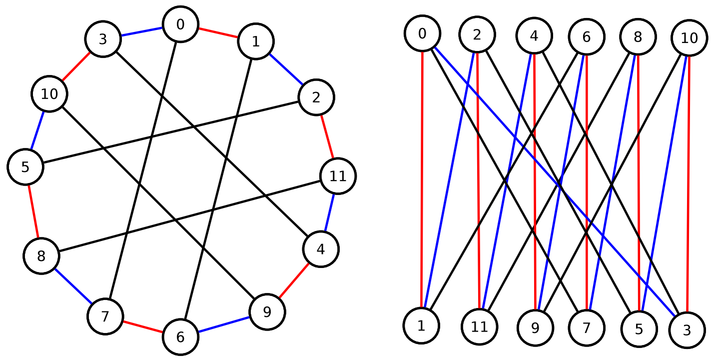



COMP6661

# COMBINATORIAL ALGORITHMS

*Instructor:*
Dr. Harutyunyan

# Graph Theory

## Undirected & Directed Graphs

A graph $G$ is represented by $\boldsymbol{G = (V, E)}$ where

- $\boldsymbol{V}$ is the set of vertices
- $\boldsymbol{E}$ is the set of edges

*Definition* Graphs can be regrouped as follows:

- **Undirected graphs**: Edges of the graph are undirected
  - **Simple graph**: No multiple edges or loops are allowed
  - **Multigraph**: Multiple edges are allowed but loops are not allowed
  - **Pseudograph**: Multiple edges and loops are allowed
- **Directed graphs**: Every edge has a direction
  - **Directed graph**: Loops are allowed but multiple edges in the same direction are not allowed
  - **Directed multigraph**: Loops and multiple directed edges are allowed

*Definition* In undirected graphs, vertex $u$ and vertex $v$ are called **adjacent** *iff* $\{u, v\}$ is an edge in $G$. We say $\{u, v\}$ is **incident** on vertices $u$ and $v$. The **degree $\boldsymbol{d(v)}$** of a vertex $v$ is the number of edges incident on $v$.

*Theorem 1* ***(Handshaking)*** *For an undirected graph $G = (V, E)$ where $|E| = e$,*

$$2e = \sum_{v \in V} d(v)$$

*(true even for graphs with multiple edges and loops)*

*Proof* It follows from the fact that each edge contributes 2 to the sum of degrees of vertices since it's incident to exactly 2 (possibly equal i.e. loop) vertices □

*Theorem 2* *An undirected graph has an even number of vertices of odd degree*

*Proof* $2e = \sum_{v \in V} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v)$ where
$V_1$: set of odd degree vertices
$V_2$: set of even degree vertices
The second term of RHS is even, hence $\sum_{v \in V_1} d(v)$ must also be even. But for all vertex $v$ in $V_1$, $d(v)$ is odd; hence for $\sum_{v \in V_1} d(v)$ to be even, $|V_1|$ must be even. $\square$

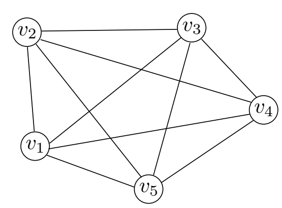
*Definition* In directed graphs, $(u, v)$ is an edge, $u$ is the **initial vertex** (**adjacent to** $v$), and $v$ is the **terminal vertex** (**adjacent from** $u$). Also
$\boldsymbol{d^-(v)}$ is **in-degree** of vertex $v$ (i.e. # of edges terminating at $v$)
$\boldsymbol{d^+(v)}$ is **out-degree** of vertex $v$ (i.e. # of edges originating at $v$)

*Theorem 3* *Let $G = (V, E)$ be a directed graph. Then*

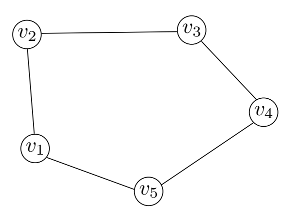
$$\sum_{v \in V} d^-(v) = \sum_{v \in V} d^+(v) = |E|$$

## Graph Terminologies

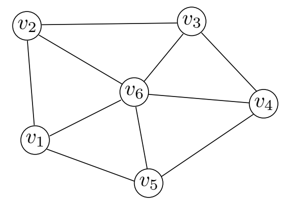
Complete Graphs on $n$ vertices $\boldsymbol{K_n}$: a simple graph with exactly one edge between any pair of distinct vertices



