A BRIEF INTRODUCTION TO ROBUSTNESS ANALYSIS

via a practical approach to μ-analysis

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OUTLINE

ROBUSTNESS ANALYSIS: context and objective

Observation: Control laws design is usually based on a (linear) mathematical model Σ_0, which significantly simplifies/alters reality (neglected dynamics, badly-known physical phenomena...).

Assumption: The behavior of a physical system can be accurately described by a (possibly infinite) set of simple models $(\Sigma_i)_i$.

Objective

Check whether the control laws designed using the single linear time-invariant model Σ_0 ensure good performances on the whole set of models $(\Sigma_i)_i$. If this is true, it can be guaranteed that good performances will be obtained on the real system.

ROBUSTNESS ANALYSIS: context and objective

A linear time-invariant (LTI) model is not a perfect representation of the real behavior of a physical system because of:

- high-frequency uncertainties (neglected dynamics)
- uncertainties on the parameters which characterize the system (mass, inertia, aerodynamic coefficients. . .)
- time-varying parameters:
  - fast variations $\rightarrow$ mass of a launcher during atmospheric flight
  - slow variations $\rightarrow$ mass, velocity, altitude of a transport aircraft
- nonlinear phenomena:
  - aerodynamic phenomena at high angles
  - actuators saturations
  - transmission delays
  - intrinsically nonlinear phenomena

Robustness with respect to these phenomena must be ensured!

EXAMPLE: clearance of flight control laws

Before an aircraft can be tested in flight, it has to be proven to the authorities that the flight control system is reliable.

Classical industrial approach = Monte-Carlo simulations:

1. choose many random samples, each of them being composed of random operating points (*e.g.* mass configurations, CoG position. . .) and random inputs (*e.g.* pilot inputs, wind. . .),
2. perform a closed-loop simulation for each sample,
3. perform statistical analysis on the resulting output samples to get the probabilities that some stability, performance, loads and comfort criteria are satisfied.

Advantage:

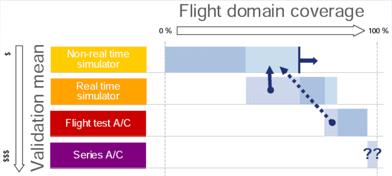
- easy to implement

Drawbacks:

- exponential-time approach ⇒ high computational complexity
- statistical approach ⇒ worst cases can be missed

EXAMPLE: clearance of flight control laws

New trend: develop some inexpensive tools so as to determine quickly the most critical parametric configurations without simulations.



Some techniques such as μ, IQC-based of Lyapunov-based analysis can be efficient alternatives to Monte-Carlo simulations.

Advantages:

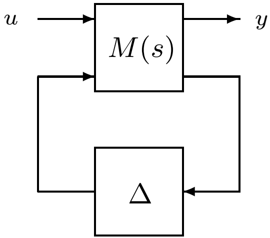
- polynomial-time approach $\Rightarrow$ reduction of the computational burden
- deterministic approach, the criteria can be checked for all admissible operating points $\Rightarrow$ impossible to miss worst cases

Drawbacks:

- requires a preliminary modeling step
- fewer criteria can be checked than with a simulation-based approach

ROBUSTNESS ANALYSIS: standard form

To apply a deterministic approach, the physical system must usually be described by a Linear Fractional Representation (LFR), or standard form.



Several elements can be isolated in Δ:

- time-invariant uncertainties (parametric and dynamic)
- time-varying parameters
- non-linearities (saturations, deadzones, sector non-linearities. . .)

ROBUSTNESS ANALYSIS: focus on μ-analysis

This presentation focuses on μ-analysis, which allows to consider:

- time-invariant uncertainties (parametric and dynamic)
- ~~time-varying parameters~~
- ~~non-linearities (saturations, deadzones, sector non-linearities. . .)~~

Only time-invariant uncertainties can be considered, but several important practical issues can be addressed (see next slide).

Moreover, μ-analysis and Monte-Carlo simulations are complementary:

1. build a high-fidelity LFR of the considered system,
2. check the stability and performance criteria of interest over the admissible set of parametric uncertainties using μ-analysis techniques,
3. use traditional methods such as simulations to investigate only the worst-case configurations that have been identified at step 2.

