SOFT COMPUTING

MODULE-4: Fuzzy Sets

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SYLLABUS

- Introduction
- Fuzzy set operations
- Fuzzy relations
- Fuzzy membership functions
- Fuzzification
- Defuzzification

SET

- **Definition of Mathematics:**
- No universally accepted definition till the notion of set was introduced by Cantor in 1873.

- **Set:** It is a **well-defined collection** of **distinct objects** called elements
- **Russell's Paradox** led to antimonies to this definition

- Example of a **Barber in a town**
- More general: Set of all sets

SET CONTD…

- **Alternate Approaches:**
- Assumption as a known concept
- Axiomatic approach

- **Why do we use the Intuitive Definition?**
- Deficiency of sets, mentioned above, has rather philosophical than practical meaning, since sets used practically in mathematics are free from the above discussed faults

- **Antinomies are associated with very artificial sets**
- **So, there is no harm to mathematics with the definition.**

VAGUENESS

- No solution to find alternate to classical set theory
- Another issue discussed in connection with the notion of a set is **vagueness (impreciseness)**
- **Example:**
- The notion of a beautiful painting is vague
- **Reason:**
- We are unable to classify uniquely all paintings into two classes: **beautiful** and **not beautiful**.
- **NOTE:** Almost all concepts we are using in natural languages are vague
- **THEREFORE, COMMON SENSE REASONING BASED ON NATURAL LANGUAGE MUST BE BASED ON VAGUE CONCEPTS AND NOT ON CLASSICAL LOGIC.**

VAGUENESS CONTD…

- Vagueness is usually associated with the boundary region approach
- That is the **existence of objects which cannot be uniquely classified relative to a set or its complement**
- First formulated in 1893 by the father of modern logic, German logician, Gottlob Frege
- Vagueness is
  - **not allowed in classical mathematics based on set theory**;
  - **interesting for philosophy;**
  - a difficult problem for natural language, cognitive science, artificial intelligence, machine learning, philosophy and computer science

SOME IMPORTANT OBSERVATIONS

- So far as laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality. **-Albert Einstein (1921)**

- All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to the terrestrial life but only to an imagined celestial existence.

  -Bertrand Russell (1923)

- **THESE OBSERVATIONS PUT A QUESTION MARK ON THE APPLICABILITY OF CRISP SET THEORY IN REAL LIFE SITUATIONS**

SOME EXAMPLES

- We consider some examples where the modeling using crisp sets is unrealistic

- **Young Person**
- **Examination Results**
- **Physical Test for candidates**
- **Trustworthy Customer**

CHARACTERISTIC FUNCTION FOR SETS

- Associated with any set A defined over a universal set U is the characteristic function χ_A defined by

$$\chi_A(x) = \begin{cases} 1, & if\ \ x \in A; \\ 0, & otherwise. \end{cases}$$

- **EXAMPLE:**
- Let U = {1, 2, ….10} and A = {1, 3, 5, 7, 9}.
- Then
- $\chi_A(3) = 1$ and $\chi_A(4) = 0.$

FUZZY SETS

- The **dichotomous nature of the characteristic function** was extended to define the membership function which is the origin of fuzzy sets

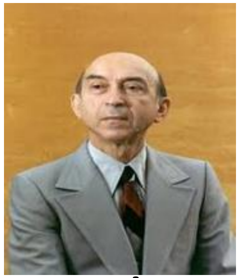
- **Membership function is gradual(Degree)**

- In a **fuzzy set context** we say that the **membership of an element is graded**

- Instead of the values for a characteristic function which are 0 or 1, the values for a **membership function can assume any value lying in the interval [0, 1]**

UNCERTAINTY BASED MODELS AND ROUGH SET

[Lotfi Abraham Zadeh 1921 – 2017]



- One of the earliest models of uncertainty is Fuzzy set introduced by L. A. Zadeh in 1965
- It introduced the concept of graded membership of elements in a set instead of dichotomous membership in crisp sets introduced by G. Cantor in 1820
- It is more realistic and has better modelling power
- Given a universal set U, a fuzzy subset X of U is defined through a membership function associated with X, denoted by μ_X defined as $\mu_X: U \rightarrow [0,1]$ such that for any $x \in U$, $\mu_X(x) = \alpha$, where $\alpha \in [0,1]$

MEMBERSHIP FUNCTION

- We shall denote the **membership function of a fuzzy set** A defined over a universal set U by μ_A , which is defined as

$$\mu_A : U \to [0,1]$$

such that for each $x \in U$,

$$\mu_A(x) = \alpha, \quad 0 \le \alpha \le 1.$$

- Each fuzzy set is **completely** and **uniquely** defined by one **particular membership function**.

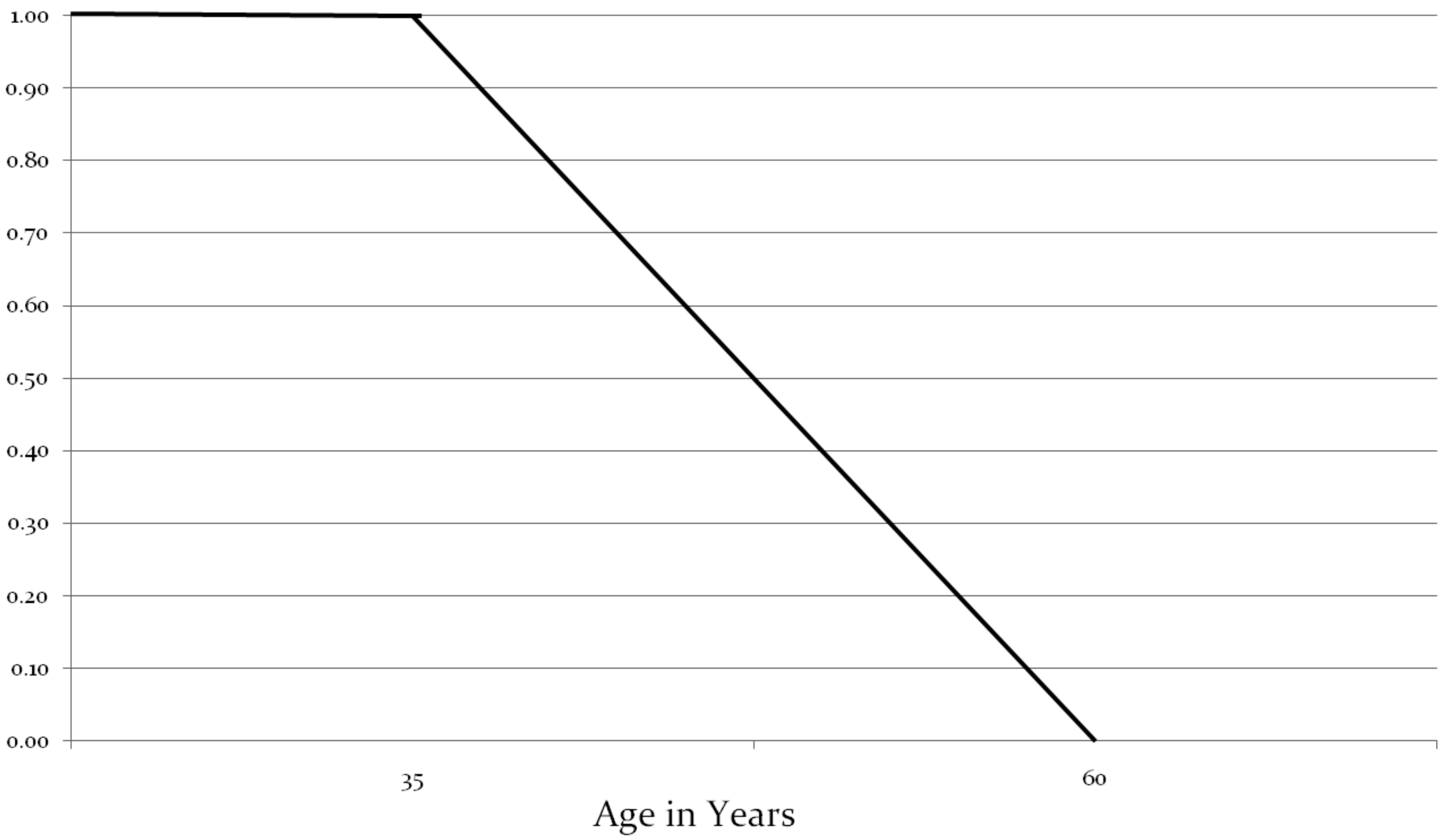
- The fuzzy membership functions are **not only dependent upon the concept** but also upon the **context**.

AN EXAMPLE

- Let us consider a **young person**
- Using **crisp set concept** we have to find a **cut off age** (say 35) up to which a person can be called as young
- **When a person crosses 35 years** of age, he/she is no more considered as young, which **sounds illogical**
- However, we can **represent the same concept as a fuzzy one** and define its membership function say by

$$\mu_{young}(x)=\begin{cases}1, & x\leq 35;\\ 1-(x-35)/25, & 35<x<60;\\ 0, & otherwise\end{cases}$$

GRAPHICAL REPRESENTATION



THE PARADIGM SHIFT

- Scientific Paradigm:
- Introduced by Thomas Kuhn in 1962
- Book: **The structure of scientific revolutions**

- **PARADIGM:**
- **A set of theories, standards, principles and methods that are taken for granted by the scientific community in a given field**

- **SCIENTIFIC REVOLUTIONS ARE INTERWOVEN WITH PERIODS OF PARADIGM SHIFTS**

CHARACTERISTICS OF PARADIGM SHIFT

- Each paradigm shift is initiated by emerging problems
- These are difficult or impossible to deal with in the current paradigm
- Each paradigm, when proposed, is **initially rejected** in various forms by most scientists in the given field
- Those who support the new paradigm are either very young or very new to the field

THE AGONIES

- Professor Rudolf Kalman (one of the foremost contributors to system theory and control said in 1972)

…I would like to comment briefly on professor Zadeh's presentation. His proposals could be severely, ferociously, even brutally criticized from a technical point of view.

This would be out of place here. But a blunt question remains: Is Professor Zadeh presenting important ideas or is he indulging in wishful thinking?

... **Let me say quite categorically that there is no such thing as a fuzzy scientific concept, in my opinion."**

– Expressed by Zadeh (1996)

THE AGONIES CONTD…

- **Professor William Kahan** (A colleague of Prof. Zadeh in 1975)
- ….Fuzzy theory is **wrong, wrong and pernicious** **(harmful)**.
- I cannot think of any problem that could not be solved better by ordinary logic.
- What Zadeh is saying is the sort of things 'Technology got us into this men and now it can't get us out'.
- Well, technology did not get us into this men, greed and weakness and ambivalence **(co-existence of opposite feeling in the mind of a person towards a person)** got us into this men.
- What we need is more logical thinking, not less.
- **The danger of fuzzy theory is that it will encourage the sort of imprecise thinking that has brought us so much trouble**

- Expressed by Zadeh (1996)

WHAT IS ACHIEVED?

- What is gained through fuzzification is:
- **greater generality**
- **higher expressive power**
- **an enhanced ability to model real world problems**
- and most importantly, a methodology for exploiting the tolerance for imprecision
- a methodology which serves to achieve tractability, robustness and lower solution cost

  Speech by L.A.Zadeh (1996) presented on the occasion of the award of Doctorate Honoris Causa, University of Oviedo, Spain:
- **The Birth and Evolution of Fuzzy Logic (FL), Soft Computing (SC) and Computing with Words (CW): A personal Perspective**

REPRESENTATION OF FUZZY SETS

- There are **several different ways** in which a **fuzzy set can be represented.**
- **I.** A fuzzy set is **denoted by an ordered set of pairs**, the first element of which denotes the **element** and the second the **degree of membership**.
- Let X be a set. Then a fuzzy set A in X is denoted by the collection of pairs $\{(x, \mu_A(x))/x \in X\}$.
- **Example**
- Let U = {1,2,3,4,5,6}. Then A = {(1, 0.6),(2, 0.3),(4, 0.7),(5,1)}

  is a fuzzy set on U.

