COMP 6651: Solutions to Assignment 9

Fall 2020

Submission through Moodle is due by November 22nd at 23:55

1. (a) $Alex - \mathcal{NP}$ is the class of all languages. Consider the verifier $A(x, y) : return\ 1$, i.e., it ignores its inputs and always returns 1 (yes). Consider an arbitrary language L. We claim that it satisfies the definition of Alex with $d = 1$ and verifier A (independent of language L), since $\forall x \in L$ there exists $y = \epsilon$ (empty string) with $|y| = 0 \leq |x|^1$ such that $A(x, y) = 1$.

   (b) $Joana - \mathcal{NP}$ is the class of all languages. Consider the verifier $A(x, y) : return\ 0$, i.e., it ignores its inputs and always returns 0 (no). Consider an arbitrary language L. We claim that it satisfies the definition of Joana with $d = 1$ and verifier A (independent of language L), since $\forall x \notin L$ no matter which y is used with $|y| \leq |x|^1$ we have $A(x, y) = 0$.

   (c) $Steve - \mathcal{NP}$ is the class of all languages. Consider the verifier $A(x, y) : return\ y_1$, i.e., it always returns y_1 (first bit of certificate y) and ignores other parts of the input. Consider an arbitrary language L. We claim that it satisfies the definition of Alex with $d = 0$ and verifier A (independent of language L), since $\forall x \in L$ define $y = 1$ if $x \in L$ and define y if $x \notin L$. Observe that certificate $|y| = 1 \leq |x|^0$ and $x \in L$ if and only if $A(x, y) = y_1 = y = 1$.

2. (a) *Optimization version*:

   Input: $G = (V, E)$ – simple undirected graph

   Output: $k \in \mathbb{Z}$ – the size of a minimum vertex cover in G

   Decision version (= $MVC - DEC$*)*:

   Input: $G = (V, E)$ – simple undirected graph; $k \in \mathbb{Z}$

   Output: yes, if there exists a vertex cover in G of size $\leq k$; no, otherwise.

   Search version:

   Input: $G = (V, E)$ – simple undirected graph; $k \in \mathbb{Z}$

   Output: $S' \subseteq V$ – a vertex cover in G of size $\leq k$ if it exists; "impossible", otherwise.

   (b) **Claim:** $CLIQUE \leq_p MVC - DEC$.

   Proof: Given an instance (G, k) to $CLIQUE$, the reduction outputs $(G^c, n - k)$, where G^c is the complement graph and $n = |V(G)|$ is the number of vertices. This can clearly be computed in polynomial time. To show correctness we need to prove:

   G has a clique of size at least k if and only if G^c has a vertex cover of size at most $n - k$.

   If S is a clique of size at least k in G then S is an independent set in G^c and therefore every edge in G^c has one endpoint in $V - S$. Thus, $V - S$ is a vertex cover in G^c of size $|V| - |S| \leq n - k$.

   If S is a vertex cover of size at most $n - k$ in G^c then $V - S$ is an independent set in G^c. Thus, $V0S$ is a clique in G. The size of $V - S$ is $|V| - |S| \geq n - (n - k) = k$.

3. Recall that a simple undirected graph $G = (V, E)$ is k-colorable if there exists a coloring $c : V \to \{1, 2, \ldots, k\}$ such that for every edge $\{u, v\} \in E$ we have $c(u) \neq c(v)$. Define $k - COL = \{\langle G \rangle :$

G is k-colorable$\}$. Prove that $3 - COL \leq_p 4 - COL$. Give a reduction and prove its correctness and that it runs in polynomial time.

Claim: $3 - COL \leq_p 4 - COL$.

Proof: Let $G = (V, E)$ be a simple undirected graph (an instance of $3 - COL$ problem). The reduction outputs a graph $G' = (V', E')$ such that

- $V' = V \cup \{s\}$, where s is a new vertex not appearing in V,
- $E' = E \cup \{\{s, v\} : v \in V\}$.

In words, G' is G plus an extra vertex that is connected by an edge to all other vertices. Clearly, G' can be computed in polynomial time. To show correctness, we need to prove that

G is 3-colorable if and only if G' is 4-colorable.

If G is 3-colorable, let $c : V \to \{1, 2, 3\}$ be a valid 3-coloring for G. Then define $c' : V' \to \{1, 2, 3, 4\}$ as follows: $c'(v) = c(v)$ for $v \in V$ and $c(v') = 4$, i.e., assign the extra vertex s a new color. It is evident that c' is a valid 4-coloring of G'.

If G' is 4-colorable, let $c' : V' \to \{1, 2, 3, 4\}$ be a valid 4-coloring for G'. If $c'(s) \neq 4$ we can permute the colors so that s becomes colored with 4. Thus, we assume without loss of generality that $c'(s) = 4$. Then c' restricted to set V provides a valid 3-coloring of G.