INTRODUCTION TO MATRIX

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A matrix of size $m \times n$ is a collection of $m.n$ elements from a nonempty set S, arranged in rectangular array with n columns and m rows, and it is denoted by

$$A_{m,n} := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where m, n are positive integers and $a_{ij} \in S$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and a_{ij} is the element in the i-th row and j-th column, which is called ij-th element of the matrix A. When $m = n$, the matrix A is square matrix. Some examples of matrices

$$A := \begin{bmatrix} 1 & 5 & 3 & 2 & 5 \\ 13 & 2 & 7 & 1 & 1 \\ 8 & 11 & 0 & 5 & 4 \end{bmatrix} B := \begin{bmatrix} 1+x & x^2 & x^5 \\ x^3 & x^2+x^4 & 2 \end{bmatrix} C := \begin{bmatrix} 2 & 1+i & 3i & -i \\ i & 0 & 1+\sqrt{2}i & \sqrt{3} \end{bmatrix}$$

here A is a 3×5 matrix whose elements are from the set of real numbers, B is a 2×3 matrix whose elements are from the set of polynomials in x, and C is 2×4 matrix whose elements are from the set of complex numbers.

Most of the times we consider matrices whose elements are from a field, particularly set of real numbers (or complex numbers).

Zero Matrix: A matrix of size $m \times n$ is called zero matrix if all the entries of the matrix is zero and it is denoted by $O_{m,n}$.

Identity Matrix: A square matrix of size $n \times n$ is called identity matrix if ii-th entries of the matrix is 1 for $1 \leq i \leq n$ and all others entries are zero, and this matrix is denoted by I_n

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Let n be a positive integer. We denote the vector $(0, 0, \ldots, 1, \ldots, 0)$ in $\mathbb{R}^n$ $(or\, \mathbb{C}^n)$ as e_i, the vector in $\mathbb{R}^n$ $(or\, \mathbb{C}^n)$, has 1 in i-th position and 0 elsewhere. Here we use e_i as a column vector of size $n \times 1$, (where $i1$-th entry is 1 and 0 elsewhere) or a row vector of size $1 \times n$, (where $1i$-th entry is 1 and 0 elsewhere), depending upon the context.

For example I_n written as

$$\begin{bmatrix} e_1 & e_2 & \ldots & e_n \end{bmatrix}$$

[1] where $e_1, e_2, e_3, \ldots, e_n$ are column vectors of size $n \times 1$. Also I_n can be written as

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$e_1, e_2, e_3, \ldots, e_n$ are rows vectors of size $1 \times n$.

Addition of two matrices:

We define addition of two matrices of same size by adding its elements in same position.

$$A := \begin{bmatrix} 2 & 1 & 9 & 0 \\ -1 & 5 & 7 & 9 \\ 5 & -6 & 11 & 8 \end{bmatrix} B := \begin{bmatrix} 4 & 2 & -9 & -4 \\ 0 & 9 & -8 & 2 \\ 1 & 3 & 0 & 3 \end{bmatrix}$$

addition of the matrices A and B is

$$A + B := \begin{bmatrix} 2+4 & 1+2 & 9-9 & 0-4 \\ -1+0 & 5+9 & 7-8 & 9+2 \\ 5+1 & -6+3 & 11+0 & 8+3 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 0 & -4 \\ -1 & 14 & -1 & 11 \\ 6 & -3 & 11 & 11 \end{bmatrix}$$

Matrix addition is commutative that is $A + B = B + A$ and it is followed from the commutative property of the field. Matrix addition is associative that is $A + (B + C) = (A + B) + C$ and it is followed from the associative property of the field.

Analogously we can define substruction of two matrices of same size. We can multiply a matrix by a scalar. Suppose A be a matrix of size $m \times n$ and c be a scalar then the multiplication cA defined as follows

$$c \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

[1]To avoid any ambiguity by

$$\begin{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \end{bmatrix} \text{ we mean } \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

similarly by

$$\begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{bmatrix} \text{ we mean } \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Multiplication of matrices

We define matrix multiplication of two matrices A and B where number of columns in A equals number of rows in B. Let A be a matrix of size $m \times n$ and B be of size $n \times p$,

$$A := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} B := \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

Product of A and B is AB and defined as follows

$$AB := \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{22} + \cdots + a_{2n}b_{n1} & \cdots & a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2n}b_{np} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + a_{22}b_{2p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

The ij-th element of the matrix AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{np}$, which is a dot product of the i-th row vector of A and j-th column vector of B.

