

## Optimal rendez-vous trajectory

Let us consider a space station  $S$  orbiting around the earth (orbit center:  $O$ ) and a servicing vehicle  $M$  in the orbital plane. The objective is to dock the (active) chaser  $M$  on the (passive) target  $S$ .

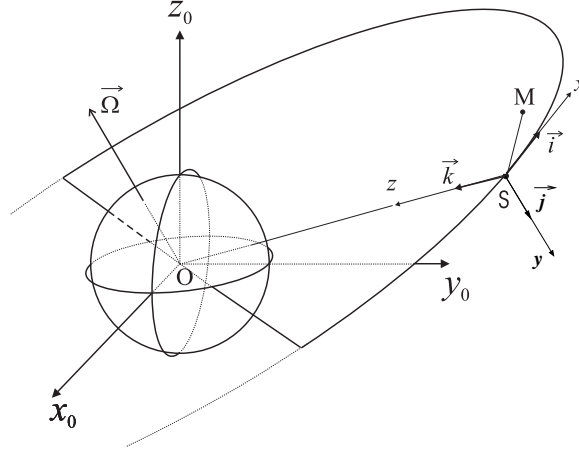


Figure 1: Frame definition

The differential equations governing the relative dynamics of  $M$  w.r.t.  $S$  in the orbital plane (simplified HILL-CLOHESSEY-WHITSHIRE equations) projected in the local orbital frame  $\{S, \vec{i}, \vec{j}, \vec{k}\}$  reads:

$$\begin{aligned}\ddot{x} - 2\omega\dot{z} &= \varphi_x \\ \ddot{z} + 2\omega\dot{x} - 3\omega^2 z &= \varphi_z\end{aligned}$$

where  $x$  and  $z$  are the coordinates of  $M$  in the local orbital plane.

The 2 control inputs are the specific propulsion forces  $\varphi_x$  and  $\varphi_z$  (resp. tangential and radial) of  $M$ .

1) Give a state space representation of this system:

$$\dot{X} = \mathbf{A}X + \mathbf{B}u$$

where  $X^T = (z, x, \dot{z}, \dot{x})$  and  $u^T = (\varphi_z, \varphi_x)$ .

2) Is the system controllable using:

- a) the radial thrust  $\varphi_z$  only ?
- b) the tangential thrust  $\varphi_x$  only ?

In the following, it is assumed that only one control input can be used. Which one ( $\varphi_x$  or  $\varphi_z$ ) ?

3) At the initial time  $t = 0$ , the space vehicle  $M$  has an initial state  $X(0) = X_0$ , assumed to be measured. The objective is to dock  $M$  on  $S$  at a given time  $t = T$  while minimizing the fuel consumption of  $M$ . The problem is thus to find the state optimal trajectories  $\hat{X}(t)$  and the optimal control  $\hat{u}(t)$ <sup>1</sup> such that  $X(T) = \mathbf{0}$  while minimizing the performance index:

$$\mathcal{C} = \int_0^T \frac{1}{2} u^2 dt$$

3.1) In a first step, one wishes to develop a generic MATLAB function solving the two point boundary-value problem in the general case of a linear system, a quadratic index and a finite time horizon. Fill the code of the following `twopbv.m` function:

<sup>1</sup>In this case  $u$  is a dimension 1 signal since only one control input is used.

```

function [K_t,P_t,phi_t]=twopbvp(T,t,a,b,q,r)
%TWOPBVP Two point boundary-value Problem (LQ problem with finite horizon T
% and null final state).
%
% * K_t=twopbvp(T,t,A,B,Q,R) compute, at current time t (in [0, T]),
% the optimal gain K_t for the LQ problem:
%
% - System:
%
%  $\dot{x} = Ax + Bu$  with negative state feedback:  $u(t)=-K_t(t)*x(t)$ ,
%
% - Performance index
%
%  $J = 0.5 \int_0^T \{x'Qx + u'Ru\} dt$ 
%
% - Hard constraint:
%
%  $x(T)=0$ 
%
% * [K_t,P_t]=twopbvp(T,t,A,B,Q,R) computes also P_t: the (semi-
% definite positive) solution at current time t of the associated
% Riccati equation:
%
%  $\dot{P}_t = -P_t A - A' P_t + P_t B R^{-1} B' P_t - Q$ 
%
% * [K_t,P_t,phi_t]=twopbvp(T,t,...) computes also the transistion
% matrix phi_t at current time t on the optimal trajectory,
% such that:
%
%  $x(t)=\phi_t x(0)$ 
%
error(nargchk(6,6,nargin));
error(abcchk(a,b));

if t>T, disp('Input argument problem: t>T !!!');K_t=[];P_t=[];phi_t=[];return,end

[m,n] = size(a);[mb,nb] = size(b);[mq,nq] = size(q);[mr,nr] = size(r);
if (m ~= mq) | (n ~= nq)
    error('A and Q must be the same size');
end
if (mr ~= nr) | (nb ~= mr)
    error('B and R must be consistent');
end

H=[a -b*inv(r)*b';-q -a'];
eH_t=expm(H*(T-t));
...

```

3.2) Numerical application:  $\omega = \frac{2\pi}{T_{orb}} (rd/s)$ ,  $T = \frac{1}{4}T_{orb}$  and  $T_{orb} = 5400(s)$ .  
For the 4 following initial states:

1.  $z(0) = \dot{z}(0) = \dot{x}(0) = 0$  et  $x(0) = 1000(m)$ ,
2.  $z(0) = \dot{z}(0) = \dot{x}(0) = 0$  et  $x(0) = -1000(m)$ ,
3.  $x(0) = \dot{x}(0) = \dot{z}(0) = 0$  et  $z(0) = 1000(m)$ ,
4.  $x(0) = \dot{x}(0) = \dot{z}(0) = 0$  et  $z(0) = -1000(m)$ ,

compute  $\mathcal{E}$ ,  $\hat{u}(t)$ ,  $\hat{X}(t)$  and plot, using the given user-function plotresults.m:

- the 4 optimal state trajectories  $\hat{z}(t), \hat{x}(t), 100\hat{z}(t), 100\hat{x}(t)$  in the same graph,
- the optimal control response  $\hat{u}(t)$ ,
- the optimal trajectory in the local orbital plane  $\hat{x} = F(\hat{z})$ .

Comment these responses.

3.3) Knowing that:

- the chaser  $M$  has a total energy of 1 (with the same unit than the performance index  $\mathcal{C}$ ),
- at  $t = 0$ ,  $M$  is exactly on the same orbit than  $S$  ( $z(0) = 0$ ) with a null relative velocity ( $\dot{x}(0) = \dot{z}(0) = 0$ ),

what is the maximal distance  $|x(0)|_{max}$  for the rendez-vous to be possible?

3.4) go back to questions 3.2) et 3.3) considering now that the 2 control inputs ( $\varphi_z$  and  $\varphi_x$ ) can be used and the new performance index:

$$\mathcal{C} = \int_0^T \frac{1}{2} (\varphi_z^2 + \varphi_x^2) dt .$$