  Note that the elements 3 and 6 have membership values 0.

REPRESENTATION OF FUZZY SETS CONTD…

- A fuzzy set A can be represented as

$$A = \mu_A(x_1)/x_1 \quad + \quad \mu_A(x_2)/x_2 \quad + \quad \dots \quad + \quad \mu_A(x_n)/x_n \quad + \quad \dots$$
$$= \sum_i \mu_A(x_i)\Big/x_i \,.$$

- The ‘+’ , ‘/’ and '$\sum$' symbols have not been used with their usual meanings, these are only representational.

- For example, the fuzzy set example above can be represented as

$$A = .6/1 + .3/2 + .7/4 + 1/7.$$

CHARACTERISTICS OF FUZZY MEMBERSHIP FUNCTIONS

- The **core of a membership function** for some fuzzy set *A* is defined as that region of the universe that is characterized by complete and full membership in the set A.
- That is, the **core comprises of** those elements *x of the* universe such that $\mu_A(x)$ = 1.

- The **support of a membership function** for some fuzzy set A is defined as that region of the universe that is characterized by **nonzero membership in the set** A.
- That is, the **support comprises of** those elements x of the universe such that $\mu_A(x)$ >0.

CHARACTERISTICS OF FUZZY MEMBERSHIP FUNCTIONS CONTD…

- The **boundaries of a membership function for some fuzzy set** *A* are defined as that region of the universe containing elements that **have a nonzero membership but not complete membership**
- That is, the **boundaries** comprise those elements x of the universe such that $0 < \mu_A(x) < 1$.
- A **normal fuzzy set** is one whose **membership function has at least one element x** in the universe whose membership **value is unity.**
- In fuzzy sets, where **one and only one element has a membership equal to one**, the element is typically referred to as the prototype of the set, or the **prototypical element**.

CHARACTERISTICS OF FUZZY MEMBERSHIP FUNCTIONS CONTD…

- A **convex fuzzy set** is described by a membership function whose membership values are
- **strictly monotonically increasing, or**
- **strictly monotonically decreasing,** or
- whose membership values are **strictly monotonically increasing** then **strictly monotonically decreasing** with increasing values for elements in the universe.

- if for any elements x, y, and z in a fuzzy set *A*, the relation
- $x < y < z$ implies that $\mu_A(y) \geq \min[\mu_A(x), \mu_A(z)]$
- *then A* is said to be a **convex fuzzy set**

CHARACTERISTICS OF FUZZY MEMBERSHIP FUNCTIONS CONTD…

- A special property of two convex fuzzy sets, say A and B, is that the **intersection of these two convex fuzzy sets is also a convex fuzzy set**
- That is, for A and B, which are both convex, A $\cap$B is also convex.
- The **crossover points of a membership function** are defined as the elements in the universe for which a particular fuzzy set A has values equal to 0.5, that is, for which $\mu_A(x)$= 0.5.

- The **height of a fuzzy set A** is the maximum value of the membership function, that is, hgt(A) = max{$\mu_A(x)$ }.

CHARACTERISTICS OF FUZZY MEMBERSHIP FUNCTIONS CONTD…

- If the hgt(A) < 1, the fuzzy set is said to be **subnormal**

- The hgt(A) may be viewed as the **degree of validity** or **credibility of information expressed by A**

- The most common forms of membership functions are those **that are normal and convex.**

- However, **many operations on fuzzy sets**, hence operations on membership functions, **result in fuzzy sets that are subnormal and non-convex**

EXAMPLES

- Let U = $\{x_1, x_2, x_3, x_4, x_5\}$. We define

  $X = \{(x_1, 0.3), (x_2, 0.5), (x_3, 0.8)\}$

- Then
- Crossover points of μ_X is the set $\{x_2\}$
- hgt(X) = 0.8
- X is subnormal as its height is less than 1
- X is convex as $x_1 < x_2 < x_3$ and we have
- $\mu_X(x_2) = 0.5 \geq 0.3 = \min\{\mu_X(x_1), \mu_X(x_3)\}$

BASIC OPERATIONS ON FUZZY SETS

- **Union of Sets**: For any two sets A and B we define their union as

$$A \bigcup B = \{x \in A \; or \; x \in B\}$$

- This can be rewritten using characteristic functions as

$$\chi_{A \bigcup B}(x) = \max\{\chi_A(x), \chi_B(x)\},$$

- Extending this **union of two fuzzy sets** is defined as

$$\mu_{A \bigcup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

BASIC OPERATIONS ON FUZZY SETS CONTD…

- The intersection of two sets A and B is defined as

$$A \bigcap B = \{x \in A \; and \; x \in B\}$$

- Using characteristic functions this can be rewritten as

$$\chi_{A \bigcap B}(x) = \min\{\chi_A(x), \chi_B(x)\}$$

- Extending this **intersection of two fuzzy sets** is defined as

$$\mu_{A \bigcap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

BASIC OPERATIONS ON FUZZY SETS CONTD…

- For any two sets A and B the **complement of B in A** is defined as

$$A \setminus B = \{x \in A \; and \; x \notin B\}$$

- Using characteristic functions it can be rewritten as

$$\chi_{\mathrm{A\setminus B}}(x) = \max\{0, \chi_A(x) - \chi_B(x)\}.$$

- Extending this intersection of two fuzzy sets is defined as

$$\mu_{\mathrm{A\setminus B}}(x) = \max\{0, \mu_A(x) - \mu_B(x)\}.$$

BASIC OPERATIONS ON FUZZY SETS CONTD…

- The **complement of a set A in a universal set U** is given by

$$A^C = \{x \in U \,/\, x \notin A\}$$

- Using the characteristic function it can be rewritten as

$$\chi_{A^C}(x) = 1 - \chi_A(x)$$

- Extending this the complement of a fuzzy set is defined as

$$\mu_{A^C}(x) = 1 - \mu_A(x)$$

COMPARISON OF SETS

- For any two sets A and B we say that $A \subseteq B$ iff $x \in A \Rightarrow x \in B$

- Using characteristic functions it can be rewritten as
- $A \subseteq B \Leftrightarrow \chi_A(x) \leq \chi_B(x)$ for all $x \in U$
- Extending this, for any two fuzzy sets A and B we say that
- $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in U$

- So, we say that A = B iff $\mu_A(x) = \mu_B(x)$ for all $x \in U$
- We say that A is a proper subset of B iff $A \subseteq B$ and
- $\mu_A(x) < \mu_B(x)$ for atleast one $x \in U$

EXAMPLES

- Suppose U = {1, 2, 3, 4, 5},
- A = { (2,1), (3, 0.5), (4, 0.2),(5,0.3)} and
- B = {(2,0.5), (3, 0.2), (4, 0.7),(5,0.4)}

- Then $A^C = \{(1,1),(2,0),(3,0.5),(4,0.8),(5,0.7)\}$

$$A \bigcup B = \{(2,1),(3,0.5),(4,0.7),(5,0.4)\}$$

$$A \bigcap B = \{(2,0.5),(3,0.2),(4,0.2),(5,0.3)\}$$

$$A \setminus B = \{(2,0.5),(3,0.3),(4,0),(5,0)\}$$

PROPERTIES OF FUZZY SETS

- For any two fuzzy sets A and B
- **Commutativity:** $A \cup B = B \cup A$
  $A \cap B = B \cap A$
- **Associativity:** $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributivity:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Idempotency:** $A \cap A = A, A \cup A = A$
- **Identity:** $A \cup \phi = A, A \cap U = A, A \cap \phi = \phi, A \cup U = U$

A RESULT

- For $m, n, p \in \mathbb{N}$, we have
- $\max\{m, \min\{n, p\} = \min\{\max\{m, n\}, \max\{m, p\}\}$
- $\min\{m, \max\{n, p\} = \max\{\min\{m, n\}, \min\{m, p\}\}$

- **Proof:** We consider two different cases
- **Case 1:** $n \geq p$ and
- **Case 2:** $n < p$.
- We provide the proof in Case 1.
- The proof of Case 2 is similar.

PROOF CONTD…

- **Proof of Case 1:**
- We have the following three possibilities in this case:
- (i) $m \geq n >= p$ (ii) $n > m \geq p$ and (iii) $n \geq p \geq m$

- **Case 1 (i):**
- Here, we get $m \geq n \geq p$. So,
- LHS = max {m, min {n, p}} = max {m, p} = m.
- RHS = min {max {m, n}, max {m, p}} =min {m, m} = m.
- So, LHS = RHS.

PROOF CONTD…

- **Case 1 (ii):** We have
- LHS = max {m, min {n, p}} = max {m, p} = m.
- RHS = min {max {m, n}, max {m, p}} = min {n, m} = m.
- So, LHS = RHS.

- **Case 1 (iii):** Here, we have
- LHS = max {m, min {n, p}} = max {m, p} = p.
- RHS = min {max {m, n}, max {m, p}} = min {n, p} = p.
- So, LHS = RHS.

- This completes the proof.

PROPERTIES OF FUZZY SETS CONTD…

- **Transitivity:** If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
- **Involution:** $(A^C)^C = A$
- **Axiom of excluded middle:** $A \bigcup A^C \neq U$
- **Axiom of contradiction:** $A \bigcap A^C \neq \phi$
- **NOTE: The above two properties hold for crisp sets**
- **De Morgan's Law:** $(A \bigcap B)^C = A^C \bigcup B^C$ **and** $(A \bigcup B)^C = A^C \bigcap B^C$

CLASSICAL RELATIONS

- Let A and B be two sets. Then a relation from A to B is a subset of the Cartesian product A x B
- A binary relation on a set A is a subset of A x A
- Universal relation on A is A x A
- Identity relation on A is $id_A = \{(x,x) \mid \mathrm{x} \in \mathrm{A}\}$
- In general a n-ary relation on A is a subset of

$$A^n = \underbrace{A \times A \ldots \times A}_{n-times}$$

MATRIX REPRESENTATION OF CRISP RELATIONS

- **Let A = {1, 2, 3} B = {x, y, z}**
- **R = { (1, x), (2, y), (3, z)}. The R can be represented as a matrix as**

$$\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

- **S = {(1,y), (1, z), (2, x), (2, y), (3, x), (3, y), (3, z)}**
- **S can be represented as**

$$\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

SPECIAL RELATIONS

- **Define R and S as two separate relations on the Cartesian universe X × Y**
- **define the null relation and the complete relation as the relation matrices O and E, respectively.**
- **An example of a 4 × 4 form of the O and E matrices is given here is**

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

RELATION IN TERMS OF CHARACTERISTIC FUNCTION

- **The characteristic function is used to assign values of relationship R as the mapping of the Cartesian space X × Y to the binary values of (0, 1):**

$$\chi_R(x,y)=\begin{cases}1, & if\,(x,y)\in R,\\ 0, & if\,(x,y)\notin R.\end{cases}$$

- **Let R and S as two separate relations on the Cartesian universe X × Y. Then**

$$\chi_{R\cup S}(x,y)=\max\{\chi_R(x,y),\chi_S(x,y)\}$$

$$\chi_{R\cap S}(x,y)=\min\{\chi_R(x,y),\chi_S(x,y)\}$$

$$\chi_{R^C}(x,y)=1-\chi_R(x,y)$$

COMPOSITION OF RELATIONS

- **There are two common forms of the composition operation**
- **one is called the *max–min composition***
- **the other the *max–product composition***
- **Let R be a relation from X to Y and S be a relation from Y to Z and we define T = R ∘ S*,***

