LO

Assignment

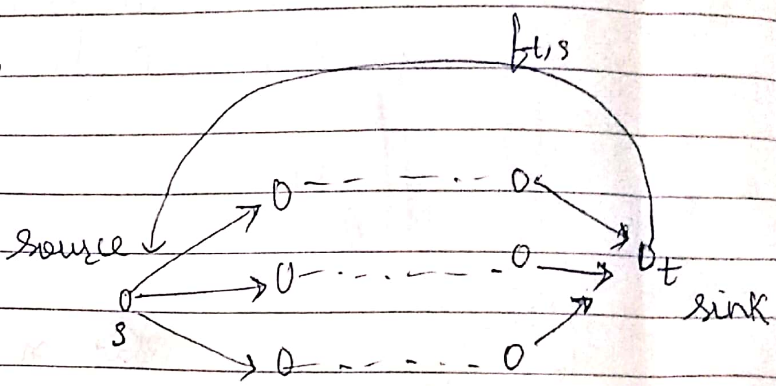
① Show that max-flow is equal to min-cut for any given flow network ?

Max-flow : objective := $\max \left\{ \sum_{u:(s,u)\in E} f_{s,u} \right\}$ st

① $\forall v \neq s,t \quad \sum_{u:(u,v)\in E} f_{u,v} = \sum_{w:(v,w)\in E} f_{v,w}$

② $f_{u,v} \leq c_{u,v}$

③ $f_{u,v} \geq 0$



The flow conservation holds at every vertex except sink and source but since outgoing flow at source must be same as the incoming at sink, we can introduce edge from t to s whose flow will be same as the value of total flow in the network

∴ LP reduces to

objective := $\max f_{t,s}$ st

① $\forall v \quad \sum_{u:(u,v)\in E} f_{u,v} \leq \sum_{u:(v,u)\in E} f_{v,u}$

② $f_{u,v} \leq c_{u,v}$

③ $f_{u,v} \geq 0$

let's try and construct a dual for the above problem!

dual: objective $:= \left(\min \sum_{(i,j)\in E} c_{ij} x_{ij}\right)$ st c_{ij} = capacity of $(i,j)\in E$

① $x_{ij} = \{0,1\}$

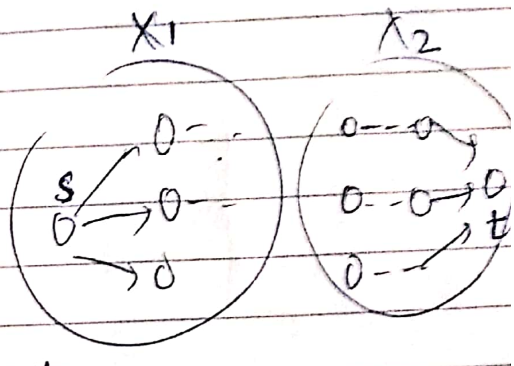
② $x_{ij} \geq 0 \quad \forall (i,j)\in E$

③ $x_{ij} \geq p_i - p_j \quad \forall (u,v)\in E$

④ $p_s - p_t \geq 1$

⑤ $p_i \geq 0 \quad \forall i \in V$

Min-flow:



define p_u st $p_u = 1$ if $u \in X1$ and $p_u = 0$ if $u \in X_2$ - where X1 is the set containing s after the cut and X_2 is set of vertices containing t after the cut.

so

objective function can be written as

$\min \sum_{(u,v)\in E} c_{ij} x_{ij}$ st

① $x_{ij} \geq p_i - p_j$

② $x_{ij} \geq 0$

③ $p_u \geq 0$

④ $p_s - p_t \geq 1$

→ thus only the edges included in the minimum possible cut shall have $x_{ij} = 1$

Clearly, the dual for the max-flow and the LP for the min-cut problem come out to be the same ; So have same optimal solution.

If the capacities are all integer, the min-cut will also be an integer and thus, max-flow will also be an integer for the given flow-networks.