Gravitational Lensing

Rajath Shashidhara

12th July, 2015

Bending of light around a massive object. Using General relativity, the value of deflection angle due to a mass distribution can be calculated. For most astronomical situations, linearized theory of general relativity is sufficient to describe gravitational lensing.

Important phenomena due to lensing -

- Multiple images.
- Distortion of images.
- Magnified or Diminished images.
- Time delay between images.

According to general relativity, path of light is the null geodesic of the curved spacetime. Using the weak-field metric, $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$ where $\eta_{\mu\nu}$ is the minkowski metric of flat spacetime and $h_{\mu\nu}$ is the perturbation due to presence of mass distribution.

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) (d\vec{x})^2$$

where Φ is the newtonian gravitational potential.

The deflection angle is given by

$\hat{\vec{\alpha}} = \int \vec{\nabla}_\perp \Phi d\lambda$ along the path of light which can be approximated by an integrating along a straight line between the source and the observer.

For a point mass,

$$|\hat{\vec{\alpha}}(b)| = \frac{4GM}{c^2 b}$$

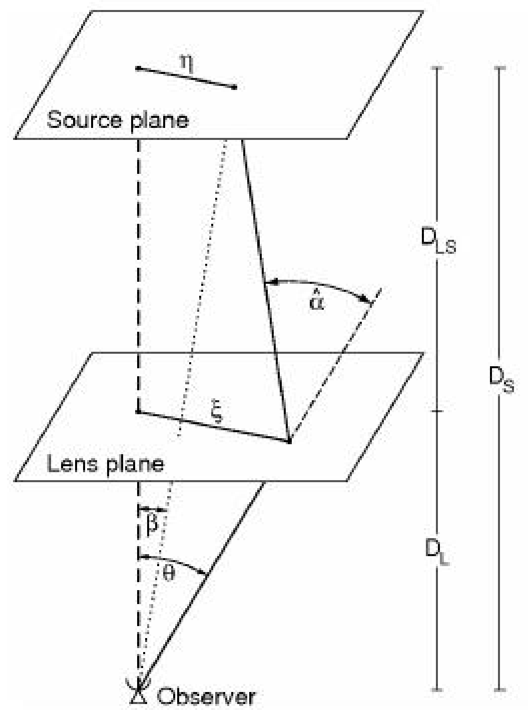
For n-point mass system,

$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum \frac{4G}{c^2} M_i \frac{\vec{\xi} - \vec{\xi_i}}{|\vec{\xi} - \vec{\xi_i}|^2}$$

For continuous matter distribution,

$$\sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z) dz$$

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\zeta})\sigma(\vec{\zeta})}{|\vec{\xi} - \vec{\zeta}|^2} d^2\zeta$$



The Lens equation obtained using simple geometry is $\vec{\theta} D_S = \vec{\beta} D_S + \hat{\vec{\alpha}} D_{LS}$. In reduced co-ordinates are, the equation is written as

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}$$

where $\vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S} \hat{\vec{\alpha}}(\vec{\theta})$.

1 Grid Search Method

In most cases, the lens equation is not analytically solvable. In fact, it might not be possible to evaluate the integral of deflection analytically. Numerical methods are inevitable in such situations.

We introduce a coordinate grid $\vec{x}_{ij}$on the lens plane. Since, we know $\vec{\alpha}(\vec{x}_{ij})$, either as an analytic formula or evaluated using a numerical integral, we can map the points on the lens plane to points on the source plane given by $\vec{y}_{ij} = \vec{x}_{ij} - \vec{\alpha}(\vec{x}_{ij})$. The grid in the image plane is divided into triangles because a triangle in the image plane is always mapped onto a triangle in the source plane and it is simple to determine if a point is contained inside a triangle. To determine the images of $\vec{y}_s$, iterate over every triangle in the lens plane and check if $\vec{y}_s$ is contained in the mapped triangle in the source plane. In order to improve accuracy, grid search may be performed recursively on the cells containing the images.

