Periodically Driven Aubry-André-Harper Models

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2012B5A7589P

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December 6, 2016

Introduction

Berry's Discovery

Cyclic Adiabatic evolution

- Gauge freedom in Quantum Mechanics.
- Parameter space $\mathbf{R}$ and Vector line bundle $|n(\mathbf{R})\rangle$.
- Single-valued Eigenvectors in Parameter space.
- Assumptions : Adiabatic evolution, No degeneracies/level crossings, Schrodinger evolution

Expand as

$$|\psi(t)\rangle = \sum c_n(t) e^{-\frac{i}{\hbar}\int_0^t E_n(t)dt} |n(t)\rangle$$

If at $t = 0$, $|\psi(0)\rangle = |n(0)\rangle$, from time-dependent perturbation theory

$$|\psi(T)\rangle = e^{i\gamma_n(T)} e^{i\theta_n(T)} |\psi(0)\rangle$$

(Quantum Adiabatic Theorem)

Geometric properties

From adiabatic theorem

$$\gamma_n(T) = i\int_0^T \langle n(t)|\frac{\partial}{\partial t}n(t)\rangle\ dt = i\oint_c \langle n(\mathbf{R})|\boldsymbol{\nabla}_{\boldsymbol{R}}|n(\mathbf{R})\rangle\cdot d\mathbf{R}$$

By Stokes theorem,

$$\gamma_n(T) = i\int_S \sum_{m\neq n}\frac{\langle n(\mathbf{R})|\boldsymbol{\nabla}_{\boldsymbol{R}}\hat{H}(\mathbf{R})|m(\mathbf{R})\rangle\wedge\langle m(\mathbf{R})|\boldsymbol{\nabla}_{\boldsymbol{R}}\hat{H}(\mathbf{R})|n(\mathbf{R})\rangle}{(E_n(\mathbf{R})-E_m(\mathbf{R}))^2}\cdot d\mathbf{S}$$

- Path-integral. Gauge Invariant. Reparameterization invariant.
- Not single-valued on parameter space $\Longrightarrow$ Non-integrable; Cannot be expressed as a scalar field over parameter space.
- Physically observable. (Aharanov-Bohm effect)
- Analogy to Parallel transport of vectors on curved manifold.

Analogy to Electromagnetism

Berry Connection

$$\mathbf{A}_n(\mathbf{R}) = i \langle n(\mathbf{R}) | \boldsymbol{\nabla}_{\boldsymbol{R}} | n(\mathbf{R}) \rangle \tag{1}$$

Berry Curvature

$$\mathbf{B}_n(\mathbf{R}) = i\, \boldsymbol{\nabla}_{\boldsymbol{R}} \wedge \langle n(\mathbf{R}) | \boldsymbol{\nabla}_{\boldsymbol{R}} | n(\mathbf{R}) \rangle = \boldsymbol{\nabla}_{\boldsymbol{R}} \wedge \mathbf{A}_n(\mathbf{R}) \tag{2}$$

Berry phase is then expressed as

$$\gamma_n(C) = \oint_C \mathbf{A}_n(\mathbf{R}) \cdot \, d\mathbf{R} = \int_S \mathbf{B}_n(\mathbf{R}) \cdot \, d\mathbf{S} \tag{3}$$

Analogy to Electromagnetism

Berry Connection

$$\mathbf{A}_n(\mathbf{R}) = i \langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle$$

Berry Curvature

$$\mathbf{B}_n(\mathbf{R}) = i \, \nabla_{\mathbf{R}} \wedge \langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle = \nabla_{\mathbf{R}} \wedge \mathbf{A}_n(\mathbf{R})$$

A gauge transformation of the states $|n(\mathbf{R})\rangle \to e^{i\delta(\mathbf{R})} |n(\mathbf{R})\rangle$
transforms $\mathbf{A}_n(\mathbf{R}) \to \mathbf{A}_n(\mathbf{R}) - \nabla_{\mathbf{R}}\delta(\mathbf{R})$
But, $\mathbf{B}_n(\mathbf{R})$ is unchanged.

Bargmann invariants

Definition

Gauge invariant quantity defined over an ordered set of n states

$$\Delta = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\ldots\langle\psi_{n-1}|\psi_n\rangle\langle\psi_n|\psi_1\rangle \tag{4}$$

Consider two infinitesimally separated points on the curve C in parameter space, then

$$e^{i\Delta\gamma} = \frac{\langle n(\mathbf{R})|n(\mathbf{R}+\delta\mathbf{R})\rangle}{|\langle n(\mathbf{R})|n(\mathbf{R}+\delta\mathbf{R})\rangle|}$$

$$\Delta\gamma \approx -i\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle\cdot\delta\mathbf{R}$$

We have established that

$$\arg(\langle n(\mathbf{R})|n(\mathbf{R}+\delta\mathbf{R})\rangle) \approx -\mathbf{A}_n(\mathbf{R})\cdot\delta\mathbf{R} \tag{5}$$

Bargmann invariants

We have established that

$$\arg(\langle n(\mathbf{R})|n(\mathbf{R}+\delta\mathbf{R})\rangle) \approx -\mathbf{A}_n(\mathbf{R})\cdot\delta\mathbf{R}$$

which means that

$$\gamma(C) = \oint_C \mathbf{A}_n(\mathbf{R})\cdot\delta\mathbf{R} = -\oint_C \arg(\langle n(\mathbf{R})|n(\mathbf{R}+\delta\mathbf{R})\rangle)$$

$$= -\lim_{N\to\infty} \arg(\prod_{j=0}^{N-1} \langle n(\mathbf{R}(t+j\Delta t))|n(\mathbf{R}(t+(j+1)\Delta t))\rangle)$$

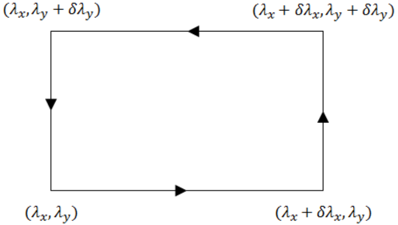
where $\Delta t = \frac{T}{N} \ni \mathbf{R}(0) = \mathbf{R}(T)$.

