Hide

Linearization

One of the main uses of the derivative is linearization, which uses the first two terms (the constant and linear terms) of the Taylor series as an approximation. In many applications, this gives a very good approximation, as we will see in some examples.

• Differentiation rules | Know Page | Higher derivatives •

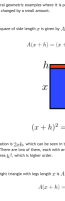
Linear variation visualization

There are several geometrie examples where it is possible to see the linear variation as the change in area as a parameter is changed by a small amount.

SQUARE

The area of a square of side length x is given by $A(x) = x^2$. When that is varied by a small amount h, the result is

$$A(x+h) = (x+h)^2 = x^2 + 2xh + h^2.$$



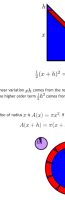
$$(x+h)^2 = x^2 + 2xh + h^2$$

The linear variation is $2xh$, which can be seen in the diagram as the rectangles along the right and top edges of the square. There are two of them, each with area xh. The final bit of area, the purple square in the diagram, has area h^2, which is higher order.

TRIANGLE

The area of a right triangle with legs length x is $A(x) = \frac{1}{2}x^2$. When the leg is varied by h, the result is

$$A(x+h) = \frac{1}{2}x^2 + xh + \frac{1}{2}h^2.$$



$$\frac{1}{2}(x+h)^2 = \frac{1}{2}x^2 + xh + \frac{1}{2}h^2$$

Visually, the linear variation xh comes from the red parallelogram, of base x, and height h, running along the hypotenuse. The higher order term $\frac{1}{2}h^2$ comes from the small purple triangle at the tip of the triangle.

DISC

The area of a disc of radius x is $A(x) = \pi x^2$. When it is increased by h, the result is

$$A(x+h) = \pi(x+h)^2 = \pi x^2 + 2\pi xh + \pi h^2.$$

$$\pi(x+h)^2 = \pi x^2 + 2\pi xh + \pi h^2$$

Visually breaking this into the linear variation and higher order variation is a little trickier. The best way to imagine taking the ring formed by the revised area and breaking it into rectangles and wedges. In the limit, the wedges can be rearranged into a stack of rectangles, and the rectangles can be arranged to form a strip of length $2\pi x$ (the circumference of the lower circle) and width h.

Linear approximation

The equation underlying any linear approximation should be familiar, since it is just the first order Taylor series about $x = a$, after making the substitution $h \to x - a$:

$$f(x+h) \approx f(x) + f'(x)h.$$

This will be a good linear approximation provided that f_0 is small, i.e., the point of is close to the input we are trying to approximate. In general, one wants to pick a to be an input where it is easy to compute $f(a)$ and $f'(a)$ which is as close to the desired input as possible.

Example

Using a linear approximation, estimate $\sqrt{25.01}$. Hide

The function here is $f(x) = \sqrt{x}$. Possible choices for a are perfect squares, since it is easy to compute the square root of those. The nearest perfect square is $\underline{\underline{a}} = 25$, so we choose $a = 25$. Then, $h = x - a = -0.01$. Then

$$f(x) \approx f(a) + f'(a)h$$
$$\sqrt{25.01} \approx \sqrt{25} + \frac{1}{2\sqrt{25}}(-0.01)$$
$$= 5 - \frac{1}{50}$$
$$= \frac{249}{50}$$
$$= 4.98$$

which is very close to the calculator's answer of 4.99851 . . . .

Example

Using a linear approximation, estimate $\sqrt{101}$. Is this an over-approximation or an under-approximation? Hide

As above, $a = 100$ is the closest perfect square to x that is easy to compute $\sqrt{x}$ and derivatives. Then $h = 1$, so the linear approximation is

$$f(a+h) \approx f(a) + f'(a)h$$
$$= \sqrt{100} + \frac{1}{2\sqrt{100}} \cdot 1$$
$$= 10 + \frac{1}{20}$$
$$= 10.2$$

This is an over-approximation. One way to see why is to consider the graph of $\sqrt{x}$, which is concave down, so the linear approximation is above the true value. Another argument is to consider the next term of the Taylor series, which is negative since the second derivative of $\sqrt{x}$ is negative.

For comparison, the value according to a calculator is $\sqrt{101} \approx 10.198$.

Example

Approximate π^{10}. Hint: $\pi^2 \approx 9.86$. Hide

From the hint, $\pi^{10} = (\pi^2)^5 \approx (9.86)^5$. Thus, we are trying to approximate $f(x) = x^5$ at $x = 9.86$. The nearest easy input is $a = 10$, so we find

$$f(9.86) \approx f(10) + f'(10)(-.14)$$
$$= 10^5 + 10(10)^4(-.14)$$
$$= 10^5(1 - .14)$$
$$= 8.6 \cdot 10^5.$$

The true answer is approximately $9.77 \cdot 10^5$, so this estimate is within 1%.

