Active Portfolio Construction

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August 24, 2019

Abstract

This article describes a quantitative framework to optimally express a researcher's view/alpha as a portfolio of investable products. We provide rigorous definitions of the terms "alpha" and "optimality". If needed, this framework can be widgetized. WIth the use of the widget, we take away discretion on aspects of portfolio construction from the researcher, allowing the researcher to focus on developing views that explain the return of assets in the market.

1 Terminology

Assume that the eligible universe of investable assets is a set of N stocks $S_1, S_2, ..., S_N$.

Assume that B is the benchmark index that portfolio P is intended to track.

We use the following notation to express the daily return of each stock S_i.

		Mean	Variance
Total Return R_t	$R_t = \frac{P_t+Div_t}{P_{t-1}}$		
Excess Return r_t	$r_t = R_t - RFR_t$	μ	σ^2 per stock, Σ in matrix form
Residual Return θ_t	$\theta_t = r_t - \beta r_{B,t}$	α	ω^2 per stock, $\Sigma - \beta\beta^T\sigma_B^2$ in matrix form
Active Return δ_t	$\delta_t = r_t - r_{B,t}$	$\alpha + (\beta - 1)\mu_B$	$\omega^2 + (\beta - 1)^2\sigma_B^2$ per stock

Assume that the weight of each stock in the benchmark index is given by an N dimensional vector w_B such that $w_B^T e = 1$, ie, the benchmark is fully invested.

As the beta β of each stock is measured with respect to the benchmark B, we can easily see that benchmark beta is 1, ie, $w_B^T \beta = 1$.

Assume that the weight of each stock in the active portfolio is given by an N dimensional vector w_P.

We can now define the active weights of portfolio P as

$$w_{PA} = w_P - w_B$$

2 Goal

The goal is to calculate w_{PA} given a researcher's view/alpha.

For a long-short active portfolio construction, we assume that w_{PA} satisfies the following constraints:

1. Long-short neutrality: $w_{PA}^T e = 0$. This ensures that portfolio P is fully invested, ie, $w_P^T e = 1$.
2. Beta neutrality: $w_{PA}^T \beta = 0$. This ensures that portfolio beta matches benchmark beta matches 1, ie, $w_P^T \beta = w_B^T \beta = 1$.

3 Defining Alpha

As defined in the table above, alpha is the expected residual return of a given stock in a given time period, ie, $\alpha = E(\theta)$. Over long periods of time, the return of a stock over the beta-adjusted benchmark return is expected to be zero, ie, $\alpha = 0$.

However, over the short term, researchers might have access to information that they believe helps predict the short term outperformance of a stock over the beta-adjusted benchmark return.

The first step in defining alpha given a researcher view is to convert the information into a numerical score, say g. For example, in the context of factor portfolios, if a researcher has a view that book-to-price predicts future outperformance, the score would be the B/P ratio for each stock i at each time period t, ie, $g_{i,t} = (\frac{B}{P})_{i,t}$.

Using the BLUE framework per stock in time series, we can now get

$$\alpha = E(\theta|g) = E(\theta) + \frac{Cov(\theta, g)}{Var(g)}(g - E(g))$$

$$\alpha = IC \cdot \omega \cdot z_{TS}$$

where, IC is called the information coefficient and is defined as $IC = Corr(\theta, g) = Corr(\theta, z_{TS})$ and $z_{TS} = \frac{g-E(g)}{Stdev(g)}$ is the time series z-score of g.

The above defintion of α is defined per stock in a time series context. However, in most cases, researchers intend to pick relative winners and losers in the cross section of investable assets for a given time period. Hence, we need to redefine α in a cross sectional context. Under the assumption that $E(g) = 0$ both in time series and cross section, and $Stdev_{TS}(g) \propto \omega$, we get

$$\alpha = IC \cdot \omega \cdot z_{TS} = IC \cdot \omega \cdot \frac{g}{Stdev_{TS}(g)} = IC \cdot \omega \cdot \frac{g}{c \cdot \omega} = IC \cdot \frac{g}{c}$$

Assume now that $Stdev_{CS}(g) = c'$. We get

$$\alpha = IC \cdot \frac{c'}{c} \cdot \frac{g}{Stdev_{CS}(g)}$$

$$\alpha \propto IC \cdot z_{CS}$$

$$\boxed{\alpha = IC_{adj} \cdot z_{CS}}$$

where the proportionality constant has been moved into IC.

Note that the cross sectional mean of g $E_{CS}(g)$ used to convert g to z_{CS} should be benchmark weighted, ie, $E_{CS}(g) = w_B^T g$ in order to ensure that benchmark alpha is 0, ie, $w_B^T \alpha = 0$.

4 Portfolio Construction

We will construct the portfolio as a two step process.

4.1 Optimal Portfolio

We will first construct an optimal portfolio in the absence of tradeability constraints.

We start with the standard mean-variance framework in which optimality is defined as follows:

$$\max_{w_P} \mu_P - \lambda\sigma_P^2$$

where $\mu_P = w_P^T\mu$ and $\sigma_P^2 = w_P^T\Sigma w_P$.

Under the constraints $w_{PA}^T e = 0$ and $w_{PA}^T\beta = 0$, it can be shown that $\mu_P = \mu_B + w_{PA}^T\alpha$ and $\sigma_P^2 = \sigma_B^2 + w_{PA}^T\Sigma w_{PA}$.

