A SIMPLE MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL PROBLEM

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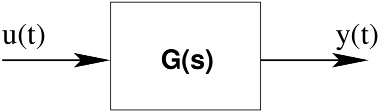
The $\mathcal{H}_2$ norm



Frequency-domain definition

$$\|G(s)\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Trace}\left(G^*(j\omega)G(j\omega)\right) d\omega \right)^{1/2}$$

$\|G(s)\|_2$ is only finite for **stable** and **strictly proper** (*i.e.* $D = 0$) systems. Otherwise, $\|G(s)\|_2 = \infty$.

Interpretation:

- If $G(s)$ is a SISO system, $\|G(s)\|_2^2$ is the energy of y when u is an impulse.
- In the general case, $\|G(s)\|_2^2$ is the variance of y when u is a centered normalized white noise, *i.e.* a random signal such that $U(j\omega)U^*(j\omega) = I$.

Practical computation of the $\mathcal{H}_2$ norm

Let $G(s)$ be a stable and strictly proper transfer: $G(s) = C(sI - A)^{-1}B$. The Parseval's theorem is applied, noting that $\mathcal{F}(Ce^{At}B) = G(j\omega)$, where $\mathcal{F}(.)$ denotes the Fourier transform.

Time-domain definition

$$\|G(s)\|_2 = \left(\text{Trace} \int_0^{+\infty} B^T e^{A^T t} C^T C e^{At} B\right)^{1/2}$$

$Q_o = \int_0^{\infty} e^{A^T t} C^T C e^{At}\, dt$ is called the **Observability Gramian**. It is a definite positive (symmetric) matrix solution of the Lyapunov equation:

$$A^T Q_o + Q_o A + C^T C = 0$$

$$\Rightarrow \; \|G(s)\|_2^2 = \text{Trace}(B^T Q_o B) \qquad (\text{Matlab function } \textbf{norm})$$

Noises in theory and practice

Using **centered normalized white noises** allows to derive many important theoretical results. But such signals **do not exist**. Indeed, they are characterized by a **constant power spectral density** (PSD = power distribution as a function of frequency):

$$\phi(\omega) = 1 \quad \forall \omega \geq 0$$

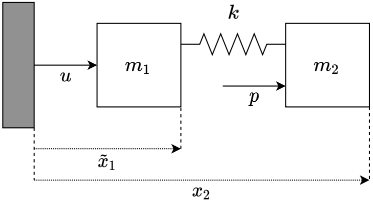
which implies that their power $P = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} \phi(\omega) d\omega$ is infinite!

In practice, noises have a **finite power**, for example $\phi(\omega) = \frac{1}{1+\omega^2}$.

- Replacing $j\omega$ by s gives the complex spectrum: $\phi(s) = \frac{1}{1-s^2}$.
- The later can then be factorized: $\phi(s) = \underbrace{F(s)}_{stable} \times \underbrace{F(-s)}_{unstable} = \frac{1}{1+s} \times \frac{1}{1-s}$.
- By filtering a centered normalized white noise by $F(s)$, a random signal with the same PSD as the considered noise is finally obtained.

An academic spring-mass example

Let us consider a spring-mass system, where u is the control signal (force) and p is a disturbance (force). The measurement y of the position of the mass m_2 is perturbed by a random signal b ($y = x_2 + b$) with a power spectral density (PSD) $\phi(\omega) = \frac{1}{1+\omega^2}$.



Let $x_1 = \tilde{x}_1 + \Delta x_{eq}$

- compression if $x_1 - x_2 > 0$
- extension if $x_1 - x_2 < 0$

Specifications:

- actuator fatigue alleviation: minimize the variance of $\dot{u}$ (the time-derivative of the control signal) in response to the measurement noise b
- disturbance rejection: $|\mathcal{T}_{p \to x_2}(j\omega)| < A \;\; \forall \omega \geq 0$