Example

Approximate $\sqrt{e}$. Hint: $\sqrt{e^3} \approx 20.1$, $\sqrt{e^5} \approx 150.9$. Hide

From the first hint, $\sqrt{e^3} = (\sqrt{e})^3 \approx (20.1)^{\frac{2}{3}}$, so consider the linear approximation for $f(x) = x^{\frac{2}{3}}$ near $a = 20$

$$f(x) \approx f(20) + f'(20)(x - 20)$$
$$x^{\frac{2}{3}} \approx 20^{\frac{2}{3}} + 10 \cdot 20^{\frac{1}{3}}(x - 20)$$

So it follows that

$$e^{\frac{3}{2}} \approx (20.1)^{\frac{2}{3}}$$
$$\approx 20^{\frac{2}{3}} + 10 \cdot 20^{\frac{1}{3}} \cdot .1$$
$$\approx 20^{\frac{2}{3}}(1 + 1.35/1.2)$$
$$\approx 2^{\frac{2}{3}} \cdot 10^{\frac{2}{3}} \cdot 1.65$$
$$\approx 4.45 \cdot 10^{\frac{2}{3}}$$

(the last step used the second hint that $2^{\frac{2}{3}} \approx 1.6$). The true answer is approximately $e^{\frac{3}{2}} \approx 1.648 \cdot 10^{\frac{2}{3}}$ so the error is less than 2%.

Newton's method

Consider a continuous, differentiable function f, the goal is to find a root r, so that $f(r) = 0$. Suppose x is an initial guess of a root. Then $x + \underline{\underline{h}}$ will be a root for some small value of $\underline{\underline{h}}$. Linearizing, and ignoring the higher order terms, gives

$$f(x+h) = f(x) + f'(x)h + O(h^2)$$
$$= f(x) + f'(x)h.$$

Since we supposed that $x + \underline{\underline{h}}$ was a root of f, it follows that $f(x + \underline{\underline{h}}) = 0$. Therefore, setting the above equal to 0 and solving for $\underline{\underline{h}}$, gives

$$f(x) + f'(x)h = 0 \quad \Rightarrow \quad \underline{\underline{h}} = -\frac{f(x)}{f'(x)}.$$

By taking this new guess of a root, and repeating the above process, one (hopefully) gets a better and better approximation of a root.

More formally, this can be stated a difference equation. Given an initial guess, called x_0, of a root of the function, one uses the update rule

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to get x_1, and then x_2, and so on.

The resulting sequence hopefully converges to a root of f. Graphically, what is happening is as follows:

1. Pick a guess x_0.
2. Find the tangent line to f through the point $(x_0, f(x_0))$.
3. Let x_1 be the point where the tangent line intersects the x-axis.
4. Repeat steps 1 and 2 (use the figure).

CAVEAT

This sequence is only defined if $f'(x_n)$ exists and is non-zero for every x_n in the sequence. Even if the sequence is defined, it may not converge to anything. But if the sequence is defined and it does converge, say to $\underline{\underline{r}}$, then $\underline{\underline{r}}$ is a root of f.

Example

Find the positive root for approximating $\sqrt{2}$, the reciprocal of a number is. Hide

First, we must find a function which has $\sqrt{2}$ as a root. We might try $f(x) = x - \frac{1}{x}$, but unfortunately (after a little algebra) this leads to the update rule

$$x_{n+1} = \frac{1}{x_n}.$$

which is not particularly helpful, since that is the quantity we are trying to approximate.

Another try would be $f(x) = x - \frac{2}{x}$. Now, find the derivative $f'(x) = \frac{2}{x^2}$, so the update rule is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n - \frac{2}{x_n}}{\frac{2}{x_n^2}}$$
$$= x_n - \frac{x_n^2}{2} + \frac{2}{2x_n}$$
$$= x_n - \frac{x_n}{2} + \frac{1}{x_n}$$
$$= \frac{x_n}{2} + \frac{1}{x_n}$$

Using the update rule with $x_0 = 3$ yields $\underline{\underline{x_1}}$ Hide

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0}$$
$$= \frac{3}{2} + \frac{1}{3}$$
$$= \frac{9}{6} + \frac{2}{6}$$
$$= \frac{11}{6}$$

Updating one more time gives

$$x_2 = \frac{x_1}{2} + \frac{1}{2x_1}$$
$$= \frac{11}{12} + \frac{3}{11}$$
$$= \frac{121}{132} + \frac{36}{132}$$
$$= \frac{157}{132}$$
$$\approx 1.1916.$$

The calculator gives that $\sqrt{2} \approx 1.1916$, so we get a good approximation with only a little bit of work. The cool aspect the update rule several more times to get an even better approximation.

