LINEAR ALGEBRA

I
CSD001P5M
Samir Kar
Indian Institute of Technology, Jammu.

Most fundamental problem in linear algebra is to find out a set of solutions of a system of linear equations. Here we will discuss details about system of linear equations.

Consider the problem of finding n numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ (or $\mathbb{C}$), which satisfy the following m equations

$$(S.L.E)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & \cdots \text{ eq: 1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 & \cdots \text{ eq: 2} \\ \qquad\qquad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m & \cdots \text{eq: m} \end{cases}$$

where $a_{ij},\ 1 \le i \le m,\ 1 \le j \le n, b_1, b_2, \dots, b_m$ are given elements from $\mathbb{R}(or\ \mathbb{C})$. (S.L.E) is called a system of m linear equations in n unknowns. For $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$), which satisfies each of the equations in $(S.L.E)$, is called a solution of the system. If $b_1 = b_2 = \cdots = b_n = 0$, then the system $(S.L.E)$ is called homogeneous system of linear equations.

The most fundamental method to find the solutions of the linear equation is elimination method. For example

$$\begin{cases} x_1 + x_2 = 5 & \cdots \text{ eq: 1} \\ 3x_1 - 2x_2 = 5 & \cdots \text{ eq: 2} \end{cases}$$

If we consider (3.eq:1+(-1)eq:2), then we have $5x_2 = 10$, i.e, $x_2 = 2$. After putting this value $x_2 = 2$ in any of the equations (eq: 1 or eq: 2), we obtain $x_1 = 3$.

Eliminating unknowns by multiplying equations by non-zero scalars and then adding to produce equations in which some of the variables are not present. Consider $c_1, c_2, \dots, c_m$ are m scalars and multiply the j-th equation in (S.L.E.) by c_j and then add. Finally we get

$$(L.C)\begin{cases} (c_1a_{11} + c_2a_{21} + \cdots + c_ma_{m1})x_1 + \cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_ma_{mn})x_n \\ \qquad = c_1b_1 + \cdots + c_mb_m \end{cases}$$

Now if we want to eliminate one variable, let's say, x_1 we can choose $c_1, c_2, \dots, c_m$ such a way that the coefficient of x_1, that is, $c_1a_{11} + c_2a_{21} + \cdots + c_ma_{m1}$ become zero, as you have seen in the previous example. (L.C) is called a linear combination of the equations in (S.L.E). One can verify that any solution of the system of linear equations (S.L.E) will also be a solution of the new equation (L.C). Whereas converse is not true.

Two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.

Theorem 0.1. *Let (1-S.L.E), (2-S.L.E) and (3-S.L.E) be three systems of linear equations. Suppose (1-S.L.E) and (2-S.L.E) are equivalent systems, and (2-S.L.E) and (3-S.L.E) are equivalent systems, then (1-S.L.E) and (3-S.L.E) are equivalent systems.*

Theorem 0.2. *Two equivalent systems of linear equations have same set of solutions.*

Proof. Consider the following two equivalent systems of linear equations

$$(S.L.E.i)\begin{cases} a_{11}x_1 + a_{12}x_2 & + \cdots + a_{1n}x_n = b_1 & \cdots \text{ eq: } 1 \\ a_{21}x_1 + a_{22}x_2 & + \cdots + a_{2n}x_n = b_2 & \cdots \text{ eq: } 2 \\ & \cdots & \\ a_{m1}x_1 + a_{m2}x_2 & + \cdots + a_{mn}x_n = b_m & \cdots \text{eq: m} \end{cases}$$

$$(S.L.E.ii)\begin{cases} c_{11}x_1 + c_{12}x_2 & + \cdots + c_{1n}x_n = d_1 & \cdots \text{ eq: m+1} \\ c_{21}x_1 + c_{22}x_2 & + \cdots + c_{2n}x_n = d_2 & \cdots \text{ eq: m+2} \\ & \cdots & \\ c_{p1}x_1 + c_{p2}x_2 & + \cdots + c_{pn}x_n = d_p & \cdots \text{eq: m+p} \end{cases}$$

Suppose $(y_1, y_2, \cdots, y_n)$ is a solution of (S.L.E.i). Since (S.L.E.i) is equivalent to (S.L.E.ii), eq: m+1 is a linear combination of equations in (S.L.E.i) and hence any solution of (S.L.E.i) is also a solution of eq: m+1. Therefore $(y_1, y_2, \cdots, y_n)$ satisfies eq: m+1. Similarly $(y_1, y_2, \cdots, y_n)$ satisfies eq: m+2, eq: m+3, …, eq: m+p. This shows that set of solutions of (S.L.E.i) $\subseteq$ set of solutions of (S.L.E.ii).

