Minimal Broadcast Networks

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ABSTRACT

Broadcast refers to the process of information dissemination in a communication network whereby a message, originated by one member, is transmitted to all members of the network. A minimal broadcast network is a communication network in which a message can be broadcast in minimum time regardless of originator. This paper describes several classes of minimal broadcast networks. An algorithm is presented which constructs minimal broadcast networks which have approximately the minimum number of lines possible.

INTRODUCTION

Information dissemination refers to any process whereby a set of *messages*, generated by a set of *originators*, is transmitted to a set of *receivers* within a communication *network*. Specific classes of information dissemination processes may be defined by placing constaints upon the sets of messages, originators, and receivers and upon the topology and transmission characteristics of the network. For example, transmission of a message from one originator to one receiver represents the fundamental communication process of message transfer. In this report, our focus will be upon the broadcasting process. *Broadcast* is the process of information dissemination whereby a message, originated by one member, is transmitted to all members of the network.

We model a communication network by a graph G = (V,E), consisting of a set V of vertices, which represent communication sites or *members* of the network, and a set E of edges, which represent communication (or leased) *lines* between the members.

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Information, as messages, may be transmitted by *calls* placed over lines of the network. Several constraints are assumed, which complete the model. We assume that each call has two participants and that each call requires one time unit. Furthemore, a member may be a participant in at most one call during any given time unit. Finally, a member may only call another member to which it is directly connected by a line of the network.

Given a set of n members, several criteria of goodness suggest themselves with respect to the process of broadcast and the construction of optimal communication networks for broadcast. These include the cost of the supporting network, the cost of the broadcast process, and the time for completion of the process. These criteria may be represented respectively by the following three minima:

1) the minimum number of communication lines required to complete the broadcast process,

2) the minimum number of calls required to complete the broadcast process,

3) the minimum number of time units required to complete the broadcast process.

Fortunately, all three numbers are readily determined. For broadcast to be completed, a network must be connected. Thus, at least n-1 communication lines are required. Similarly, the minimum number of calls required to broadcast a message from the originator to n-1 receivers is clearly n-1. All broadcast networks and calling schemes to be discussed here require only the minimum number of calls. Implicit in these first two minima is the assumption that each line is of equal cost, both to install and maintain as part of the network and to traverse by a call during the broadcast process. Such an assumption is reasonable for a model of a local communication network or a computer with parallel processors.

A straightforward inductive argument proves that the maximum number of members who can be informed of a message after t time units is 2^t (including the message originator). The maximum number of people are informed if during each time unit each informed member calls a different, uninformed member. The number of informed members can at most double during each time unit. This result directly implies that the minimum number of time units required to broadcast a message throughout a set of n members is $\lceil \log_2 n \rceil$. Let us define a *minimal broadcast network* to be a communication network with n members in which a message can be broadcast in minimum time (i.e., $\lceil \log_2 n \rceil$ time units) regardless of message originator. Such networks are of natural interest in the design of distributed communication networks and parallel computation systems. Interest is also revived in a modified version of our first goodness criterion, now stated as: the minimum number of communication lines required in a minimal broadcast network with n members.

The remainder of this paper considers the problem of constructing minimal broadcast networks. An algorithm is presented which, for any positive integer n, constructs a minimal broadcast network with n members. These networks have approximately the minimum number of lines possible. A comparison is made between the numbers of lines required in different classes of minimal broadcast networks.

THREE CLASSES OF MINIMAL BROADCAST NETWORKS

The inductive argument proving that after t time units at most 2^t members can be informed of a message can be used as the basis of a calling scheme for broadcast in a completely connected network (i.e. a complete graph). Consider the n members of the network to be positioned at equal intervals about a circle. Let the members be numbered sequentially in the clockwise direction from 0 to n-1. Figure 1 illustrates this situation for n=12. Initially, only the message originator (mo) is informed of the message. During the first time unit, mo calls its nearest clockwise neighbor, member (mo+1) mod n. Now two members are informed. During the second time unit, the two informed members call their clockwise neighbors once removed. In other words, mo calls (mo+2) mod n and (mo+1) mod n calls (mo+1+2) mod n. Generally, during time unit t, each informed member i, calls uninformed member $(i+2^{t-1})$ mod n. This process continues for $\lceil \log_2 n \rceil$ time units. For $n \neq 2^k$ (k a positive integer), only $n-2^{\lfloor \log_2 n \rfloor}$ of the $2^{\lfloor \log_2 n \rfloor}$ members which are informed prior to the last time unit need place calls during the last time unit. Let these be the message originator and its $n-2^{\lfloor \log_2 n \rfloor}-1$ nearest clockwise neighbors. Figure 2 shows the calls which would be made under this calling scheme during a message broadcast originated by member 5 in a network with 12 members. Each directed arc represents a call made by the tail member to the head member. Each arc is labelled by an integer indicating the time unit during which the call would be made.