- **(max-min)** $$\chi_T(x,z) = \bigvee_{y \in Y} (\chi_R(x,y) \wedge \chi_R(y,z))$$

- **(max-product)** $$\chi_T(x,z) = \bigvee_{y \in Y} (\chi_R(x,y) \cdot \chi_R(y,z))$$

EXAMPLE

- **Let X = {x1, x2, x3}, Y = {y1, y2, y3, y4} and Z = {z1, z2}.**
- **R = { (x1, y1), (x1, y3), (x2, y4)}**
- **S = { (y1, z2), (y3, z2)}**
- **The relation matrices are given by**
- R = $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **and S =** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ **. Then T =** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\chi_T(x1, z1) =$ max[min*(1, 0), min(0, 0), min(1, 0), min(0, 0)] = 0*

$\chi_T(x1, z2) =$ max[min*(1, 1), min(0, 0), min(1, 1), min(0, 0)] = 1*

FUZZY RELATIONS

- **A fuzzy relation from a set X to a set Y is a fuzzy subset of X × Y**
- **Let R an S be two fuzzy relations on X × Y. Then**
- **Union** $\mu_{R \cup S}(x,y) = \max\{\mu_R(x,y), \mu_S(x,y)\}$
- **Intersection** $\mu_{R \cap S}(x,y) = \min\{\mu_R(x,y), \mu_S(x,y)\}$
- **Complement** $\mu_{R^C}(x,y) = 1 - \mu_R(x,y)$
- **Subrelation** $R \subseteq S \quad iff \;\; \mu_R(x,y) \le \mu_S(x,y)$

PROPERTIES OF FUZZY RELATIONS

- The properties of commutativity, associativity, distributivity, involution, and idempotency all hold for fuzzy relations

- De Morgan's principles hold for fuzzy relations

- $R \bigcup R^C \neq E$,where E is the complete relation

- $R \bigcap R^C \neq O$,where O is the null relation

FUZZY CARTESIAN PRODUCT

- **Let A be a fuzzy set on universe X and B be a fuzzy set on universe Y, then the Cartesian product between fuzzy sets A and B will result in a fuzzy relation R, which is contained within the full Cartesian product space,**

$$A \times B = R \subseteq X \times Y$$

$$\mu_R(x, y) = \mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

- **Each of the fuzzy sets could be thought of as a vector of membership values; each value is associated with a particular element in each set.**

EXAMPLE OF FUZZY CARTESIAN PRODUCT

- Let A be a fuzzy set (vector) having four elements
- Then A can be represented as a column vector of size 4×1

- Let B be a fuzzy set (vector) having five elements
- Then B can be represented as a row vector size of 1×5

- The resulting fuzzy relation R will be represented by a matrix of size 4×5, that is, R will have four rows and five columns

EXAMPLE OF FUZZY CARTESIAN PRODUCT

- **Let X = {x1, x2, x3} and Y = {y1, y2}**
- **Let A and B be two fuzzy sets defined over X and Y respectively by**
- **A = {(x1,0. 2), (x2, 0.5), (x3,1)} and B = { (y1, 0.3), (y2, 0.9) }.**
- **A can be represented as a column vector** $\begin{bmatrix} 0.2 \\ 0.5 \\ 1 \end{bmatrix}$
- **B can be represented as a row vector [0.3 0.9]**
- **The fuzzy Cartesian product** $A \times B$ **is given by a 3 x 2 matrix**

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix}$$

COMPOSITION OF FUZZY RELATIONS

- Composition of fuzzy relations can be defined just as it is for crisp (binary) relations
- Suppose **R is a fuzzy relation** on the Cartesian space **X × Y**
- **S is a fuzzy relation on Y × Z**
- Then the **composition** SoR **= T is a fuzzy relation on X × Z**
- **Then fuzzy max–min composition** is defined in terms of the set-theoretic notation and membership function-theoretic notation in the following manner:

$$\mu_T(x,z) = \bigvee_{y \in Y} (\mu_R(x,y) \wedge \mu_S(y,z))$$

- Neither **CRISP** nor **FUZZY COMPOSITION** are **commutative** in general, that is $R \circ S \neq S \circ R$

EXAMPLE

- **Let X = {x1, x2}, Y = {y1, y2}, and Z = {z1, z2, z3}.**
- R and S be given by

$$\begin{array}{c c} & \begin{array}{cc} Y1 & y2 \end{array} \\ \begin{array}{c} X1 \\ x2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \qquad \begin{array}{c c} & \begin{array}{ccc} z1 & z2 & z3 \end{array} \\ \begin{array}{c} Y1 \\ y2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

- The resulting **composition R o S,** which relates **elements of X** to **elements of Z** is defined on the Cartesian space **X x Z** as

$$\begin{array}{c c} & \begin{array}{ccc} z1 & z2 & z3 \end{array} \\ \begin{array}{c} X1 \\ x2 \end{array} & \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

$$\mu_T(x1, z1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

EXAMPLE

- **If we take the max product composition then**

$$T = \begin{array}{c} \\ x1 \\ x2 \end{array}\begin{array}{c} \begin{array}{ccc} z1 & z2 & z3 \end{array} \\ \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \end{array}$$

CRISP EQUIVALENCE RELATION

- A relation R on a universe X can also be thought of as a relation from X to X
- A relation R is an equivalence relation if it has the following three properties:

➢ reflexivity

➢ symmetry

➢ transitivity

- **Reflexive:** For all $x \in X$ **,** $xRx \text{ or } (x, x) \in R$
- **Symmetric:** $xRy \Rightarrow yRx$**,i.e.** $(x, y) \in R \Rightarrow (y, x) \in R, \forall x, y \in X$.
- **Transitive:** x R y and y R z $\Rightarrow$x R z or

$$(x, y), (y, z) \in R \Rightarrow (x, z) \in R$$

EQUIVALENCE RELATION IN TERMS OF CHARACTERISTIC FUNCTION

- R is **reflexive** if and only if

$$\chi_R(x,x) = 1, \forall x \in X$$

- R is **symmetric** if and only if

$$\chi_R(x,y) = 1 \Rightarrow \chi_R(y,x) = 1, \forall x, y \in X$$

- R is **transitive** if and only if

$$\chi_R(x,y) = 1 \, and \; \chi_R(\mathrm{y},\mathrm{z}) = 1 \Rightarrow \chi_R(\mathrm{y},\mathrm{z}) = 1, \forall x, y, z \in X$$

EXAMPLES

- **POSITIVE EXAMPLES:**
- Parallelism among straight lines in a plane
- Similarity among triangles in a plane
- Congruence among triangles in a plane
- Over Z, x R y iff x – y is even
- “Works in the same building as” among workers in a company
- “Reads in the same class as” among students in a college
- **NEGATIVE EXAMPLES:**
- “x R y iff x is a brother of y” among people in a town
- “x R y iff x is a friend of y” among people in a country

TOLERANCE RELATION

- A relation R on a set X is said to be a tolerance (proximity) relation iff it is only reflexive and symmetric but not transitive
- **Example:**
- X = {1, 2, 3} and R = { (1,1), (2,2), (3,3), (1,2), (2,1)}

- **NOTE:** If R is a tolerance relation on X of cardinality n then

$$\underbrace{R \circ R \circ \ldots\ldots \circ R}_{n-1 times}$$

- is always an equivalence relation

FUZZY EQUIVALENCE RELATION

- A fuzzy relation R on a single universe X is a **fuzzy equivalence relation** if all three of the following properties hold:

- **Fuzzy reflexive:** $\mu_R(x, x) = 1, \forall x \in X$

- **Fuzzy symmetric:** $\mu_R(x, y) = \mu_R(y, x), \forall x, y \in X$

- **Fuzzy transitive:**

- $\mu_R(x, y) = \lambda_1, \mu_R(\mathrm{y}, \mathrm{z}) = \lambda_2 \Rightarrow \mu_R(x, \mathrm{z}) = \lambda$, where $\lambda \geq \min\{\lambda_1, \lambda_2\}$.

FUZZY PROXIMITY RELATION

- A fuzzy relation R on a single universe X is a **fuzzy proximity relation** iff it is fuzzy reflexive and fuzzy symmetric

- **NOTE:** If R is a fuzzy tolerance relation on X of cardinality n then

$$\underbrace{R \circ R \circ \ldots\ldots \circ R}_{n-1 times}$$

- is always a fuzzy equivalence relation

AN EXAMPLE

- Consider the fuzzy relation from X to X, where |X| = 5:

$$\begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

- It is easy to verify that the relation is fuzzy reflexive and fuzzy symmetric **(All the diagonal elements are 1 and the matrix is symmetric)**

EXAMPLE CONTD…

- However, the relation is not fuzzy transitive, for

$$\mu_R(x_1, x_2) = 0.8 \text{ and } \mu_R(x_2, x_5) = 0.9$$

- Whereas

$$\mu_R(x_1, x_5) = 0.2 \leq \min[0.8, 0.9] = \min[\mu_R(x_1, x_2), \mu_R(x_2, x_5)]$$

- Here taking the 4th power of the relation we get

$$R^4 = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.5 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

EXAMPLE CONTD…

- It can be verified that R^4 is transitive also. For example, we can verify the case where it was not true earlier for R**.**

$$\mu_{R^4}(x_1, x_2) = 0.8 \text{ and } \mu_{R^4}(x_2, x_5) = 0.9$$

$$\mu_{R^4}(x_1, x_5) = 0.8 \geq \min[0.8, 0.9] = \min[\mu_{R^4}(x_1, x_2), \mu_R(x_2, x_5)]$$

- The other cases can be verified similarly

NOTE: It is not necessary that we should go up to R^{n-1} always to get the transitivity.
For example, here $R^3 = R^4$. So, we got the transitivity property satisfied earlier.

VALUE ASSIGNMENTS TO FUZZY RELATIONS

- The most definitive way for determining value assignments for relations is actually a family of procedures termed **similarity methods introduced by Zadeh in 1971**

- We shall consider two such methods coming under this broad category
- **1. Cosine Amplitude**
- **2. Max-Min method**

COSINE AMPLITUDE METHOD

- **Let** $X = \{x_1, x_2, \ldots x_n\}$ **. Suppose each** x_i **is an m-dimensional vector given by**

$$x_i = (x_{i1}, x_{i2}, \ldots, x_{im}), \mathrm{i} = 1, 2, \ldots \mathrm{n}$$

- **Then we know that** $x_i \cdot x_j = |x_i| \cdot |\mathrm{x}_j| \cos\theta$

- $x_i \cdot x_j = \sum_{k=1}^{m} x_{ik} x_{jk}$ **,** $|x_i| = \left(\sum_{k=1}^{m} x_{ik}^2 \right)^{1/2}$ **,** $|x_j| = \left(\sum_{k=1}^{m} x_{jk}^2 \right)^{1/2}$

- **So,** $$\cos\theta = \frac{\left|\sum_{k=1}^{m} x_{ik} x_{jk}\right|}{\left(\sum_{k=1}^{m} x_{ik}^2 \right)^{1/2} \cdot \left(\sum_{k=1}^{m} x_{jk}^2 \right)^{1/2}} = r_{ij}$$

MAX-MIN METHOD

- **Another popular method, which is computationally simpler than the cosine amplitude method, is known as the max–min method. In this case the matrix elements** r_{ij}

$$r_{ij} = \frac{\sum_{k=1}^{m} \min(x_{ik}, x_{jk})}{\sum_{k=1}^{m} \max(x_{ik}, x_{jk})}, \quad i, j = 1, 2, \ldots n$$

