6651 Comments on lecture 1

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1. computability: a problem is computable means that it can be solved by some algorithm.
2. Examples of uncomputable (sometimes people say undecidable):

   Halting problem: whether a computer program halts (i.e., stops) on a given input.

   Hilbert's 10th problem: whether an equation with integer coefficients has integer solutions.
3. we briefly mentioned the important P vs NP problem. We will discuss it a little more later. Here is a good and short video on P vs NP explaining what it is: you should be able to get a rough idea after watching this video.
4. In class we saw $4n = O(n+8)$, this is because $4n \leq 4(n+8)$ always holds for $n \geq 1$. So, in this case, we have $c = 4$ and $n_0 = 1$.

   We also mentioned $4n = O(n-8)$ holds as well. The reason is as follows:

   Proof. We will show $4n \leq 5(n-8)$ for "sufficiently large" n. Indeed, if we cancel $4n$ from both sides, we get $0 \leq n - 40$, equivalently, $n \geq 40$. So, the inequality holds as long as $n \geq 40$. Hence, we have found $c = 5, n_0 = 40$, so that
   $$\forall\ n \geq n_0 = 40, \quad 4n \leq 5(n-8).$$
   That is, "eventually", $4n$ is bounded above by a *scaled version of* $n - 8$. Hence, we can write $4n = O(n-8)$. $\square$
5. Examples of functions $f(n)$ and $g(n)$ satisfying: (1) $f(n) = \Theta\big(g(n)\big)$, (2) $f(n)/g(n)$ has no limit as $n \to \infty$.

   - In the class we give an example $f(n) = n$ and $g(n) = n \sin n$. A student later pointed out there was a mistake. Indeed, because $\sin n \leq 1$, $n \sin n \leq n$ always holds, so $g(n) = O\big(f(n)\big)$ holds. But on the other hand, $g(n) = \Omega\big(f(n)\big)$ or equivalently, $f(n) = O\big(g(n)\big)$ does not hold, this is because the value of $\sin n$ can be both positive and negative, and sometimes $\sin n$ can get arbitrarily close to 0. So, when $\sin n$ is negative, or when $\sin n$ is extremely close to 0, in both cases $f(n) = n$ cannot be bounded above by a *positive constant scaled version* of $g(n) = n \sin n$. So, $f(n) = O(g(n))$ does NOT hold. As a result, $f(n) = \Theta\big(g(n)\big)$ does not hold.

     This bug can be easily fixed as follows: modify the function $g(n)$ as follows:
     $$f(n) = n, \qquad g(n) = n(2 + \sin n).$$
     Note that
     $$1 \leq 2 + \sin n \leq 3,$$
     so,
     $$n \leq n(2 + \sin n) \leq 3n.$$
     In other words, $f(n) \leq g(n) \leq 3f(n)$. So, both $g(n) = \Omega\big(f(n)\big)$ and $g(n) = O\big(f(n)\big)$ hold, that is, $g(n) = \Theta(f(n))$ or equivalently $f(n) = \Theta(g(n))$ holds. On the other hand, $g(n)/f(n) = 2 + \sin n$, this function has no limit when $n \to \infty$.

- A student gave the following example in class:

$$f(n) = n, \qquad g(n) = \begin{cases} n, & n \text{ is odd,} \\ 2, & n \text{ is even.} \end{cases}$$

We pointed out in class this example is not correct. But we can also easily modify the functions a little bit to make it work, as follows:

$$f(n) = n, \qquad g(n) = \begin{cases} n, & n \text{ is odd,} \\ 2n, & n \text{ is even.} \end{cases}$$

Make sure you understand why the modified version works!

References