Theorem

Geometric Phase = Bargmann invariant of states lying on C.

Bargmann invariants

Consider an infinitesimal square on a 2D parameter space



$$\oint_Q \mathbf{A}_n(\boldsymbol{\lambda}) \cdot \delta\boldsymbol{\lambda} = -\arg(\langle n(\boldsymbol{\lambda}) | n(\boldsymbol{\lambda} + \delta\lambda_x \hat{\mathbf{x}})\rangle \langle n(\boldsymbol{\lambda} + \delta\lambda_x \hat{\mathbf{x}}) | n(\boldsymbol{\lambda} + \delta\lambda_x \hat{\mathbf{x}} + \delta\lambda_y \hat{\mathbf{y}})\rangle$$
$$\langle n(\boldsymbol{\lambda} + \delta\lambda_x \hat{\mathbf{x}} + \delta\lambda_y \hat{\mathbf{y}}) | n(\boldsymbol{\lambda} + \delta\lambda_y \hat{\mathbf{y}})\rangle \langle n(\boldsymbol{\lambda} + \delta\lambda_y \hat{\mathbf{y}}) | n(\boldsymbol{\lambda})\rangle)$$

$$\oint_Q \mathbf{A}_n(\boldsymbol{\lambda}) \cdot \delta\boldsymbol{\lambda} = \int_Q \mathbf{B}_n(\boldsymbol{\lambda}) \cdot d\mathbf{S}_{\boldsymbol{\lambda}}$$

Bargmann invariants

$$\oint_Q \mathbf{A}_n(\boldsymbol{\lambda}) \cdot \delta\boldsymbol{\lambda} = \mathbf{B}_n(\boldsymbol{\lambda})\delta\lambda_x\delta\lambda_y$$

Theorem

$$\mathbf{B}_n(\boldsymbol{\lambda})\delta\lambda_x\delta\lambda_y = -\arg\big(\langle n(\boldsymbol{\lambda})|n(\boldsymbol{\lambda}+\delta\lambda_x\hat{\mathbf{x}})\rangle \langle n(\boldsymbol{\lambda}+\delta\lambda_x\hat{\mathbf{x}})|n(\boldsymbol{\lambda}+\delta\lambda_x\hat{\mathbf{x}}+\delta\lambda_y\hat{\mathbf{y}})\rangle \\ \langle n(\boldsymbol{\lambda}+\delta\lambda_x\hat{\mathbf{x}}+\delta\lambda_y\hat{\mathbf{y}})|n(\boldsymbol{\lambda}+\delta\lambda_y\hat{\mathbf{y}})\rangle \langle n(\boldsymbol{\lambda}+\delta\lambda_y\hat{\mathbf{y}})|n(\boldsymbol{\lambda})\rangle\big)$$

Surface integral of Berry Curvature can be converted into sum of 4-point Bargmann invariant at each point.

Chern numbers

Topological Invariant

Theorem (Gauss-Bonnet)

The surface integral of Gaussian curvature over a 2-dimensional closed Riemannian manifold is equal to $2\pi\chi$, where χ is the Euler characteristic of the manifold.

- Euler characteristic of any manifold is an integer.
- Two homotopic surfaces have the same Euler characteristic.
- Intuitively, they are related to number of holes in the surface.

Berry curvature behaves like Gaussian curvature. This topological invariant is also called the first Chern number.

TKNN invariant

Particles on a Lattice

On a rectangular 2D lattice, translation of the form $\mathbf{r} \to \mathbf{r} + \mathbf{R}$ where $\mathbf{R} = ma\,\hat{\mathbf{x}} + nb\,\hat{\mathbf{y}} \ni m, n \in \mathbb{Z}$, does not affect the Hamiltonian.

From Bloch's theorem the wavefunctions are of the form

$$|\psi_{\mathbf{k}}(\mathbf{r})\rangle = e^{i\mathbf{k}\cdot\mathbf{r}}\,|u_{\mathbf{k}}(\mathbf{r})\rangle \tag{6}$$

such that $|u_{\mathbf{k}}(\mathbf{r}+\mathbf{R})\rangle = |u_{\mathbf{k}}(\mathbf{r})\rangle$.

$\mathbf{k}$ is the crystal momentum confined to limits of the Brillouin zone

$$\frac{-\pi}{a} \leq k_x \leq \frac{\pi}{a} \text{ and } \frac{-\pi}{b} \leq k_y \leq \frac{\pi}{b}$$

Each band in the spectrum is parameterized by $\mathbf{k}$ on a torus $\mathbf{T}^2$.

TKNN invariant

Particles on a Lattice

The Brillouin zone torus is a closed 2D parameter space. Berry connection on this parameter space is

$$A_x^\alpha(\mathbf{k}) = i \langle u_\mathbf{k}^\alpha | \frac{\partial}{\partial k_x} | u_\mathbf{k}^\alpha \rangle \quad A_y^\alpha(\mathbf{k}) = i \langle u_\mathbf{k}^\alpha | \frac{\partial}{\partial k_y} | u_\mathbf{k}^\alpha \rangle \tag{7}$$

Berry curvature

$$B_z^\alpha = \frac{\partial A_x^\alpha}{\partial k_y} - \frac{\partial A_y^\alpha}{\partial k_x} \tag{8}$$

where α is the band index.
Integral of Berry curvature over $\mathbf{T}^2$ is

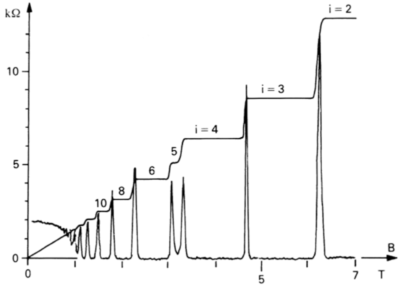
$$\int_{\mathbf{T}^2} d^2k \; B_z^\alpha = 2\pi C_\alpha \tag{9}$$

C_α is an integer - a Chern number, also called the TKNN invariant.

