 9 and Equation 10, the P controller gain is calculated and given in Table IV.

TABLE IV. Velocity controller gains

Gain	Expression	Value
$k_{p,\, v_x}$	$\frac{3}{T_{rep}}$	1
$k_{p,\, v_z}$	$\frac{3}{T_{rep}}$	1

The resulting model of the P velocity controller is shown in Figure 9.

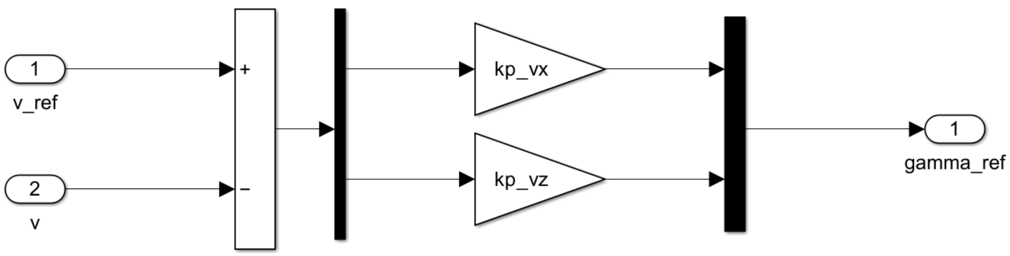


Figure 9. Model of the P velocity controller

The resulting model of the feedback linearized system with the cascaded PD acceleration and P velocity controllers is shown in Figure 10, and its simulated response is shown in Figure 11.

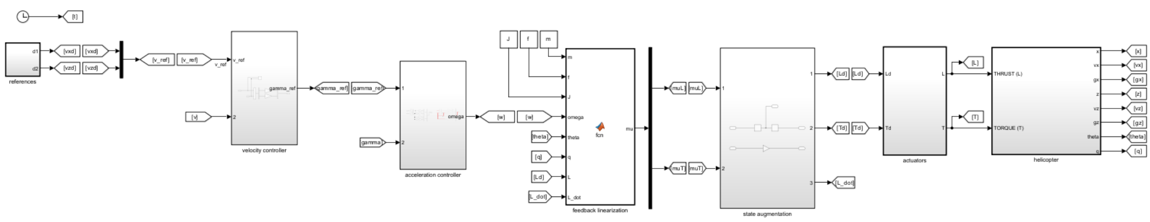


Figure 10. Model of the feedback linearized system with the cascaded PD acceleration and P velocity controllers