Construction of the weighted standard form $P_W(s)$

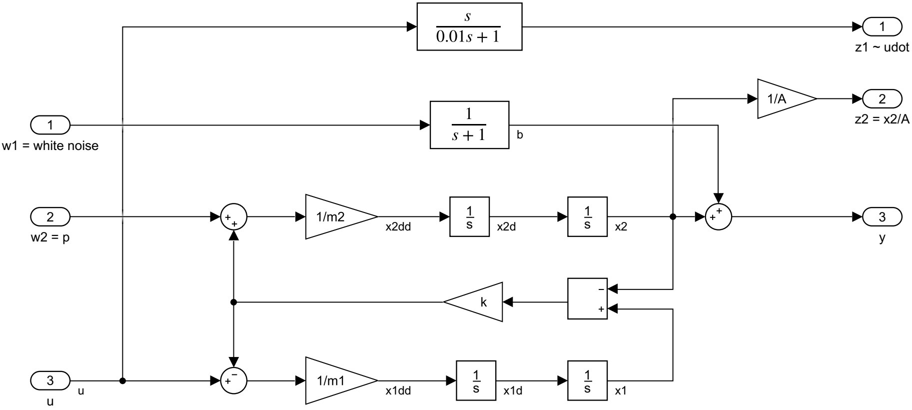
System equations:

$$\begin{aligned}\ddot{x}_1 &= \frac{1}{m_1}\left(u - k\left(x_1 - x_2\right)\right)\\ \ddot{x}_2 &= \frac{1}{m_2}\left(p + k\left(x_1 - x_2\right)\right)\\ y &= x_2 + b\end{aligned}$$

Inclusion of the specifications:

- addition of the transfer function $F(s) = \frac{1}{s+1}$, so that $b = F(s)w_1$ is a random signal with PSD $\phi(\omega) = \frac{1}{1+\omega^2}$ obtained from the centered normalized white noise w_1
- addition of a pseudo-derivator $H(s) = \frac{s}{0.01s+1}$, so that $z_1 = H(s)u$ is a good approximation of $\dot{u}$ inside the system bandwidth
- addition of a static weighting function $W = A^{-1}$, so that $|\mathcal{T}_{p\rightarrow x_2}(j\omega)| < A$ $\forall\omega \geq 0$ is equivalent to $\|\mathcal{T}_{w_2\rightarrow z_2}(s)\|_\infty < 1$, where $w_2 = p$ and $z_2 = Wx_2$

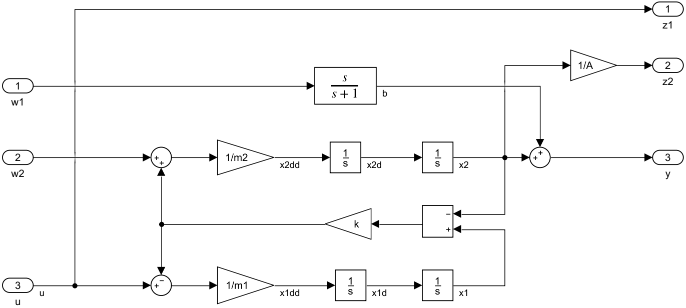
Construction of the weighted standard form $P_W(s)$



Improved weighted standard form

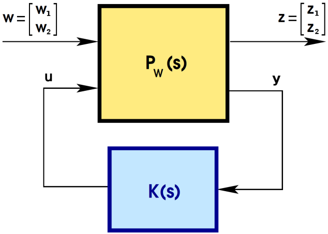
$$\begin{aligned}\mathcal{T}_{w_1\to z_1}(s) &= \frac{s}{0.01s+1}\mathcal{T}_{y\to u}(s)\frac{1}{s+1}\\ &= \frac{1}{0.01s+1}\mathcal{T}_{y\to u}(s)\frac{s}{s+1}\end{aligned}$$

The filter $\frac{1}{0.01s+1}$ was only introduced for regularization and is no longer needed $\Rightarrow \mathcal{T}_{w_1\to z_1}(s) = \mathcal{T}_{y\to u}(s)\frac{s}{s+1}$ is preferred.



Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem

Let $\mathcal{F}_l(P_k(s), K(s))$ denote the closed-loop transfer function $\mathcal{T}_{w_k \to z_k}(s)$.



Control problem

Find a stabilizing controller $\widehat{K}(s)$ such that:

$$\widehat{K}(s) = \arg \min_{K(s) \in \mathcal{K}} \|F_l(P_1(s), K(s))\|_2 \qquad \textbf{(soft constraint)}$$

where $\mathcal{K} = \{K(s) : \|F_l(P_2(s), K(s))\|_\infty \leq 1\}$ **(hard constraint)**