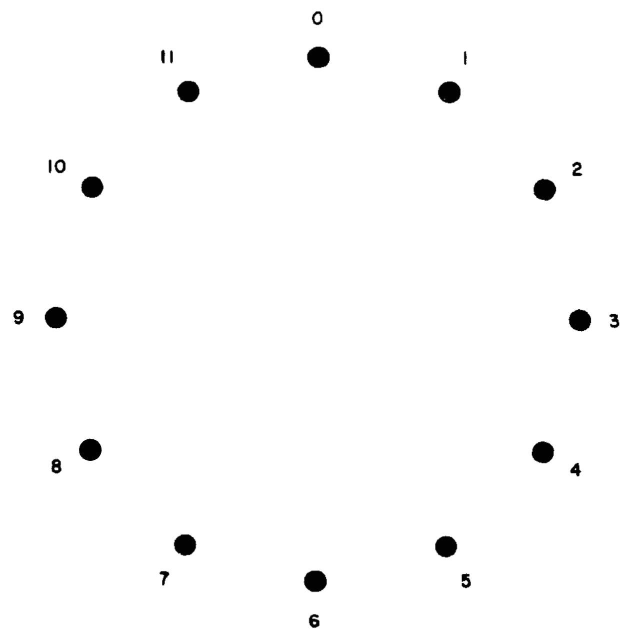


Fig. 1

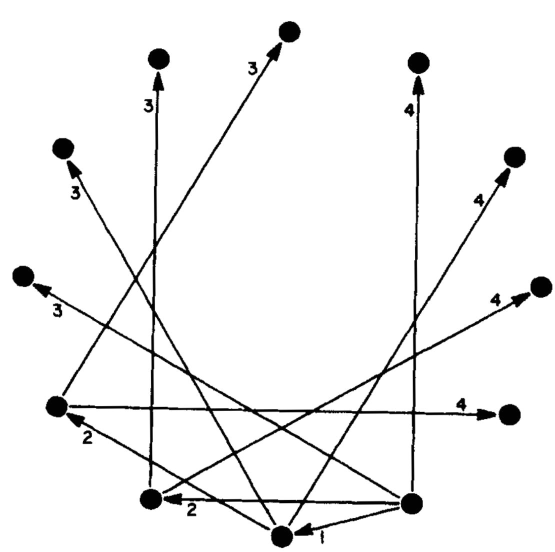


Fig. 2

The above calling scheme demonstrates that a completely connected network is a minimal broadcast network. A closer look indicates that not all lines of the network are used by the calling scheme. The calling scheme can be used to construct a new class of minimal broadcast networks. In a minimal broadcast network, each member must be able to serve equally well as message originator. In other words, each member must be able to place the necessary call during each potential time unit. Therefore, each member must have communication lines to $\lceil \log_2 n \rceil$ other members in order that the above calling scheme always realize the minimum time. A member i must have communication lines defined by the following set of edges:

$$\{(i,j) \mid j=i+2^k \bmod n,\ 0 \leq k \leq \lceil \log_2 n \rceil -1\}.$$

These minimal broadcast networks are a subclass of the class of graphs known as star polygons. This is not the first time that a subclass of star polygons has satisfied a goal of communication network design. Boesch and Felzer [1] have reported a subclass of star polygons which satisfy a definition of invulnerability, an important concept in network reliability.

Figure 3 illustrates the first 12 of these star polygon minimal broadcast networks. Each network is labelled by the number of members and, in parentheses, by the number of lines that it has. Any communication network which contains such a star polygon over all of its members is also a minimal broadcast network. Therefore, communication networks equivalent to complete graphs are minimal broadcast networks. Of interest here is that a communication network need not be a complete graph in order to be a minimal broadcast network. This result has import for the number of lines required in minimal broadcast networks. Given a set of n members, the number of lines required by a star polygon minimal network, constructed as described above, is as follows:

$$n \lceil \log_2 n \rceil - n/2 \qquad \text{for } n=2^k \text{ (k a positive integer);}$$

$$n \lceil \log_2 n \rceil - n \qquad \text{for } n-2^{\lfloor \log_2 n \rfloor} = 2^k \quad (k \geq 0);$$

$$n \lceil \log_2 n \rceil \qquad \text{otherwise.}$$

These numbers are significantly less than the (n(n-1))/2 edges in the undirected complete graph with n vertices. See Table 1 for a comparison of number of lines for selected values of n.