Quantum Hall Effect

Phenomenon

Experiments by von Klitzing et al., showed that Hall conductivity exists on quantized plateaus as magnetic field is increased.



Quantum Hall Effect

Phenomenon

- Conductivity on these plateaus take the value $\sigma_{x,y} = \frac{e^2}{2\pi\hbar} v \quad v \in \mathbb{Z}$.
- The centre of each plateau occurs when the magnetic field $B = \frac{2\pi\hbar n}{ve}$ where n is the electron density. This is the magnetic field at which v Landau levels are filled.
- Despite the presence of disorders, the observations do not change.

Explanation in terms of filled Landau levels, Edge modes and Spectral Flow was given by *Laughlin*.
Thouless et al., explained Integer Quantum Hall effect using the Kubo formula.

Kubo formula for Electrical Conductivity

Linear Response Theory

Kubo formula is a result of linear response theory. The linear correlation between applied electric field (stimulus) and the resulting current density (response) is given by the Kubo formula for Electrical Conductivity.

$$J_x = \sigma_{xy} E_y \tag{10}$$

$$\sigma_{xy} = i\hbar \sum_{\alpha,\beta | E_\alpha < E_F < E_\beta} \int_{\mathbf{T}^2} \frac{d^2k}{(2\pi)^2} \frac{\langle u_k^\alpha | J_y | u_k^\beta \rangle \langle u_k^\beta | J_x | u_k^\alpha \rangle - \langle u_k^\alpha | J_x | u_k^\beta \rangle \langle u_k^\beta | J_y | u_k^\alpha \rangle}{(E_\beta(\mathbf{k}) - E_\alpha(\mathbf{k}))^2} \tag{11}$$

E_F is the Fermi energy.

Kubo formula for Electrical Conductivity

TKNN invariant

Using the definition of current density in terms of group velocity of wavepackets

$$\tilde{H} = e^{-i\mathbf{k}\cdot\mathbf{x}} H e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \mathbf{J} = \frac{e}{\hbar}\frac{\partial \tilde{H}}{\partial \mathbf{k}}$$

the Kubo formula can be recast as

$$\sigma_{xy} = \frac{ie^2}{\hbar}\sum_{\alpha}\int_{\mathbf{T}^2}\frac{d^2k}{(2\pi)^2}\ \langle \partial_y u_{\mathbf{k}}^{\alpha}|\partial_x u_{\mathbf{k}}^{\alpha}\rangle - \langle \partial_x u_{\mathbf{k}}^{\alpha}|\partial_y u_{\mathbf{k}}^{\alpha}\rangle \tag{12}$$

where $\partial_x = \frac{\partial}{\partial k_x}$ and $\partial_y = \frac{\partial}{\partial k_y}$.
Amazingly, the integral is exactly same as the one for TKNN invariant.

$$\sigma_{xy} = -\frac{e^2}{2\pi\hbar}\sum_{\alpha} C_{\alpha} \tag{13}$$

TKNN result

Theorem

Hall Conductivity is a topological invariant.

The Hall conductivity cannot change continuously. It takes discrete jumps. Any deformation that does not change the underlying topology of the vector line bundle does not affect Hall Conductivity. Caution : This result is only valid at absolute zero $T = 0K$. We have no way to extend this result to time-varying Hamiltonians as of now.

Floquet Theory

Statement

Theorem (Floquet Theory)

Solutions to time-dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle$$

where $\hat{H}(t)$ *is periodic* $\quad \forall t \quad \hat{H}(t+T) = \hat{H}(t)$ *are of the form*

$$\left|\psi_\alpha(t)\right\rangle = e^{-i\epsilon_\alpha t}\left|\phi_\alpha(t)\right\rangle$$

- There are $n = dim(H)$ independent solutions indexed by α.
- $\left|\psi_\alpha(t)\right\rangle$ – *Floquet states.*
- $\left|\phi_\alpha(t)\right\rangle$ – *Floquet modes.*
- ϵ_α – *Quasienergies.*

Floquet Theory

Corollaries

$$\begin{aligned} |\psi_\alpha(t+T)\rangle &= e^{-i\epsilon_\alpha(t+T)}\,|\phi_\alpha(t+T)\rangle \\ &= e^{-i\epsilon_\alpha T}e^{-i\epsilon_\alpha t}\,|\phi_\alpha(t)\rangle \\ &= e^{-i\epsilon_\alpha T}\,|\psi_\alpha(t)\rangle \end{aligned}$$

- ϵ_α is real. (Normalization)
- If $\hat{U}(t_2, t_1)$ is the time evolution operator, then
$$\hat{U}(t+T, t)\,|\psi_\alpha(t)\rangle = e^{-i\epsilon_\alpha T}\,|\psi_\alpha(t)\rangle$$
Floquet states at any time t form a complete orthonormal basis.
- ϵ_α may be replaced by $\epsilon_{\alpha n} = \epsilon_\alpha + n\omega$ without affecting the above equations. Restrict $\epsilon_\alpha \in \left[\frac{-\omega}{2}, \frac{\omega}{2}\right)$, called the *Floquet Brillouin zone*.

Floquet Theory

Corollaries

$$\begin{aligned}\hat{U}(t_2, t_1) &= \sum_{\alpha_2} |\psi_{\alpha_2}(t_2)\rangle \langle\psi_{\alpha_2}(t_2)| \hat{U}(t_2, t_1) \sum_{\alpha_1} |\psi_{\alpha_1}(t_1)\rangle \langle\psi_{\alpha_1}(t_1)| \\ &= \sum_{\alpha_1,\alpha_2} |\psi_{\alpha_2}(t_2)\rangle \langle\psi_{\alpha_2}(t_2)|\psi_{\alpha_1}(t_2)\rangle \langle\psi_{\alpha_1}(t_1)| \\ &= \sum_{\alpha} e^{-i\epsilon_\alpha(t_2-t_1)} |\phi_\alpha(t_2)\rangle \langle\phi_\alpha(t_1)|\end{aligned}$$

We can therefore express any $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = \sum_{\alpha} \langle\phi_\alpha(t_0)|\psi(t_0)\rangle\, e^{-i\epsilon_\alpha(t-t_0)} |\phi_\alpha(t)\rangle$$

i.e., the contribution of each floquet mode remains constant as the state evolves in time.

