Poisson Distribution

A type of probability distribution useful in describing the number of events that will occur in a specific period of time or in a specific area or volume is the Poisson distribution.

The following are some of the examples, which may be analysed using Poisson distribution.

1. The number of alpha particles emitted by a radioactive source in a given time interval.
2. The number of telephone calls received at a telephone exchange in a given time interval.
3. The number of defective articles in a packet of 100 .
4. The number of printing errors at each page of a book.
5. The number of road accidents reported in a city per day.

The Poisson Distribution

Properties of Poisson process

- The number of outcomes in one time interval is independent of the number that occurs in any other disjoint interval [Poisson process has no memory]
- The probability that a single outcome will occur during a very short interval is proportional to the length of the time interval and does not depend on the number of outcomes occurring outside this time interval.

Poisson distribution

The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval t, is

$$f(x, \lambda t) = P(X = x) = \frac{e^{-\lambda t}.(\lambda t)^x}{x!}, x = 0, 1, \dots \dots$$

where λ is the average number of outcomes per unit time and $e = 2.71828 \dots$

Moment generating function:

The MGF of the Poisson variate X with parameter λ given as

$$M_x(t) = E\left[e^{tx}\right] = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\mu}\mu^x}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{\left(\mu e^t\right)^x}{x!}$$

$$= e^{-\mu} e^{\mu e^t} \because \sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a$$

$$= e^{\mu\left(e^t - 1\right)}$$

Both the mean and the variance of the Poisson distribution $p(x; \lambda t)$ are λt. (Home work!!)

If we denote mean (λt) as μ, then we can write the probability distribution as

$$P(X = x) = \frac{e^{-\mu}.\mu^{x}}{x!}, x = 0, 1, \dots \dots$$

Ex: During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Answer:

Using the Poisson distribution with x = 6 and λt = 4, we have

$$p(6;4) = \frac{e^{-4}4^6}{6!} = 0.1042$$

Poisson Distribution as Limiting Form of Binomial Distribution

Poisson distribution is a limiting case of binomial distribution under the following conditions:

(i) n, the number of trials is indefinitely large, i.e., $n \rightarrow \infty$.

(ii) p, the constant probability of success in each trial is very small, i.e., $p \rightarrow 0$.

(iii) $n\,p(=\mu)$ is finite or $p = \frac{\mu}{n}$ and $q = 1 - \frac{\mu}{n}$.

Ex: In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles.
What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

Answer:

This is essentially a binomial experiment with $n = 8000$ and $p = 0.001$. Since p is very close to 0 and n is quite large, we shall approximate with the Poisson distribution using

$$\mu = (8000)(0.001) = 8.$$

Hence, if X represents the number of bubbles, we have

$$P(X < 7) = \sum_{x=0}^{6} b(x; 8000, 0.001) \approx p(x; 8) = 0.3134.$$

Ex: The number of monthly breakdowns of a computer is a random variable having Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month.

(a) Without a breakdown
(b) With only one breakdown and
(c) With at least one breakdown

Answer:

Let X denotes the number of breakdowns of the computer in a month.

X follows a Poisson distribution with mean $\mu = 1.8$.

$$P(X = x) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-1.8}(1.8)^x}{x!}$$

(a) $P(X = 0) = e^{-1.8} = 0.1653$
(b) $P(X = 1) = e^{-1.8}(1.8) = 0.2975$
(c) $P(X \geq 1) = 1 - P(X = 0) = 0.8347$

Poisson frequency distribution: If an experiment satisfying the requirements of Poisson distribution is repeated N times, the expected frequency distribution of getting x successes is given by

$$NP(X = x) = N\frac{e^{-\lambda t}.(\lambda t)^x}{x!}, x = 0, 1, \dots \dots$$

Ex: Fit a binomial distribution for the following data:

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80

Solution: Fitting a binomial distribution means assuming that the given distribution is approximately binomial and hence finding the probability mass function and then finding the theoretical frequencies.

To find the binomial frequency distribution $N * n_{C_x} p^x q^{n-x}$, which fits the given data, we require N, n and p. We assume N = total frequency = 80 and n = no. of trials = 6 from the given data.

To find p, we compute the mean of the given frequency distribution and equate it to np (mean of the binomial distribution).

x :	0	1	2	3	4	5	6	Total
f :	5	18	28	12	7	6	4	80
fx :	0	18	56	36	28	30	24	192

$$\bar{x} = \frac{\sum f x}{\sum f} = \frac{192}{80} = 2.4$$

i.e., $np = 2.4$ or $6p = 2.4$

$\therefore$ $p = 0.4$ and $q = 0.6$

If the given distribution is nearly binomial, the theoretical frequencies are given by the successive terms in the expansion of $80(0.6 + 0.4)^6$. Thus we get,

	x:	0	1	2	3	4	5	6
Theoretical	f:	3.73	14.93	24.88	22.12	11.06	2.95	0.33

Converting these values into whole numbers consistent with the condition that the total frequency is 80, the corresponding binomial frequency distribution is as follows:

x:	0	1	2	3	4	5	6	Total
f:	4	15	25	22	11	3	0	80

Ex: Fit a Poisson distribution for the following distribution:

x:	0	1	2	3	4	5	*Total*
f:	142	156	69	27	5	1	400

Solution Fitting a Poisson distribution for a given distribution means assuming that the given distribution is approximately Poisson and hence finding the probability mass function and then finding the theoretical frequencies.

To find the probability mass function

$$P\{X = r\} = \frac{e^{-\lambda} \cdot \lambda^r}{r!} \qquad r = 0, 1, 2, \ldots, \infty$$

of the approximate Poisson distribution, we require λ, which is the mean of the Poisson distribution.

We find the mean of the given distribution and assume it as λ.

x :	0	1	2	3	4	5	Total
f :	142	156	69	27	5	1	400
fx :	0	156	138	81	20	5	400

$$\bar{x} = \frac{\sum f x}{\sum f} = \frac{400}{400} = 1 = \lambda$$

The theoretical frequencies are given by

$$\frac{N e^{-\lambda} \cdot \lambda^r}{r!} \text{ where } N = 400\text{, obtained from the given distribution.}$$

$$= \frac{400\, e^{-1}}{r!}, \quad r = 0, 1, 2, \ldots, \infty$$

Thus, we get

	x:	0	1	2	3	4	5
Theoretical	f:	147.15	147.15	73.58	24.53	6.13	1.23

The theoretical frequencies for $x = 6, 7, 8, \ldots$ are very small and hence neglected.

Converting the theoretical frequencies into whole numbers consistent with the condition that the total frequency = 400, we get the following Poisson frequency distribution which fits the given distribution:

	x:	0	1	2	3	4	5
Theoretical	f:	147	147	74	25	6	1

Problem 1: There are 50 telephone lines in an exchange. The probability of them being busy is 0.1. What is the probability that all the lines are busy?

Problem 2: The probability that a bomb dropped from an envelope will strike a certain target is $\frac{1}{5}$. If 6 bombs are dropped, find the probability that (i) exactly 2 will strike the target and (ii) at least 2 will strike the target.

Problem 3: Suppose that P(X = 2) = $\frac{2}{3}$ P(X = 1), find P(X=0).

Problem 4: Probability of getting no misprint in a page of book is exp(-4). What is the probability that a page contains more than two misprints?

Problem 5: Six coins are tossed 6400 times using Poisson distribution. What is the approximate probability of getting six heads 10 times?

Problem 6: Fit a Poisson distribution to the following data and calculate the expected value and expected (theoretical) frequencies.

x	0	1	2	3	4
f	122	60	15	2	1