Numerical formulation for simulating a Lid Driven Cavity

February 3, 2022

1 Introduction

The Lid driven cavity is a well known benchmark problem for viscous incompressible fluid flow. The Incompressible Navier Stokes Equations (NS) are used to solve the problem. The formulation below is given for unsteady non-dimensionalized NS equations.

2 Mathematical Formulation

In 2D, the NS equations consist of 3 equations:-

Continuity equation

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 \tag{1}$$

x-momentum equation

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} * (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{2}$$

y-momentum equation

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re} * (\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{3}$$

As it can be seen, the NS equations are highly coupled and an equation for pressure doesn't exist explicitly. To obtain one, the following is done. We define

$$F = \frac{1}{Re} * (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} \tag{4}$$

$$G = \frac{1}{Re} * (\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) - \frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} \tag{5}$$

Using these in the momentum equations, we obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + F \tag{6}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + G \tag{7}$$

Further we define,

$$M_x = \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(-\frac{\partial p}{\partial x} + F) \tag{8}$$

$$M_y = \frac{\partial v}{\partial t} = \frac{\partial}{\partial y}(-\frac{\partial p}{\partial y} + G) \tag{9}$$

$$\frac{\partial M_x}{\partial x} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial t}) = \frac{\partial}{\partial x}(\frac{-\partial p}{\partial x} + F) = \frac{\partial}{\partial t}(\frac{\partial u}{\partial x}) \tag{10}$$

$$\frac{\partial M_y}{\partial y} = \frac{\partial}{\partial y}(\frac{\partial v}{\partial t}) = \frac{\partial}{\partial y}(\frac{-\partial p}{\partial y} + G) = \frac{\partial}{\partial t}(\frac{\partial v}{\partial y}) \tag{11}$$

Adding 10 and 11, we get

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \tag{12}$$

Using continuity 1, the LHS goes to 0. Therefore,

$$-\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{13}$$

$$\nabla^2 p = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \tag{14}$$

14 is a Poisson equation for solving Pressure.

3 Discretization

The Finite Difference Method is used for spatial discretization along with explicit scheme for time discretization. A staggered grid is used i.e. in a cell the velocities would be at the boundaries and pressure at center of the cell.

Discretization for Equation 4 and 5

$$F_{i,j} = \frac{1}{Re} * (\frac{u_{i,j+1} - 2 * u_{i,j} + u_{i,j-1}}{(\delta x)^2} + \frac{u_{i-1,j} - 2 * u_{i,j} + u_{i+1,j}}{(\delta y)^2}) - \frac{(uu)_e - (uu)_w}{\delta x} - \frac{(uv)_n - (uv)_s}{\delta y} \tag{15}$$

$$u_e = \frac{u_{i,j+1} + u_{i,j}}{2}, u_w = \frac{u_{i,j-1} + u_{i,j}}{2}, u_n = \frac{u_{i+1,j} + u_{i,j}}{2}$$

$$u_s = \frac{u_{i,j} + u_{i-1,j}}{2}, v_n = \frac{v_{i,j} + v_{i,j+1}}{2}, v_s = \frac{v_{i-1,j} + v_{i-1,j+1}}{2}$$

$$G_{i,j} = \frac{1}{Re} * (\frac{v_{i,j+1} - 2 * v_{i,j} + v_{i,j-1}}{(\delta x)^2} + \frac{v_{i-1,j} - 2 * v_{i,j} + v_{i+1,j}}{(\delta y)^2}) - \frac{(uv)_e - (uv)_w}{\delta x} - \frac{(vv)_n - (vv)_s}{\delta y} \tag{16}$$

$$v_e = \frac{v_{i,j+1} + v_{i,j}}{2}, v_w = \frac{v_{i,j-1} + v_{i,j}}{2}, v_n = \frac{u_{i+1,j} + v_{i,j}}{2}$$

$$v_s = \frac{v_{i,j} + v_{i-1,j}}{2}, u_e = \frac{u_{i+1,j} + u_{i,j}}{2}, u_w = \frac{u_{i+1,j-1} + u_{i,j-1}}{2}$$

Discretization of Pressure Poisson equation 14,

$$\nabla^2 p = \frac{p_{i,j+1} - 2 * p_{i,j} + p_{i,j-1}}{(\delta x)^2} + \frac{p_{i-1,j} - 2 * p_{i,j} + p_{i-1,j}}{(\delta y)^2} \tag{17}$$

$$\frac{\partial F}{\partial x} = \frac{F_{i,j} - F_{i,j-1}}{\delta x} \tag{18}$$

$$\frac{\partial G}{\partial y} = \frac{G_{i,j} - G_{i-1,j}}{\delta y} \tag{19}$$