AN EXAMPLE

- **Suppose there was an earth quack in India affecting 5 regions**
- **Suppose the buildings in the area can be categorized into one of the three categories; no damage, medium damage and serious damage.**
- **Suppose the damage ratio in each region is given by**

$$\begin{array}{c c} & \begin{array}{ccccc} x1 & x2 & x3 & x4 & x5 \end{array} \\ \begin{array}{c} c1 \\ c2 \\ c3 \end{array} & \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.7 & 0.4 \\ 0.6 & 0.4 & 0.6 & 0.2 & 0.6 \\ 0.1 & 0.4 & 0.3 & 0.1 & 0 \end{bmatrix} \end{array}$$

EXAMPLE CONTD…(USING COSINE METHOD)

- **We shall use the cosine formula to derive the membership values between the 5 regions. Here n = 5 and m = 3**
- **So,**

$$R = \begin{bmatrix} 1 & & & & \\ 0.836 & 1 & & SYM & \\ 0.914 & 0.934 & 1 & & \\ 0.682 & 0.6 & 0.441 & 1 & \\ 0.982 & 0.74 & 0.818 & 0.774 & 1 \end{bmatrix}$$

- **For example,**

$$r_{12} = \frac{0.3 \times 0.2 + 0.6 \times 0.4 + 0.1 \times 0.4}{[(0.3^2 + 0.6^2 + 0.1^2).(0.2^2 + 0.4^2 + 0.4^2)]^{1/2}} = \frac{0.34}{[0.46 \times 0.36]^{1/2}} = 0.836$$

EXAMPLE CONTD…(USING MAX-MIN METHOD)

- **Here,**

$$R = \begin{bmatrix} 1 & & & & \\ 0.538 & 1 & & SYM & \\ 0.667 & 0.667 & 1 & & \\ 0.429 & 0.333 & 0.250 & 1 & \\ 0.818 & 0.429 & 0.538 & 0.429 & 1 \end{bmatrix}$$

- **For example,**

$$r_{12} = \frac{\min(0.3,0.2) + \min(0.6,0.4) + \min(0.1,0.4)}{\max(0.3,0.2) + \max(0.6,0.4) + \max(0.1,0.4)} = \frac{0.2 + 0.4 + 0.1}{0.3 + 0.6 + 0.4} = 0.538$$

FUZZIFICATION

- Fuzzification is the process of **making a crisp set fuzzy**
- We do this by simply recognizing that **many of the quantities** that **we consider to be crisp and deterministic are actually not** deterministic at all; they carry considerable uncertainty
- If the **form of uncertainty** happens to arise **because of imprecision, ambiguity, or vagueness,** then the **variable is probably fuzzy** and **can be represented by a membership function.**
- The **representation of imprecise data** as fuzzy sets **is a useful but not mandatory step** when those data are used in fuzzy systems

FUZZIFICATION CONTD…

- **Example 1:**
- Suppose in the **reading of a voltage**, we say it is **low voltage**
- **We need not measure it precisely**
- When we measure it **precisely it may be 0.3**
- That is **the membership value** of the **current voltage in the fuzzy set representing low voltage is 0.3**
- The membership function for "low" may be given by

$$\mu_{Low}(x) = \begin{cases} 0, & if \;\; x \geq 80; \\ \dfrac{80 - x}{x - 20}, & if \;\; 50 \leq x < 80; \\ 1, & if \;\; x \leq 50. \end{cases}$$

EXAMPLES OF FUZZIFICATION

- **Example 2:**
- We can say that somebody is young
- He may be 37 years
- But when we measure it precisely we say he is 0.92 young

- **Example 3:**
- In finding the height of a person, we may say that he is tall.
- Actually, his height may be 5 feet 10 inches.
- So, taking the height of tall persons, we may say he is 0.95 tall.

DEFUZZIFICATION

- We begin by considering a fuzzy set A, then define a **lambda-cut set,** A_λ where $0 \leq \lambda \leq 1$.
- The set A_λ is a **crisp set** called the lambda (λ)-cut set of the fuzzy set A, where $A_\lambda = \{x \mid \mu_A(x) \geq \lambda\}$.
- Any particular fuzzy set A can be transformed into **an infinite number of λ-cut sets**, because there are **an infinite number of values λ** in the interval [0, 1].

EXAMPLE

- Let X = {a, b, c, d, e, f} and a fuzzy set A on X be defined as
- A = { (a, 1), (b, 0.9), (c, 0.6), (d, 0.3), (e, 0.01), (f,0)}
- A_1 = {*a*}
- $A_{0.9}$ = {a, b}
- $A_{0.6}$ = {a, b, c}
- $A_{0.7}$ = {a, b}
- $A_{0.3}$ = {a, b, c, d}
- $A_{0.001}$ = {a, b, c, d, e},
- A_0 = *X.*

PROPERTIES OF LAMBDA CUTS

1. $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$

2. $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$

3. $(A^C)_\lambda \neq A_\lambda$ *except for a value of λ = 0.5*

4. For any *λ ≤ α, where 0 ≤ α ≤ 1, it is true that* $A_\alpha \subseteq A_\lambda$*, where* A_0 *= X*

SOME PROOFS

- We have $\mu_{A\bigcup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$
- So, $x \in (A \bigcup B)_\lambda \Leftrightarrow \max\{\mu_A(x), \mu_B(x)\} \geq \lambda$

$$\Leftrightarrow \mu_A(x) \geq \lambda \; or \; \mu_B(x) \geq \lambda$$

$$\Leftrightarrow x \in A_\lambda \; or \; x \in B_\lambda$$

$$\Leftrightarrow x \in A_\lambda \bigcup B_\lambda$$

Again $\mu_{A\bigcap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$

- So, $x \in (A \bigcap B)_\lambda \Leftrightarrow \min\{\mu_A(x), \mu_B(x)\} \geq \lambda$

$$\Leftrightarrow \mu_A(x) \geq \lambda \; and \; \mu_B(x) \geq \lambda$$

$$\Leftrightarrow x \in A_\lambda \; and \; x \in B_\lambda$$

$$\Leftrightarrow x \in A_\lambda \bigcap B_\lambda$$

λ-CUTS FOR FUZZY RELATIONS

- A fuzzy relation can be converted to a crisp relation in the following manner
- R_λ = {(x, y) | $\mu_R(x, y)$ ≥ λ} as a λ-cut relation of the fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

EXAMPLE CONTD…

- Taking $\lambda = 1$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Taking $\lambda = 0.6$

$$R_{0.6} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

PROPERTIES OF λ-CUTS OF FUZZY RLATIONS

- For any two fuzzy relations R and S, we have

(i) $(\mathrm{R} \bigcup \mathrm{S})_\lambda = R_\lambda \bigcup S_\lambda$

(ii) $(\mathrm{R} \bigcap \mathrm{S})_\lambda = R_\lambda \bigcap S_\lambda$

(iii) $(R^C)_\lambda \neq (R_\lambda)^C$

(iv) For any λ ≤ α, 0 ≤ α ≤ 1, then $R_\alpha \subseteq R_\lambda$

DEFUZZIFICATION TO SCALARS

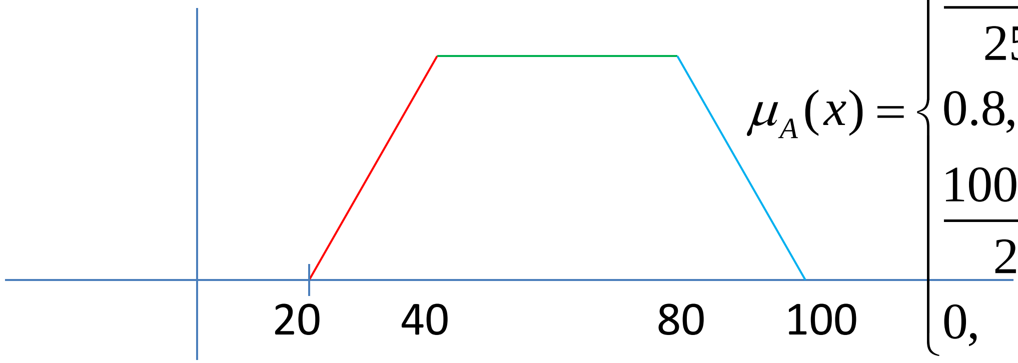
- There may be situations where the **output of a fuzzy process needs to be a single scalar quantity** as opposed to a fuzzy set

- **Defuzzification** is the **conversion of a fuzzy quantity to a precise quantity,** just as fuzzification is the conversion of a precise quantity to a fuzzy quantity

- The **output of a fuzzy process** can be the **logical union of two or more fuzzy membership functions** defined on the universe of discourse of the output variable

SOME TYPES OF MEMBERSHIP FUNCTIONS

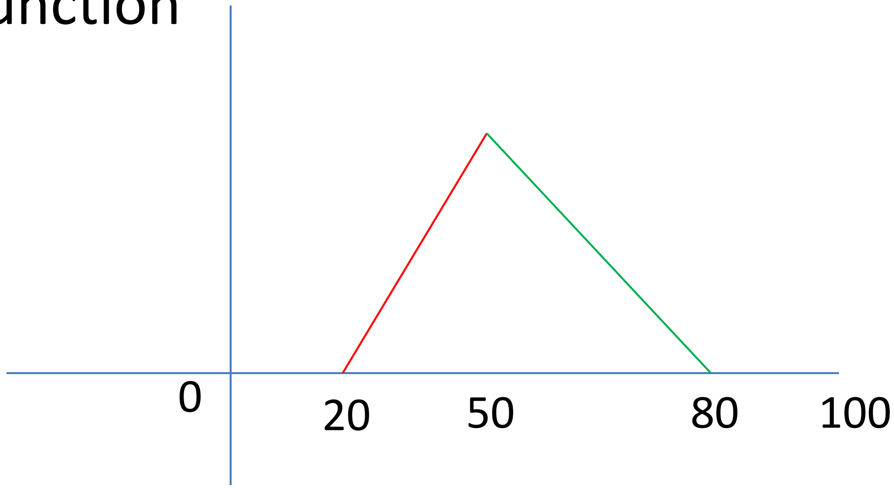
- Trapezoidal membership function



$$\mu_A(x) = \begin{cases} 0, & 0 \le x \le 20; \\ \dfrac{x-20}{25}, & 20 \le x \le 40; \text{(Red line)} \\ 0.8, & 40 \le x \le 80; \text{(Green line)} \\ \dfrac{100-x}{25}, & 80 \le x \le 100; \text{(Blue line)} \\ 0, & x \ge 100. \end{cases}$$

- Triangular membership function

$$\mu_A(x) = \begin{cases} 0, & 0 \le x \le 20; \\ \dfrac{x-20}{30}, & 20 \le x \le 50; \\ \dfrac{80-x}{30}, & 50 \le x \le 80; \\ 0, & 80 \le \mathrm{x} \le 100. \end{cases}$$



DEFUZZIFICATION TO SCALARS CONTD…

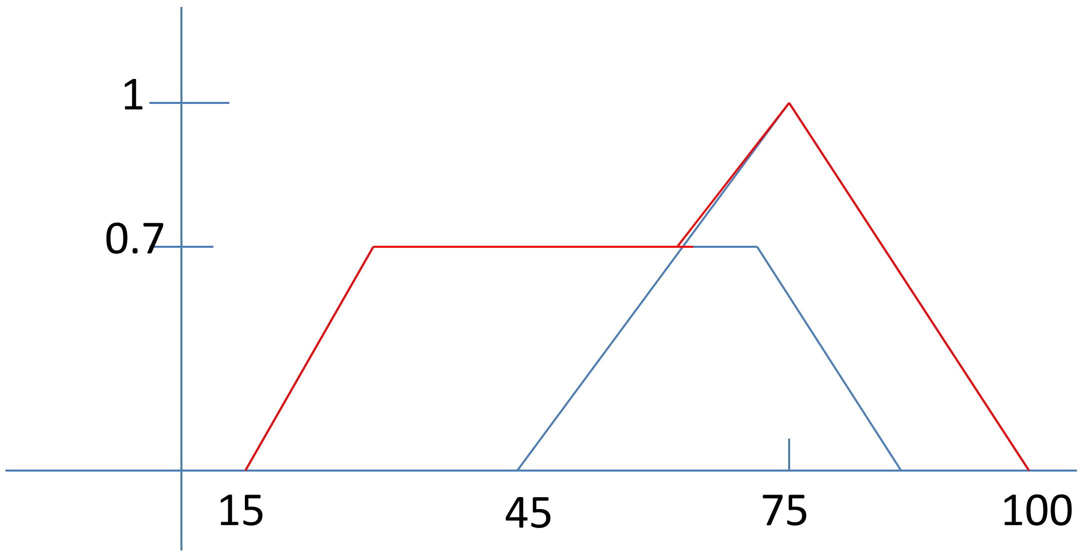
- Let the two fuzzy outputs be given by C_1 and C_2. Let these be given by **a trapezoidal subnormal function** and **normal triangular function** respectively

- **Their union will be a non-uniform figure** (a composition of the two figures)
- The number of output functions **may be more than 2**

- The problem is **how to express such an output as a single scalar**

COMPOSITION OF TWO MEMBERSHIP FUNCTIONS

- Composition of a subnormal trapezoidal fuzzy membership function and a normal triangular fuzzy membership function



DIFFERENT APPROACHES TO EXPRESS THE OUTPUT AS A SCALAR

- **Max membership principle:**
- *Also known as the* **height method***,* this scheme is **limited to peaked output functions.** This method is given by the algebraic expression
- $\mu_C(z^*) \geq \mu_C(z)$, for all z ∈ Z
- where $z*$ is the defuzzified value
- **Weighted average method:** The weighted average method is the most frequently used in fuzzy applications since it is one of the more computationally efficient methods.

