LINEAR ALGEBRA

CSD001P5M

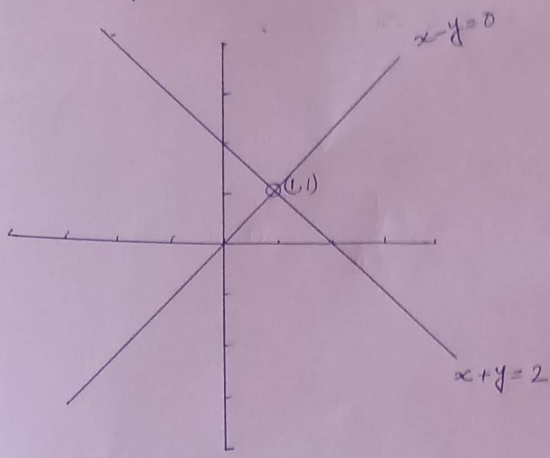
Indian Institute of Technology, Jammu.

One may find some mistakes in this note and mistakes are there for a certain purpose. This note is only for the course CSD001P5M (2020-batch).

Errors Submission of Linear Algebra (Determinant)

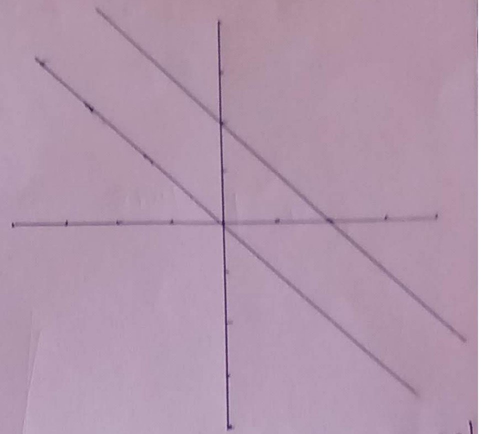
Course : CSD001P5M

$$\left.\begin{aligned} x+y&=2 \\ x-y&=0 \end{aligned}\right\} \text{(S.L.E-I)}$$



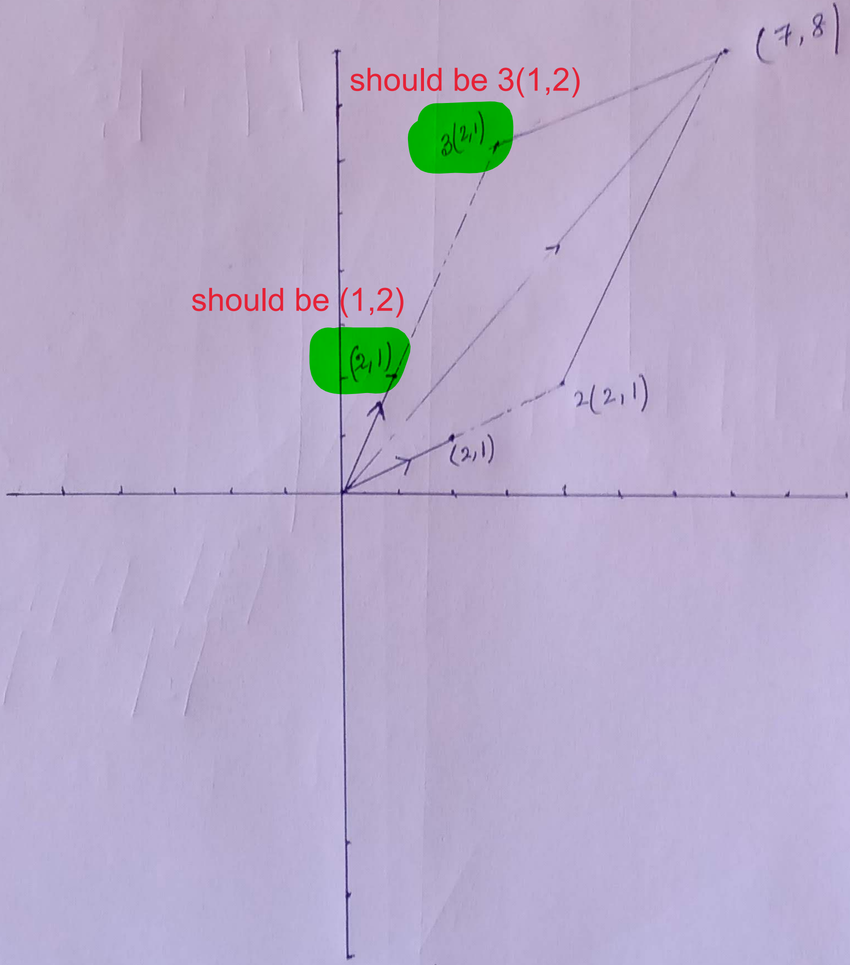
Row picture of (S.L.E-I)
Intersection point of the ~~lines in (xx)~~. two lines is (1,1).