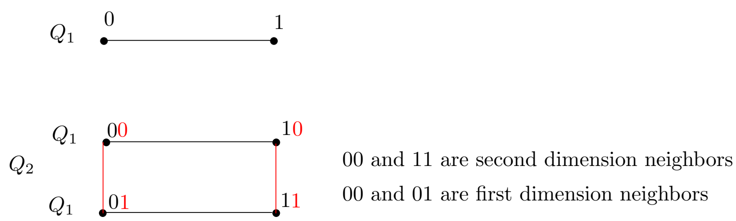
Cycles $\boldsymbol{C_n}$, $n > 3$: simple graph with vertices $v_1, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$



Wheels $\boldsymbol{W_n}$, $n > 3$: add $(n+1)$-st vertex to $C_n$ and connect it to each of $n$ vertices in $C_n$

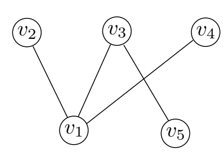



$n$-Cubes $\boldsymbol{Q_n}$, $n > 3$: simple graph with vertices representing $2^n$ bit strings of length $n$, $n \geq 1$ such that adjacent vertices have bit strings differing in exactly one bit position



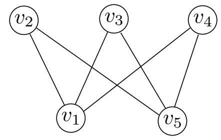

**Note**:
Problem 1: How many binary numbers we can write with $n$ numbers ?
Answer: $2 \times 2 \times \cdots \times 2 = 2^n$
Problem 2: Find the number of subsets of a set of $n$ elements
Answer: Let $S = \{a_1, a_2, \ldots, a_n\}$ be the set of $n$ elements
A subset of $S$ is $S_1 = \{a_1, \ldots, a_4\}$
$S_1$ can also be presented as $S_1 = \{11110\ldots0\}$ (put 1 if element is in subset 0 otherwise)
Point: Problem1 and Problem2 are equivalent and their answer represent the number of vertices in a hypercube

Bipartite graphs: simple graphs such that $V$ can be partitioned into 2 disjoint subsets $V_1$ and $V_2$ such that each edge connects a vertex in $V_1$ and a vertex in $V_2$, and no edges connect 2 vertices that are both in $V_1$ or in $V_2$




*Hypercube $Q_n$ is a bipartite graph for all $n \geq 1$*

Complete bipartite graphs $K_{m,n}$: Let $V_1$ and $V_2$ be two partitions of vertex set of $K_{m,n}$ such that $|V_1| = m$ and $|V_2| = n$. There is an edge between two vertices *iff* one vertex is in $V_1$ and the other in $V_2$




## Trees

*Definition* A **tree** is a connected undirected graph with no simple circuits

*Theorem 4* *An undirected graph is a tree iff there is a unique simple path between any two of its vertices*

*Definition* A rooted tree is called **$m$-ary tree** if every vertex has no more than $m$ children. The tree is called a **full $m$-ary tree** if every internal vertex has exactly $m$ children. An $m$-ary tree with $m = 2$ is a **binary tree**

*Theorem 5* *A tree with $n$ vertices has $n - 1$ edges*

*Theorem 6* *A full $m$-ary tree with*

1. *$n$ vertices has $i = \frac{n-1}{m}$ internal vertices and $l = \frac{(m-1)n+1}{m}$ leaves*
2. *$i$ internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves*
3. *$l$ leaves has $n = \frac{ml-1}{m-1}$ vertices and $i = \frac{l-1}{m-1}$ internal vertices*

## Graph Connectivity

*Definition* A **path** of length $n$ from $u$ to $v$ in an undirected graph is a sequence of edges $e_1, e_2, \dots, e_n$ which starts at $u$ and ends at $v$. A path is **simple** if it does not contain the same edge twice

*Definition* If $u = v$, the path from $u$ to $u$ is a **circuit**

*Definition* ***(Connectedness)*** An undirected graph is **connected** if there exists a path between every pair of vertices

*Theorem 7* *There is a simple path between every pair of vertices in a connected undirected graph*

## Connectedness in Directed Graphs

*Definition* A directed graph $G = (V, E)$ is **strongly connected** if there exists a path from $a$ to $b$ and from $b$ to $a$, whenever $a, b \in V$.