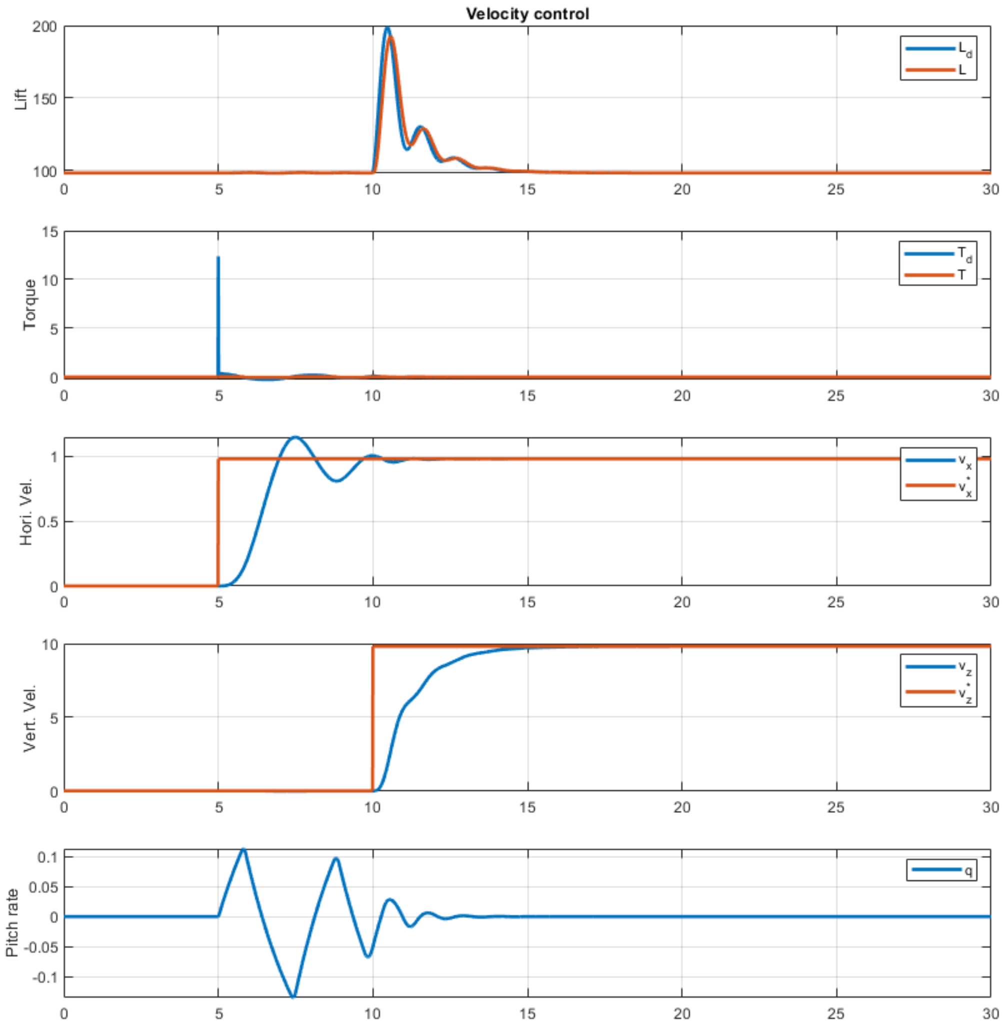


Figure 11. Simulated response of the feedback linearized system with the cascaded PD acceleration and P velocity controllers

6 Position Control

A PI controller is used to regulate the position control. A proportional controller is generally sufficient for achieving the required response time of *10 s* under the assumption that no wind is present . The structure of the P controller developed is similar to that for the velocity control explained in the previous section. An additional I controller with a marginal value of gain K_I is incorporated to improve controller performance in case of disturbances due to wind, if any. The controller gains used for the PI position control are tabulated below in Table V. The P gains are computed from the relation $K_p = \frac{3}{T_{rep}}$. The I gains are chosen to be small enough so to not be of effect unless in the presence of wind. The time response curves on action of this controller are depicted in Figure 14. As expected, the horizontal and vertical positions settle after about 10 seconds after the application of a step input, as specified in the requirements.