$$\left.\begin{aligned} x+y&=2 \\ x+y&=0 \end{aligned}\right\} \text{(S.L.E-II)}$$



Row picture of (S.L.E-II)
There is no intersection point of the two lines

$$\left.\begin{aligned} 2x+y&=7 \\ x+2y&=8 \end{aligned}\right\} \text{(S.L.E-III)} \Rightarrow x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix}$$



Column picture of (S.L.E-III)

$x=2$
$y=3$

$$2\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix}$$

Determinant: Let $n \in \mathbb{Z}^+$, determinant is a function from a set of square matrices of size $n \times n$ to $\mathbb{R}\,(\,or\,\mathbb{C})$, satisfying the following properties

$$(*)\begin{cases}(i)det(I_n) = 1.\\ (ii)det \text{ is linear in rows.}\\ (iii)\text{If two rows of a matrix A are equal, then } det(A) = 0.\end{cases}$$

det is linear in rows means, for $1 \leq i \leq n$,

$$det(A) = det\begin{bmatrix}\vdots\\ c_1a_i + c_2b_i\\ \vdots\end{bmatrix} = c_1.det\begin{bmatrix}\vdots\\ a_i\\ \vdots\end{bmatrix} + c_2.det\begin{bmatrix}\vdots\\ b_i\\ \vdots\end{bmatrix} = c_1det(A') + c_2det(A''),$$

where c_1 and c_2 are scalars, a_i and b_i denote the i-th rows of A' and A'' respectively, A, A', A'' are same matrices except i-th row, and i-th row of A is c_1 times i-th row of A' plus c_2 times i-th row of A''.

From the definition of determinant it is clear that the determinant value of a matrix, which has a zero row, is zero. Observe the following consequence results.

(a). If A' is obtained from A by adding a scalar multiple times j-th row of A to i-th row, then $det(A) = det(A')$.

Suppose $A := \begin{bmatrix}\vdots\\ a_i\\ \vdots\\ a_j\\ \vdots\end{bmatrix}$, where i-th row is denoted by a_i and j-th row is denoted by a_j. Now $A' = \begin{bmatrix}\vdots\\ a_i + ca_j\\ \vdots\\ a_j\\ \vdots\end{bmatrix}$, observe that

$$det(A') = det\begin{bmatrix}\vdots\\ a_i + ca_j\\ \vdots\\ a_j\\ \vdots\end{bmatrix} = det\begin{bmatrix}\vdots\\ a_i\\ \vdots\\ a_j\\ \vdots\end{bmatrix} + cdet\begin{bmatrix}\vdots\\ a_j\\ \vdots\\ a_j\\ \vdots\end{bmatrix} = det\begin{bmatrix}\vdots\\ a_i\\ \vdots\\ a_j\\ \vdots\end{bmatrix} = det(A)$$

Note: If $A = I_n$, then $A' = E_{ij(c)}$. Therefore $det(E_{ij(c)}) = det(I_n) = 1$. This shows that the determinant value of an elementary matrix of type II is one.

(b). If A' is obtained from A by interchanging i-th row of A and j-th row of A, then $det(A') = -det(A)$. Therefore we have $det(E_{ij}) = -det(I_n) = -1$, and the determinant value of an elementary matrix of type I is -1.

(c). If A' is obtained from A by multiplying i-th row by a scalar c (it can be zero also), then $det(A') = c.det(A)$. Therefore we have $det(E_{i(c)}) = cdet(I_n) = c$ (here c is nonzero).