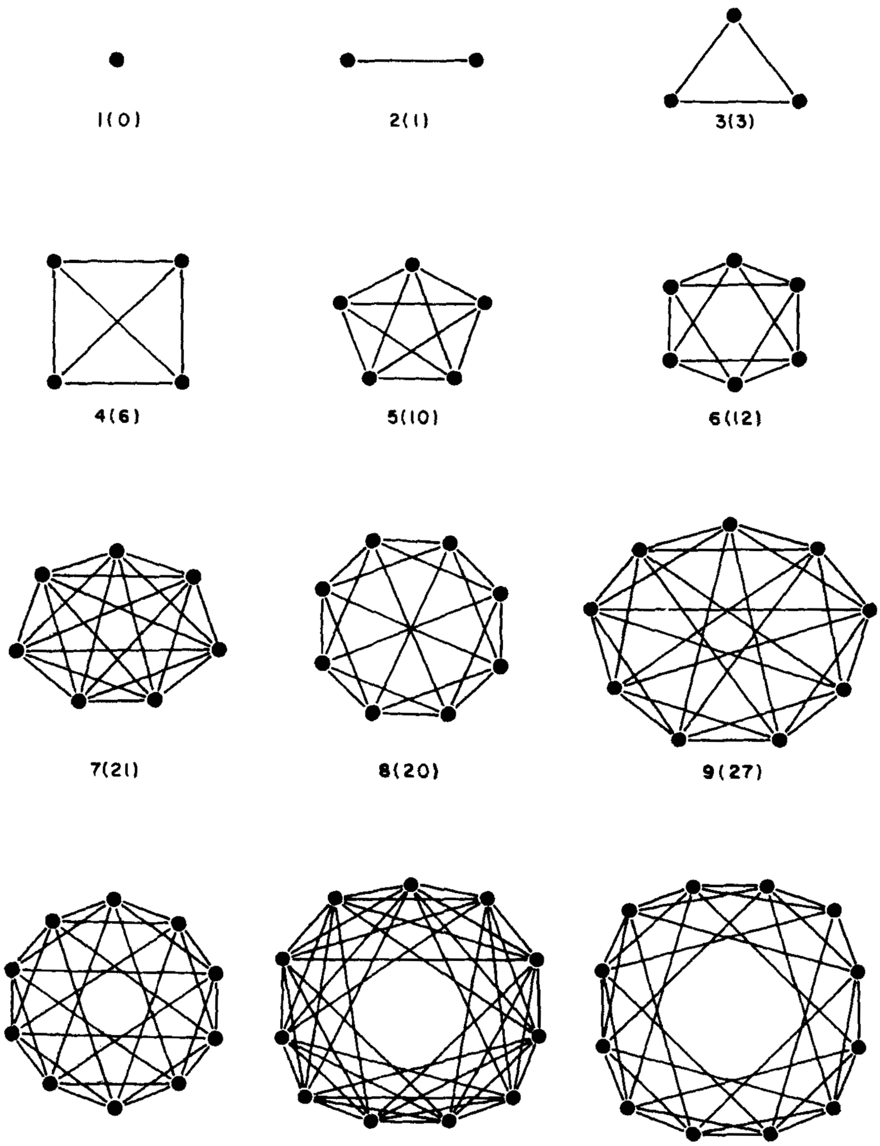


Fig. 3

Table 1

Number of Lines in
Minimal Broadcast Network

n	complete graphs	minimal broadcast star polygons	recursively-constructed networks
4	6	6	4
6	15	12	6
8	28	20	12
12	66	36	18
16	120	56	32
24	276	96	48
32	496	144	80
48	1128	240	120
64	2016	352	192
65	2080	390	153

Though the above results appear favorable with respect to the number of communication lines required, they are not optimal. The minimal broadcast network of nine members shown in Figure 3 requires 27 communication lines. Consider a communication network of nine members realized by adding a new member to the minimal broadcast network of eight members shown in Figure 3. Let the new member be added by connecting it to one member of the original eight, as shown in Figure 4. This new network requires only 21 communication lines. Is it a minimal broadcast network? Broadcast requires a minimum of four time units in a network of nine members. If the new member originates the message, it makes the only call it can during the first time unit. The message is then broadcast throughout the original eight members in an additional three time units. If a member of the original network of eight members is message originator, the original eight can complete broadcast in three time units. A fourth time unit is required for the new member to be called. Regardless of message originator, broadcast can be completed in the minimal four time units. Therefore, the network shown in Figure 4 is a minimal broadcast network.