Floquet Theory

Macromotion and Micromotion

The time-evolution operator over one-time period $\hat{U}(t_0 + T, t_0)$ is the *Macromotion/Strobosscopic* operator. We designate $\hat{H}^F_{t_0}$ from $\exp\left(-iT\hat{H}^F_{t_0}\right) = \hat{U}(t_0 + T, t_0)$ as the *Floquet Hamiltonian*. Floquet Hamiltonian satisfies

$$\hat{H}^F_{t_0} |\phi_\alpha(t_0)\rangle = \epsilon_\alpha |\phi_\alpha(t_0)\rangle \tag{14}$$

The *Micromotion* operator is

$$|\phi_\alpha(t_2)\rangle = \hat{U}_F(t_2, t_1) |\phi_\alpha(t_1)\rangle \tag{15}$$

$$\hat{U}_F(t_2, t_1) = \sum_\alpha |\phi_\alpha(t_2)\rangle \langle\phi_\alpha(t_1)| \tag{16}$$

which describes the time evolution of periodic floquet modes.

Floquet Theory

- Floquet Hamiltonian is time-independent but parameterized by initial time t_0. It holds all the information regarding the system.
- By diagonalizing the Floquet Hamiltonian, the Quasienergies and the Floquet modes can be determined.
- However, we have not yet described a procedure to calculate the Floquet Hamiltonian from the original Hamiltonian.
- A myriad of approximation schemes to obtain the Floquet Hamiltonian have been designed. We discuss two such methods.

Floquet Theory

Effective Hamiltonian

The Floquet modes satisfy the following eigenvalue equation

$$\left[\hat{H} - i\frac{\partial}{\partial t}\right]|\phi_\alpha\rangle = \epsilon_\alpha |\phi_\alpha\rangle$$

This is an alternative definition to Floquet Hamiltonian. We call this the *Quasienergy operator*.

$$\hat{Q} = \left[\hat{H} - i\frac{\partial}{\partial t}\right] \tag{17}$$

Floquet Hamiltonian is parameterized by initial time. Define a static Hamiltonian without any initial time parameter without losing the physical interpretation of macromotion.

Effective Hamiltonian

Let $\hat{U}_F(t)$ be a unitary transformation, such that

$$\hat{H}_F = \hat{U}_F(t)\hat{Q}(t)\hat{U}_F^\dagger(t) \tag{18}$$

is time-independent.
Under this transformation

- Macromotion operator $\hat{H}_F$ is independent of initial time.
- Floquet modes $|\phi_\alpha^F\rangle = \hat{U}_F(t)\,|\phi_\alpha(t)\rangle$ are time-independent.
- Micromotion operator $\hat{U}_F(t_2, t_1) = \hat{U}_F^\dagger(t_2)\hat{U}_F(t_1)$.
- Time-evolution operator $\hat{U}(t_2, t_1) = \hat{U}_F^\dagger(t_2)e^{-i\hat{H}_F(t_2-t_1)}\hat{U}_F(t_1)$

Effective Hamiltonian

1. Enforce periodicity on $\hat{U}_F(t) = \hat{U}_F(t+T)$.
2. Rewrite $\hat{U}_F(t)$ as $e^{i\hat{K}(t)}$ where $\hat{K}(t)$ is called the *Kick operator*.
3. Expand Hamiltonian in Fourier series $\hat{H}(t) = \hat{H}_{(0)} + \sum_{j=1}^{\infty} \hat{H}_{(j)} e^{ij\omega t} + \hat{H}_{(-j)} e^{-ij\omega t}$
4. Perturbation ansatz : $\hat{H}_F(t) = \sum_{j=0}^{\infty} \frac{1}{\omega^j} \hat{H}_F^{(j)}$.
5. Perturbation ansatz : $\hat{K}(t) = \sum_{j=0}^{\infty} \frac{1}{\omega^j} \hat{K}^{(j)}$.

In the high-frequency limit, contributions from higher order terms is negligible. With this ansatz, expressions for $\hat{H}_F$ and $\hat{K}_F$ can be obtained.

$\hat{H}_F$ is called the Effective Hamiltonian.

Brillouin-Wigner Perturbation

Concept

Let us introduce some terminology

- **Reference states** : **R** is a complete set of orthonormal states of the Hilbert space.
- **Model State** : One chosen state $|\phi_0\rangle$ from **R**.
- **Model Space** : One dimensional complex vector space with Model State as the basis.
- **Orthogonal Space** : Hilbert Space − Model Space
- **Projection Operator** : Projects to Model space $P = |\phi_0\rangle\langle\phi_0|$. P transports a vector from Hilbert space to Model space. $|\phi\rangle = P|\psi\rangle$
- **Orthogonal Projection Operator** : $Q = 1 - P$.
- **Wave Operator** : Reconstructs Hilbert Space wavefunction from Model space wavefunction. $|\psi\rangle = \Omega|\phi\rangle$

Brillouin-Wigner Perturbation

Concept

We intend to solve

$$\hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle$$

If $\left|\phi\right\rangle = P\left|\psi\right\rangle$, then

$$\hat{H}_{eff}\left|\phi\right\rangle = E\left|\phi\right\rangle \tag{19}$$

where

$$\hat{H}_{eff} = P\hat{H}\Omega P \tag{20}$$

Diagonalizing the Effective Hamiltonian $\hat{H}_{eff}$ in turn solves the original eigenvalue problem.