Exercise 0.1. Show that *condition*$(*)$ in the determinant function is equivalent to

$$(**)\begin{cases}(i)det(I_n) = 1.\\ (ii)det \text{ is linear in rows.}\\ (iii)\text{If two adjacent rows of a matrix A are equal, then } det(A) = 0.\end{cases}$$

Now, can we calculate the value of the determinant for a square matrix A of size 2×2?
$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$det(A) = det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = det\begin{bmatrix} a(1,0)+b(0,1) \\ c \quad d \end{bmatrix} = a.det\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b.det\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix},$$

$$det\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} = det\begin{bmatrix} 1 \quad 0 \\ c(1,0)+d(0,1) \end{bmatrix} = c.det\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d.det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = d,$$

$$det\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} = det\begin{bmatrix} 0 \quad 1 \\ c(1,0)+d(0,1) \end{bmatrix} = c.det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d.det\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = -c.$$

Hence $det(A) = ad - bc$.

Now, what about the determinant value of a matrix of size 3×3?

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$det(A) = det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = det\begin{bmatrix} a_{11}(1,0,0)+a_{12} & (0,1,0)+a_{13} & (0,0,1) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11}.det\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{12}.det\begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{13}.det\begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$det\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{21}.det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{22}.det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{23}.det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{22}\left(a_{31}.\text{det}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{32}.\text{det}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_{33}.\text{det}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$+a_{23}.\left(a_{31}.\text{det}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + a_{32}.\text{det}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + a_{33}.\text{det}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$= a_{22}a_{33} - a_{23}a_{33}.$ a22 a33-a23 a32 ->1st and 3rd terms will be 0, so only 2nd term i.e a32 will left.

Finally we have $det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$.

Suppose $A := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$,

$$det(A) = \sum_{k_1,k_2,\ldots,k_n} a_{1k_1}a_{2k_2}\cdots a_{nk_n} det\begin{bmatrix} e_{k_1} \\ e_{k_2} \\ \vdots \\ e_{k_n} \end{bmatrix} = \sum_{\sigma} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} det\begin{bmatrix} e_{\sigma(1)} \\ e_{\sigma(2)} \\ \vdots \\ e_{\sigma(n)} \end{bmatrix},$$ where σ

is a bijective map from the set $\{1,2,\ldots,n\}$ onto itself. One can observe that determinant value of the matrix $\begin{bmatrix} e_{\sigma(1)} \\ e_{\sigma(2)} \\ \vdots \\ e_{\sigma(n)} \end{bmatrix}$ is unique for a fixed bijective map σ (the value is either 1 or -1, depends on the bijective map σ, for more details see "Hoffman and Kunze —*Linear Algebra*"). Hence this determinant function is an unique function which satisfies *condition* $(*)$.

Lemma 0.2. *Let A be square matrix of size $n \times n$ and E be an elementary matrix of size $n \times n$, then $det(EA) = det(E).det(A)$.*

Proof. Suppose E is an elementary matrix of type I, which is obtained by interchanging i-th row and j-th row of the identity matrix of size $n \times n$. Hence $det(E) = -det(I_n) = -1$. Now EA is a matrix, which is obtained by interchanging i-th row and j-th row of the matrix A. Therefore $det(EA) = -det(A) = det(E).det(A)$. Similarly we obtain $det(EA) = det(E)det(A)$, when E is an elementary matrix of type II or type III. □

Theorem 0.3. *Let A and B be square matrices of same size. Then $det(AB) = det(A)det(B)$.*

Proof. **Case – I** : Suppose A is an invertible matrix of size $n \times n$ (i.e, A is row-equivalent to I_n), then there exist k elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = I_n.$$

Now, $det(B) = det(E_k E_{k-1} \cdots E_1 AB) = det(E_k)det(E_{k-1}) \cdots det(E_1)det(AB)$.

Hence $det(AB) = \frac{1}{det(E_k)det(E_{k-1})\cdots det(E_1)} det(B) = det(A)det(B)$.

Case – II : Suppose A is not an invertible matrix, then there exist k elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \ldots E_1 A = R,$$

where R is a row-reduced echelon matrix, whose bottom row is zero. We have

$$0 = det(R) = det(E_k E_{k-1} \ldots E_1 A) = det(E_k)det(E_{k-1}) \cdots det(E_1)det(A).$$

Therefore $det(A) = 0$. Now,

$$0 = det(RB) = det(E_k E_{k-1} \ldots E_1 AB) = det(E_k)det(E_{k-1}) \cdots det(E_1)det(AB).$$

Therefore $det(AB) = 0 = det(A)det(B)$. □

Theorem 0.4. *A square matrix is invertible if and only if its determinant is different from zero. If A is invertible, then $det(A^{-1}) = (detA)^{-1}$*

Proof. Analyse the proof of Theorem 0.3 and try to prove the claim. □

Exercise 0.5. Let A and B be square matrices of same size, then $(AB)^t = B^t A^t$ and $(A^t)^t = A$.