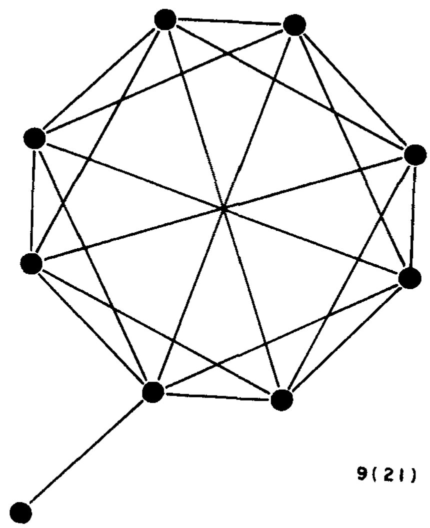


Fig. 4

This example demonstrates that the previous result as to the number of lines required for a minimal broadcast network of n members is not the best or minimum result. A better result can be obtained by a recursive construction algorithm which generalizes upon the method that has been employed above to produce the improved network of nine members. The algorithm constructs a minimal broadcast network of n members by combining minimal broadcast networks of fewer members. The process of combining the component networks is subject to furthur specification. A formal definition of the algorithm is as follows:

CONSTRUCT_MINIMAL_BROADCAST_NETWORK(n)

CASE 0: $(n \leq 2)$

Step 0.1) If n=1, the network is a single node.

If n=2, the network is 2 vertices connected by a single communication line.

Case 1: $(n \leq 3*2^{\lceil \log_2 n \rceil - 2})$

Step 1.1) Select integers i,j, and k such that

$i,j,\ k \leq 2^{\lceil \log_2 n \rceil - 2}$, $i+j+k = n$, $i \geq j \geq k$,

and $i - k \leq 1$.

Step 1.2) CONSTRUCT_MINIMAL_BROADCAST_NETWORK(i) ;

CONSTRUCT_MINIMAL_BROADCAST_NETWORK(j) ;

CONSTRUCT_MINIMAL_BROADCAST_NETWORK(k) .

Step 1.3) If n is even then connect each member of the three components to a different member of a different component. (This requires n/2 lines). If n is odd then do as above for n-1 of the members; then connect the remaining member to a member of a different component to which no member of its component is already connected. (This requires $\lceil n/2 \rceil$ lines).

CASE 2: $(3*2^{\lceil \log_2 n \rceil - 2} < n \leq 2^{\lceil \log_2 n \rceil})$

Step 2.1) Select integers i and j such that $i, j \leq 2^{\lceil \log_2 n \rceil - 1}$, $i+j = n$, $i \leq j$, and $j - i \leq 1$.

Step 2.2) CONSTRUCT_MINIMAL_BROADCAST_NETWORK(i) ;

CONSTRUCT_MINIMAL_BROADCAST_NETWORK(j) .

Step 2.3) Connect each member of the i member component to a different member of the j member component. (This requires $i = \lfloor n/2 \rfloor$ lines.)

END

The algorithm is recursive, requiring the construction of component minimal broadcast networks by execution of the same algorithm. The reduction of the problem continues until the fixed point networks of one or two members are reached. Step 1.1 can be realized by setting i to $\lceil n/3 \rceil$, then setting j to $\left\lceil \frac{n-i}{2} \right\rceil$, and then setting k to n-(i+j). Step 1.3 can be realized by a cyclic process which selects a component with the largest number of members as yet unconnected to a member of a different component. The process connects a member of that component to an unconnected member of a remaining component with the next largest number of unconnected members. An arbitrary choice is made between components with an equal number of unconnected members. Step 2.1 can be realized by setting i to $\lfloor n/2 \rfloor$, then setting j to n-i. The recursive construction algorithm can be implemented so as to require time which is linearly related to the number of lines in the resultant network.

Figure 5 presents examples of minimal broadcast networks constructed by this algorithm for n $\leq$ 12. As in Figure 3, each network is labelled by the number of members and, in parentheses, the number of lines. In most cases the largest components are arranged vertically. Notice that a planar minimal broadcast network can be constructed for every n $\leq$ 12.