Ω is obtained as

$$\Omega = \left(1 - \frac{Q\hat{H}}{E}\right)^{-1} P \tag{21}$$

Brillouin-Wigner Perturbation

Floquet Hamiltonian

$\hat{H}$, $|\phi\rangle$ are periodic in time. They are expanded in Fourier series. Let

$$\mathcal{H}_{m,n} = \frac{1}{T}\int_0^T e^{i(m-n)t}\hat{H}(t)\,dt \tag{22}$$

$$\mathcal{M}_{m,n} = m\delta_{m,n} \tag{23}$$

$$|\phi_\alpha^m\rangle = \frac{1}{T}\int_0^T e^{imt}\,|\phi_\alpha\rangle\;dt \tag{24}$$

Then

$$(\mathcal{H} - \mathcal{M}\omega)\,|\phi_\alpha\rangle = \epsilon_\alpha\,|\phi_\alpha\rangle \tag{25}$$

Apply Brillouin-Wigner Perturbation theory to solve above equation.

Brillouin-Wigner Perturbation

Effective Hamiltonian

- Choose Fourier modes as the Reference states.
- Project to static fourier mode. $\mathcal{P} = \delta_{m,n}\delta_{m,0}$
- Obtain Wave operator Ω and $\hat{H}_{eff}$.
- Further simplified by defining E independent Ω called Ω_{BW}. Obtain $\hat{H}_{BW} = \mathcal{P}\mathcal{H}\Omega_{BW}\mathcal{P}$. $\hat{H}_{BW}$ is time-independent because we are in Fourier basis.
- Ω_{BW} is expanded in a $1/\omega$ series.

Details are omitted for brevity.

Aubry-André-Harper Model

Problem Description

- Motion of electrons (spinless, non-relativistic) in a periodic 2-dimensional rectangular lattice subject to a constant magnetic field perpendicular to the plane of the lattice.
- Landau-level problem on a lattice.

Hamiltonian in the presence of magnetic field is obtained by replacing $\hat{\mathbf{p}}$ by $\hat{\mathbf{p}} - e\mathbf{A}$.

$$\hat{H}(x, y) = \frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A})^2 + \hat{V}(x, y) \tag{26}$$

Aubry-André-Harper Model

Bloch's theorem in the presence of Magnetic field

- Uniform Magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$
- Magnetic Vector Potential in Landau gauge $\mathbf{A} = Bx\hat{\mathbf{y}}$

Hamiltonian has no translational invariance $\mathbf{A}(\mathbf{r}) \neq \mathbf{A}(\mathbf{r} + \mathbf{R})$. The Translation Operators do not commute with the Hamiltonian rendering Bloch's theory useless.

Aubry-André-Harper Model

Bloch's theorem in the presence of Magnetic field

For uniform magnetic field

$$\mathbf{A}(\mathbf{r}+\mathbf{R}) = \mathbf{A}(\mathbf{r}) + \nabla\mathcal{G}(\mathbf{r},\mathbf{R})$$

Therefore, the operation of translation operator on the Hamiltonian is equivalent to a gauge transformation

$$\hat{T}_{\mathbf{R}}\left(\frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r}))^2\right) = \left(\frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r}) - e\nabla\mathcal{G}(\mathbf{r},\mathbf{R}))^2\right)\hat{T}_{\mathbf{R}}$$

$$\left(\frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r}) - e\nabla\mathcal{G}(\mathbf{r},\mathbf{R}))^2\right) = e^{\frac{ie}{\hbar}\mathcal{G}(\mathbf{r},\mathbf{R})}\left(\frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r}))^2\right)e^{\frac{-ie}{\hbar}\mathcal{G}(\mathbf{r},\mathbf{R})}$$

Magnetic translation operators

$$\hat{\mathcal{T}}_{\mathbf{R}} = e^{\frac{-ie}{\hbar}\mathcal{G}(\mathbf{r},\mathbf{R})}\hat{T}_{\mathbf{R}}$$

commutes with $\hat{H}$.

Aubry-André-Harper Model

Bloch's theorem in the presence of Magnetic field

For Magnetic Translation operators to form a group

$$\hat{\mathcal{T}}_{\mathbf{R}}\hat{\mathcal{T}}_{\mathbf{R}'} = e^{\frac{-ie}{\hbar}BR'_x R_y}\hat{\mathcal{T}}_{\mathbf{R}+\mathbf{R}'}$$

we need

$$\alpha = \frac{e}{h}Bd^2 = \frac{p}{q} \text{ such that } p, q \in \mathcal{Z}^+ \text{ and } gcd(p,q) = 1$$

and define *Magnetic Translation vectors*

$$\mathcal{R} = qmd\hat{\mathbf{x}} + nd\hat{\mathbf{y}} \text{ such that } m, n \in \mathcal{Z}$$

then, a subset of magnetic translation operators, that translate by units of magnetic unit cell form a group

$$\hat{\mathcal{T}}_{\mathcal{R}}\hat{\mathcal{T}}_{\mathcal{R}'} = e^{-i2\pi\frac{e}{h}Bm_1n_2qd^2}\hat{\mathcal{T}}_{\mathcal{R}+\mathcal{R}'} = e^{-i2\pi pm_1n_2}\hat{\mathcal{T}}_{\mathcal{R}+\mathcal{R}'} = \hat{\mathcal{T}}_{\mathcal{R}+\mathcal{R}'}$$

Aubry-André-Harper Model

Bloch's theorem in the presence of Magnetic field

This redefinition of Translation vectors also defines the Magnetic Brillouin Zone

$$k_x \in \left(-\frac{\pi}{q}, \frac{\pi}{q}\right) \text{ and } k_y \in (-\pi, \pi)$$

Now we can write the Bloch's theorem like equation

$$\hat{\mathcal{T}}_{\mathcal{R}} \left|\psi_{\mathbf{k}}(\mathbf{r})\right\rangle = e^{i\mathbf{k}\cdot\mathcal{R}} \left|\psi_{\mathbf{k}}(\mathbf{r})\right\rangle$$

$$\left|\psi_{\mathbf{k}}(\mathbf{r})\right\rangle = e^{i\mathbf{k}\cdot\mathbf{r}} \left|u_{\mathbf{k}}(\mathbf{r})\right\rangle$$

However, $\left|u_{\mathbf{k}}(\mathbf{r}+\mathcal{R})\right\rangle \neq \left|u_{\mathbf{k}}(\mathbf{r})\right\rangle$. Rather,

$$\left|u_{\mathbf{k}}(\mathbf{r}+\mathcal{R})\right\rangle = e^{-i2\pi\frac{e}{h}Bqmdy} \left|u_{\mathbf{k}}(\mathbf{r})\right\rangle$$

This is called the *Generalized bloch condition*.

