COL703 Assignment-1

Rajat Bhardwaj
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1 Introduction and Elimination rules for $\phi \leftrightarrow \psi$

1.1 Elimination rules

$$\frac{\phi \leftrightarrow \psi}{\phi \rightarrow \psi} e1_{\leftrightarrow}$$

$$\frac{\phi \leftrightarrow \psi}{\psi \rightarrow \phi} e2_{\leftrightarrow}$$

Let us interpret $\phi \leftrightarrow \psi$ as $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ and try to derive the elimination rules

1.	$(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$	premise
2.	$(\phi \rightarrow \psi)$	E1

This gives us the elimination rule 1.

1.	$(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$	premise
2.	$(\psi \rightarrow \phi)$	E1

This gives us the elimination rule 2.

1.2 Introduction Rules

$$\frac{\phi \rightarrow \psi, \psi \rightarrow \phi}{\phi \leftrightarrow \psi} I_{\leftrightarrow}$$

Let us interpret $\phi \leftrightarrow \psi$ as $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ and try to derive the Introduction rules

1.	$\phi \rightarrow \psi$	premise
2.	$\psi \rightarrow \phi$	premise
3.	$(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$	I1,2

This gives us the Introduction rule.

2 Validity of the following using natural deduction

2.1 $(p \rightarrow r) \wedge (q \rightarrow r) \vdash p \wedge q \rightarrow r$

1.	$(p \rightarrow r) \wedge (q \rightarrow r)$	premise
2.	$p \rightarrow r$	$\wedge E1$
3.	$p \wedge q$	assumption
4.	p	$\wedge E3$
5.	r	$\rightarrow E2, 4$
6.	$p \wedge q \rightarrow r$	$\rightarrow I3-5$

2.2 $p \to q \wedge r \vdash (p \to q) \wedge (p \to r)$

1.	$p \to q \wedge r$	premise
2.	p	assumption
3.	$q \wedge r$	$\to E1, 2$
4.	q	$\wedge E3$
5.	$p \to q$	$\to I2 - 4$
6.	p	assumption
7.	$q \wedge r$	$\to E1, 6$
8.	r	$\wedge E7$
9.	$p \to r$	$\to I6 - 8$
10.	$(p \to q) \wedge (p \to r)$	$\wedge I5, 9$

3 Adequacy of connectives

3.1 Proof that $\{\neg, \wedge\}$ is adequate

We know that all the Boolean formulas are (or can be reduce to) of the form having only $\{\wedge, \vee, \neg\}$ connectives. Now we will prove that the logic formulas having connectives $\{\wedge, \vee, \neg\}$ can be reduced to the formulas having connectives $\{\vee, \neg\}$ Let us say we have an $\wedge$ in our boolean formula and and instance of this can be represented as

$$X \wedge Y$$

Now we will show that it is equivalent to $\neg((\neg X) \vee (\neg Y))$
Applying double negation to $X \wedge Y$

$$\neg(\neg(X \wedge Y))$$

using de morgan's law on the inner negation we have

$$\neg((\neg X) \vee (\neg Y)))$$

This can be done on all the $\wedge$ in our boolean formula, thus the boolean formula having the connectives $\{\wedge, \vee, \neg\}$ can be reduced to the formulas having connectives $\{\vee, \neg\}$

3.2 Proving that $\{\leftrightarrow, \neg\}$ is inadequate

We will show that any boolean expression having $\{\leftrightarrow, \neg\}$ will have even number of T's and F's in their truth table and will fail to capture those expressions that have odd number of T's and F's in their truth table(having 2 or more variables)
We will show this using induction on the length of the boolean statement
Base case - let our boolean statement be of the form P i.e. an atomic statement, now P can be true in half of the entries and false in the other half of entries of the truth table. Now since we have more than 1 variable our truth table rows will be equal to 2^n where n > 1, thus it will be a multiple of 4, thus half of it, i.e. $\frac{2^n}{2} = 2^{n-1}$ will be a multiple of 2.
Induction - Let our statement be ϕ now there are 2 possibilities considering the connectives $\{\leftrightarrow, \neg\}$
Case - 1 : $\phi = \neg\psi$
Using induction hypothesis on the length of the boolean statement (since $len(\psi) < len(\phi)$), there will be an even number of T's and F's in the truth table for ψ and thus even number (only opposite number) of T's and F's for ϕ in the truth table
Case - 2 : $\psi = \psi_1 \leftrightarrow \psi_2$
By the inductive hypothesis, ψ_1 and ψ_2 both have an even number of Trues and False in their truth table, now in

$$\psi = \psi_1 \leftrightarrow \psi_2$$

Let us find out the number of rows in which it is T and F.
Let ψ_1 be true in A rows, now out of these A rows, ψ_2 is true in B rows thus ψ is false in A - B rows. Thus ψ is true in B rows and False in A - B rows. Now ψ_1 is false for C rows. Out of those C rows,

