Approximation Algorithms for Weighted Vertex Cover

CS 511

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Weighted Vertex Cover: Problem Definition

Input: An undirected graph $G = (V, E)$ with vertex weights $w_i \geq 0$.

Problem: Find a minimum-weight subset of nodes S such that every $e \in E$ is incident to at least one vertex in S.

Weighted Vertex Cover: Some Facts

- WVC is NP-hard.
- WVC can be 2-approximated.
  - Proved next.
- A 2-approximation algorithm for WVC does not provide any sort of approximation guarantee for maximum-weight independent set.

Weighted Vertex Cover: IP Formulation

$$
\begin{array}{llcll}
\text{minimize} & \sum_{i \in V} w_i x_i & & & \\
\text{subject to} & x_i + x_j & \geq & 1 & \text{for every edge } (i,j) \in E \\
 & x_i & \in & \{0,1\} & \text{for every vertex } i \in V
\end{array}
\qquad (1)
$$

Observation

- Any feasible solution x to (1) yields a cover $S = \{i \in V : x_i = 1\}$.
- If x^* is optimal solution to (1), then $S^* = \{i \in V : x_i^* = 1\}$ is a minimum weight vertex cover.

Weighted Vertex Cover: LP Relaxation

$$\begin{array}{llcll} \text{minimize} & \sum_{i \in V} w_i x_i & & & \\ \text{subject to} & x_i + x_j & \geq & 1 & \text{for every edge } (i,j) \in E \\ & x_i & \geq & 0 & \text{for every vertex } i \in V \end{array} \tag{2}$$

Observation

The optimum value of LP relaxation (2) is at most equal to the optimum value of integer program (1), because the LP has fewer constraints.

The LP-Rounding Algorithm

1. Compute the optimum solution x^* to LP relaxation (2).
2. Let $S = \{i \in V : x_i^* \geq 1/2\}$.
3. Return S.

Theorem

The LP-Rounding Algorithm is a 2-approximation algorithm for MWVC.

Proof.

1. S is a vertex cover. Consider an edge $(i, j) \in E$. Since $x_i^* + x_j^* \geq 1$, either $x_i^* \geq 1/2$ or $x_j^* \geq 1/2 \Rightarrow (i, j)$ is covered.
2. If S^* is an optimum vertex cover, then $w(S) \leq 2w(S^*)$.

$$\begin{aligned}
w(S^*) &\geq \sum_{i=1}^{n} w_i x_i^* && \text{since LP is a relaxation of ILP} \\
&\geq \sum_{i \in S} w_i x_i^* && \text{since } S \subseteq \{1, \ldots, n\} \\
&\geq \frac{1}{2} \sum_{i \in S} w_i && \text{since } x_i^* \geq 1/2 \text{ for all } i \in S \\
&= \frac{1}{2} w(S)
\end{aligned}$$

$\square$

Weighted Vertex Cover: Dual LP

$$
\begin{array}{llcll}
\text{maximize} & \sum_{e \in E} y_e & & & \\
\text{subject to} & \sum_{e=(i,j) \in E} y_e & \leq & w_i & \text{for every node } i \in V \\
 & y_e & \geq & 0 & \text{for every edge } e \in E
\end{array}
\tag{3}
$$

Intuition for Duality

- Edge e pays price $y_e \geq 0$ to be covered.
- Goal: Collect as much money as possible from the edges.
- Fair price condition: For every $i \in V$,

$$\sum_{e=(i,j)\in E} y_e \leq w_i .$$

The Dual Gives a Lower Bound

Lemma (Fairness Lemma)

Let $(y_1, y_2, \ldots, y_{|E|})$ be any feasible solution to the dual LP and S^* be a minimum-weight vertex cover. Then,

$$\sum_{e \in E} y_e \leq w(S^*).$$

Proof.

Let z_D^*, z_P^*, and z_{IP}^*, be the optimal objective values of the dual LP, the primal LP, and the primal ILP for WVC. Then,

$$\sum_{e \in E} y_e \underset{\text{dual optimality}}{\leq} z_D^* \underset{\text{strong duality}}{=} z_P^* \underset{\text{integrality}}{\leq} z_{IP}^* = w(S^*).$$

$\square$

Using Duality

- Any dual solution y gives a lower bound on the optimal solution to WVC — don't need to solve dual to optimality.
- However, y should be easy to convert into a vertex cover S.
- Further, S should not be too far from optimum.
- We can find such a y by a simple and fast method, without using an LP solver.

The Pricing Method

Definition

Vertex i is tight if $\sum_{e=(i,j)\in E} y_e = w_i$.

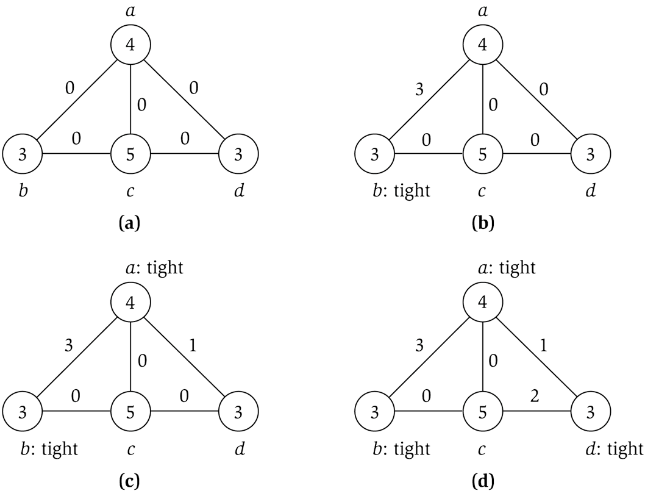
Pricing-Method(G, w)

for each $e \in E$
$\quad y_e = 0$

while there is an edge (i, j) such that neither i nor j are tight
$\quad$ select such an edge e
$\quad$ increase y_e as much as possible while preserving dual feasibility

$S = \{i \in V : i \text{ is tight}\}$
return S

The Pricing Method: Example



Source: Kleinberg & Tardos, *Algorithm Design* (Fig. 11.8).

Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.

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Proof.

1. Running time is polynomial.
   Reason: At least one new node becomes tight after each iteration of while loop.
2. The set S returned is a vertex cover.
   Reason: At termination, for each edge $e = (u, v)$, at least one of u and v is tight $\Longrightarrow$ at least one of u and v is in S.

□

Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.

Proof.

3. Let S^* be an optimum vertex cover. Then $w(S) \leq 2w(S^*)$.
   Reason:

$$
\begin{aligned}
w(S) &= \sum_{i \in S} w_i \\
&= \sum_{i \in S} \sum_{e=(i,j)} y_e && \text{since the nodes in } S \text{ are tight} \\
&\leq \sum_{i \in V} \sum_{e=(i,j)} y_e && \text{since } S \subseteq V \text{ and } y_e \geq 0 \text{ for all } e \\
&= 2 \sum_{e \in E} y_e && \text{because each edge is counted twice} \\
&\leq 2w(S^*) && \text{by the Fairness Lemma}
\end{aligned}
$$

□