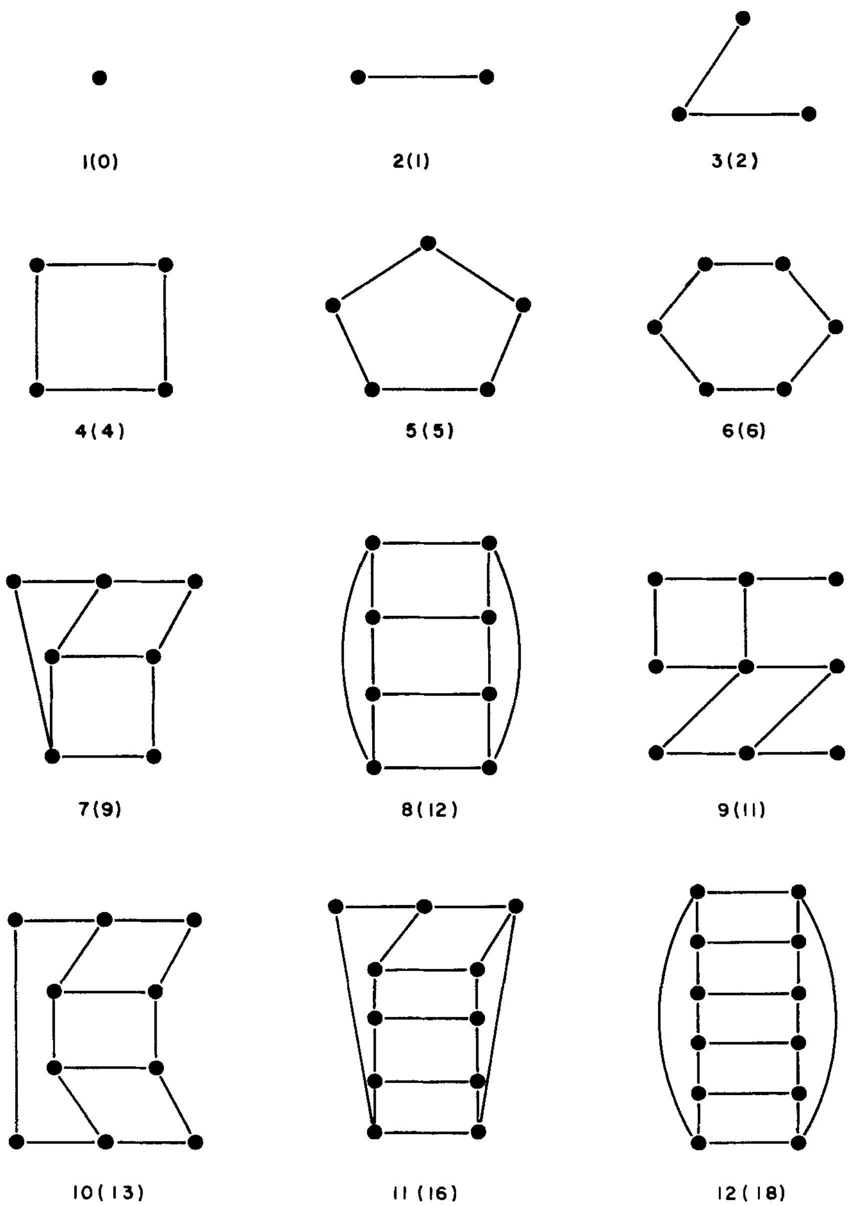


Fig. 5

Theorem 1: For any $n > 0$, the network produced by the recursive construction algorithm is a minimal broadcast network.

Proof: The networks produced for one or two members (Case 0 networks) are minimal broadcast networks by inspection (and by the prior, star polygon scheme). It must be shown that those produced by combining components (Case 1 and 2 networks) are also minimal broadcast networks. The proof is realized by defining calling schemes in such networks which produce minimum broadcast time regardless of message originator. The proof is inductive. Given n, we shall assume that the algorithm does construct minimal broadcast networks for any i, j, or k less than n.

Case 1 networks: The network of n members is realized by interconnecting three component networks. Broadcast proceeds in three phases:

Phase 1: The message originator calls a member external to its component to which it is connected. This requires one unit of time. There is always at least one such call possible at this point.

Phase 2: The two components with an informed member complete broadcast independently and in parallel. This requires at most $\lceil \log_2 n \rceil - 2$ time units, by the inductive assertion.

Phase 3: Each member of the third component is called by an informed member of another component. This requires one time unit.

Case 2 networks: The network of n members is realized by interconnecting two component networks of i and j members respectively, $i \le j$. There are two subcases to consider here. In each subcase broadcast proceeds in two phases.

Subcase a): The originator is in the j member component.

Phase 1: Broadcast is completed in the component of the originator. This requires at most $\lceil \log_2 n \rceil - 1$ time units, by the inductive assertion.

Phase 2: Those members of the informed component which are connected to a member of the other component each call the member in the uninformed component to which they are connected. This requires one time unit.

Subcase b): The originator is in the i member component.

Phase 1: The originator calls the member in the j member component to which it is linked. This requires one time unit.

Phase 2: Broadcast is completed in the two components independently and in parallel. This requires at most $\lceil \log_2 n \rceil - 1$ time units, by the inductive assertion.

Thus, regardless of message originator, broadcast can be completed in the minimum, $\lceil \log_2 n \rceil$ time units in all networks constructed by the algorithm. Therefore, the algorithm constructs a minimal broadcast network for any given $n > 0$.

The networks produced by the recursive construction algorithm clearly have fewer lines than the star polygon networks developed earlier. For $n = 2^k$ (k a positive integer), the constructed network consists of two component networks of n/2 members interconnected by n/2 lines. This recurrence relation can be solved to yield $(n/2) \lceil \log_2 n \rceil$ lines in these special case, n member networks. The result also follows from noting that for $n = 2^k$, the recursively constructed network is k regular. In the special case that $n = 3*2^{\lceil \log_2 n \rceil - 2}$, the constructed network consists of 3 component networks of $2^{\lceil \log_2 n \rceil - 2}$ members, interconnected by n/2 lines. This relation can likewise be solved, indicating that for this special case the number of lines is equal to $(n/2) \lceil \log_2 n \rceil - n/2$.