According to Type II, Eij(c) not equals to Eij(c)

Suppose E_{ij} is an elementary matrix of type I then $(E_{ij})^t = E_{ij}$, $E_{ij(c)}$ is an elementary matrix of type II then $(E_{ij(c)})^t = E_{ij(c)}$ and $E_i(c)$ is an elementary matrix then $(E_{i(c)})^t = E_{i(c)}$. Hence $det(E^t) = det(E)$ when E is an elementary matrix.

Theorem 0.6. *The determinant of a matrix A is equal to the determinant of its transpose A^t.*

Proof. **Case-I** Suppose A is an invertible matrix of size $n \times n$, then there exist k elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = I_n.$$

Now we have $I_n = A^t E_1^t \cdots E_k^t$, therefore [redacted]

$$1 = det(I_n) = det(A^t)det(E_1^t) \cdots det(E_k^t) = det(A^t)det(E_1) \cdots det(E_k).$$

Hence $det(A) = det(A^t)$.

Case-II Suppose A is not invertible, then $det(A) = 0$. In this case A^t is also not invertible (why?) and therefore $det(A^t) = 0$. Hence $det(A) = det(A^t)$. $\square$

After 1st term, a2v, a3v, a4v....are misssing!

$$f(A) = (-1)^{1+v} a_{1v} f(A_{1v}) + (-1)^{2+v} a_{2v} f(A_{2v}) + \cdots + (-1)^{j+v} a_{jv} f(A_{jv}) + \cdots + (-1)^{n+v} a_{nv} f(A_{nv})$$

Let $n \in \mathbb{Z}^+$, $A := (a_{ij})_{n\times n}$.

Let f be a function on a set of square matrices $f(A) := (-1)^{1+v} a_{1v} f(A_{1v}) + (-1)^{2+v} f(A_{2v}) + \cdots + (-1)^{j+v} f(A_{jv}) + \cdots + (-1)^{n+v} f(A_{nv})$, where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained from A by deleting i-th row and j-th column.

Claim: f is the determinant function for every $v \in \{1, 2, \ldots, n\}$. We prove the claim by showing f is linear in rows, f is zero on the set matrices, whose two adjacent rows are equal, and $f(I_n) = 1$ for every $v \in \{1, 2, \ldots, n\}$ (by using mathematical induction). Check f is linear in rows for all the matrices of size 1×1 and 2×2. Assuming f is linear in rows for all matrices of size 1×1, 2×2, …, $(n-1)\times(n-1)$, we will show f is linear in rows for the matrices of size $n\times n$.

$j \in \{1, 2, \ldots, n\}$,

$$f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1v} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2v} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1a_{j1}+c_2b_1 & c_1a_{j2}+c_2b_2 & \cdots & c_1a_{jv}+c_2b_v & \cdots & c_1a_{jn}+c_2b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nv} & \cdots & a_{nn} \end{bmatrix}\right)$$

$$= (-1)^{1+v} a_{1v} f\left(\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(v-1)} & a_{2(v+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1a_{j1}+c_2b_1 & c_1a_{j2}+c_2b_2 & \cdots & c_1a_{j(v-1)}+c_2b_{(v-1)} & c_1a_{j(v+1)}+c_2b_{(v+1)} & \cdots & c_1a_{jn}+c_2b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix}\right) +$$

(-1)^(j+v) (c1ajv+c2bv)

$$\cdots + (-1)^{j+v}(c_1a_{jv}+c_2b_j) f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(v-1)} & a_{1(v+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(j-1)1} & a_{(j-1)2} & \cdots & a_{(j-1)(v-1)} & a_{(j-1)(v+1)} & \cdots & a_{(j-1)n} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+11)(v-1)} & a_{(j+1)(v+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix}\right) + \cdots$$