Exercise 0.1. Matrix multiplication is associative, that is, A, B and C are matrices such that BC, $A(BC)$ are well defined, then $A(BC) = (AB)C$.

We can calculate the product of A and B in different ways.

Let

$$b_1 := \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}, b_2 := \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix}, \cdots, b_p := \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{np} \end{bmatrix}$$

be the $1, 2, \ldots, p$-th column vectors of B respectively. Now, the product AB can be expressed as $\begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}^{\clubsuit}$. Observe Ab_1 can be express as

$$b_{11} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{21} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{n1} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

that is,

b_{11}.first column of A+b_{21}. second column of A+$\cdots$+b_{n1}.n-th column of A.

Analogously we can define $Ab_2, Ab_3, \cdots, Ab_p$.

$$A := \begin{bmatrix} 1 & 3 & 5 & 0 \\ 5 & 1 & 3 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} B := \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 5 & 1 & 3 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 5 & 0 \\ 5 & 1 & 3 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 5 & 0 \\ 5 & 1 & 3 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3. \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + 1. \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + 2. \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} + 3. \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} & 2. \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + 2. \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + 1. \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} + 2. \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} & 4. \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + 1. \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + 3. \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} + 1. \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 13 & 22 \\ 28 & 19 & 32 \\ 11 & 7 & 12 \end{bmatrix}$$

Exercise 0.2. Suppose A is a matrix of size $m \times n$ and I is an identity matrix of size $n \times n$. Then show that $AI = A$.

Now we calculate the product A and B another way. Let
$a_1 := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$, $a_2 := \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}, \ldots, a_m := \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$
be the $1, 2, \ldots, m$-th row vectors of A respectively.

Now the product of AB can be expressed as

$$AB = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}.$$

Here $a_1 B$ can be expressed as

$$a_{11} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \end{bmatrix} + a_{12} \begin{bmatrix} b_{21} & b_{22} & \cdots & b_{2p} \end{bmatrix} + \cdots + a_{1n} \begin{bmatrix} b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix},$$

that is,

$$a_{11}.\text{first row of } B + a_{12}.\text{second row of } B + \cdots + a_{1n}.p\text{-th row of } B.$$

Analogously we can define $a_2 B, a_3 B, \cdots a_m B$.

$$A := \begin{bmatrix} 1 & 3 & 5 & 0 \\ 5 & 1 & 3 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \; B := \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 5 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1.\begin{bmatrix} 3 & 2 & 4 \end{bmatrix} + 3.\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 5.\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} + 0.\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \\ \\ 5.\begin{bmatrix} 3 & 2 & 4 \end{bmatrix} + 1.\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 3.\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} + 2.\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \\ \\ 2.\begin{bmatrix} 3 & 2 & 4 \end{bmatrix} + 0.\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 1.\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} + 1.\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 13 & 22 \\ 28 & 19 & 32 \\ 11 & 7 & 12 \end{bmatrix}$$

Exercise 0.3. Suppose A is a matrix of size $m \times n$ and I is an identity matrix of size $m \times m$. Then show that $IA = A$.

Inverse of a matrix A square matrix of size $n \times n$ is said to be invertible if there exists a matrix B of size $n \times n$ such that

$$AB = I \text{ and } BA = I,$$

then B is called an inverse of A and is denoted by A^{-1}.

For example the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is invertible and its inverse

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Exercise 0.4. Let A be a square matrix of size $n \times n$ and there exist two matrices of size $n \times n$ such that $AR = I_n$ and $LA = I_n$. Then show that $L = R$

Exercise 0.5. Let A and B be invertible matrices of same size. Then show that the product AB and the inverse A^{-1} are invertible, $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^{-1})^{-1} = A$.