So,

$$\frac{p_{i,j+1} - 2 * p_{i,j} + p_{i,j-1}}{(\delta x)^2} + \frac{p_{i-1,j} - 2 * p_{i,j} + p_{i-1,j}}{(\delta y)^2} = \frac{F_{i,j} - F_{i,j-1}}{\delta x} + \frac{G_{i,j} - G_{i-1,j}}{\delta y} \tag{20}$$

20 is solved using the Successive Over Relaxation (SOR) Method. The SOR method in general can be written as

$$a_p * f_{i,j} + a_w * f_{i,j-1} + a_e * f_{i,j+1} + a_n * f_{i+1,j} + a_s * f_{i-1,j} = b \tag{21}$$

$$f_{i,j} = (b - a_w * f_{i,j-1} + a_e * f_{i,j+1} + a_n * f_{i+1,j} + a_s * f_{i-1,j})/a_p \tag{22}$$

$$f_{i,j}^{k+1} = (1-\alpha) * f_{i,j}^{k} + (b - a_w * f_{i,j-1}^{k+1} + a_e * f_{i,j+1}^{k+1} + a_n * f_{i+1,j}^{k+1} + a_s * f_{i-1,j}^{k+1})/a_p \tag{23}$$

where superscipt k denotes value at previous iteration and k+1 at current iteration.

Applying this to the 20,

$$pn_{i,j} = (1-\alpha) * p_{i,j} + (q - a_w * pn_{i,j-1} + a_e * pn_{i,j+1} + a_n * pn_{i+1,j} + a_s * pn_{i-1,j})/a_p \tag{24}$$

where pn is pressure in current iteration, p is pressure in in previous iteration and q is RHS of 20.

Discretization to obtain velocity,

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + F \tag{25}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\delta t} = -\frac{p_{i,j+1}^{n+1} - p_{i,j}^{n}}{\delta x} + F_{i,j} \tag{26}$$

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{\delta t}{\delta x} * (p_{i,j+1}^{n} - p_{i,j}^{n}) + F_{i,j} * \delta t \tag{27}$$

Similarly,

$$v_{i,j}^{n+1} = v_{i,j}^{n} - \frac{\delta t}{\delta y} * (p_{i+1,j}^{n} - p_{i,j}^{n}) + G_{i,j} * \delta t \tag{28}$$

4 Overview of the algorithm

1. Initialize the variable fields u, v, p
2. Start the time looping
3. Compute the F and G for the whole grid using Equations 15 and 16
4. Compute pressure using SOR
5. Use the updated pressure to compute velocities using equation 27 and 28.
6. Compute the error in velocities between previous and current time step. This will to used to determine if the simulation has reached steady-state.
7. Repeat from step 3 until steady state is reached or maximum time value is attained.

5 BCs

Boundary conditions for the given problem are enforced by having ghost cells along the boundary.

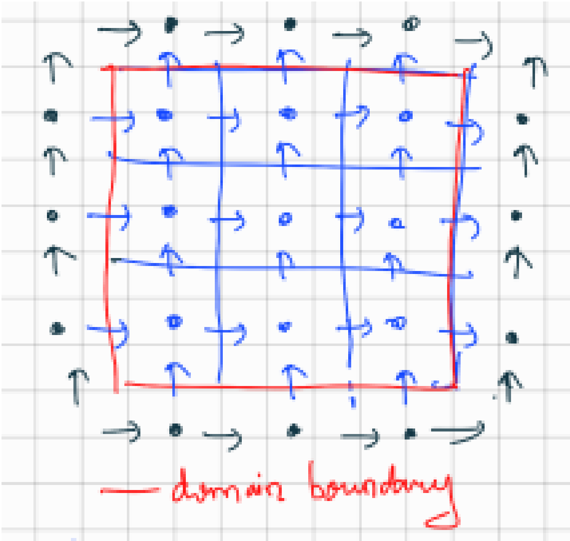

Figure 1: BC

Also, equate pressures outside and inside for a wall. Velocity perpendicular to the wall is 0. (no-slip).
With the concept of ghost cells Neumann boundary conditons can also be implemented as,

$$\frac{\partial p}{\partial k} = b \tag{29}$$

Choice of time step for the simulation is constrained by the Courant–Friedrichs–Lewy condition as explicit time discretization is used in the formulation. The Courant number is defined by

$$Co = u * \delta t / \delta x; \tag{30}$$

Co should be less than 1 for stability. In the current simulation, the maximum non-dimensional velocity is 1 and non-dimensional dx is reciprocal of number of nodes. This gives us a bound for the dt.