Theorem 2: The number of lines in a network of n members produced by the recursive construction algorithm is less than or equal to $(n/2) \lceil \log_2 n \rceil$.

Proof: The proof is by induction. Inspection of the networks shown in Figure 5 indicate that the assertion is true for $n \leq 12$. Suppose it is true for all networks with less than n members.

Case 1 networks: The n member network is constructed from three components (with i,j, and k members, respectively). The number of lines required is less than or equal to

$$(i/2)(\lceil \log_2 n \rceil - 2) + (j/2)(\lceil \log_2 n \rceil - 2) + (k/2)(\lceil \log_2 n \rceil - 2) + \lceil n/2 \rceil$$
$$= (i+j+k)(\lceil \log_2 n \rceil - 2)/2 + \lceil n/2 \rceil$$
$$= (n/2)(\lceil \log_2 n \rceil - 2) + \lceil n/2 \rceil = (n/2) \lceil \log_2 n \rceil - \lfloor n/2 \rfloor.$$

Case 2 networks: The n member network is constructed from two components (with i and j members, respectively). The number of lines required is less than or equal to

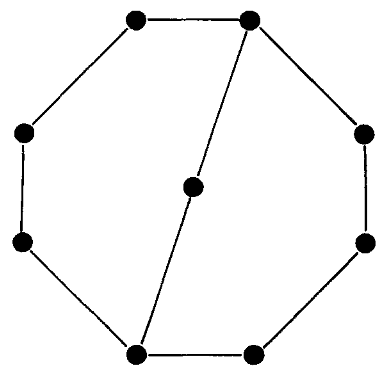
$$\begin{aligned}
&(i/2)\lceil \log_2 n\rceil -1) + (j/2)(\lceil \log_2 n\rceil -1) + i \\
&= \frac{(i+j)}{2}(\lceil \log_2 n\rceil -1) + \lfloor n/2\rfloor \\
&= (n/2)\lceil \log_2 n\rceil -n/2 + \lfloor n/2\rfloor \\
&\leq (n/2)\lceil \log_2 n\rceil .
\end{aligned}$$

Corollary: In a Case 1 network of n members produced by the recursive construction algorithm, the number of lines is less than or equal to $(n/2) \log_2 n - \lfloor n/2\rfloor$.

ADDITIONAL CONSIDERATIONS

It is interesting to observe that the number of lines in the constructed networks does not monotonically increase with increasing n. (For example, see the networks with 8 and 9 members shown in Figure 5.) For an n just above a power of 2, the increase in minimum broadcast time introduces freedom or *slack* into the calling scheme. An informed member may not make a call during a given time unit and yet minimum time can be realized. This slack in turn allows a decrease in the number of lines required to realize a minimal broadcast network. The decrease can be substantial. For example, a network of 32 members requires 80 lines. A network with 33 members, constructed from three networks of 11 members, has only 65 lines. Of course broadcast requires 6 time units in the 33 member network, but only 5 in the 32 member case.

Unfortunately, the recursive construction algorithm does not always produce minimal broadcast networks with the minimum possible number of lines. Minimal broadcast networks with nine members have been found which have only ten lines. Minimal broadcast networks with ten members have been found which have only twelve lines. Examples of such networks are shown in Figure 6. No general, direct algorithm for the construction of minimum line minimal broadcast networks is yet known. A catalog of such networks, accompanied by proofs that the number of lines in them is the minimum, has been compiled for networks with up to 17 members [2,3]. Those networks could be used as fixed point networks to further improve the performance of the recursive construction algorithm.



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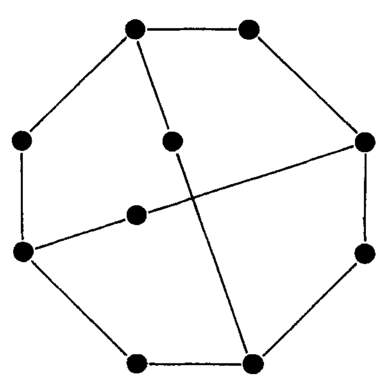

10(12)

Fig. 6

In general, the networks produced by the recursive construction algorithm approach the minimum possible number of lines. Indeed, for n equal to a power of two, the recursively constructed networks do have the minimum number of lines possible. Each member must be able to make $\log_2 n$ calls when message originator. The network must be $\log_2 n$ regular, at least. Therefore, the minimum number of lines in these cases is $(n/2)\log_2 n$. This is exactly the number of lines required by the recursive construction algorithm.