ISSUES ADDRESSED BY μ-ANALYSIS

Framework:

- LTI nominal closed-loop model (must be stable)
- LTI uncertainties (parametric uncertainties, neglected dynamics), and possibly delays and a few non-linearities

Main robustness issues:

- determine whether the poles of the uncertain closed-loop model are inside a given region of the complex plane (left half plane, unit ball, truncated sector...) $\rightarrow$ stability
- determine whether a closed-loop transfer function satisfies a frequency-domain template despite model uncertainties $\rightarrow$ performance
- determine whether the gain/phase/delay margins are sufficient despite model uncertainties

OUTLINE

PARAMETRIC UNCERTAINTIES: introduction

Parametric uncertainties appear inside the system bandwidth (frequency interval for which the model is representative of the true system) $\Rightarrow$ they are introduced into the open-loop model:

$$\left\{ \begin{array}{rcl} \dot{x} & = & A(\delta)x + B(\delta)u \\ y & = & C(\delta)x + D(\delta)u \end{array} \right.$$

- $\delta = (\delta_1, \ldots, \delta_n) \in \mathbf{R}^n$ is the vector of parametric uncertainties composed of n real scalars δ_i.
- all uncertainties are normalized, i.e. $\delta_i \in [-1,\ 1]$.
- robustness test: determine whether the closed-loop model is stable $\forall \delta_i$ such that $|\delta_i| \leq 1$
- robustness margin: compute the largest value of k for which the closed-loop model remains stable $\forall \delta_i$ such that $|\delta_i| \leq k$

PARAMETRIC UNCERTAINTIES: example

Aircraft lateral model (linearized equations):

$$
\begin{aligned}
\dot{\beta} &= Y_\beta \beta + (Y_p + sin\alpha_0)p + (Y_r - cos\alpha_0)r + \tfrac{g}{V}\phi + Y_{\delta p}\delta p + Y_{\delta r}\delta r \\
\dot{p} &= L_\beta \beta + L_p p + L_r r + L_{\delta p}\delta p + L_{\delta r}\delta r \\
\dot{r} &= N_\beta \beta + N_p p + N_r r + N_{\delta r}\delta r \\
\dot{\phi} &= p + tan\theta_0\, r
\end{aligned}
$$

Parametric uncertainties are introduced in the 14 stability derivatives:

$$Y_\beta = (1+\delta_1)Y_\beta^0 \qquad Y_p = (1+\delta_2)Y_p^0 \qquad \ldots$$

Robustness test: does the model remains stable if the stability derivatives vary by $\pm 10\%$ around their nominal values, i.e. if $\delta_i \in [-0.1,\ 0.1]$?

The uncertainties are normalized ($\delta_i \in [-1,\ 1] \Leftrightarrow$ variation of $\pm 10\%$):

$$Y_\beta = (1+0.1\delta_1)Y_\beta^0 \qquad Y_p = (1+0.1\delta_2)Y_p^0 \qquad \ldots$$

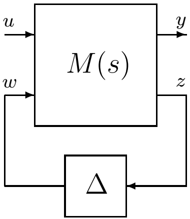
LFT MODELING: parametric uncertainties

Introduction of parametric uncertainties $\delta = (\delta_1, \dots, \delta_n) \in \mathbf{R}^n$ in the state-space representation of the considered physical system:

$$\begin{aligned} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{aligned}$$

LFR modeling:

$y = G(s,\delta)u = F_l(M(s),\Delta)u$ where $\Delta = \mathrm{diag}(\delta_1 I_{q_1}, \dots, \delta_n I_{q_n})$



$\Leftarrow$ stable nominal model (LTI)

$\Leftarrow$ structured uncertainties

LFT MODELING: general case

- The elements of $A(\delta), B(\delta), C(\delta), D(\delta)$ must be polynomial or rational functions of the parametric uncertainties δ_i.
  - tabulated data $\Rightarrow$ polynomial/rational fitting (least squares...)
  - irrational functions $\Rightarrow$ approximating polynomial/rational functions (Taylor series...)

  Systematic methods exist to convert the model into an LFR once it is written in polynomial or rational form.

- Computing a minimal representation (i.e. with the smallest possible Δ matrix) is difficult.
  - always possible if $n = 1$
  - no systematic method if $n > 1$

Minimality is critical in terms of computational time and conservatism when robustness analysis tools are applied.