ψ_2 is true for D rows, hence false in C - D rows. This means ψ is true for C - D rows and false for D rows. Thus ψ is true for total B + C - D rows and false for A - B + D.
Now we will show that B + C - D and A - B + D both are even.
Proving B + C - D is even. C is even (since ψ_1 is false for even rows). We will show that B and D both have the same parity. Now B + D = all the rows in which ψ_2 is True. Thus B + D is even, therefore B and D must have the same parity. Thus B - D is also even. This means B + C - D is also even.
Proving A - B + D is even. A is even(since ψ_1 is true for an even number of rows. , we have already shown that B and D have the same parity thus - B + D is also even. Thus A - B + D is even.
This proves that ψ is T for even number of rows and false of even number of rows
This proves the fact that any boolean formula that is made from the connectives $\{\vee, \neg\}$ will always have even number of T's and F' thus we won't be able to write the boolean formulas having odd number of T's. Thus $\{\vee, \neg\}$ is inadequate.

4 Show that the following sequents are not valid

4.1 $\neg r \to (p \wedge q), r \wedge \neg q \vdash r \to q$

The only case in which the R.H.S. is false will be when r is true and q is false. Because from the truth table, we know that $T \to F$ is False
Let us replace r with T and q with F on the left-hand side
$X = \{\{\neg r \to p \wedge q)\}, \{r \wedge \neg q\}\}$
$X[True/r, False/q] = \{\{F \to p \wedge F)\}, \{T \wedge \neg F\}\}$
$X[True/r, False/q] = \{\{F \to F)\}, \{T \wedge T\}\}$
$X[True/r, False/q] = \{\{F \to F)\}, \{T\}\}$
From the truth table, we know that $F \to F$ is true. Therefore the first clause is true
$X[True/r, False/q] = \{\{T\}, \{T\}\}$
$X[True/r, False/q] = \{\{T\}, \{T\}\}$
Thus the LHS of $\vdash$ is true and RHS of $\vdash$ is false
Therefore the sequent is not valid, because for sequant to be valid, the valuation that makes the LHS true must make the RHS true which is not the case here.

4.2 $p \to (q \to r) \vdash p \to (r \to q)$

Let us give values
p - True
q - False
r - True
Let us look at the R.H.S first
$X = p \to (r \to q)$
$X[True/p, False/q, True/r] = T \to (T \to F)$
Now from the truth table, we know that $T \to F$ is False
$X[True/p, False/q, True/r] = T \to F$
again, from the truth table, we know that $T \to F$ is False
$X[True/p, False/q, True/r] = F$
Therefore, the R.H.S evaluates to False
Now let us look at the LHS of the $\vdash$
$Y = p \to (q \to r)$
$Y[True/p, False/q, True/r] = T \to (F \to T)$
From the truth table, we know that $FtoT$ is true
$Y[True/p, False/q, True/r] = T \to T$
From the truth table, we know that $T \to T$ is true therefore $Y[True/p, False/q, True/r] = T$
Thus the LHS of $\vdash$ is true and RHS of $\vdash$ is false
Therefore the sequent is not valid, because for the sequant to be valid, the valuation that makes the LHS true must make the RHS true which is not the case here.

5 Finitely satisfiable sets

5.1 Every FSS can be extended to a maximal FSS

Proof: Let ψ be a Finitely Satisfiable Set (FSS) and $\alpha_0, \alpha_1, \alpha_3, ...$ be an enumeration of φ, We define an infinite sequence of sets $\psi_0, \psi_1, \psi_2, ...$ where $\psi = \psi_0$ as follows.

$$\psi_{i+1} = \begin{cases} \psi_i \cup \alpha_i \text{ if } \psi_i \cup \alpha_i \text{ is finitely satisfiable} \\ \psi_i \text{ otherwise} \end{cases} \tag{1}$$