Hence, the optimisation problem reduces to

$$\begin{array}{cc} \max_{w_{PA}} & w_{PA}^T\alpha - \lambda w_{PA}^T\Sigma w_{PA} \\ \text{subject to} & w_{PA}^T e = 0 \\ & w_{PA}^T\beta = 0 \end{array}$$

Instead of targeting a fixed gross exposure, we target a fixed level of active risk, ie, we choose a parameter ω_{target} such that $w_{PA}^T\Sigma w_{PA} = \omega_{target}^2$.

Our optimisation problem now reduces to

$$\begin{array}{cc} \max_{w_{PA}} & w_{PA}^T\alpha \\ \text{subject to} & w_{PA}^T\Sigma w_{PA} = \omega_{target}^2 \\ & w_{PA}^T e = 0 \\ & w_{PA}^T\beta = 0 \end{array}$$

This optimisation problem has a closed form solution given by

$$w_{PA,target} = \omega_{target}\frac{\Sigma^{-1}(\alpha - \lambda_1 e - \lambda_2\beta)}{\sqrt{(\alpha - \lambda_1 e - \lambda_2\beta)^T\Sigma^{-1}(\alpha - \lambda_1 e - \lambda_2\beta)}}$$

$$\lambda_1 = \frac{\Sigma_{\alpha\beta}\Sigma_{\beta e} - \Sigma_{\beta\beta}\Sigma_{\alpha e}}{\Sigma_{\beta e}^2 - \Sigma_{ee}\Sigma_{\beta\beta}}$$

$$\lambda_2 = \frac{\Sigma_{\alpha e}\Sigma_{\beta e} - \Sigma_{\alpha\beta}\Sigma_{ee}}{\Sigma_{\beta e}^2 - \Sigma_{ee}\Sigma_{\beta\beta}}$$

where $\Sigma_{\alpha\beta} = \alpha^T\Sigma^{-1}\beta$, $\Sigma_{\beta e} = \beta^T\Sigma^{-1}e$, $\Sigma_{\beta\beta} = \beta^T\Sigma^{-1}\beta$, $\Sigma_{\alpha e} = \alpha^T\Sigma^{-1}e$, $\Sigma_{ee} = e^T\Sigma^{-1}e$.

4.2 Tradeable Portfolio

Once we have constructed the optimal portfolio in the absence of tradeability constraints, we need to construct a tradeable portfolio.

According to the Fundamental Law of Active Management $IR = IC \cdot \sqrt{BR} \cdot TC$, the Transfer Coefficient TC is defined as the correlation between the active return of the optimal portfolio and the active return of the tradeable portfolio

We follow the same approach and maximize the correlation between the active return of the optimal portfolio and the active return of the tradeable portfolio while maintaining the same level of active risk.

The optimisation problem can be written as

$$
\begin{array}{lc}
\max_{w_{PA}} & w_{PA,target}^T \Sigma w_{PA} \\
\text{subject to} & w_{PA}^T \Sigma w_{PA} = \omega_{target}^2 \\
 & w_{PA}^T e = 0 \\
 & w_{PA}^T \beta = 0 \\
 & \text{All orthogonalization constraints} \\
 & \text{All tradeability constraints}
\end{array}
$$

5 Practical Considerations

5.1 Risk Model

This framework needs an asset level risk model Σ. In the equity space, the most commonly used asset level risk model is a fundamental factor model provided by vendors such as MSCI Barra and Axioma. At the core of the risk model lies the construction of Σ in terms of factor risk, factor exposures and specific risk.

$$\Sigma = XFX^T + \Delta$$

Every model generally come in two flavors: short term and long term. The one to use depends on the investment horizon. The short term model is more suitable to use for an investment horizon of 1 month.

Risk modeling would be a lot more accurate if we use a regional risk model per region (such as USE5 for US markets, JPE4 for Japan) instead of using a global risk model (such as GEMLT or GEM3) for all regions.

5.2 Inversion of Risk Model

We need to invert the covariance matrix Σ in order to compute the optimal portfolio which is not a trivial exercise when the number of assets is large.

Fortunately, this can be computed efficiently using the Woodbury matrix identity which expresses Σ^{-1} in terms of X, F^{-1} and Δ^{-1}.

5.3 Researcher View

The researcher needs to use his/her information and view to determine the score per stock per time period g that is then used to construct α per stock per time period.

5.4 Information Coefficient IC_{adj}

The information coefficient IC_{adj} is an exogenous parameter that controls the magnitude of α given z_{CS}. Given that it is theoretically defined as $IC = Corr(\theta, g)$ and we can reasonably expect the fraction of correct predictions for quantitatively driven active strategies to lie between 50% (no edge) and 55% (excellent edge), reasonable values of IC would lie between 0 (no edge) and 0.1 (excellent edge). However, the final decision on the value of IC_{adj} lies with the researcher.

5.5 Level of Active Risk ω_{target}

ω_{target} is the ex-ante desired level of portfolio active risk and is an exogenous parameter to be entered into the portfolio construction framework.

5.6 Choice of Benchmark

The benchmark is used in the computation of asset level betas.

$$\beta = \frac{\Sigma w_B}{\sigma_B^2}$$

For our factor portfolios, the choice of benchmark is quite clear. For example: our benchmarks would be MSCI USA for the US factor portfolios, MSCI EU, MSCI Japan etc.

We would only need to take care to find the best ETF proxy for every benchmark index as risk models only provide factor exposures for investable assets such as ETFs and stocks. Factor exposures for indices do not come readymade in risk models and would need to be computed ourselves using the index composition.

6 References

The contents of this article are heavily borrowed from slides by Dr. Ronald Kahn used for the Berkeley MFE course on Active Portfolio Management.