1.1 Grid search to find images due to a point mass lensing system

Simple code written in Python based on the procedure described above. This code can be easily extended to n-point mass system by simply replacing the expression for the deflection angle. [The location of images may not be very accurate because recursive subgridding is not used].

$$\vec{\alpha}(\vec{x}) = \frac{M\vec{x}}{|\vec{x}|^2}$$

in units where $G = 1$ and $c = 1$.

$$\vec{y}(\vec{x}) = \vec{x} - \vec{\alpha}(\vec{x}) = \vec{x} - \frac{M\vec{x}}{|\vec{x}|^2}$$

1.2 2-Point Mass system

This system is not analytically solvable for a general location of source. We choose the coordinate system such that the masses are located on the x-axis equidistant from the origin. The code for this problem is the same for 1-Point Mass with the expression of deflection angle replaced by 2-Point Mass expression. [The location of images may not be very accurate because recursive subgridding is not used].

$$\vec{\alpha}(\vec{x}) = \frac{M_1(\vec{x} - \vec{d})}{|\vec{x} - \vec{d}|^2} + \frac{M_2(\vec{x} + \vec{d})}{|\vec{x} + \vec{d}|^2}$$

in units where $G = 1$ and $c = 1$, and $\vec{d} = d\hat{x}$.

$$\vec{y}(\vec{x}) = \vec{x} - \vec{\alpha}(\vec{x}) = \vec{x} - \frac{M_1(\vec{x} - \vec{d})}{|\vec{x} - \vec{d}|^2} - \frac{M_2(\vec{x} + \vec{d})}{|\vec{x} + \vec{d}|^2}$$

Figure 1: Images of the source at (0.5,0.25) - point mass lens. Green dot - Point mass; Blue dot - Images; Red Dot - Source.

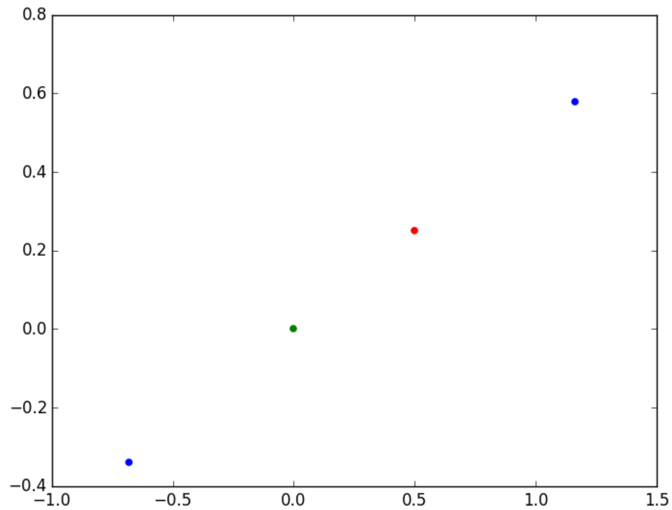


Figure 2: Einstein Ring - Point Mass. Red Dot - Point mass; Blue Dot - Images.