Gain	**Value**
K_{Px}	0.3
K_{Pz}	0.3
K_{Ix}	0.00001
K_{Iz}	0.00001

TABLE V. PI controller gains

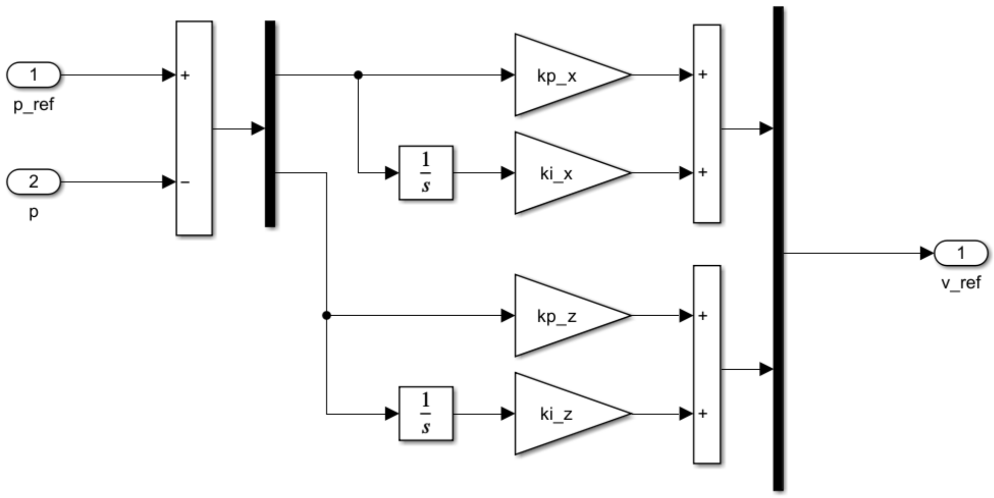


Figure 12. Model of the PI position controller

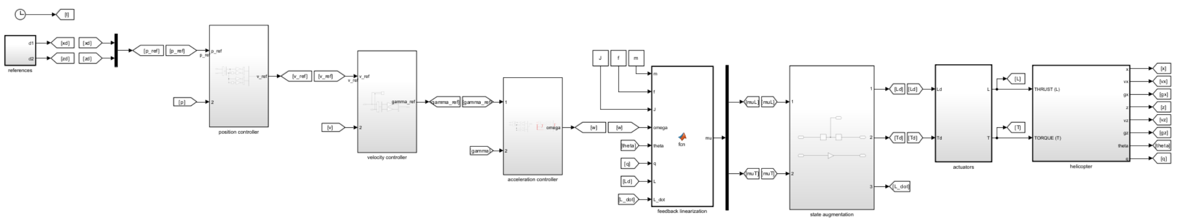


Figure 13. Model of the feedback linearized system with the cascaded PD acceleration, P velocity, and PI position controllers

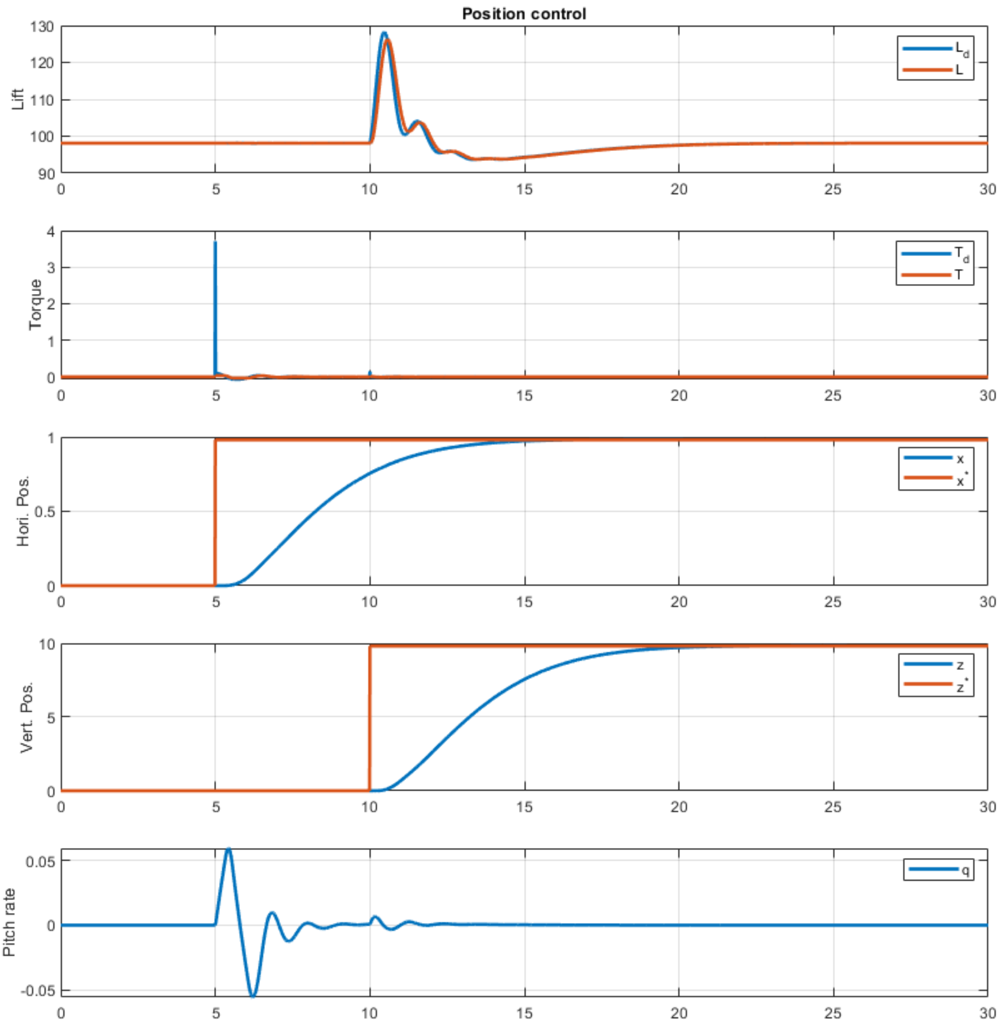
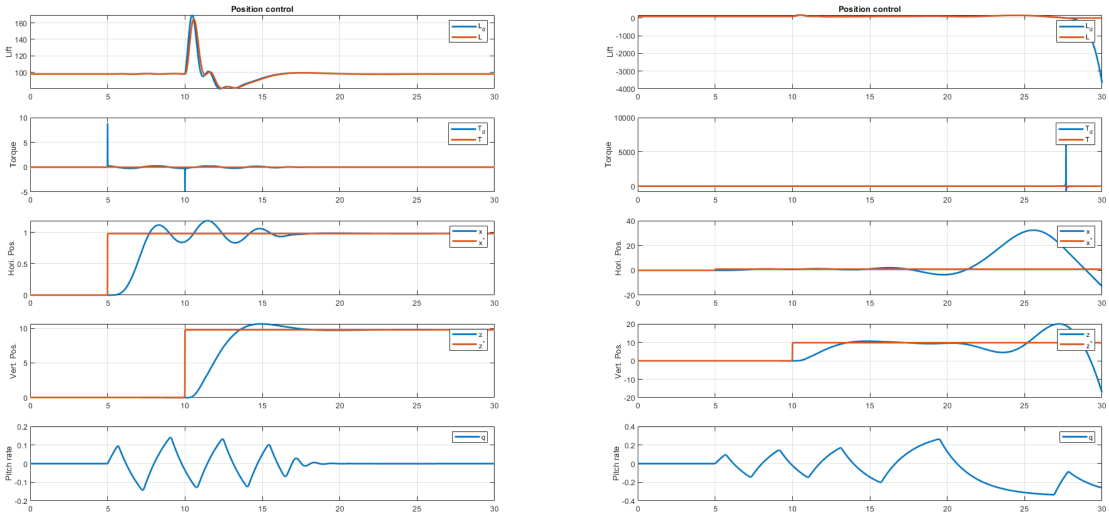


