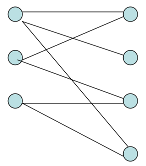
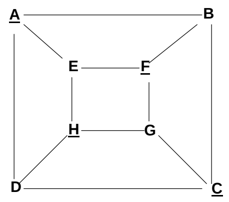
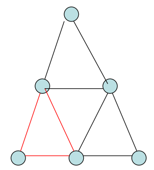
Matchings and Covers in bipartite graphs

- A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.





- Let G = (V,E) an undirected graph
- A **clique** is a set of pairwise adjacent vertices (any complete subgraph.
- The problem of finding the maximum size of a clique for a given graph is an NP-complete problem .



- Cliques arise in a number of areas of graph theory and combinatorics, including the theory of error-correcting codes.

- $\omega(G)$ – the number of graph vertices in the largest clique of G.
- For the complete graph K_n : $\omega(K_n) = n$
- Cubical graph: $\omega(G) = 2$.
- Cycle graph: $\omega(C_n) = 2$ for $n > 3$ and $\omega(C_3) = 3$

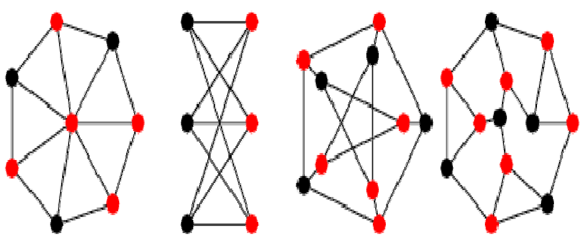
- A coclique is a set of pairwise non-adjacent vertices.
- A coclique in a graph is a clique in its complementary graph.

$\alpha(G):=\max\{|C| : C \text{ is a coclique}\}$ – coclique number

- A vertex cover is a subset $W \subset V$ such that $e \cap W \neq \varnothing$ for all $e \in G$.
- The problem of finding the minimum vertex cover for a given graph is an NP-complete problem .

$\tau(G):=\min\{|W| : W \text{ is a vertex cover}\}$ – vertex cover number.

Proposition 1: For each $U \subseteq V$ we have:
U is coclique if and only if $V \setminus U$ is a vertex cover.



- A matching is a subset $M \subset E$ such that: $e \cap e' = \varnothing$ for each $e,e' \in M$.

- The largest possible matching on a graph with n nodes consists of n/2 edges, and such a matching is called a perfect matching.
- A **perfect matching** is a matching which covers all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching.
- Although not all graphs have perfect matchings, a maximum matching exists for each graph.

- The maximum matching in a bipartite graph can be found in polynomial time.

$\upsilon(G)$:=max{|M| : M is a matching} – matching number.

- An edge cover is a subset F of E such that for each vertex v, there exist $e \in F$ such that $v \in e$.

Observation: An edge cover can exist only if G has no isolated vertices.

$\rho(G)$:=min{|F| : F is an edge cover} – edge cover number.

- A **complete bipartite graph** $G := (V_1,V_2,E)$ is a bipartite graph such that for any two vertices $v_1 \in V_1$ and $v_2 \in V_2$ (v_1,v_2) is an edge in G. A complete bipartite graph with partitions of size $|V_1|$=m and $|V_2|$=n is denoted by $K_{m,n}$.
- A complete bipartite graph $K_{m,n}$ has a vertex covering number of min{m,n} and an edge covering number of max{m,n} .
- A complete bipartite graph $K_{m,n}$ has a perfect matching of size min{m,n} .
- A complete bipartite graph $K_{m,n}$ has a coclique number of size max{m,n} .

- $\alpha(G)$:=max{|C| : C is a coclique} – coclique number
- $\tau(G)$:=min{|W| : W is a vertex cover} – vertex cover number.
- $\upsilon(G)$:=max{|M| : M is a matching} – matching number.
- $\rho(G)$:=min{|F| : F is an edge cover} – edge cover number.