LFT MODELING: example

Let us consider the following second-order model:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} u \\ y &= x_1 \end{aligned}$$

The frequency is assumed to be uncertain, i.e. $\omega = (1+\delta)\omega_0$.

The most compact representation is:

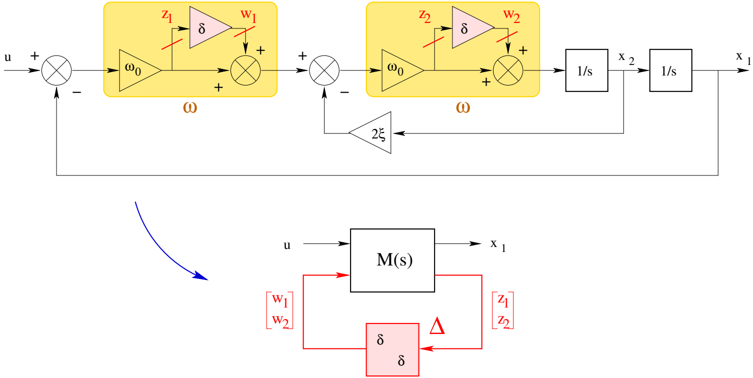
$$y = \mathcal{F}_l(M(s), \Delta)u$$

where:

$$\Delta = \mathrm{diag}(\delta, \delta)$$

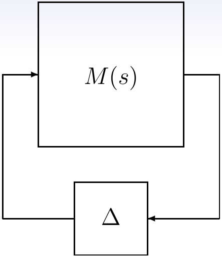
LFT MODELING: a graphical method

Creation of a block diagram from: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \omega(\omega(u - x_1) - 2\xi x_2) \\ \omega = (1+\delta)\omega_0 \end{cases}$



OUTLINE

INTRODUCTION TO μ-ANALYSIS



Robustness analysis

Compute the values of Δ for which the interconnection is stable.

With $\Delta = \mathrm{diag}(\delta_1 I_{q_1}, \ldots, \delta_n I_{q_n})$ and $\overline{\sigma}(\Delta) = \max_i |\delta_i|$, let us define:

- the structure of the uncertainties: $\boldsymbol{\Delta} = \{\mathrm{diag}(\delta_1 I_{q_1}, \ldots, \delta_n I_{q_n}\}$
- the associated unit ball (hypercube): $B(\boldsymbol{\Delta}) = \{\Delta \in \boldsymbol{\Delta} : \overline{\sigma}(\Delta) < 1\}$.

STRUCTURED SINGULAR VALUE μ_{Δ}

The interconnection $M(s) - \Delta$ is asymptotically stable iff all eigenvalues of $\mathbb{A}(\delta) = A + B\Delta C$ are inside the open left half plane.

Assumption: $M(s)$ is asymptotically stable.

Property: The roots of $\mathbb{A}(\delta)$ are continuous functions of δ. It is thus equivalent to compute the values of Δ for which:

1. the interconnection is stable,
2. the interconnection is at the limit of stability.

Limit of stability $\Leftrightarrow$ an eigenvalue λ_i of $\mathbb{A}(\delta)$ lies on the imaginary axis, i.e. $\lambda_i = j\omega \Leftrightarrow \psi(s, \delta) = \det(sI - \mathbb{A}(\delta)) = 0$ for $s = j\omega$.

$$
\begin{aligned}
\psi(s, \delta) &= \det(sI - A - B\Delta C) \\
&= \det(sI - A)\det(I - (sI - A)^{-1} B\Delta C) \\
&= \det(sI - A)\det(I - C(sI - A)^{-1} B\Delta) \\
&= \det(sI - A)\det(I - M(s)\Delta)
\end{aligned}
$$

$\Rightarrow$ limit of stability at frequency ω: $\det(I - M(j\omega)\Delta) = 0$

STRUCTURED SINGULAR VALUE $\mu_{\mathbf{\Delta}}$

Definition

The structured singular value is defined for a frequency $\omega \geq 0$ as:

$$\mu_{\mathbf{\Delta}}(M(j\omega)) = \frac{1}{\min\limits_{\Delta \in \mathbf{\Delta}} \{\overline{\sigma}(\Delta) : \det(I - M(j\omega)\Delta) = 0\}}$$

It depends on both the nominal system $M(s)$ and the structure of the uncertainties $\mathbf{\Delta}$.