If the constraint on broadcast time is relaxed, the number of lines required in the communication network may decrease. Relaxing the constraint of minimum time by one unit can have a significant effect on the required number of lines when n is equal to or slightly less than a power of 2. In these cases, there is very little slack in the calling scheme if minimum time is required. Lines are necessary to guarantee that any member can make a new call during (almost) every time unit. By accepting a broadcast time of one time unit more than the minimum, slack can be reintroduced into the calling scheme. This results in fewer lines being required in the network. For example, a network with 16 members requires 32 lines to realize the minimum broadcast time of four time units regardless of originator. If a uniform broadcast time of five time units is acceptable, a network with only 19 lines can be found, as shown in Figure 7.

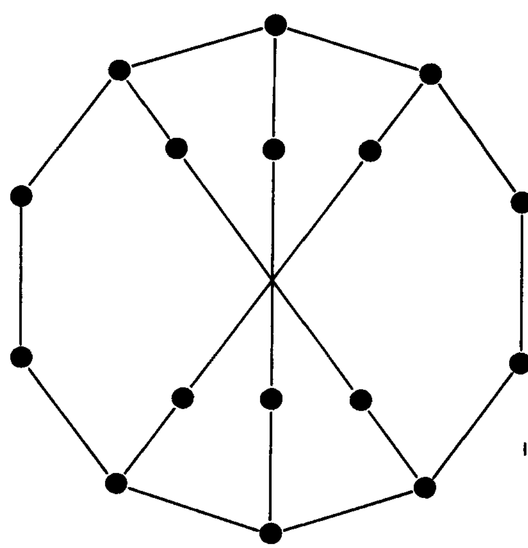

Fig. 7

There is a lower bound on the number of lines which are required for successful broadcast in a network of n members. The network must be connected. Therefore, at least n-1 lines are required. Such networks are trees. What trees produce the best results for broadcast time? What is broadcast time in such trees? In a tree, broadcast time may differ depending upon which member is message originator. Therefore, broadcast time for a tree is best characterized by three values: the minimum, the average, and the maximum number of time units required to complete the process.

A tree which has the minimum possible value for the number of time units belongs to the class of minimum broadcast trees. A *minimum broadcast tree* is defined to be a rooted tree with n vertices such that broadcast can be completed in $\lceil \log_2 n \rceil$ time units from the root member. That such trees exist has been an implicit result of the earlier discussion on minimal broadcast networks. The graph induced by a completed broadcast in a minimal broadcast network is a minimum broadcast tree. In fact, a definition of minimal broadcast network is a network such that every member is the root of a minimum broadcast tree.

A linear algorithm has been found for determining whether an arbitrary tree is a minimum broadcast tree [4]. A subclass of such trees can be constructed by a method based upon the calling scheme which led to the star polygon class of minimal broadcast networks. Let the members be numbered from 1 to n. Let member 1 be the root of the tree. Then, for each member i $(2 \leq i \leq n)$, connect i to member $i-2^{\lceil \log_2 i \rceil -1}$. This constructs the tree in layers, each successive layer being twice the size of the previous layer. Figure 8 presents a sample of such trees for $n \leq 12$. During broadcast, an informed member calls the uninformed members to which it is linked in the order of increasing member number. Message broadcast originated by member 1 or member 2 in such trees requires the minimum, $\lceil \log_2 n \rceil$ time units. The maximum number of time units required for broadcast is less than or equal to $2\lceil \log_2 n \rceil -1$. This follows from the fact that all members are within $\lceil \log_2 n \rceil -1$ lines (calls) of either member 1 or member 2. Table 2 presents a comparison between recursively constructed minimal broadcast networks and minimum broadcast trees as to number of lines and broadcast time for selected values of n.

The *broadcast center of a tree* is defined to be the set of vertices from which broadcast can be completed in the minimum number of time units for that tree. Members 1 and 2 are elements of the broadcast center of the minimum broadcast trees constructed above. A linear algorithm has been found for determining the broadcast center of an arbitrary tree [5]. This algorithm also determines an optimal calling scheme for a broadcast originated by any member of the tree.