DIFFERENT APPROACHES TO EXPRESS THE OUTPUT AS A SCALAR

- Unfortunately, it is **usually restricted to symmetrical output membership functions**
- It is given by the algebraic expression $z^* = \dfrac{\sum \mu_C(\bar{z}).\bar{z}}{\sum \mu_C(\bar{z})}$
- Where Σ denotes the algebraic sum and where $\bar{z}$ is the centroid of each symmetric membership function.
- For example, if a and b are the two central values for two fuzzy output functions with membership values 0.5 and 0.9 respectively then $z^* = (0.5 \times \mathrm{a} + 0.9 \times b) / (0.5 + 0.9)$

EXAMPLE

- In the figure given in slide 81, we have
- The **two central values are 45 and 75 respectively**
- The **weights are**, the membership values and are **0.7 and 1 respectively**
- So,

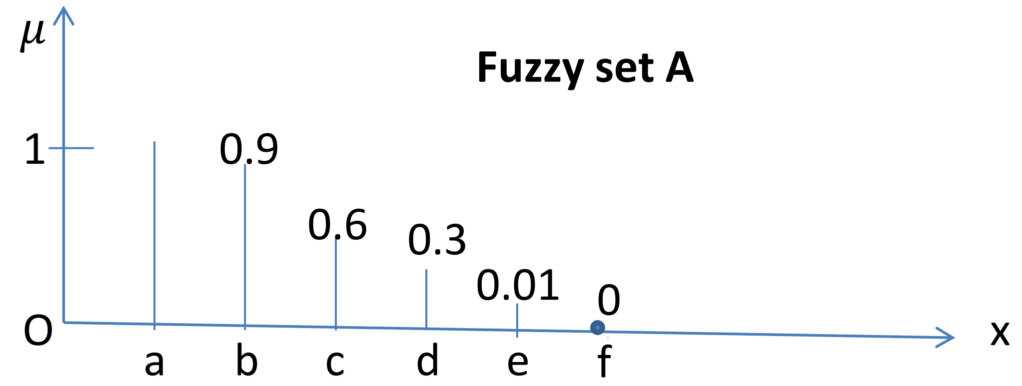
$$z^{*} = \frac{(0.7 \times 45) + 1 \times 75}{0.7 + 1} = \frac{106.5}{1.7} = 62.65$$

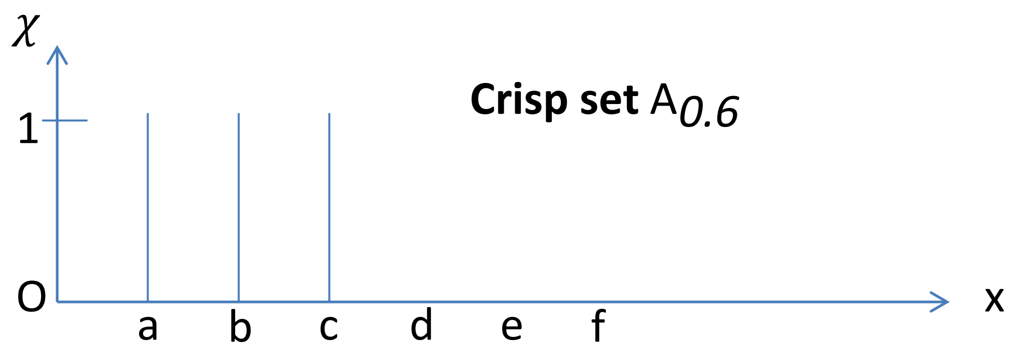
DEFUZZIFICATION

- The process of associating non-fuzzy sets with fuzzy sets
- Two types exist
- **Defuzzification to Crisp sets**
- **Defuzzification to scalars**
- Under defuzzification to crisp sets we have
- $\lambda -$ **cut sets,** $0 \leq \lambda \leq 1$
- For any fuzzy set **A,** it is denoted by $\mathbf{A}_{\boldsymbol{\lambda}}$
- $\mathrm{A}_{\lambda} = \{x|\mu_A(x) \geq \lambda\}$

DIAGRAMMATIC REPRESENTATION

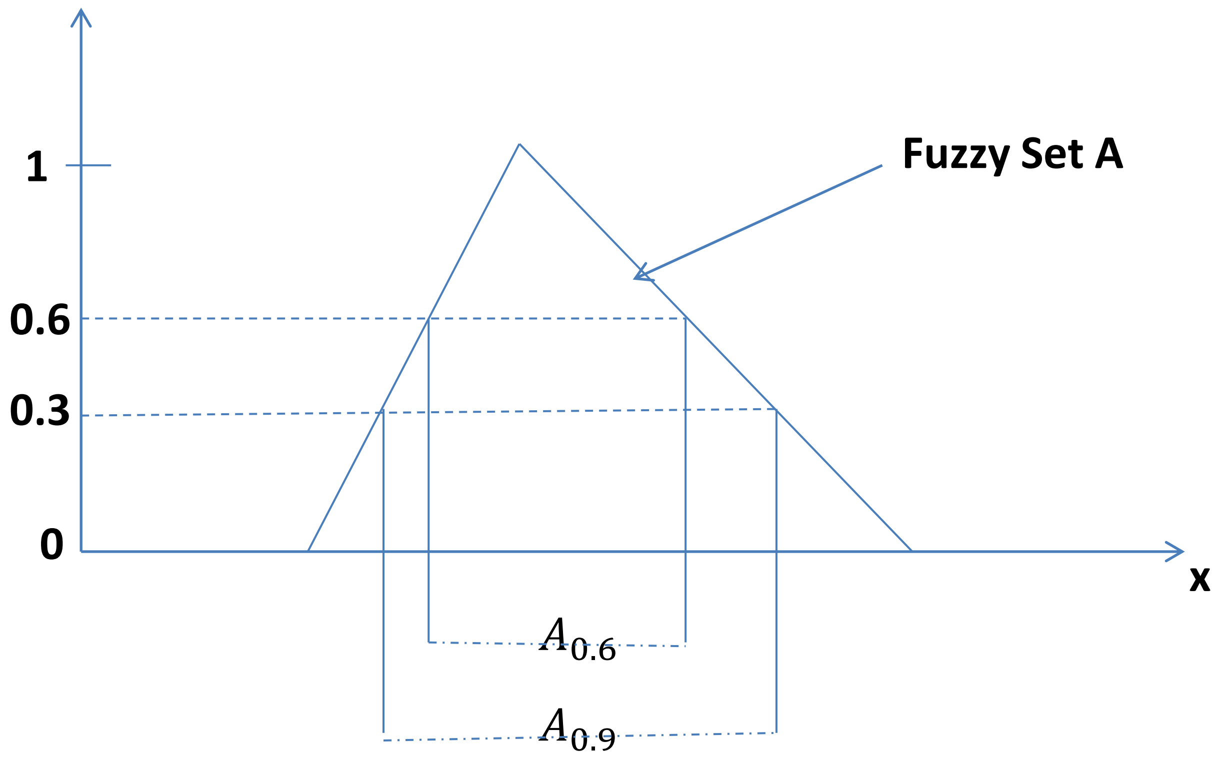
A





TWO DIFFERENT λ–CUTS FOR A CONTINUOUS VALUED FUZZY SET

a



DEFUZZIFICATION TO SCALARS

- In certain cases a fuzzy process needs to be a single scalar quantity as opposed to a fuzzy set
- As mentioned defuzzification is the process of conversion of a fuzzy quantity to a precise quantity
- The output of a fuzzy process can be the logical union of two or more fuzzy membership functions defined on the universe of discourse of the output variable

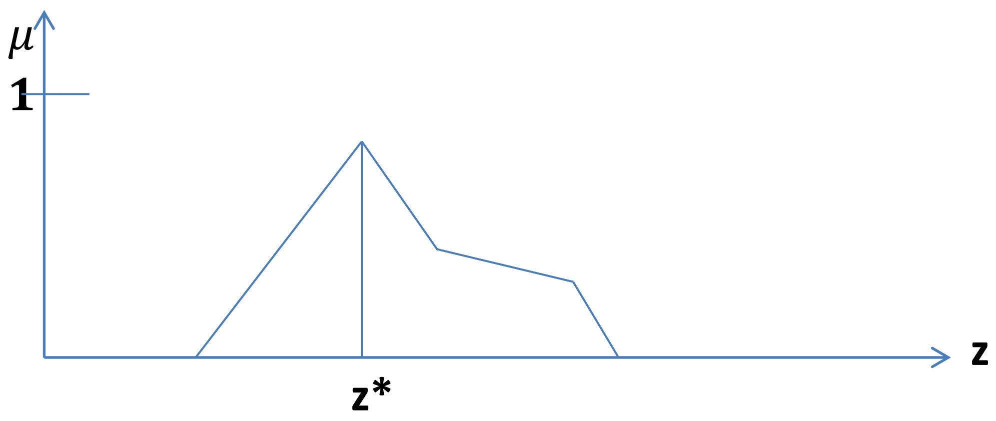
METHODS OF CONVERSION TO SCALARS

- There are several methods
- We select only SEVEN of these under two categories
- Group of FOUR and group of THREE

- **Group of FOUR**
  - **Max membership principle**
  - **Centroid method**
  - **Weighted average method**
  - **Mean-max membership**

MAX MEMBERSHIP PRINCIPLE

- Also known as **height method**
- **Limitation:** Applicable only to peaked output functions
- The output is the point z* (defuzzified value)
- Where $\boldsymbol{\mu_C(z\,*) \geq \mu_C(z), \forall z \in Z}$
-