Example

Find the update rule for finding $\sqrt[3]{x}$. Use the update rule once with the initial guess of $x_0 = 5$ to estimate $\sqrt[3]{125}$. Hide

The function in this case is $f(x) = x^3 - x$. Then $f'(x) = 3x^2$, and so the update rule is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^3 - x_n}{3x_n^2}$$
$$= x_n - \frac{x_n^3}{3x_n^2} + \frac{x_n}{3x_n^2}$$
$$= \frac{2x_n}{3} + \frac{1}{3x_n}$$

Using the update rule with $x_0 = 5$ yields $\underline{\underline{x_1}}$ Hide

$$x_1 = \frac{x_0}{3} + \frac{1}{3x_0}$$
$$= \frac{5}{3} + \frac{1}{15}$$
$$= \frac{25}{15} + \frac{1}{15}$$
$$= \frac{26}{15}$$

Updating one more time gives

$$x_2 = \frac{x_1}{3} + \frac{1}{3x_1}$$
$$= \frac{26}{45} + \frac{5}{26}$$
$$= \frac{676}{45} + \frac{225}{676}$$
$$= \frac{676}{45} + \frac{45}{676}$$
$$= \frac{676^2 + 45^2}{45 \cdot 676}$$
$$\approx 5.2156.$$

The calculator gives that $\sqrt[3]{125} \approx 5.2156$, so we get a good approximation with only a little bit of work. The cool aspect the update rule several more times to get an even better approximation.

Example

Find the update rule for finding $\sqrt[3]{x}$. Use the update rule once with the initial guess of $x_0 = 5$ to estimate $\sqrt[3]{125}$. Hide

The function in this case is $f(x) = x^3 - x$. Then $f'(x) = 3x^2$, and so the update rule is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^3 - x_n}{3x_n^2}$$
$$= x_n - \frac{x_n^3}{3x_n^2} + \frac{x_n}{3x_n^2}$$
$$= \frac{2x_n}{3} + \frac{1}{3x_n}$$

Using this to estimate $\sqrt[3]{125}$ with the initial guess $x_0 = 5$, we find:

$$x_1 = \frac{1}{3} \cdot x_0 + \frac{1}{3x_0}$$
$$= \frac{10}{3} + \frac{1}{15}$$
$$= \frac{50}{15} + \frac{1}{15}$$
$$= \frac{51}{15}$$
$$= 6.600$$

If we compare this to the answer from a calculator, we find that $\sqrt[3]{125} \approx 5.848$, and so even after just one step, we are within 1% of the true answer.

EXERCISES

- Use a linear approximation to estimate $\sqrt[3]{26.9}$. Round your answer to four decimal places. Hint: what is $\sqrt[3]{27}$
- Use a linear approximation to estimate the cosine of an angle of $\frac{\pi}{3}$. Round your answer to four decimal places. Hint: remember that $\frac{\pi}{3} = \frac{\pi}{6}$ and focus $\frac{\pi}{6} = \frac{\pi}{3}$
- The golden ratio $\frac{1+\sqrt{5}}{2}$ is a root of the polynomial $x^2 - x - 1$. If you use Newton's method to estimate its value, what is the appropriate update rule for the sequence x_n?
- To approximate $\sqrt[3]{x}$ using Newton's method, what is the appropriate update rule for the sequence x_n?
- You want to build a square pen for your new chickens, with an area of $\frac{1}{100}(\text{yd})^2$. Not having a calculator handy, you decide to use linearization method to approximate the length of one side of the fence. If your first guess is $\frac{10}{25}$, what is the next approximation? Hint: $\frac{1}{100} = \frac{4}{25} - \frac{9}{100}$
- Suppose you are a mechanical engineer designing a pendulum clock for a grandfather's clock. The mechanic department tells you that you need to make a cube shaped box. The engineering department says that you will need a box with a volume of $\frac{1}{100}(\text{yd})^3$. What are the dimensions of the cubical box? Starting with a guess of $\frac{10}{25}$ for the length of the side of the cube, what approximation does Newton's method of linearization method give you? Round your answer to two decimal places.
- Without using a calculator, approximate $\frac{1}{\sqrt{10.1}}$. Here are some hints. First, $\frac{1}{\sqrt{10}} = \frac{1}{3.16}$ and $\frac{1}{3.16} \approx \frac{1}{3.2}$ in scientific notation. What does linear approximation give as an estimate of $\frac{1}{\sqrt{10.1}}$? Round to the nearest 0.01.
- A strong beast of burden $\underline{\underline{B}}$ needs pulse the weight of a clove dangling on its edge. The free end of the leaved moves down a distance $D = P\ell^3/(3EI)$ where P is the weight of the clove, E is a constant of elasticity (that depends on the material from which the beast is manufactured), and I is a moment of inertia. (These last two quantities will again make an appearance in Lecture 13 and on 4C, but do not worry about what they mean now. . .) Suppose our beast has a length $\ell = 2.5\,\text{yd}$, and that it takes a deflection of $D = 25\,\text{cm}$ under the weight of the clove. Use a linear approximation to estimate the deflection due to a heavier load P its length was increased by 20 cm