The Max-Flow Min-Cut Theorem

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1 Definitions

A *network* is a directed graph (digraph) $D = (V, A)$ with a *capacity function* $c : A \to \mathbb{R}$ assigning arcs to nonnegative real values. V can be partitioned into three sets: the *sources* X, the *sinks* Y, and the *intermediates* I. X and Y must be nonempty.

To a network we may associate a *flow* $f : V \to \mathbb{R}$ assigning arcs to nonnegative real values such that $0 \leq f(a) \leq c(a)$ for any $a \in A$ and $f_{\text{in}}(v) = f_{\text{out}}(v)$ for all $v \in I$, where

$$f_{\text{in}}(v) = \sum_{uv \in A} f(uv) \text{ and } f_{\text{out}}(v) = \sum_{vu \in A} f(vu).$$

In other words, the flow over any arc is no more than its capacity, and the inflow is equal to the outflow on intermediates. (One might also require that the sources have greater outflow than inflow, and vice-versa for sinks; indeed this is always the intention.) The *value* of a flow f, denoted $\operatorname{val} f$, is defined as

$$\operatorname{val} f = \sum_{x \in X} f_{\text{out}}(x) - f_{\text{in}}(x)$$

or, by an abuse of notation, $\operatorname{val} f = f_{\text{out}}(X) - f_{\text{in}}(X)$. Given a network, the natural optimization problem is: *what is the maximum value attained by any flow?*

A *cut* $(S, \bar{S})$ in a network is the set of arcs $\{s\bar{s} \in A \mid s \in S, \bar{s} \in \bar{S}\}$ where $X \subseteq S \subseteq V - Y$ and $\bar{S} = V - S$. The *capacity* of a cut K, denoted $\operatorname{cap} K$, is defined as

$$\operatorname{cap} K = \sum_{a \in K} c(a).$$

Finally, for each cut $K = (S, \bar{S})$ we can define an *anticut* $\bar{K} = (\bar{S}, S) = \{\bar{s}s \in A \mid s \in S, \bar{s} \in \bar{S}\}$.

2 Max-Flow Min-Cut Theorem

Our aim will be to show that the maximum of all flow values (ie, the value of the *maximum flow*), is equal to the minimum of all cut capacities (ie, capacity of the *minimum cut*). This is not at all hard to prove.

Lemma 1. *Given a network, for any flow f and cut K on the network,* $\operatorname{val} f \leq \operatorname{cap} K$.

Proof. Let $K = (S, \bar{S})$. As S is comprised of sources and intermediates, clearly

$$\operatorname{val} f = f_{\text{out}}(X) - f_{\text{in}}(X) = f_{\text{out}}(S) - f_{\text{in}}(S)$$

since intermediates contribute nothing to the flow value. Now consider an arc with both endpoints in S: its flow is counted in both $f_{\text{out}}(S)$ and $f_{\text{in}}(S)$, and thus makes no net impact on the flow value. Therefore the only arcs flows which positively impact $\operatorname{val} f$ are those originating in S and terminating in $\bar{S}$, which are precisely the flows over the cut K. We conclude that

$$\operatorname{val} f \leq \sum_{a \in K} f(a) \leq \sum_{a \in K} c(a) = \operatorname{cap} K.$$

□

The following corollary is a quite obvious consequence of Lemma 1, so we present it without proof.

Corollary 2. *Given a network, let f^* be the maximum flow and K^* the minimum cut on the network. Then* $\operatorname{val} f^* \leq \operatorname{cap} K^*$.

The next step is somewhat more complicated; it requires some ideas about how to construct a flow that is in some sense "maximal" (indeed, maximum, although we will not know that until later).

Consider a *u, v-path* as a set of intersecting arcs connecting vertex u to vertex v, taken without regard to arc direction. Given u, v-path P and flow f on a network, let the *f-augment* of P, denoted $\iota_f(P)$, be defined as

$$\iota_f(P) = \min_{a \in P} \iota_f(a)$$

where

$$\iota_f(a) = \begin{cases} c(a) - f(a), & \text{if } a \text{ points towards } v \\ f(a), & \text{if } a \text{ points towards } u \end{cases}.$$

An x, y-path is *f-augmenting* iff x is a source, y is a sink, and the f-augment is positive. Given an f-augmenting path P, we can very simply construct a new flow f' with $\operatorname{val} f' = \operatorname{val} f + \iota_f(P)$: for every arc on P, increase the flow by $\iota_f(P)$ if it is a forward arc, or decrease the flow by $\iota_f(P)$ if it is a reverse arc.

Lemma 3. *Given a network, there exist flow f and cut K on the network such that* $\operatorname{val} f = \operatorname{cap} K$.

Proof. Let f be a flow such that there are no f-augmenting paths in the network. (We can construct such a flow by starting with the zero flow; we then find an augmenting path, adjust the flow as described above, and repeat until no more augmenting paths can be found.) Let S be a set of containing the network sources and all vertices v such that there exists an x,v-path from some source x to v with positive f-augment. Note, then, that $\bar{S}$ contains the network sinks, and let $K = (S, \bar{S})$.

Suppose, for the sake of contradiction, that $f(a) < c(a)$ for some $a \in K$. Let $a = s\bar{s}$, and note that an x,s-path from source x to s with positive f-augment may be extended to an $x,\bar{s}$-path also with positive f-augment. Thus $\bar{s} \in S$, which is a contradiction, and we conclude $f(a) = c(a)$ for all $a \in K$. Similarly, $f(a') = 0$ for all $a' \in \bar{K}$.

In the proof of Lemma 1 we demonstrated that only the flows over the cut K positively impact $\operatorname{val} f$; by the same argument, only the flows over the anticut negatively impact $\operatorname{val} f$. We conclude $\operatorname{val} f = \operatorname{cap} K$. □

We now have all the necessary elements to prove our main result.