CENTROID METHOD

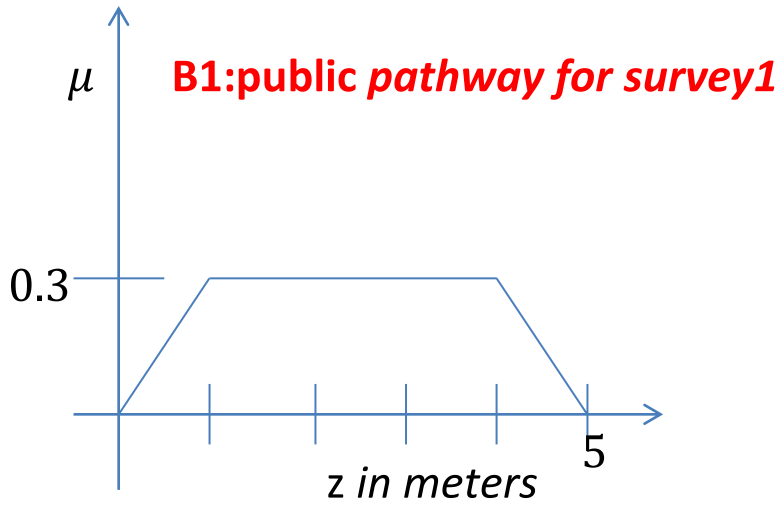
- Also called as **Centre of area** or **centre of gravity**
- **Most prevalent** and **physically appealing** of all the defuzzification methods
- Developed by **Sugeno 1985** and **Lee 1990**
- Given by the algebraic expression
- $$z^* = \frac{\int \mu_C(z).z dz}{\int \mu_C(z) dz}$$

EXAMPLE-1

- A railroad company intends to lay a new rail line in a particular part of a county.
- The **whole area through which the new line is passing must be purchased for right-of-way considerations**
- It is **surveyed in 3 stretches** and the data are collected for analysis
- The surveyed data for the road are given by **three fuzzy sets B1, B2 and B3**
- For the railroad to purchase the land, it must have an **assessment of the amount of land to be purchased**

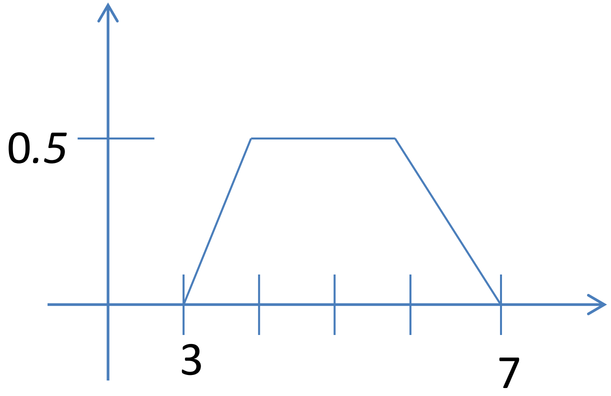
EXAMPLE-1 CONTD…

- The **three surveys** on right-of-way width **are ambiguous**
- However, because some of the land along the proposed railway route is already public domain, will not need to be purchased
- The three fuzzy sets are shown in the figures next

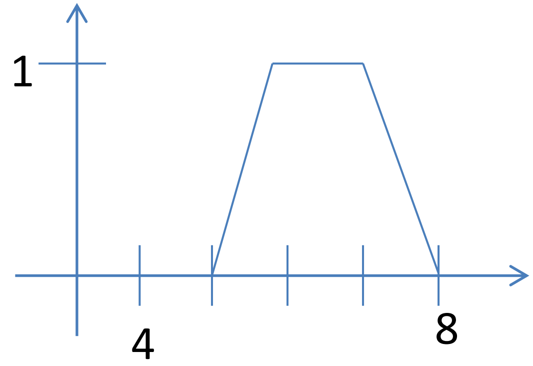


EXAMPLE-1 CONTD…

- Fuzzy set B2: **Public right-of-way for survey 2**:

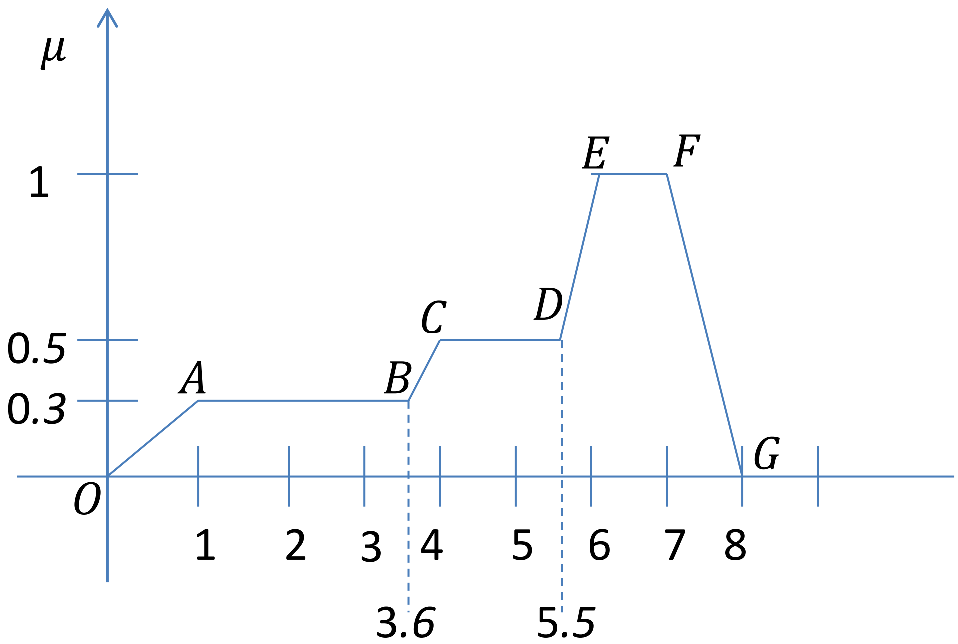


- Fuzzy set B3: **Public right-of-way for survey 3:**



EXAMPLE-1 CONTD…

- **We want to aggregate these three survey results** to find the single most nearly representative right-of-way width to allow the railroad to make its **initial estimate of the right-of-way purchasing cost.**
- The union of the three surveys is given by



COMPUTATION OF THE BOUNDARY EQUATIONS

- We use the 2 point formula to find the equations of the boundary lines
- Equation of the line ‘OA’ (It joins (0, 0) to (1, 0.3))
- $y - 0 = [(0 - 3)/(1 - 0)].(x - 0)$ i.e. y = (0.3).x
- Equation of the line ‘AB’ (It is parallel to X-axis)
- Y = 0.3
- Equation of the line ‘BC’ (It joins (3.6,0.3) to (4, 0.5))

$$y - 0.3 = \left(\frac{0.5 - 0.3}{4 - 3.6}\right) \times (x - 3.6)$$

- That is y = (x-3)/2

COMPUTATION OF THE BOUNDARY EQUATIONS

- Equation of the line ‘CD’ (It is parallel to the X-axis)
- y = 0.5
- Equation of the line ‘DE’ (It joins the points (5.5,0.5) to (6, 1))
- $(y-0.5)=\left(\dfrac{1-0.5}{6-5.5}\right)\times(x-5.5)$ i.e. y = x – 5
- Equation of the line ‘EF’ (It is parallel to the X-axis)
- y = 1
- Equation of the line ‘FG’ (It joins the points (7, 1) to (8, 0)
- $(y-1)=\left(\dfrac{0-1}{8-7}\right)\times(x-7)$ i.e. y = 8 - x

EXAMPLE-1 CONTD…

- We shall use the centroid method to find z*
- z*using the centroid method is given by

$$z* = \frac{\int \mu_B(z).zdz}{\int \mu_B(z)dz}$$

- So, z* =

$$\frac{\int_0^1 \Big((0.3z)zdz + \int_1^{3.6}(0.3)zdz + \int_{3.6}^4 \Big((z-3)/2)zdz + \int_4^{5.5}(0.5)zdz + \int_{5.5}^6(z-5)zdz + \int_6^7 zdz + \int_7^8(8-z)zdz}{\int_0^1 \Big((0.3z)dz + \int_1^{3.6}(0.3)dz + \int_{3.6}^4 \Big((z-3)/2)dz + \int_4^{5.5}(0.5)dz + \int_{5.5}^6(z-5)dz + \int_6^7 dz + \int_7^8(8-z)dz}$$

EXAMPLE-1 CONTD…

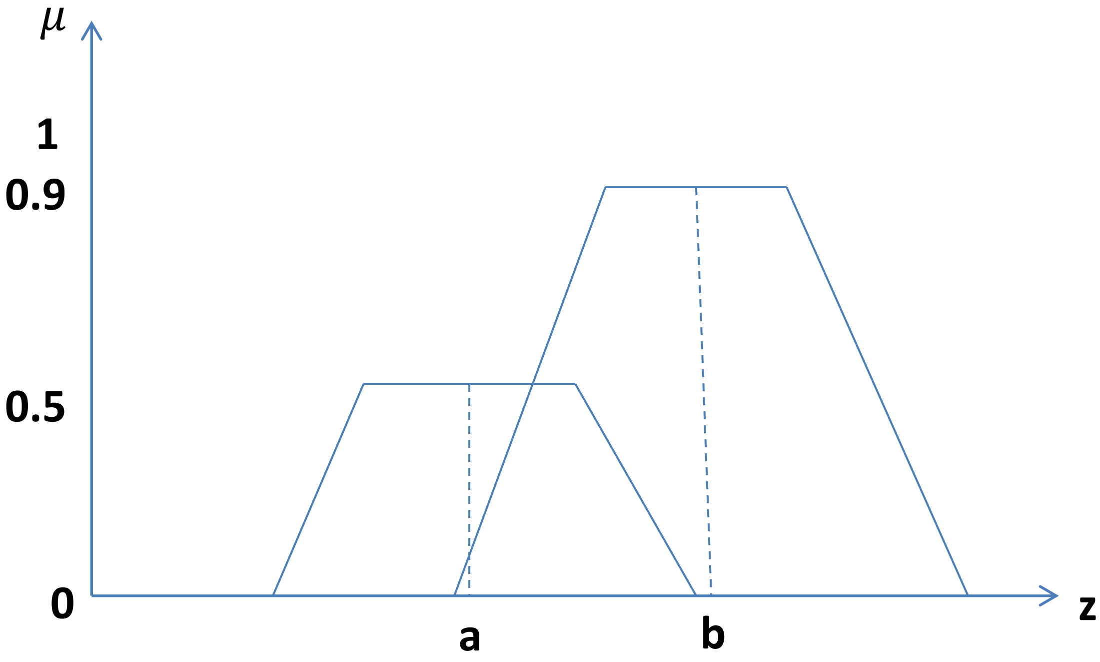
- $$= \frac{\left[0.3\frac{z^3}{3}\right]_0^1 + \left[0.3\frac{z^2}{2}\right]_1^{3.6} + \left[\frac{z^3}{6} - \frac{3z^2}{4}\right]_{3.6}^4 + \left[0.5\frac{z^2}{2}\right]_4^{5.5} + \left[\frac{z^3}{3} - 5.\frac{z^2}{2}\right]_{5.5}^6 + \left[\frac{z^2}{2}\right]_6^7 + \left[8.\frac{z^2}{2} - \frac{z^3}{6}\right]_7^8}{\left[0.3\frac{z^2}{2}\right]_0^1 + [0.3.z]_1^{3.6} + \left[\frac{z^2}{4} - \frac{3z}{2}\right]_{3.6}^4 + [0.5.z]_4^{5.5} + \left[\frac{z^2}{2} - 5.z\right]_{5.5}^6 + [z]_6^7 + \left[8.z - \frac{z^2}{4}\right]_7^8}$$
- = 4.9m

WEIGHTED AVERAGE METHOD

- This method is most frequently used in fuzzy application since it is one of the more computationally efficient methods
- It is usually restricted to symmetrical output membership functions
- The output z* is given by $$z^* = \frac{\sum \mu_C(\bar{z}).\bar{z}}{\sum \mu_C(\bar{z})}$$
- Where $\bar{z}$ is the centroid of each symmetric membership function
-

