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A Minimum Broadcast Graph on 26 Vertices

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Abstract—Broadcasting is the process of information dissemination in a communication network in which a message, originated by one member, is transmitted to all members of the network. A broadcast graph is a graph which permits broadcasting from any originator in minimum time. The broadcast function $B(n)$ is the minimum number of edges in any broadcast graph on n vertices. In this paper, we construct a broadcast graph on 26 vertices with 42 edges to prove $B(26) = 42$. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Broadcast graph, Communication network, Minimum broadcast graph.

1. INTRODUCTION

Broadcasting is the process of distributing information from an originator to all other vertices of a communication network. The problem we address considers that only one piece of information is to be distributed, each communication involves exactly two adjacent vertices and takes one unit of time, and no vertex is involved in two or more simultaneous communications.

Let G be a graph of order n, representing a communication network. It is easy to see that at least $\lceil \log_2 n \rceil$ time units are required to complete broadcasting under the above assumptions, since during each time unit, the number of informed vertices can at most double. For $u \in V(G)$, we define the *broadcast time of* u, denoted by $b(u)$, to be the minimum number of time units required to complete broadcasting from vertex u, the *broadcast time of* G, denoted by $b(G)$, to be the maximum broadcast time of any vertex in G, i.e., $b(G) = \max\{b(u) \mid u \in V(G)\}$. If $b(G) = \lceil \log_2 n \rceil$, then G is called a *broadcast graph*. A broadcast originated by a vertex u determines a spanning tree rooted at u called a *broadcast tree for* u. The *broadcast function* $B(n)$ is the minimum number of edges in any broadcast graph with n vertices. A broadcast graph with n vertices and $B(n)$ edges is called a *minimum broadcast graph* or *mbg*.

From the point of view of application, minimum broadcast graphs represent the cheapest possible communication networks such that broadcasting can be accomplished from any vertex as fast as theoretically possible.

A survey of the history of these problems and a list of references can be found in [1]. Minimum broadcast graphs are difficult to construct and there is no known method for constructing an *mbg*

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for arbitrary n. In fact, even determining the value of $b(u)$ in an arbitrary graph is **NP-complete** (see [2]). For $B(n)$, only the values of $B(n)$ for $n \leq 22$ or $n = 31, 63$ or $n = 2^k$, $2^k - 2$ if $k \geq 5$ have been determined.

We now consider $n = 26$. So far it is known that $B(26) \leq 43$ (see [3]). We prove that $B(26) = 42$ by constructing a *mbg* on 26 vertices with 42 edges. We refer to [4] for notations and terminology not defined here.

2. SOME PRELIMINARY PROPERTIES

PROPOSITION 2.1. *If G is a minimum broadcast graph on 26 vertices, then its minimum degree $\delta = 3$.*

PROOF.

(a) If there exists in G a vertex v of degree 1, i.e, $\deg v = 1$, let u be the only neighbor of v. Then in a broadcasting originated by v, v must first send the message to u, in the remaining four time units, broadcast from u can at most inform 2^4 vertices including u, so the total number of informed vertices in five time units would be at most $2^4 + 1$, a contradiction. Similarly, if $\deg v = 2$, the total number of informed vertices in five time units would be at most 25, a contradiction too.

(b) If $\delta > 3$, then $|E(G)| > 43$, a contradiction.

Thus, the proof is completed.

PROPOSITION 2.2. *If G is a minimum broadcast graph on 26 vertices, $v \in G$ is a vertex of degree 3, then v must be adjacent to a vertex of degree at least 4.*

PROOF. If not, then v can, at most, adjacent to three vertices with degree 3. In this case, broadcast originated by v can at most inform 25 vertices in five time units which is shown in Figure 1. A number beside a vertex indicates the time unit of the vertex receiving the message. ∎

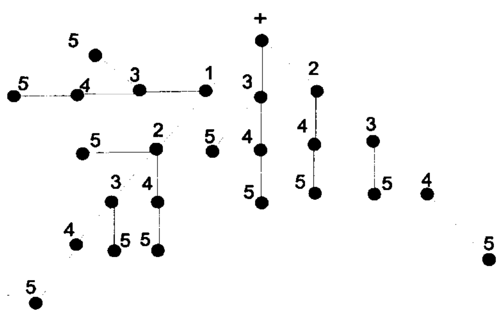

Figure 1. An extremal broadcast tree.