Complete first row missing w.r.t to previous steps

a11 .... a12 ....a1(v-1)...........a1(v+1).....................a1n

$$+(-1)^{n+v} a_{nv} f\left(\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(v-1)} & a_{2(v+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1a_{j1}+c_2b_1 & c_1a_{j2}+c_2b_2 & \cdots & c_1a_{j(v-1)}+c_2b_{(v-1)} & c_1a_{j(v+1)}+c_2b_{(v+1)} & \cdots & c_1a_{jn}+c_2b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(v-1)} & a_{(n-1)(v+1)} & \cdots & a_{(n-1)n} \end{bmatrix}\right)$$

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$$= c_1(-1)^{1+v} a_{1v} f \left(\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(v-1)} & a_{2(v+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{j(v-1)} & a_{j(v+1)} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix} \right) + \cdots$$

$$+ c_1(-1)^{j+v} a_{jv} f \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(v-1)} & a_{1(v+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(j-1)1} & a_{(j-1)2} & \cdots & a_{(j-1)(v-1)} & a_{(j-1)(v+1)} & \cdots & a_{(j-1)n} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+11)(v-1)} & a_{(j+1)(v+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix} \right) + \cdots$$

a(j+1)(v-1)

$$+ c_1(-1)^{n+v} a_{nv} f \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(v-1)} & a_{1(v+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{j(v-1)} & a_{j(v+1)} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(v-1)} & a_{(n-1)(v+1)} & \cdots & a_{(n-1)n} \end{bmatrix} \right)$$

$$+ c_2(-1)^{1+v} a_{1v} f \left(\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(v-1)} & a_{2(v+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_{v-1} & b_{(v+1)} & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix} \right) + \cdots$$

bv

$$+ c_2(-1)^{j+v} b_j f \left(\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(v-1)} & a_{2(v+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(j-1)1} & a_{(j-1)2} & \cdots & a_{(j-1)(v-1)} & a_{(j-1)(v+1)} & \cdots & a_{(j-1)n} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+11)(v-1)} & a_{(j+1)(v+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix} \right) +$$

a(j+1)(v-1)

$$+ c_2(-1)^{n+v} a_{nv} f \left(\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(v-1)} & a_{2(v+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_{v-1} & b_{v+1} & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(v-1)} & a_{(n-1)(v+1)} & \cdots & a_{(n-1)n} \end{bmatrix} \right)$$

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$$= c_1 f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1v} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2v} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jv} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nv} & \cdots & a_{nn} \end{bmatrix}\right) + c_2 f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1v} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2v} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_v & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nv} & \cdots & a_{nn} \end{bmatrix}\right)$$

This shows that f is linear in row for every $v \in \{1, 2, \ldots, n\}$.

$--------$

Now, we want to prove $f(A) = 0$, whenever A has two identical adjacent rows (using mathematical induction). Check f is zero on the matrices of size 2×2, which have two two identical adjacent rows. Assuming f is zero on the set of square matrices of size 2×2, $\cdots$, $(n-1) \times (n-1)$, and each matrix has two identical adjacent rows.

$$f(A) = f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1v} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2v} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jv} & \cdots & a_{jn} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+1)v} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nv} & \cdots & a_{nn} \end{bmatrix}\right), \text{ where } a_{ij} = a_{(i+1)j}$$

for all $j = 1, 2, \cdots, n$.

$$f(A) = (-1)^{1+v} a_{1v} f\left(\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(v-1)} & a_{2(v+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{j(v-1)} & a_{j(v+1)} & \cdots & a_{jn} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+1)(v-1)} & a_{(j+1)(v+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix}\right) +$$

$$\cdots + (-1)^{j+v} a_{jv} f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(v-1)} & a_{1(v+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(j-1)1} & a_{(j-1)2} & \cdots & a_{(j-1)(v-1)} & a_{(j-1)(v+1)} & \cdots & a_{(j-1)n} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+1)(v-1)} & a_{(j+1)(v+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix}\right) +$$

$$(-1)^{j+1+v} a_{(j+1)v} f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(v-1)} & a_{1(v+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(j-1)1} & a_{(j-1)2} & \cdots & a_{(j-1)(v-1)} & a_{(j-1)(v+1)} & \cdots & a_{(j-1)n} \\ a_{j1} & a_{j2} & \cdots & a_{j(v-1)} & a_{j(v+1)} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(v-1)} & a_{n(v+1)} & \cdots & a_{nn} \end{bmatrix}\right) +$$

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$$\cdots + (-1)^{n+v} a_{nv} f\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(v-1)} & a_{1(v+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{j(v-1)} & a_{j(v+1)} & \cdots & a_{jn} \\ a_{(j+1)1} & a_{(j+1)2} & \cdots & a_{(j+1)(v-1)} & a_{(j+1)(v+1)} & \cdots & a_{(j+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(v-1)} & a_{(n-1)(v+1)} & \cdots & a_{(n-1)n} \end{bmatrix}\right)$$

$= 0.$ Here, f(In) should be equals to 1. because determinant value of an Identity matrix is 1.