WEIGHTED AVERAGE METHOD

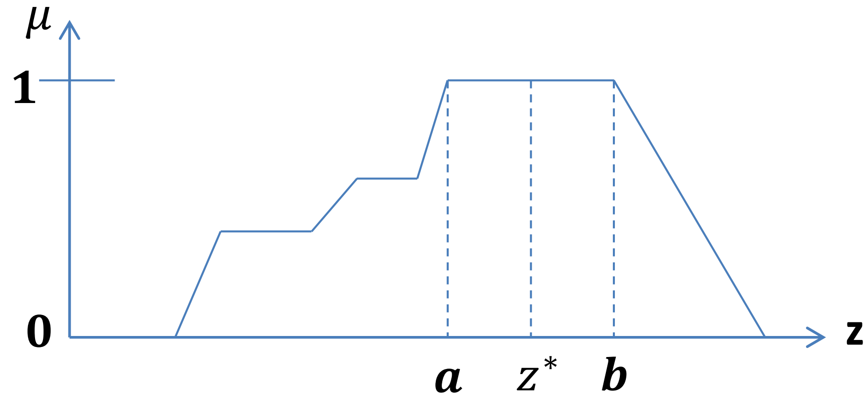
- a



$$z* = \frac{a.(0.5) + b(0.9)}{0.5 + 0.9}$$

MEAN MAX MEMBERSHIP METHOD

- Also called as **middle of maxima**
- It is closely related to Maximum membership principle
- The difference being the location of the maximum membership can be a plateau rather than a single point
- Introduced by **Sugeno 1985 and also Lee 1990**
- Here $\boldsymbol{z}^* = \frac{\boldsymbol{a}+\boldsymbol{b}}{\mathbf{2}}$, where a and b are the end points of the plateau



EXAMPLE-1
(MEAN MAX MEMBERSHIP METHOD)

- In the **mean max membership method** for defuzzification z* is given by (a + b)/2, where **a and b are the minimum and the maximum values where the maximum membership occurs**
- This method (also called **middle-of-maxima**)
- Is closely related to the first method (**Max membership principle**),
- Except that the **locations of the maximum membership can be non-unique** (i.e., the maximum membership can be a plateau rather than a single point).
- This method is given by the expression (Sugeno, 1985; Lee, 1990)

EXAMPLE-1
(WEIGHTED AVERAGE METHOD)

- The weighted average method is the most frequently used in fuzzy applications since it is one of the more computationally efficient methods.
- Unfortunately, it is usually restricted to symmetrical output membership functions.
- It is given by the algebraic expression

$$z* = \frac{\sum \mu_C(\bar{z}).\bar{z}}{\sum \mu_C(\bar{z})}$$

- $\bar{z}$ is the **centroid of each symmetric membership function**.

EXAMPLE-1 CONTD…

- In this case there are 3 symmetric regions
- So, the defuzzified value is

$$z^* = \frac{(0.3 \times 2.5) + (0.5 \times 5) + (1 \times 6.5)}{0.3 + 0.5 + 1}$$

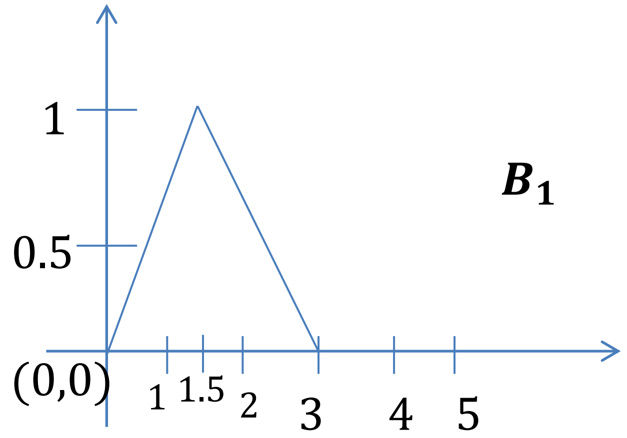
$$= \frac{.75 + .25 + 6.5}{1.8} = 5.41$$

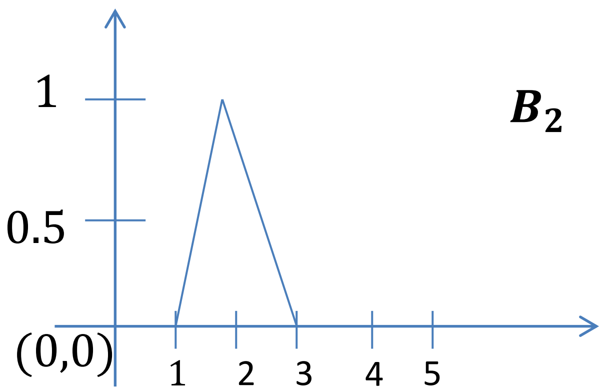
EXAMPLE-2

- Many products such as tar, petroleum jelly and petroleum are extracted from crude oil
- In a **newly drilled oil well three sets of oil samples** are taken and tested for their viscosity
- The **results are given in the form of three fuzzy sets B1, B2 and B3**
- All defined on a universe of normalised viscosity as shown in the figures below
- We want to find the most nearly representative viscosity value for all three oil samples
- Hence find z* for the three fuzzy viscosity sets

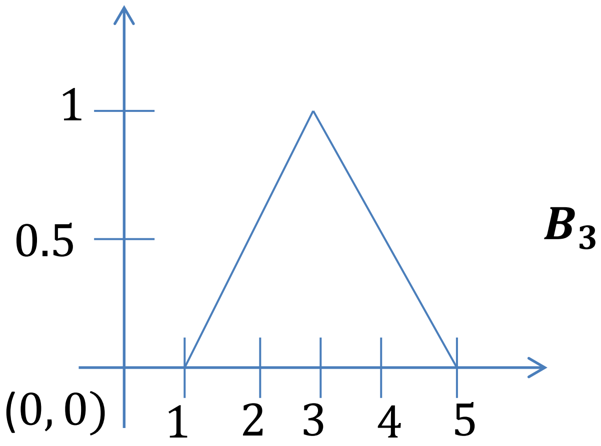
EXAMPLE-2 CONTD…

- The three membership functions are given by

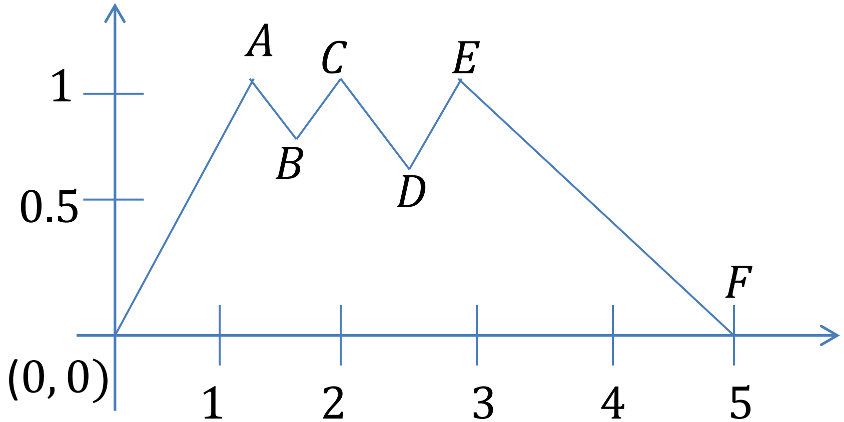




EXAMPLE-2 CONTD…



- The logical union of the 3 fuzzy sets is given as



EQUATIONS OF BOUNDARY LINES

- The line OA is joining the two points (0, 0) and (1.5, 1)
- Its equation is
- $(y-0) = \frac{1-0}{1.5-0} \times (x-0)$ i.e. y = (2/3)x = 0.67 x

- Similarly all other equations can be obtained.
- So, $$z^* = \frac{\int z.\mu_B(z)dz}{\int \mu_B(z)dz}$$

$$\frac{\int_0^{1.5}(0.67z)zdz + \int_{1.5}^{1.8}(2-0{,}67z)zdz + \int_{1.8}^{2}(z-1)zdz + \int_2^{2.33}(3-z)zdz + \int_{2.33}^{3}(0.5z-0.5)zdz + \int_3^5(2.5-0.5z)zdz}{\int_0^{1.5}(0.67z)dz + \int_{1.5}^{1.8}(2-0{,}67z)dz + \int_{1.8}^{2}(z-1)dz + \int_2^{2.33}(3-z)dz + \int_{2.33}^{3}(0.5z-0.5)dz + \int_3^5(2.5-0.5z)dz}$$

- =2.5m.

COMPUTATIONS

- The other computations are similar to those in example-1

(5)CENTRE OF SUMS METHOD

- This process involves the algebraic sum of individual output fuzzy sets, instead of their union
- **Advantages:**
- This is **faster than many defuzzification methods** that are currently in use
- This is **not restricted to symmetric membership functions**
- **Drawbacks:**

➢ The **intersecting areas are added twice**

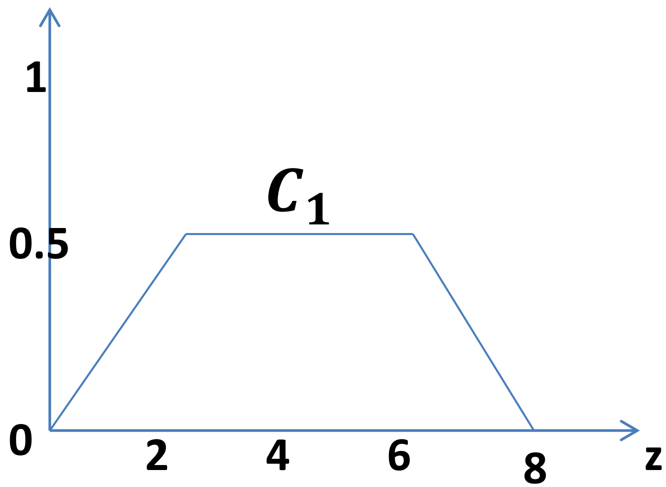
➢ The method also **involves finding the centroids of the individual membership functions**

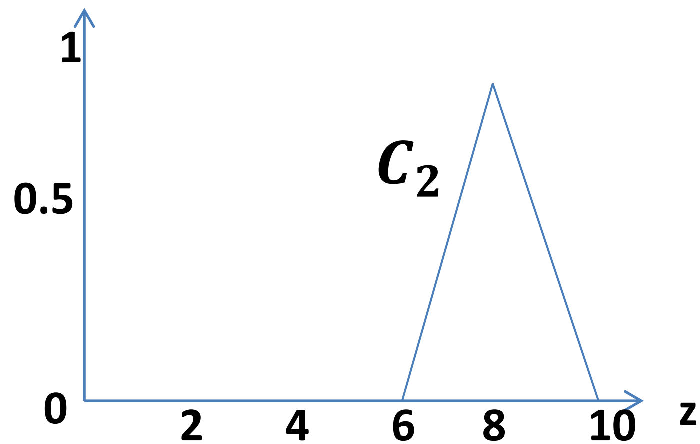
(5)CENTRE OF SUMS METHOD CONTD…

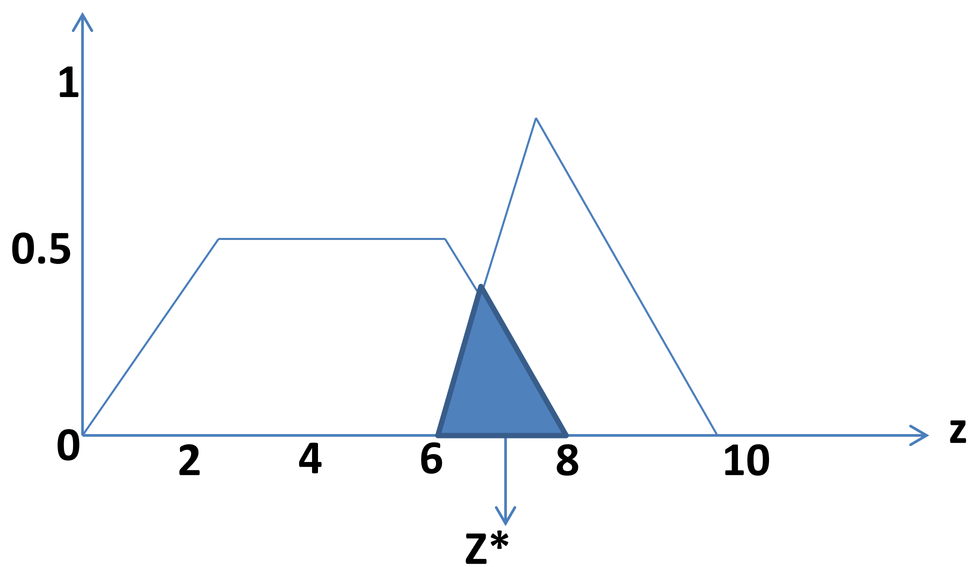
- The defuzzified value z* is given by
  - $$z^* = \frac{\sum_{k=1}^{n} \mu_{C_k}(z) \int_z \bar{z} dz}{\sum_{k=1}^{n} \mu_{C_k}(z) \int_z dz}$$
- Here $\bar{z}$ is the **distance to the centroid of each of the respective membership** functions
- This method is **similar to the weighted average method**
- Here the weights are the **areas of the respective membership functions**
- In the weighted average method the weights are the **individual membership values**