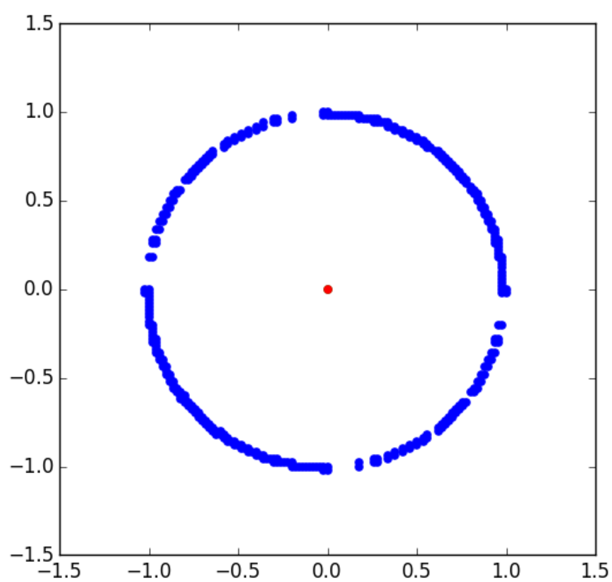
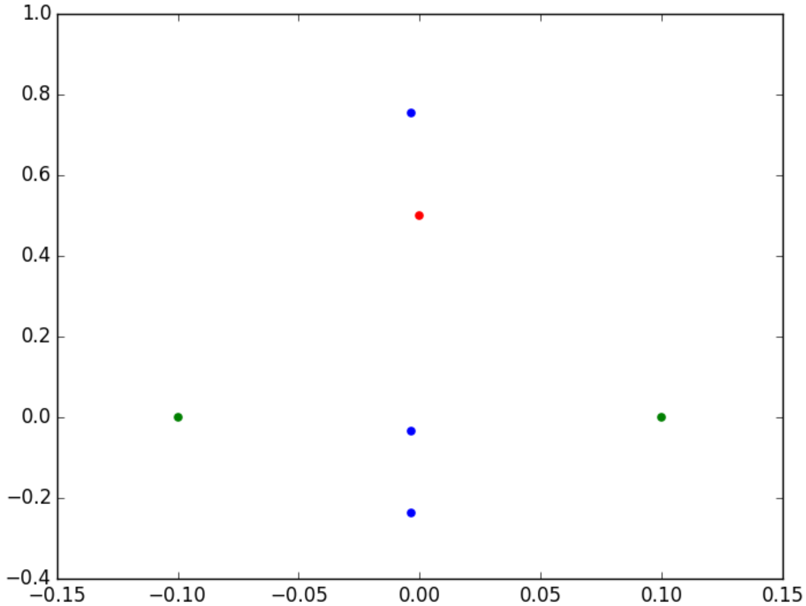


Figure 3: Images of the source at (0.0,0.5) - two point mass lens. Green dot - Point mass; Blue dot - Images; Red Dot - Source.



2 Elliptical Lens

The lens is bounded by an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The mass distribution is described by $\rho(x, y, z)$ inside the ellipsoid and zero outside the boundary of the ellipsoid. Using the thin lens approximation, surface mass density is used for calculating the angular deflection due to gravitational lensing.

$$\sigma(x, y) = \int \rho(x, y, z) dz$$

The angular deviation is given by

$$\vec{\alpha}(\vec{x}) = \int \frac{\sigma(\vec{\nu})(\vec{\nu} - \vec{x})}{|\vec{\nu} - \vec{x}|^2} d^2\nu$$

The code for the above problem solves the case of uniform mass distribution. It can be extended to a general case in a straight forward manner by setting $\sigma(x, y)$ if an analytical expression is known or by evaluating $\sigma(x, y) = \int \rho dz$ in the bounds of the ellipsoid, numerically. Critical curves and causatics are also generated by this code.

The numerical integration package used is Cubature.

To run the code, install the latest version of Julia by following the instructions provided here. Run the code by calling "julia -p <No-of-cores-in-your-computer> code.jl".

3 References

1. Introduction to Gravitational Lensing, Massimo Meneghetti.
2. Gravitational Lenses. P. Schneider J. Ehlers E. E. Falco.
3. LECTURES ON GRAVITATIONAL LENSING, RAMESH NARAYAN AND MATTHIAS BARTELMANN.
4. Singularity Theory and Gravitational Lensing, A. O. Petters, H. Levine, J. Wambsganss.
5. The Sas Fee Lectures on Strong Gravitational Lensing, C.S.Kochanek.
6. NUMERICAL METHODS IN GRAVITATIONAL LENSING, MATTHIAS BARTELMANN.
7. Computational Methods for Gravitational Lensing, Charles R. Keeton.
8. GLAMER Part I: A Code for Gravitational Lensing Simulations with Adaptive Mesh Refinement, R. Benton Metcalf & Margarita Petkova.
9. From the Fundamental Theorem of Algebra to Astrophysics: A "Harmonious" Path, Dmitry Khavinson and Genevra Neumann.
10. On modeling galaxy-scale strong lens systems, Charles R. Keeton.