Other way suppose $(z_1, z_2, \cdots, z_n)$ is a solution of (S.L.E.ii). Since (S.L.E.i) is equivalent to (S.L.E.ii), eq: 1 is a linear combination of equations in (S.L.E.ii) and hence any solution of (S.L.E.ii) is also a solution of eq: 1. Therefore $(z_1, z_2, \cdots, z_n)$ satisfies eq: 1. Similarly $(z_1, z_2, \cdots, z_n)$ satisfies eq: 2, eq: 3, …, eq: m. This shows that, set of solutions of (S.L.E.ii) $\subseteq$ set of solutions of (S.L.E.i).

If any system ((S.L.E.i)/(S.L.E.ii)) does not have a solution, then another system (S.L.E.ii)/(S.L.E.i) also does not have solution (Why?).

Hence two equivalent systems of linear equations have same set of solutions. □

By eliminating variables we try to form some equivalent system of linear equations (much simpler one compare to given system of linear equations) and we will find a set of solutions of the new equivalent system, and hence this set of solutions of the new equivalent system is also the set of solutions of the given system of linear equations. Now here question is that how do we proceed to find much simpler equivalent system of linear equations.

Elimination method

Consider (S.L.E), a general system of linear equations.

Step-1: If $a_{11} = 0$, we will interchange some equation whose coefficient of x_1 is non-zero (what will you do if no such equation exists in the (S.L.E)–Think!!!) with eq: 1, and relabeling this equation as a eq: 1. So nothing will change in the solutions if we interchange the equations.

Step 2: If $a_{11} \neq 0$, we replace eq: 2 by eq: 2+$(\frac{-a_{21}}{a_{11}})$.eq: 1, eq: 3 by eq: 3+$(\frac{-a_{31}}{a_{11}})$.eq: 1,..., and eq: m by eq: m+$(\frac{-a_{m1}}{a_{11}})$.eq: 1. Let's say we get the following system of equations

$$(S.L.E.2)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & \cdots \text{ eq: m+1} \\ \qquad\quad b_{22}x_2 + \cdots + b_{2n}x_n = c_2 & \cdots \text{ eq: m+2} \\ \qquad\quad b_{32}x_2 + \cdots + b_{3n}x_n = c_3 & \cdots \text{ eq: m+3} \\ \qquad\qquad \cdots \\ \qquad\quad b_{m2}x_2 + \cdots + b_{mn}x_n = c_m & \cdots \text{eq: m+m} \end{cases}$$

Here each equation in (S.L.E. 2) is a linear combination of equations in (S.L.E). Now eq: 1, (that is, eq: m+1) is a linear combination of equations in (S.L.E 2). As eq: m+2 is obtained by adding eq: 2 with $(\frac{-a_{21}}{a_{11}})$ times eq: 1, it is equivalent to saying eq: 2 can be obtained by adding eq: m+2 with $(\frac{a_{21}}{a_{11}})$ times eq: 1. Since eq: m+1 is eq: 1, eq: 2 can be obtained by adding eq: m+2 with $(\frac{a_{21}}{a_{11}})$ times eq: m+1.

Informally we can think like, eq: m+2 $\equiv$ eq: 2+ $(\frac{-a_{21}}{a_{11}})$.eq: 1, that is, eq:2 $\equiv$eq: m+2 + $(\frac{a_{21}}{a_{11}})$.eq: m+1.

Hence eq: 2 is a linear combination of equations in (S.L.E. 2). Similarly we have eq: 3, eq: 4,$\cdots$, eq: m are also linear combination of equations (S.L.E. 2). Hence (S.L.R) and (S.L.E. 2) are equivalent system of linear equations.

♦♦ This non-zero element a_{11} is called a pivot element. In some cases (mainly when we develop more theory about system of linear equations) we want this pivot element to be 1. How can we get $a_{11} = 1$? **Step-3:** we just multiply first equation by $\frac{1}{a_{11}}$. If we multiply an equation by non-zero scalar, new system and given systems are equivalent systems of linear equations.