Proposition 2: The following inequalities hold:

$$\alpha(G) \le \rho(G) \text{ and } \upsilon(G) \le \tau(G).$$

Observation: Strict inequalities are possible (for example the case of C_3).

Theorem (Gallai's theorem)

For any graph G=(V,E) without isolated vertices one has:

$$\alpha(G) + \tau(G) = |V| = \upsilon(G) + \rho(G).$$

Theorem (Konig's matching theorem)

For any bipartite graph G=(V,E) one has:

$$\tau(G) = \upsilon(G).$$

That is, the maximum cardinality of a matching in a bipartite graph is equal to the minimum cardinality of a vertex cover.

Theorem (Konig's edge cover theorem)

For any bipartite graph G=(V,E) one has:

$$\alpha(G) = \rho(G).$$

That is, the maximum cardinality of a coclique in a bipartite graph is equal to the minimum cardinality of an edge cover.

Cardinality bipartite matching algorithm

We focus on the problem of finding a maximum-sized matching in a bipartite graph.

- Let M be a matching in a graph G=(V,E).
- A path $P=(v_0,v_1,\dots,v_t)$ in G is called *M-augmenting* if

i) t is odd and $v_0,v_1,\dots,v_t$ are all distinct;
ii) $v_1v_2, v_3v_4, \dots, v_{t-2}v_{t-1} \in M$;
iii) $v_0, v_t \notin M$.

• If $P=(v_0,v_1,\ldots,v_t)$ is an M – augmenting path, then

$$M' := M \,\Delta\, E_P$$

is a matching satisfying $|M'|=|M|+1$.

• In fact, it is not difficult to show that:

Theorem: Let G=(V,E) be a graph and let M be a matching in G. Then either M is a matching of maximum cardinality, or there exists an M-augmenting path.

So in any graph, if we have an algorithm for finding an M augmenting path for any matching M, then we can find a maximum cardinality matching: we iteratively find matchings $M_0, M_1, \ldots$,until we have a matching M_k s.t. there does not exist any M_k - augmenting path.

Matching augmenting algorithm for bipartite graphs

Input: a bipartite graph G=(V,E) and a matching M,

Output: a matching M´ satisfying $|M'|>|M|$ (if there is one),

Description of the algorithm: Let G have colour classes U and W. Orient each edge e={u,w} of G (with $u \in U$ and $w \in W$) as follows:

if $e \in M$ then orient e from w to u,

if $e \notin M$ then orient e from u to w.

Let D be the directed graph arising in this way. Consider The sets $U' := U \setminus M$ and $W' := W \setminus M$.

- Now an M – augmenting path (if it exists) can be found by finding a directed path in D from any vertex in U´ to any vertex in W´.
- Hence in this way we can find a matching larger larger than M

Application: **The Marriage Problem**

Suppose that in a group of *n* single women and *n* single men who desire to get married, each participant indicates who among the opposite sex would be acceptable as a potential spouse. This situation could be represented by a bipartite graph in which the vertex classes are the set of *n* women and the set of *n* men, and a woman *x* is joined by an edge to a man *y* if they like each other.

Could we marry everybody to someone they liked?

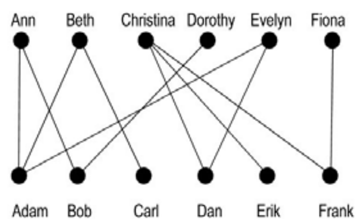


Figure 1

• Every woman can be married to at most one man, and every man to at most one woman. Therefore, a possible set of marriages can be represented as a subset *M* of the edges, no two of which are adjacent (matching).

• Thus, the marriage problem can be stated in graph-theoretic terms as asking if a given bipartite graph *G* has a perfect matching.