(5)CENTRE OF SUMS METHOD CONTD…

A







EXAMPLE-2 AND CENTRE OF SUM METHOD

- There are 3 graphs
- Here **we use the area of the trapezium** as
- ½(sum of the two parallel sides)x(distance between the parallel sides)
- The **three centroids are 2.5, 5 and 6.5** respectively
- So, the computations for the graphs in the **numerator of the formula are:**
- (2.5)x(0.5)x(0.3)x(3+5); (5)x(0.5)x(0.5)x(2+4) and (6.5)x(0.5)x1x(3+1) respectively
- In the **denominator we will have the same factors without the centroid values** (2.5 in the first, 5 in the second and 6.5 in the third)

EXAMPLE-2 AND CENTRE OF SUM METHOD

- So,
- $$z^* = \frac{(2.5\times0.5\times0.3\times8)+(5\times0.5\times0.5\times6)+(6.5\times0.5\times1\times4)}{(0.5\times0.3\times8)+(0.5\times0.5\times6)+(0.5\times1\times4)}$$
  $$= \frac{3+7.5+13}{1.2+1.5+2} = \frac{23.5}{4.7} = 5$$

CENTER OF LARGEST AREA METHOD

- If the **output fuzzy set has at least two convex sub regions** then the **centre of gravity of the convex fuzzy sub region with the largest area is used to obtain the defuzzified value** z* of the output
- The formula is

$$z^* = \frac{\int \mu_{C_m}(z).\, z dz}{\int \mu_{C_m}(z) dz}$$

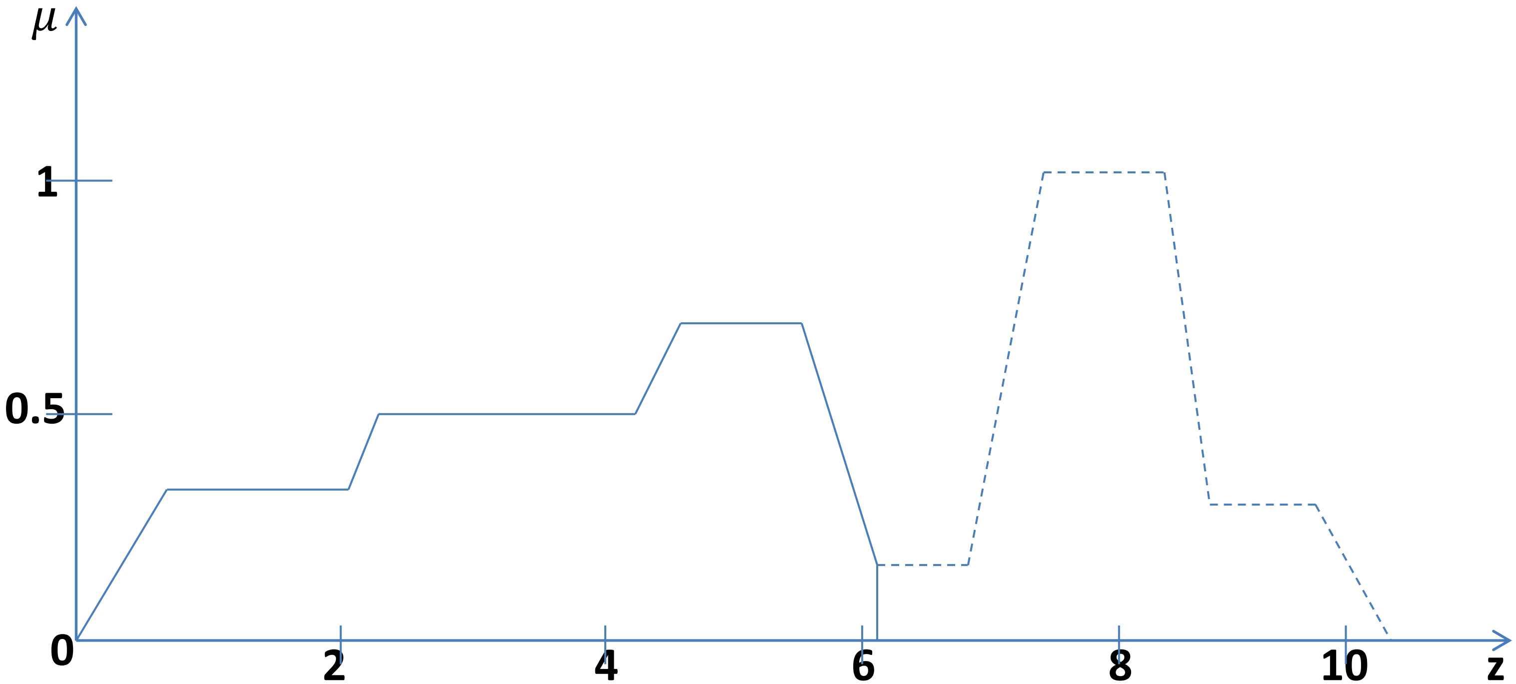
- Here C_m **is the convex sub region that has the largest area making up C_k**

(6)CENTER OF LARGEST AREA METHOD CONTD…

- This **condition applies** in the case when the **overall output** C_k **is non-convex**
- When C_k is convex, z* is the same quantity as determined by the centroid method or the centre of largest area method as then **there will be only one convex region**

(6)CENTER OF LARGEST AREA METHOD CONTD…

a



EXAMPLE-2 AND CENTRE OF LARGEST AREA METHOD

- As the whole area is convex, according to the centre of largest area method, the output is same as that of the centroid method.
- So, $z^* = 2.5$

(7)FIRST (OR LAST) OF MAXIMA

- This method uses the **overall output or union of all individual output fuzzy sets $\boldsymbol{C_k}$** to determine the smallest value (or the largest value) of the domain with maximized membership degree in C_k
- The formula for the output is determined as
- The **largest height** in the union (hgt(C_k)) is determined

$$hgt(C_k) = \sup_{z \in Z} \mu_{C_k}(z)$$

- The **first of the maxima** is found as

$$z^* = \inf_{z \in Z} \{z \in Z | \mu_{C_k}(z) = hgt(C_k)\}$$

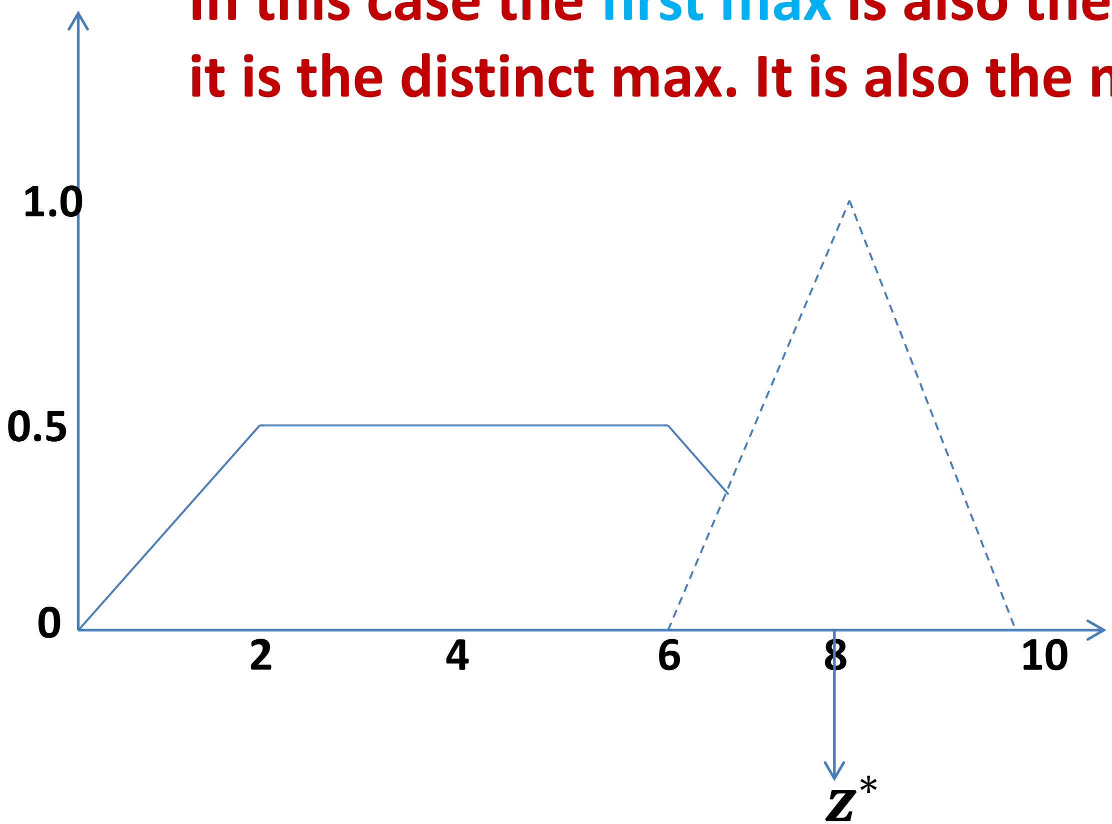
- (**Alternatively**) The **last of the maxima** is found as

$$z^* = \sup_{z \in Z} \{z \in Z | \mu_{C_k}(z) = hgt(C_k)\}$$

(7)FIRST (OR LAST) OF MAXIMA

a

In this case the first max is also the last max as it is the distinct max. It is also the mean max



COMPUTATIONS WITH EXAMPLE-2

- The maximum value occurs in the last trapezium and it is a plateau
- So, the first max = 6
- The last max = 7

SUMMARY

- Q. Of the seven defuzzification methods presented, which is the best?
- Ans. It is context or problem dependent
- Five Criteria of Hellendoorn and Thomas (1993):
- **Continuity:** A small change in the input of a fuzzy process should not produce a large change in the output
- **Disambiguity:** The defuzzification method should always result in a unique value for z*; i.e. there should not be any ambiguity in the defuzzified value
- **Plausibility:** z* should lie approximately in the middle of the support region of C_k and have a high degree of membership in C_k

SUMMARY

- **Computational Simplicity:** The more time consuming a method is, the less value it should have in a computation system
- **Weighting Method:** Which weighs he output fuzzy sets. It compares the output values and finds the difference. This criteria is not easy.

- **NOTE:** Other methods available are intended to seem as superior to the simple methods presented here.