We can clearly see from the above definition that ψ_i is FSS for all i. And $\psi_{i+1} \subseteq \psi_i$
Each set in this sequence is FSS, we define

$$\psi^+ = \bigcup_{i \geq 0} \psi_i$$

We claim that ψ^+ is maximal FSS extending ψ. Thus we have to show that ψ^+ is FSS and and ψ^+ is maximal.
If ψ^+ is not finitely satisfiable then this means one of the finite subset f ψ^+ would be unsatisfiable and it would have been made from a ψ_i which would contradict the fact that all ψ_i are satisfiable. This means that ψ^+ is finitely satisfiable.
Proving ψ^+ is maximal - Now if any α_i is such that $\psi^+ 1 \cup \{\alpha_i\}$ is finitely satisfiable. Since $\alpha_i \notin \psi^+$ this means that α_i was not added in the $i + 1^{th}$ step. This means that $\psi_i \cup \{\alpha_i\}$ is not finitely satisfiable. Which means $\psi^+ = \bigcup_{i \geq 0} \psi_i$ is not finitely satisfiable which is a contradiction. Thus ψ^+ is maximal.

5.2 If X is a maximal FSS then for every formula $\alpha, \alpha \in X$ iff $\neg\alpha \notin X$

Now clearly X cannot contain both α and $\neg\alpha$ since $\{\alpha, \neg\alpha\}$ is unsatisfiable.
Now let us assume that X contains neither α and $\neg\alpha$
Now let there be a finite subset of X such that, $X_1 \subseteq X$ such that $X_1 \cup \{\alpha\}$ is unsatisfiable and another finite subset $X_2 \subseteq X$ such that $X_2 \cup \{\neg\alpha\}$ is unsatisfiable.
Claim : $X_1 \cup X_2$ is unsatisfiable:
Proof - We know that X_1 is finite and satisfiable, now since $X_1 \cup \{\alpha\}$ is unsatisfiable this means that whatever valuation that makes X_1 true, makes α false. and whatever valuation that makes X_2 true makes $\neg\alpha$ false. This means that

$$X_1 \models \neg\alpha$$

$$X_2 \models \alpha$$

It is clear from the above $X_1 \cup X_2$ is unsatisfiable
This contradicts the fact that X is finitely satisfiable since $X_1 \cup X_2$ is a finite subset of X and is unsatisfiable. Therefore we arrive at the conclusion that it is not possible for both α and $\neg\alpha$ to be both simultaneously present or simultaneously absent in X.

5.3 If X is a maximal FSS then for all formulas $\alpha, \beta, (\alpha \vee \beta) \in X$ iff ($\alpha \in X$ or $\beta \in X$

Let $\alpha \vee \beta \in X$.
To prove : $\alpha \in X$ or $\beta \in X$
Proof : Let us assume that $\alpha \notin X$ and $\beta \notin X$ now we can say that $X \cup \{\alpha\}$ is finitely unsatisfiable and $X \cup \{\beta\}$ is finitely unsatisfiable This means that , there exists some finite subset of X, $X_1 \subseteq X$ such that $X_1 \cup \{\alpha\}$ is unsatisfiable and another finite subset X_2 such that $X_2 \cup \beta$ is unsatisfiable. This means that there is no valuation such that both X_1 and α can be made true simultaneously as well as no common valuation such that X_2 and β can be made true simultaneously this means that $X_1 \cup X_2 \cup \{\alpha \vee \beta\}$ is unsatisfiable because if we try to make both X_1 and X_2 to be true then both α and β will become false. This means that $X_1 \cup X_2 \cup \{\alpha \vee \beta\}$ is a finite subset of X and is unsatisfiable. Now since $X_1 \cup X_2$ is satisfiable (property of maximal FSS) we get that $\alpha \vee \beta \notin X$ which is a contradiction. Thus it is not true that $\alpha \notin X$ and $\beta \notin X$ hence $\alpha \in X$ or $\beta \in X$
Let $\alpha \in X$ or $\beta \in X$

To prove : $\alpha \vee \beta \in X$
Proof:
w.l.o.g $\alpha \in X$
Let us assume that $\alpha \vee \beta \notin X$
Now since $\alpha \vee \beta \notin X$ there exists a finite set X_1 in X such that $X_1 \cup \{\alpha \vee \beta\}$ is unsatisfiable this means that there does not exist a valuation that satisfies both X_1 and $\alpha \vee \beta$, we can also say that there does not exist a valuation that doesn't satisfy both X_1 and α or satisfy both X_1 and β (using the property of $\vee$). This means that $X_1 \cup \{\alpha\}$ is unsatisfiable. This means that $\alpha \notin X$ which is a contradiction. This $\alpha \vee \beta \in X$