Tight-Binding Model

The wavefunction is expanded as a linear combination of a set of localized states - Wannier/single atom states.

$$|\psi(\mathbf{r})\rangle = \sum_{\mathbf{R}} a_{\mathbf{R}} |\phi_{\mathbf{R}}(\mathbf{r})\rangle$$

where $\langle \phi_{\mathbf{R}'} | \phi_{\mathbf{R}} \rangle = \delta_{m'm}\delta_{n'n}$.
Generic Tight-Binding model is written as

$$W_{1,0}(a_{m+1,n} + a_{m-1,n}) + W_{0,1}(a_{m,n+1} + a_{m,n-1}) = Ea_{m,n} \quad (27)$$

In matrix form

$$\hat{H}_0 = \sum_{m,n} W_{1,0} |m+1, n\rangle \langle m, n| + W_{0,1} |m, n+1\rangle \langle m, n| + h.c.$$

Tight-Binding Model

Peirels Substitution

In the presence of magnetic field, we expand as

$$|\psi(\mathbf{r})\rangle = \sum_{\mathbf{R}} a_{\mathbf{R}} e^{2\pi \frac{ie}{h} \int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A} \cdot d\mathbf{r}'} |\phi_{\mathbf{R}}(\mathbf{r})\rangle$$

This leads to the tight-binding Hamiltonian

$$\hat{H} = \sum_{m,n} W_{1,0} |m+1, n\rangle \langle m, n| + W_{0,1} e^{2\pi i \alpha m} |m, n+1\rangle \langle m, n| + h.c.$$

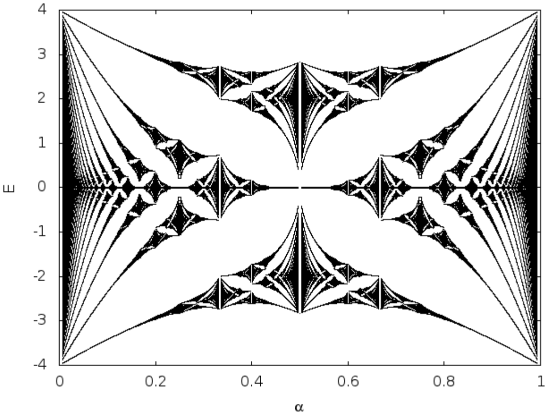
where $\alpha = \frac{eBd^2}{h}$ is the magnetic flux through an unit cell. Coefficients in the above equation, involve only m and do not depend on n. Plane-wave behaviour in y-direction $a_{mn} = e^{in\theta} a_m$.

$$a_{m+1} + a_{m-1} + \lambda \cos(2\pi m\alpha + \theta) a_m = E a_m \tag{28}$$

This is known as the Harper's equation for $\lambda = 2$.

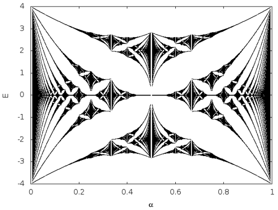
Harper Model

Spectrum



Harper Model

Spectrum



- α and $N + \alpha$ produce the same spectrum. α may be restricted to $[0, 1]$ for this reason.
- The energy eigenvalues are symmteric with respect to zero. i.e., if $\epsilon \in spectrum(\alpha)$, then $-\epsilon \in spectrum(\alpha)$.
- $|\epsilon| \leq 4$.
- The energy eigenvalues of irrational α is homeomorphic to a cantor set.
- The graph has a recursive structure.

Localization/Delocalization

Inverse Participation Ratio

IPR is defined as

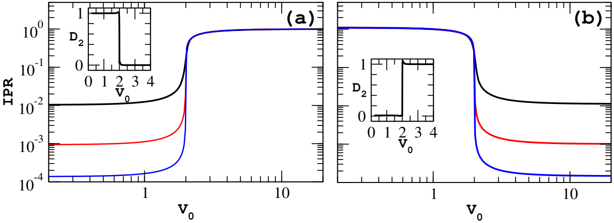
$$IPR = \frac{\sum_{n=1}^{L} |a_n|^4}{(\sum_{n=1}^{L} |a_n|^2)^2} \tag{29}$$

a_n's are the coefficients of the eigenstates in some basis.
IPR lies in the range 1 to $1/L$, where 1 indicates a perfectly localized state and $1/L$ indicates a perfectly delocalized state.

Aubry-André-Harper Model

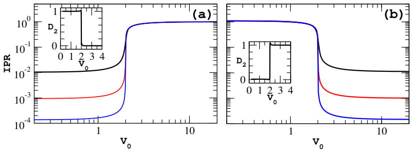
Localization/Delocalization

Figure: The metal-to-insulator transition of the AAH Hamiltonian for the systems ground state. Plot (a) shows the IPR versus V_0 or λ in real space for L = 144, 1597 and 10 946 (top to bottom) with $\alpha_0 = (\sqrt{(5)} - 1)/2$ (inverse of golden ratio). The inset shows the variation of D_2 with V_0 which also exhibits a transition. Plot (b) exhibits the mirror behavior in the dual space



Aubry-André-Harper Model

Localization/Delocalization



AAH model has a duality transformation

$$|m\rangle = \frac{1}{\sqrt{L}} \sum_n e^{-i2\pi\alpha_0 mn} |n\rangle$$

Wavefunctions localized in real space are delocalized in the dual space and vice versa.