N.B: We replace eq: 2 by eq: 2+$(\frac{-a_{21}}{a_{11}})$.eq: 1, that is not the unique way we can replace eq: 2, we can also replace eq:2 by a_{11}.eq:2+$(-a_{21})$.eq: 1, similarly for other equations. In this case also we get (S.L.E. 2) is an equivalent system of (S.L.E).

***** It may happen that some of the equations among eq: m+2, eq: m+3, $\cdots$, eq: m+m, become zero equations *. We can remove those zero equations (nothing will change in the solutions as long as we are in equivalent systems) and we proceed further.

*Here by zero equation we mean $0 = 0$ that is come from $0.x_1 + 0.x_2 + \cdots + 0.x_n = 0$.

Now we apply the same process † for the following system of $m-1$ linear equations (S.L.E.3) (In some cases, as *, we may have less than $m-1$ linear equations),

$$(S.L.E.3)\begin{cases} b_{22}x_2 + b_{23}x_3 & + \cdots + b_{2n}x_n = c_2 & \cdots \text{ eq: m+2} \\ b_{32}x_2 + b_{33}x_3 & + \cdots + b_{3n}x_n = c_3 & \cdots \text{ eq: m+3} \\ & \cdots & \\ b_{m2}x_2 + b_{m3}x_3 & + \cdots + b_{mn}x_n = c_m & \cdots \text{eq: m+m} \end{cases}$$

and we obtain an equivalent system of (m-1) linear equations (S.L.E. 4) to (S.L.E. 3)

$$(S.L.E.4)\begin{cases} b_{22}x_2 + b_{23}x_3 + \cdots + b_{2n}x_n = c_2 & \cdots \text{ eq: 2m+2} \\ \qquad c_{33}x_3 + \cdots + c_{3n}x_n = d_3 & \cdots \text{ eq: 2m+3} \\ \cdots & \\ \qquad c_{m3}x_3 + \cdots + c_{mn}x_n = d_m & \cdots \text{eq: 2m+m} \end{cases}.$$

.

Now, the following system of m linear equations

$$(S.L.E.5)\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 & \cdots \text{ eq: 2m+1} \\ \qquad b_{22}x_2 + b_{23}x_3 + \cdots + b_{2n}x_n = c_2 & \cdots \text{ eq: 2m+2} \\ \qquad\qquad c_{33}x_3 + \cdots + c_{3n}x_n = d_3 & \cdots \text{ eq: 2m+3} \\ \qquad \cdots & \\ \qquad\qquad c_{m3}x_2 + \cdots + b_{mn}x_n = d_m & \cdots \text{eq: 2m+m} \end{cases}$$

is equivalent to (S.L.E. 2). Hence (S.L.E. 5) is equivalent to (S.L.E.). If we proceed same way we can finally get an equivalent system like,

$$(S.L.E.f)\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n = b_1 & \cdots \text{ eq: 1} \\ \qquad b_{22}x_2 + b_{23}x_3 + \cdots\cdots + b_{2n}x_n = c_2 & \cdots \text{ eq: m+2} \\ \qquad\qquad c_{33}x_3 + \cdots\cdots + c_{3n}x_n = d_3 & \cdots \text{ eq: 2m+3} \\ \qquad \cdots\cdots\cdots\cdots & \\ \qquad\qquad g_{pq}x_q + \cdots + g_{pn}x_n = h_p & \cdots \text{eq: sm+p} \end{cases}$$

Hence (S.L.E. f) and (S.L.E.) have the same set of solutions. We first find the solutions of the equation eq: sm+p and substitute the values of $x_q, \cdots, x_n$ in the previous equation of eq: sm+p in (S.L.E. f) and we get values of some other variables. Using this back substitution, we able to find the set of solutions of (S.L.E. f) which is also the set of solutions of (S.L.E.).