Theorem 4 (Max-Flow Min-Cut Theorem). *For any network, the value of the maximum flow is equal to the capacity of the minimum cut.*

Proof. Let f^* be the minimum flow and K^* the minimum cut on a given network. By Lemma 1 and Lemma 3 there exists some flow f and cut K such that $\operatorname{cap} K = \operatorname{val} f \leq \operatorname{cap} K^*$, but no cut can have capacity less than the minimum cut, so in fact $\operatorname{val} f = \operatorname{cap} K^*$. Corollary 2 states $\operatorname{val} f^* \leq \operatorname{cap} K^* = \operatorname{val} f$, but no flow can have value greater than the maximum flow, so $\operatorname{val} f^* = \operatorname{cap} K^*$. □

3 Some Applications

In this section we discuss some simple applications of the Max-Flow Min-Cut Theorem. Here we will only sketch proofs; completely rigorous proofs can be found in almost any graph theory textbook.

First, given any digraph with at least two vertices, designate some vertex x the source and vertex y the sink, and let all arcs have unit capacity.

Then a flow on this network counts (via its value) a number of arc-disjoint directed x, y-paths, and a cut counts (via its capacity) a number of arcs whose deletion destroys all x, y-paths. The following flows immediately from the Max-Flow Min-Cut Theorem. (The statement for undirected graphs comes from the trivial observation that a graph can be represented as a digraph.)

Theorem 5 (Menger's Theorem I). *Let x, y be distinct vertices of a digraph D. The maximum number of arc-disjoint directed x, y-paths in D equals the minimum number of arcs whose deletion destroys all directed x, y-paths in D.*

Similarly, let x, y be distinct vertices of a graph G. The maximum number of edge-disjoint x, y-paths in G equals the minimum number of edges whose deletion destroys all x, y-paths in G.

There is also a variant of Menger's Theorem that treats vertices as the objects of attention.

Theorem 6 (Menger's Theorem II). *Let x, y be distinct vertices of a digraph D not joined by any arc. The maximum number of internally-disjoint directed x, y-paths in D equals the minimum number of vertices (excluding x, y) whose deletion destroys all directed x, y-paths in D.*

Similarly, x, y be distinct, nonadjacent vertices of a graph G. The maximum number of internally-disjoint x, y-paths in G equals the minimum number of vertices (excluding x, y) whose deletion destroys all x, y-paths in G.

Proof sketch. Consider the digraph version; the graph version will follow directly from this. Construct digraph D' from $D = (V, A)$ as follows:

1. split every vertex $v \in V - \{x, y\}$ into vertices v', v'' joined by arc $v'v''$; then

2. for each arc originating in $u \in V - \{x, y\}$, replace u with u'', and for each arc terminating in $v \in V - \{x, y\}$, replace v with v'.

It can be verified that arcs in D' correspond to vertices in D in such a way that Menger's Theorem I applied to D' to give the desired result. □

Finally, we use Menger's Theorem II to prove the following theorem due to König. Let us first recall that, for any matching M, a vertex v is *M-saturated* iff some edge of M is incident on v; otherwise v is *M-unsaturated.*

Theorem 7 (König's Theorem). *The number of edges in a maximum matching is equal to the number of vertices in a minimum covering in a bipartite graph.*

Proof. Let G be a bipartite graph with maximum matching M^* and minimum covering K^*. We construct the graph N from G by introducing vertices x, y such that x is adjacent only to every vertex of one partite set and y is adjacent only to every vertex of the other partite set. (x will act as the "source" and y as the "sink.")

Consider a set $\mathcal{P}$ of internally-disjoint x,y-paths in N. We can exhibit a function $\phi : \mathcal{P} \to M^*$ by the following method: for any $P \in \mathcal{P}$, if the second vertex P_2 of P is M^*-saturated, let $\phi(P)$ be the associated match; otherwise P_2 is M^*-unsaturated, but then the third vertex P_3 of P must be M^*-saturated (otherwise M^* could be enlarged by P_2P_3, violating the maximality condition), and we can let $\phi(P)$ be the associated match. We now prove that ϕ is injective (one-to-one). Take $P, Q \in \mathcal{P}$ such that $\phi(P) = \phi(Q) = m \in M^*$, and suppose, for the sake of contradiction, that $P \neq Q$. P_2, Q_2 cannot both be M^*-saturated as they are in the same partite set and thus nonadjacent. Similarly, they may not both be M^*-unsaturated, as P_3, Q_3 would both be saturated and in the same partite set. Without loss of generality, say P_2 is saturated while Q_2 is not, but note that neither P nor Q contains $m = P_2Q_3$ as they must be internally-disjoint. But then M^* can be extended by removing m and adding P_2P_3 and Q_2Q_3, which contradicts the maximality condition. We conclude that $P = Q$, and thus ϕ is injective. Hence $|\mathcal{P}| \leq |M^*|$.

Consider a set $J \subset V(G)$ of vertices whose deletion destroys all x,y-paths in N. Suppose, for the sake of contradiction, that $|J| < |K^*|$; then J is not a vertex cover of G, so $G - J$ leaves at least one edge in G, and this edge must span the partite sets. Thus $N - J$ has an x,y-path, which is a contradiction. We conclude that $|K^*| \leq |J|$.

To sum up, note that $|M^*| \leq |K^*|$, so

$$|\mathcal{P}| \leq |M^*| \leq |K^*| \leq |J|$$

for any $\mathcal{P}$ and J as defined above. But if we choose a maximum-size $\mathcal{P}$ and a minimum-size J, Menger's Theorem II tells us $|\mathcal{P}| = |J|$, hence $|M^*| = |K^*|$. $\square$