Aubry-André-Harper Model

Hall Conductivity

First model for which relationship between Hall Conductivity and Chern numbers was established. Tight-binding Hamiltonian in Momentum space is obtained by Fourier transform

$$|m, n\rangle = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} e^{ik_x m + ik_y n} |k_x, k_y\rangle \tag{30}$$

$$\langle m, n| = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} e^{-ik_x m - ik_y n} \langle k_x, k_y| \tag{31}$$

where $-\pi \leq k_x, k_y < \pi$ and $|k_x + 2\pi a, k_y + 2\pi b\rangle = |k_x, k_y\rangle$ where $a, b \in \mathbb{Z}$.

Aubry-André-Harper Model

Hall Conductivity

We obtain

$$H_{ij} = 2W_{1,0}\cos(k_y+2\pi\alpha m)\delta_{ij}+W_{0,1}(\delta_{i+1,j}+\delta_{i,j+1})+W_{0,1}\delta_{i,q}\delta_{j,1}e^{ik_x^0 q}+W_{0,1}\delta_{i,1}\delta_{j,q}e^{-ik_x^0 q}$$

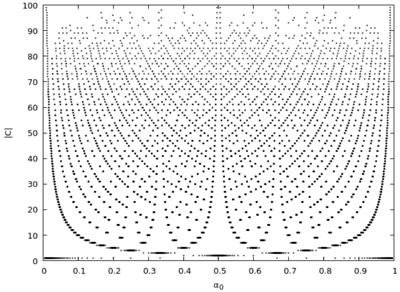
It is a $q \times q$ matrix, each eigenvalue corresponding to one of the q-subbands.

$$\hat{H}(k_x^0, k_y) = \hat{H}(k_x^0 + 2\pi/q, k_y)$$
$$\hat{H}(k_x^0, k_y) = \hat{H}(k_x^0, k_y + 2\pi)$$

The parametric dependence of $\hat{H}$ on k_x^0, k_y is described by a torus.

Aubry-André-Harper Model

Chern numbers



- Chern numbers calculated on a Discretized Magnetic Brillouin zone.
- Sum of Four-point Bargmann Invariant for each point.
- Non-abelian Berry Curvature is required to tackle the degeneracy issues.

Driven Aubry-André-Harper Model

Oscillating Magnetic Field

- The magnetic field perpendicular to the 2D lattice plane is rapidly oscillating.
- As the magnetic field oscillates, what happens to the q-subband structure?
- q is an extremely discontinuous function of time and it cannot be ascribed a closed form expression.
- Time-independent effective Hamiltonian using Floquet theory!

Driven Aubry-André-Harper Model

Effective Hamiltonian

In Landau gauge, $\mathbf{A}(t) = Bx\cos(\omega t)\hat{\mathbf{y}}$. Hamiltonian in tight-binding form

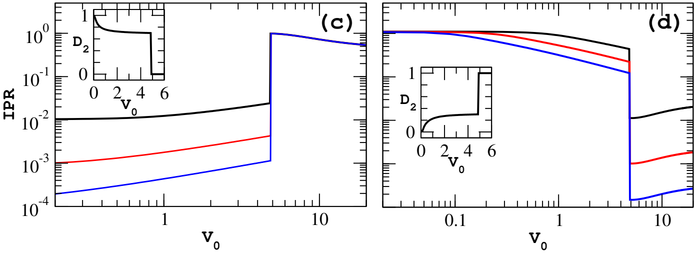
$$\hat{H} = \sum_n |n\rangle\langle n+1| + |n\rangle\langle n-1| + V_0\cos(2\pi\alpha_0 n\cos(\omega t) + \theta)|n\rangle\langle n| \qquad (32)$$

The static Effective Hamiltonian for this problem is calculated using the approach described earlier.

- In position space, the Effective Hamiltonian retains the tri-diagonal structure.
- The on-site term is a site-dependent oscillatory function because of appearance of $\mathcal{J}_i$ - Bessel functions.
- Upper and Lower diagonal terms are fixed at one.
- This model is not self-dual. The duality transformation described above does not retain the tri-diagonal structure.

Driven Aubry-André-Harper Model

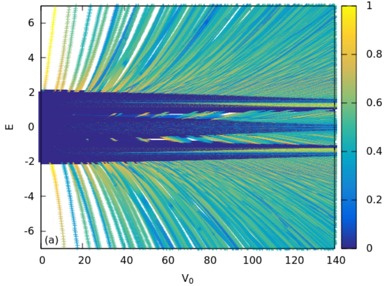
Metal-Insulator transition



Driven Aubry-André-Harper Model

Energy-dependent Mobility edge

Presence of energy-dependent mobility edge, an edge that splits the spectrum into two regions, one containing localized states and other containing delocalized states. This is a significant result as the mobility edge is atypical of 1-dimensional models.



Driven Aubry-André-Harper Model

Linearly Polarized Electric Field

- Additional Linear Electric Field along one of the axes of the lattice.
- Magnetic Translation Operators change only by a constant phase factor.
- Magnetic Translation Group can still be constructed.
- q-subband structure is also unaffected.
- Using BW perturbation we perturbatively obtain the effective Hamiltonian.

Driven Aubry-André-Harper Model

Linearly Polarized Electric Field

The magnetic vector potential corresponding to the system is

$$\mathbf{A}(t) = (Bx + A\cos(\omega t))\hat{\mathbf{y}}$$

The Hamiltonian in position space is

$$a_{n+1} + a_{n-1} + 2\lambda\cos(2\pi(\alpha_0 + \alpha\cos\omega t) + \theta)a_n = Ea_n \qquad (33)$$

where $\alpha_0 = \frac{e}{h}Bd^2$ and $\alpha = \frac{e}{h}Ad$. The Hamiltonian in momentum space is