Stability theorem (structured version of small gain theorem)

The interconnection $M(s) - \Delta$ is stable for all $\Delta \in kB(\mathbf{\Delta})$ iff:

$$\sup_{\omega \geq 0} \mu_{\mathbf{\Delta}}(M(j\omega)) \leq 1/k$$

ROBUSTNESS MARGIN k_r: definition

Definition

The robustness margin k_r is defined as:

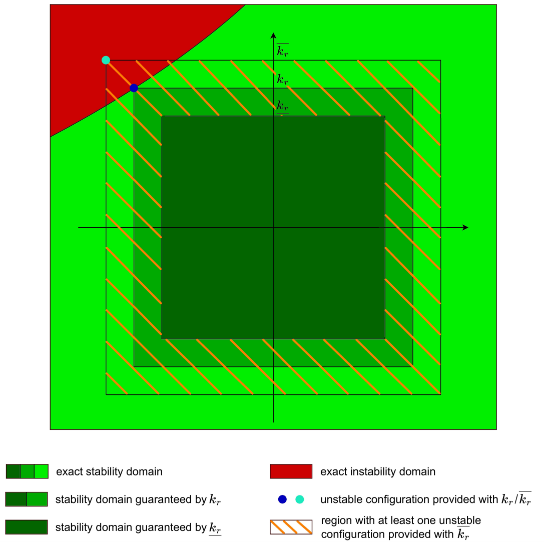
$$k_r = \frac{1}{\sup\limits_{\omega \geq 0} \mu_{\mathbf{\Delta}}(M(j\omega))}$$

The robustness margin is thus:

- the largest k such that $M(s) - \Delta$ is stable for all $\Delta \in kB(\mathbf{\Delta})$,
- the smallest k for which there exists $\Delta^* \in kB(\mathbf{\Delta})$ such that $M(s) - \Delta$ is unstable.

Computing $\max\limits_{\omega \geq 0} \mu_{\mathbf{\Delta}}(M(j\omega))$ is NP hard, so lower and upper bounds $\underline{\mu}_{\mathbf{\Delta}} = 1/\overline{k_r}$ and $\overline{\mu}_{\mathbf{\Delta}} = 1/\underline{k_r}$ are computed instead, hoping that the gap is small.

ROBUSTNESS MARGIN k_r: illustration



A lower bound $\underline{k_r}$ on k_r (μ upper bound) provides a guaranteed but pessimistic robustness margin.

An upper bound $\overline{k_r}$ on k_r (μ lower bound) provides a destabilizing uncertainty and measures the conservatism of the lower bound.

ROBUSTNESS MARGIN k_r: practical computation

Efficient algorithms are implemented in the Systems Modeling Analysis and Control (SMAC) Toolbox for Matlab, available at *http://w3.onera.fr/smac*, see the functions **muub** and **mulb** to compute $\overline{\mu}_{\mathbf{\Delta}}$ and $\underline{\mu}_{\mathbf{\Delta}}$

Example: system with uncertain parameters $p_i = (1 + 0.1\delta_i)p_i^0$.

- if $\overline{\mu}_{\mathbf{\Delta}} = 0.8$, then $\underline{k_r} = 1/0.8$, and stability is garanteed for all values of p_i inside the intervals $(1 \pm 0.1\underline{k_r})p_i^0 = [0.875p_i^0 \;\; 1.125p_i^0]$,
- if $\underline{\mu}_{\mathbf{\Delta}} = 0.6$, then $\overline{k_r} = 1/0.6$, and there exist values of p_i inside the intervals $(1 \pm 0.1\overline{k_r})p_i^0 = [0.833p_i^0 \;\; 1.167p_i^0]$ s.t. the system is unstable.

If the uncertainties are normalized:

- The system is robustly stable if and only if $k_r \geq 1$,
- The system is robustly stable if $\underline{k_r} \geq 1$,
- The system is not robustly stable if $\overline{k_r} < 1$,
- No conclusion if $\underline{k_r} < 1 < \overline{k_r}$ (stability guaranteed only on $\underline{k_r}B(\mathbf{\Delta})$).