PROPOSITION 2.3. $B(26) \geq 42$.

PROOF. Suppose G is a minimum broadcast graph on 26 vertices, G contains exactly m vertices of degree at least 4, which are collected in $V' \subset V(G)$, then $\sum_{v \in V'} \deg v \geq 4m$. On the other hand, from Propositions 2.1 and 2.2, we know that there are $26 - m$ vertices of degree 3, each must be adjacent to a vertex of degree at least 4, thus, $\sum_{v \in V'} \deg v \geq 26 - m$.

1. If $4m \geq 26 - m$, then $m \geq 6$. $|E(G)| \geq \lfloor (4 \cdot m + (26 - m) \cdot 3)/2 \rfloor \geq 42$.
2. If $26 - m > 4m$, then $m \leq 5$ and $\sum_{v \in V'} \deg v \geq 21$. Thus, $|E(G)| \geq \lfloor (\sum_{v \in V'} \deg v + (26 - m) \cdot 3)/2 \rfloor \geq \lfloor (21 + 21 \cdot 3)/2 \rfloor = 42$.

Therefore, $B(26) \geq 42$.

3. A MINIMUM BROADCAST GRAPH ON 26 VERTICES AND BROADCASTING SCHEMES

THEOREM 1. $B(26) = 42$.

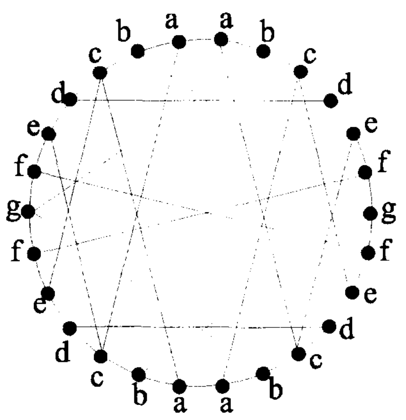

Figure 2. An *mbg* on 26 vertices.

PROOF. We present graph G shown in Figure 2 as a broadcast graph on 26 vertices with 42 edges. In fact, by the symmetry of G, we need only show seven broadcasting schemes (see Figure 3) for the broadcast graph. In Figure 2, the letter beside a vertex indicates which scheme in Figure 3