Figure 14. Simulated response of the feedback linearized system with the cascaded PD acceleration, P velocity, and PI position controllers

7 Comments

7.1 Response Time Requirements Adjustment

As expected, from Figure 15, as the response time decreases, oscillations begin to appear (at $T_{rep} = 4.2s$), and eventually, the system response diverges and becomes unstable (at $T_{rep} = 4.15s$). This could be attributed to the fact that the response times of the outer loop controllers become faster than that of the inner loop, leading to instability. For a cascaded controller, the outer loop should have a slower dynamics compared to its inner loop.



(a) (b)

Figure 15. Simulated response of the feedback linearized system with the cascaded PD acceleration, P velocity, and PI position controllers for $T_{rep} = 4.2s$ (a) and for $T_{rep} = 4.15s$ (b)

7.2 Methods to Decrease Response Time

The response time requirements of cascaded loops should be adjusted carefully considering their frequencies of dynamics. It can be attempted to decrease all of the response times simultaneously, but this method is likely to sacrifice the robustness of the system.

8 Conclusion

A cascaded position, velocity and acceleration linear controller was developed for a non-linear system, by performing NDI technique to linearize the input-output system. This linearization allowed for employing well-established and straightforward linear control techniques, and helped achieve desired system performance. Thus, this exercise demonstrates how powerful the NDI technique can be in greatly reducing the complexity of an otherwise tedious control design process for a non-linear system, and allows for applicability over a large input domain.