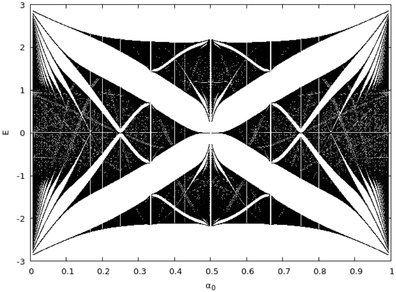
$$\begin{aligned} H(k_x, k_y, t)_{i,j} = \delta_{i+1,j} + \delta_{i,j+1} + 2\lambda\cos(k_y + 2\pi\alpha_0 j - 2\pi\alpha\cos\omega t)\delta_{i,j} \\ + e^{-iqk_x}\delta_{i,1}\delta_{j,q} + e^{iqk_x}\delta_{i,q}\delta_{j,1} \end{aligned} \qquad (34)$$

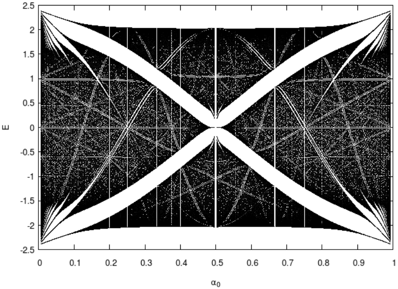
where $k_x \in [-\pi/q, \pi/q]$, $k_y \in [-\pi, \pi]$ and $i, j \in 1 \ldots q$.

Driven Aubry-André-Harper Model

Linearly Polarized Electric Field

Spectrum

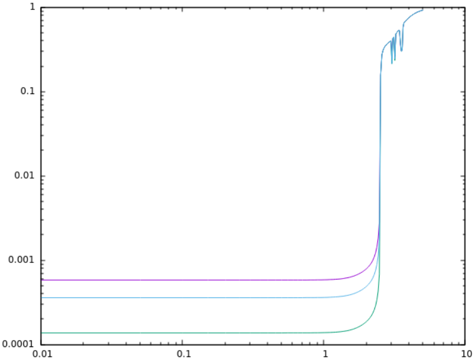




Driven Aubry-André-Harper Model

Linearly Polarized Electric Field

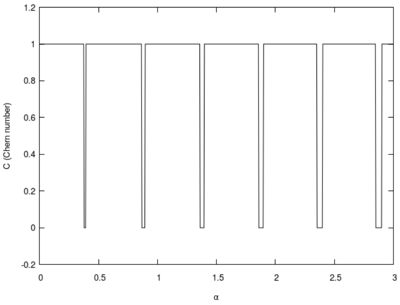
Metal-Insulator transition

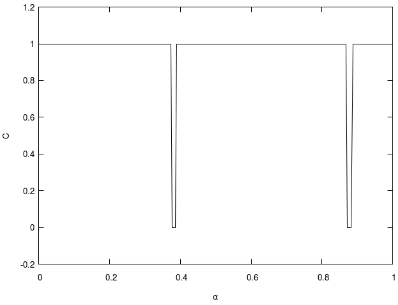


Driven Aubry-André-Harper Model

Linearly Polarized Electric Field

Topological transitions





Driven Aubry-André-Harper Model

Circularly Polarized Electric Field

As was the case for Linearly Polarized light, the Magnetic Translation Group is not affected by Circularly Polarized Light. The model still retains the q-subband structure.
The momentum space Hamiltonian is

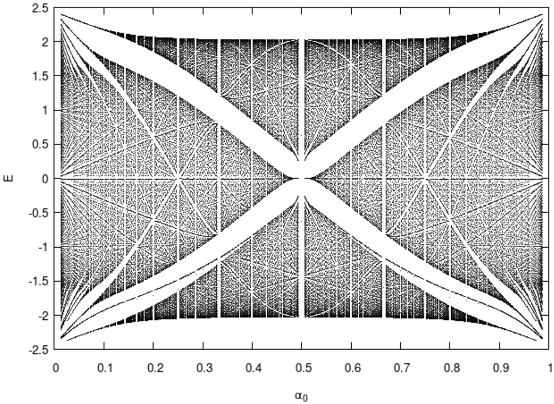
$$H(k_x, k_y, t)_{i,j} = \delta_{i+1,j} + \delta_{i,j+1} + 2\lambda \cos(k_y - 2\pi\alpha \sin \omega t + 2\pi\alpha_0 j)\delta_{i,j} + \delta_{i,1}\delta_{j,q} e^{-i(k_x - 2\pi\alpha \cos \omega t)q} + \delta_{i,q}\delta_{j,1} e^{i(k_x - 2\pi\alpha \cos \omega t)q} \quad (35)$$

Using BW perturbation we perturbatively obtain the effective Hamiltonian.

Driven Aubry-André-Harper Model

Circularly Polarized Electric Field

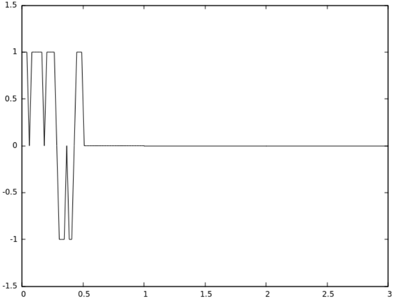
Spectrum

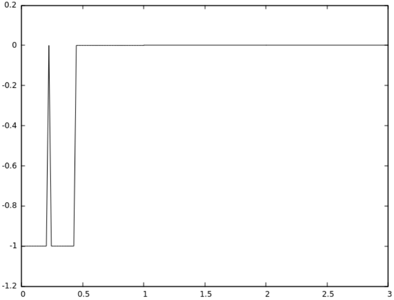


Driven Aubry-André-Harper Model

Circularly Polarized Electric Field

Topological Transitions





Summary

- Hall Conductivity is quantized.
- Integral of Berry curvature over closed 2D surface is quantized - Chern numbers.
- Chern numbers are related to Hall Conductivity.
- Floquet theory is a class of perturbation techniques used for time-periodic Hamiltonians.
- AAH Hamiltonian models electrons on 2D lattice under perpendicular uniform magnetic field.
- AAH model has a spectrum known as Hofstadter's butterfly.
- AAH model has a metal-insulator phase transition.
- Hall conductivity of AAH model can be exactly determined.
- Oscillating Magnetic Field creates a mobility edge in the spectrum of AAH.
- Linearly polarized Electric field and Circularly polarized electric field driven models exhibit topological transitions.