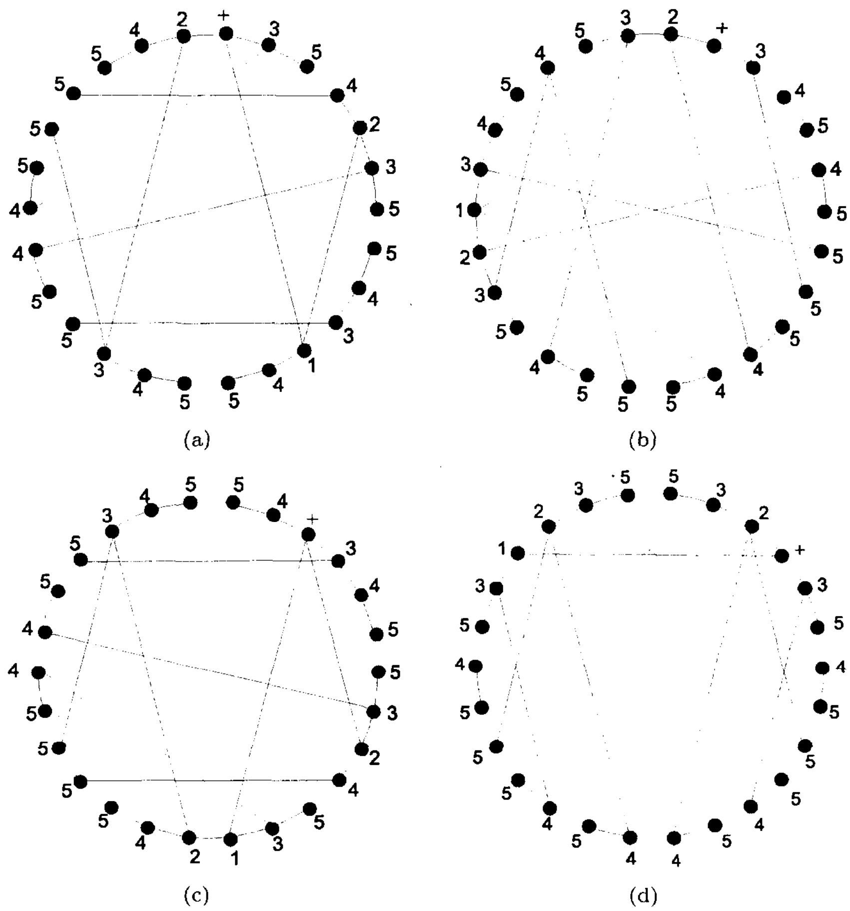

Figure 3. Broadcasting schemes.

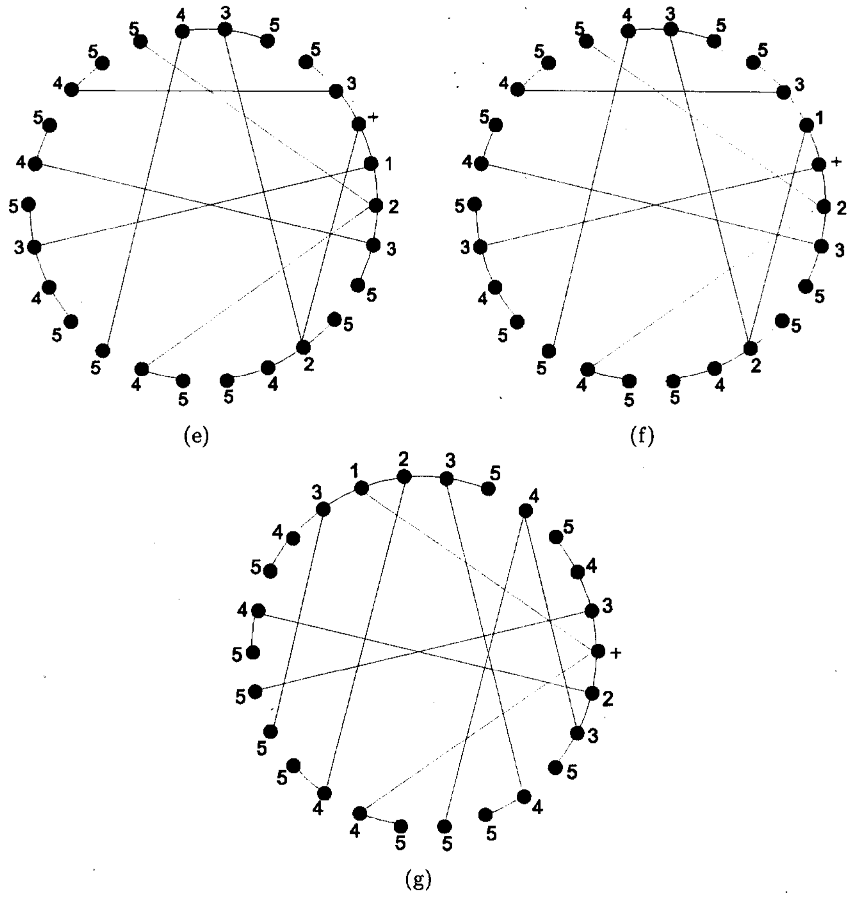

Figure 3. (cont.)

is used. In a scheme, '+' indicate originator and a label beside a vertex indicates the time unit of the vertex receiving the message. Thus, we have proved the theorem.

REFERENCES

1. S.M. Hedetniemi, S.T. Hedetniemi and A.L. Liestman, A survey of gossiping and broadcasting in communication networks, *Networks* **18**, 319–349, (1988).
2. P.J. Slater, E. Cockayne and S.T. Hedetniemi, Information dissemination in trees, *SIAM J. Comput.* **10**, 692–701, (1981).
3. J.C. Bermond, P. Fraigniaud and J.G. Peters, Antepenultimate broadcasting, *Networks* **26**, 125–137, (1995).
4. F. Harary, *Graph Theory*, Addison-Wesley, (1969).