$---------$

Now check $f(I_n) = 0$ for every $v \in \{1, 2, \ldots, n\}$.

$---------$

Hence f is the determinant function for every v in $\{1, 2, \ldots, n\}$ (i.e, $f \equiv det$). This is one formula to find out the determinant value of a square matrix and

$det(A) := (-1)^{1+v} a_{1v} det(A_{1v}) + (-1)^{2+v} det(A_{2v}) + \cdots + (-1)^{j+v} det(A_{jv}) + \cdots + (-1)^{n+v} det(A_{nv}),$

is called the co-factor expansion with respect v-th column.

$f(A) = (-1)^{1+v} a_{1v} f(A_{1v}) + (-1)^{2+v} a_{2v} f(A_{2v}) + ----$
$+ (-1)^{j+v} a_{jv} f(A_{jv}) + ---- + (-1)^{n+v} a_{nv} f(A_{nv})$

After 1st term, a2v, a3v, a4v....are missing!

co-efficients are missing here

Let $A := (a_{ij})_{n\times n}$.

$a_{11}det(A_{11}) - a_{21}det(A_{21}) + \cdots + (-1)^{n+1}a_{n1}det(A_{n1}) = det(A) = det(A^t) = a_{11}det(A^t_{11}) - a_{12}det(A^t_{12}) + \cdots + (-1)^{n+1}det(A^t_{1n}) = a_{11}det(A_{11}) - a_{12}det(A_{12}) + \cdots + (-1)^{n+1}det(A_{1n})$.

This shows that to find out the determinant value of a square matrix, we can use co-factor expansion with respect to first row of the matrix instead of first column of the matrix. Similarly we get the determinant value of a matrix using co-factor expansion with respect to v-th column (where $v \in \{1, 2, \ldots, n\}$).

Let $A := (a_{ij})_{n\times n}$ be a suare matrix. The determinant of A_{ij} is called the minor of the entry a_{ij}, where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained from A by deleting i-th row and j-th column. The cofactor of the element a_{ij} is $C_{ij} := (-1)^{i+j}det(A_{ij})$ and the adjoint of A is the matrix

$$adj(A) := \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem 0.7. *Let $A := (a_{ij})_{n\times n}$ be a matrix. Then*

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{cases} det(A) \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

This shows that $A.adj(A) = det(A)I_n$.

Proof. Suppose $i = j$, then

$$\begin{aligned} a_{i1}C_{i1}+a_{i2}C_{i2}+\ldots+a_{in}C_{in} &= (-1)^{i+1}a_{i1}det(A_{i1})+(-1)^{i+2}a_{i2}det(A_{i2})+\cdots+(-1)^{i+n}a_{in}det(A_{in}) \\ &= det(A). \end{aligned}$$

Suppose $i \neq j$ and A' is a matrix obtained from A by replacing i-th row of A with j-th row of A (therefore A' has two identical rows that are i-th row and j-th row), then

$$\begin{aligned} a_{i1}C_{j1}+a_{i2}C_{j2}+\ldots+a_{in}C_{jn} &= (-1)^{j+1}a_{i1}det(A_{j1})+(-1)^{j+2}a_{i2}det(A_{j2})+\cdots+(-1)^{j+n}a_{in}det(A_{jn}) \\ &= det(A') = 0. \end{aligned}$$

□

Cramer's Rule: If $Ax = b$ is a system of linear equations, where A is an invertible matrix of size $n \times n$, then the system has a unique solution and $x_i = \frac{det(A_i)}{det(A)}$, where A_i is a matrix obtained by replacing i-th row of A with the vector b.

Since A is invertible, we have $x = A^{-1}b = \frac{Adj(A)}{detA}b$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{detA} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Now, for $i \in \{1, 2, \ldots, n\}$, $x_i = \frac{1}{detA}(C_{i1}b_1 + C_{i2}b_2 + \cdots + C_{in}b_n) = \frac{det(A_i)}{detA}$.

Thank You, Sir!