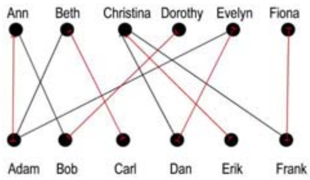


Figure 2 A perfect matching

- Let us suppose that *M* is a matching, if *M* is not a maximum matching, how could we improve it by finding a larger one?
- So suppose we have the matching Ann married to Bob, Beth to Adam, Christina to Dan, and Fiona to Frank. Dorothy, Evelyn, Carl and Erik are unmatched.
- To make progress we must be willing to rearrange our existing matchings, in order to increase their number. But how?

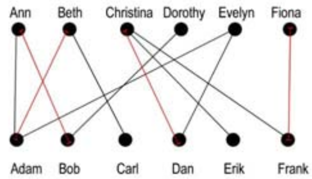


Figure 3 Another matching

- Let us start with a currently unmatched woman, say Dorothy. Now we could reason as follows: to match Dorothy we must marry her to Bob; but Bob is matched to Ann; maybe we could match Ann to someone else; well, we could match Ann to Adam instead, but Adam is already matched to Beth; so if we do that we must match Beth to someone else; we could match Beth to Carl. Carl is currently unmatched so we found a better matching!
- The new matching then is Dorothy to Bob, Ann to Adam, Beth to Carl, plus Christina to Dan, and Fiona to Frank who weren't affected by our rearrangement.

- We found a way to improve on a matching by finding a path from an unmmatched woman to an unmatched man in which every second edge is in the current matching. Such a path is called an *M - augmenting* path.

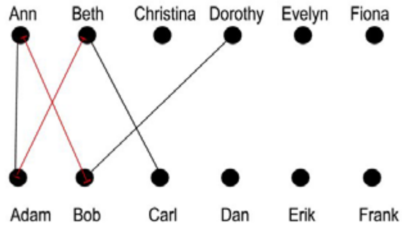
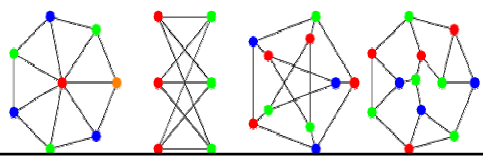


Figure 4 An alternating path in G :
Dorothy, Bob, Ann, Adam, Beth, Carl

Vertex coloring

- A vertex coloring is an assignment of labels or colors to each vertex of a graph such that no edge connects two identically colored vertices. The most common type of vertex coloring seeks to minimize the number of colors for a given graph.

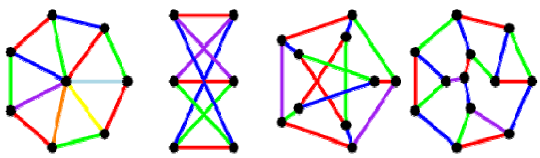
- The minimum number of colors which with the vertices of a graph G may be colored is called the chromatic number.
- It is NP-hard to decide if a graph G is k-colourable.



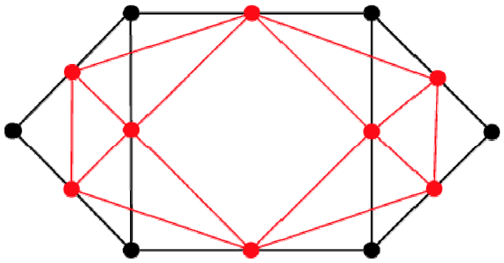
Edge coloring

An edge coloring of a graph G is a coloring of the edges of G such that adjacent edges (or the edges bounding different regions) receive different colors.

- The edge chromatic number gives the minimum number of colors with which a graph can be colored.
- Finding the minimum edge coloring is equivalent to finding the minimum vertex coloring of its line graph.



A line graph L(G) (also called an interchange graph) of a graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge iff the corresponding edges of G meet at one or both endpoints.



Applications:

- Map colouring;
- Storage of goods;
- Assignment frequencies to radio stations, car phones;
- Scheduling classes.