5.4 Every maximal FSS X generates a valuation v_X such that for every formula α, $v_x \models \alpha$ iff $\alpha \in X$

Assumption - X is made from the connectives $\{\neg, \vee\}$ since they are adequate (if X has other connectives then, X can be reduced to them, since it is trivial that the boolean logic can be written in $\vee, \wedge, \neg$ and $\wedge$ can be reduced to $\vee$ using demorgan's rule)
We will use induction on the length of α = m.
Base case - Suppose m = 1. This means that α is a terminal symbol (i.e. it does not have any $\vee, \wedge, \neg$ in it). Now, let us say that Y = $\{\alpha\}$. Now by definition, $v_x(Y) = T$ iff $Y \in X$
Now suppose m > 1. This means α can be of the following forms

1. $\alpha = \neg\beta$
2. $\alpha = \beta \vee \gamma$

Case 1 -
$\alpha = \neg\beta$
$v_X(\alpha) = T$ iff $v_X(\beta) = F$
Now,
$v_X(\beta) = F$ iff $\beta \notin X$ (induction hypothesis)
Now,
$\beta \notin X$ iff $\neg\beta \in X$ using 5.2
Now,
$\neg\beta \in X$ iff $\alpha \in X$
This proves that $v_X(\alpha) = T$ iff $\alpha \in X$

Case 2 -
$\alpha = \beta \vee \gamma$
Suppose $v_X(\alpha) = T$ this means $v_X(\beta) = T$ or $v_X(\gamma) = T$. Using the induction hypothesis, we get that $\beta \in X$ or $\gamma \in X$.
We get that $\beta \vee \gamma \in X$ using 5.3
This means that $\alpha \in X$.
Now let us prove the opposite side
Suppose $\alpha \in X$, This means $\beta \vee \gamma \in X$ This means that $\beta \in X$ or $\gamma \in X$. Using induction hypothesis, we get that $v_X(\beta) = T$ or $v_X(\gamma) = T$. Thus $v_X(\beta \vee \gamma) = T$. Thus we get that $v_X(\alpha) = T$. Hence proved

5.5 Any FSS X is simultaneously satisfiable (that is, for any FSS X, there exists v_X such that $v_X \models X$

We have already proved (in 5.4) that every formula α, $\alpha \in X$ iff $v_X \models \alpha$
Now we know that there exists a valuation v_X such that X is a FSS and for every formula $\alpha \in X$, v_X makes α true.
Claim : $v_X \models X$
Proof - Let us assume that $v_X \not\models X$, this means that there exists a formula $\alpha \in X$ such that $v_X \not\models \alpha$ This is a contradiction to the fact that for every formula α, $\alpha \in X$ iff $v_X \models \alpha$ which we have already proved
Hence we proved that $v_X \models X$

5.6 For all X and all α, $X \models \alpha$ iff there exists $Y \subseteq_{fin} X$ such that $Y \models \alpha$

To Prove - If there exists $Y \subseteq_{fin} X$ such that $Y \models \alpha$ then $X \models \alpha$
Proof - If $Y \subseteq_{fin} X$ and $Y \models \alpha$ then, we have a valuation v_X such that $v_X \models X$. Now if $v_X \models X$ then $v_X \models Y$. Now we have $Y \models \alpha$ it implies that $v_X \models \alpha$, which implies $X \models \alpha$, hence proved

Proving the other side, For all X and for all α, if $X \models \alpha$ then there exists $Y \subseteq_{fin} X$ such that $Y \models \alpha$
$X \models \alpha$ this means that $X \cup \{\neg\alpha\}$ is not satisfiable. This means that there is a finite set $Y \subseteq X \cup \{\neg\alpha\}$ such that Y is not satisfiable. Thus, $(Y \setminus \{\neg\alpha\}) \cup \{\neg\alpha\}$ is not satisfiable. Where $(Y \setminus \{\neg\alpha\}) \subseteq_{fin} X$ This implies $(Y \setminus \{\neg\alpha\}) \models \alpha$. Thus we get that $Y1 = (Y \setminus \{\neg\alpha\})$ such that $Y1 \models \alpha$. Hence proved.