†It may be the case (see example 2.S.L.E.) that all the coefficients of the variable $x_2, x_3, \ldots, x_k$ (for $k \leq n$) in (S.L.E.3) are zero, in this case will we just consider (S.L.E.3) as

$$\begin{cases} b_{2k+1}x_{k+1} + b_{2k+2}x_{k+2} & + \cdots + b_{2n}x_n = c_2 & \cdots \text{ eq: m+2} \\ b_{3k+1}x_{3k+1} + b_{3k+2}x_{k+2} & + \cdots + b_{3n}x_n = c_3 & \cdots \text{ eq: m+3} \\ & \cdots & \\ b_{mk+1}x_{k+1} + b_{mk+2}x_{k+2} & + \cdots + b_{mn}x_n = c_m & \cdots \text{eq: m+m} \end{cases}$$

accordingly we have (S.L.E.4), (S.L.E.5). In theory nothing will change and we have (S.L.E.5) equivalent to (S.L.E.2)

In the above method (called elimination method), we have seen that if we start with a system of linear equations (S.L.E.) we able to find a solution of the system. But it is not true always, for example, if we consider

$$\begin{cases} x_1 + x_2 = 5 \\ 2x_1 + 2x_2 = 5 \end{cases}$$

it does not have any solution. Hence in the method we did not address some possibilities.
— It may happen that in the equivalent system (S.L.E.2) some equation among eq:m+2, eq:m+3, $\cdots$, eq:m+m, is of the form $0 = c$, where c is some nonzero scalar. Essentially this means (S.L.E.2) does not have solution (elaborate the fact). Hence (S.L.E.) does not have solution.
— It may be the case that in the equivalent system (S.L.E.f) some variable has two different values. Essentially this means (S.L.E.f) does not have any solution. Hence (S.L.E.) does not have any solution.

For example consider

$$(1.S.L.E) \begin{cases} x_1 + x_2 + x_3 - x_4 = 3 \\ 2x_1 - x_2 + 3x_3 - x_4 = 5 \\ x_1 + 2x_2 + 3x_3 + x_4 = 8 \end{cases} \quad (1.S.L.E.f) \begin{cases} x_1 + x_2 + x_3 - x_4 = 3 \\ \quad 3x_2 - x_3 - x_4 = 1 \\ \qquad 7x_3 + 7x_4 = 14 \end{cases}$$

$$(2.S.L.E) \begin{cases} 2x_1 + 3x_2 + x_3 + x_4 + 2x_5 = 25 \\ 4x_1 + 6x_2 + 2x_3 + x_4 + x_5 = 31 \\ 6x_1 + 9x_2 + 3x_3 + 2x_4 + x_5 = 46 \end{cases} \quad (2.S.L.E.f) \begin{cases} 2x_1 + 3x_2 + x_3 + x_4 + 2x_5 = 25 \\ \qquad -x_4 - 3x_5 = 19 \\ \qquad -2x_5 = -10 \end{cases}$$

Ex: Show that the following systems of equations have same solutions (without find out the solutions)

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 + x_4 = 14 \\ x_1 + 2x_2 + x_3 + x_4 = 6 \\ x_1 + x_2 + 3x_3 = 8 \end{cases} \quad \begin{cases} 2x_1 + 3x_2 + 4x_3 + x_4 = 14 \\ \quad x_1 + x_2 + 3x_3 = 8 \end{cases}$$

We have seen that when we are changing one system of linear equation to another equivalent system of linear equation we are basically changing the co-efficients, by doing some arithmetic operations, of the variables under certain rules. Now, question is that why can not we express system of linear equation in short-hand way and we can handle the step:1 and step:2 in some better and efficient way.

Observe that (S.L.E) can be expressed as follow

$$Ax = b$$

where

$$A := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} b := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{1}$$

the matrix A is called the coefficient matrix of the system of linear equations (S.L.E) and the matrix $[A|b] := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$ is called augmented matrix of the system of linear equations (S.L.E).

It is not necessary to copy the name of the variables and the equality sign in every step.

We can think 1-st, 2-nd,..., m-th rows of augmented matrix of (S.L.E) as eq: 1, eq: $2,\cdots$, eq: m of (S.L.E) respectively.

Step: 1– Interchanging between two equations (i and j-th equations) of a system of linear equations is similar to interchange between two rows (i and j-th rows) of the augmented matrix of the system of linear equations.

Step: 2– Replace some equation (i-th equation) by adding itself with a scalar multiple of another equation (j-th equation) in a system of linear equations is similar to replace the row (i-th row) by adding itself with a scalar multiple of another row (j-th row) of the augmented matrix of the system of linear equations.

Step: 3– Multiply some equation (i-th equation) by a non-zero scalar in a system of linear equations is similar to multiply the row (i-th row) of the augmented matrix of the system linear equations.