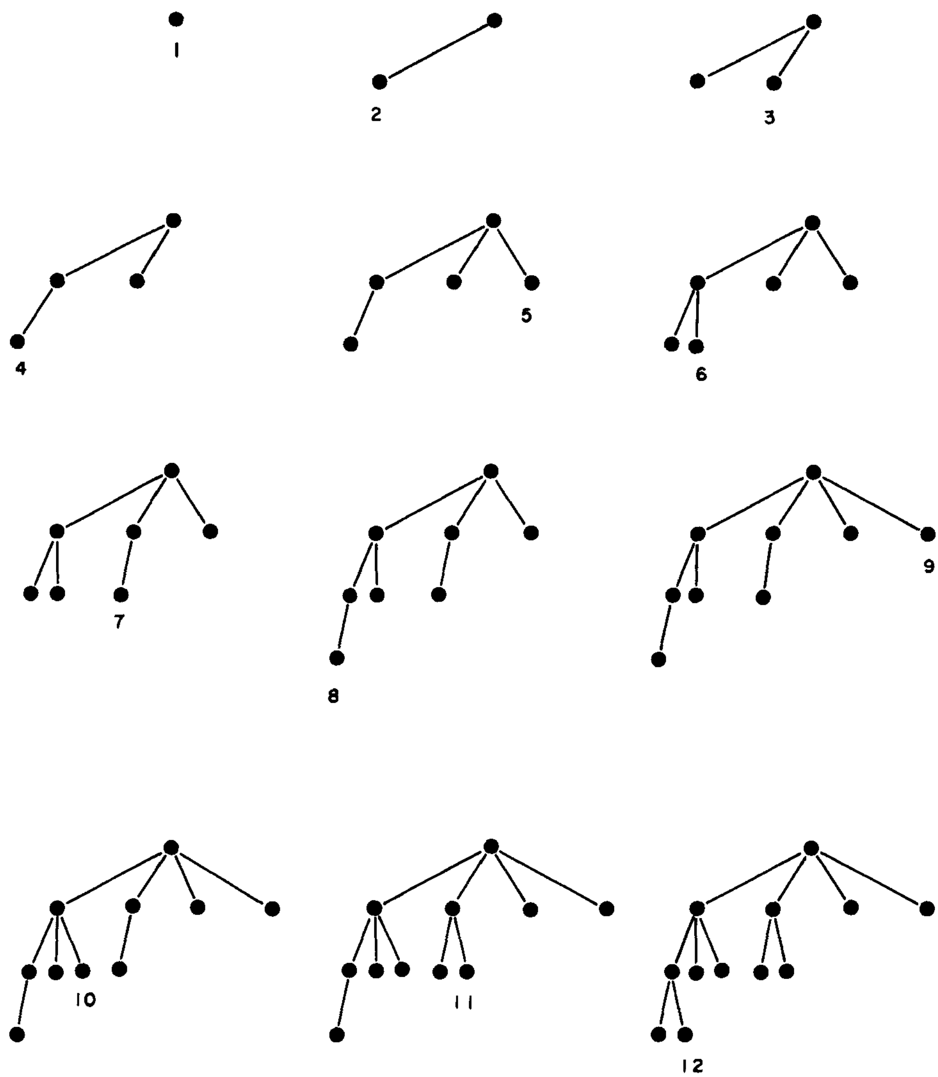

Fig. 8

Table 2

Number of Time Units
to Complete Broadcast

n	recursively constructed networks lines	time	minimun broadcast trees lines	time min, avg, max
4	4	2	3	2 , 2.5 , 3
6	6	3	5	3 , 2.67 , 4
8	12	3	7	3 , 4 , 5
12	18	4	11	4 , 5.17 , 6
16	32	4	15	4 , 5.5 , 7
24	48	5	23	5 , 6.67 , 8
32	80	5	31	5 , 7 , 9
48	120	6	47	6 , 8.17 , 10
64	192	6	63	6 , 8.5 , 11

Gossip is a related process of information dissemination which has been discussed in several recent papers. In gossip, each member acts as a message originator simultaneously. Information can be exchanged in both directions during a call, not just transmitted from one member to the other. A member can be involved in only one call at a time. Gossip is completed when all members have been informed of all messages. It has been shown that the minimum number of calls necessary to complete gossip in a network of $n \geq 4$ members is $2n-4$ [6]. Networks which realize the minimum number of calls require n lines [7]. A minimum of $2\lceil \log_2 n \rceil - 2$ time units is required to complete gossip in such networks. For even n, the minimum number of time units for gossip to be completed is $\lceil \log_2 n \rceil$ [8]. The networks which produce this time require $(n/2)\lceil \log_2 n \rceil$ lines and $(n/2)\lceil \log_2 n \rceil$ calls. The familiar tradeoff between time (time units) and resources (lines and calls) is demonstrated in these results for gossip.

CONCLUSION

Several problems relating to broadcast have yet to be solved. The problem of introducing reliability into broadcast networks is one. For $n \geq 4$, the recursively constructed networks are all two-connected, which is at least a start. Questions concerning broadcast within an arbitrary network have not been answered. For example, an algorithm which determines a best calling scheme for any member of an arbitrary network has yet to be found. Finally, the construction of a network of n members with a maximum or uniform broadcast time of k time units ($\lceil \log_2 n \rceil \leq k < n/2$) which also requires the minimum number of lines is an open problem. A simple ring (cycle) network of n members provides a uniform broadcast time of $\lceil n/2 \rceil$ time units. A minimum broadcast tree has a maximum broadcast time of $2\lceil \log_2 n \rceil -1$. The introduction of differential line costs into the model generates another class of interesting and difficult optimization problems for broadcasting.

The potential importance of message broadcast in communication and computer networks and in parallel processors has motivated this research. This paper has reported some initial results and has suggested further questions for study.

ACKNOWLEDGMENT

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