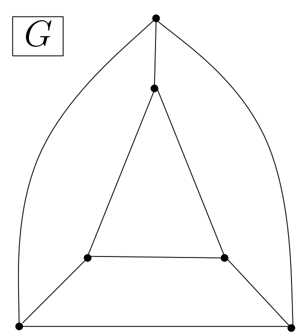
*Definition* A directed graph $G = (V, E)$ is **weakly connected** if there exists a path between any 2 vertices in the underlying undirected graph

# Planar Graphs

*Definition* A graph (or multigraph) $G$ is **planar** if $G$ can be drawn in the plane with its edges intersecting only at vertices of $G$. Such a drawing of $G$ is called an **embedding** of $G$ in the plane. An application of planar graphs is electrical circuit design with VLSI.

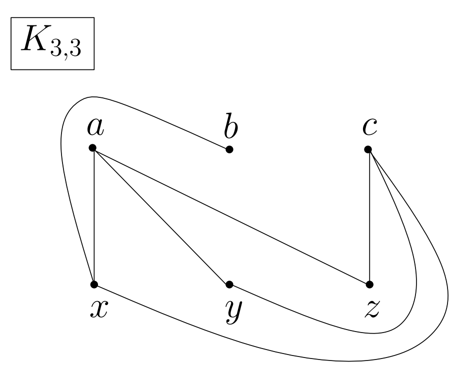
EXAMPLE

Following 3-regular graph $G$ is planar because no edges intersect except at the vertices




EXAMPLE

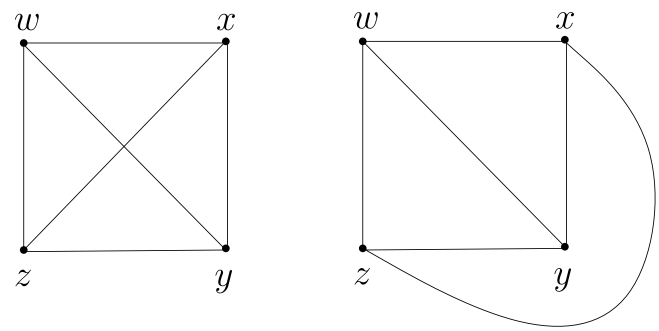
Complete bipartite graph $K_{3,3}$ is non-planar. As show by the following figure the edge $(b, y)$ will have to intersect one of the existing edges



EXAMPLE

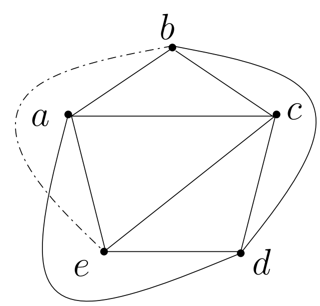
In the family of complete graphs, $K_2$,$K_2$ and $K_3$ are obviously planar. At first glance, $K_4$ seems to be non-planar. In the figure at left of $K_4$, edges $(x, z)$ and $(w, y)$ overlap at a point other than a vertex. However $K_4$ can be redrawn as shown in the figure at right and it becomes clear that $K_4$ is planar

$\boxed{K_4}$




What about $K_5$ ? As shown by the following figure, any embedding of $K_5$ will contain a pentagon (here $\{a, b, c, d, e\}$). Interior region can contain only two edges, say $(a, c)$ and $(c, e)$. Obviously the edges $(a, d)$ and $(b, d)$ are in the exterior region. We need the edge $(b, e)$ in order to have $K_5$. But $e$ is inside the region delimited by edges $(a, d)$, $(d, c)$, $(c, a)$ and $b$ is outside this region. So, the edge $(b, e)$ must intersect one of the existing edges, therefore $K_5$ is non-planar

$\boxed{K_5}$

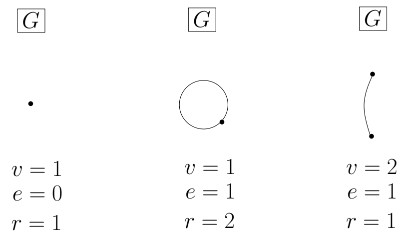



*Theorem 8* ***(Euler's Formula)*** *Let $G = (V, E)$ be a connected planar graph or multigraph with $|E| = e$ edges and $|V| = v$ vertices. Let $r$ be the number of regions in the plane determined by a planar embedding of $G$ (one of these regions has infinite area called the infinite region). Then:*

$$r = e - v + 2$$

*Proof* The proof is by induction on $e$

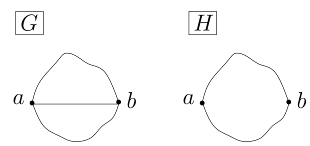
1. <u>Basis:</u> If $e = 0$ or $1$ then $G$ becomes one of the following graphs and for all of them $v - e + r = 2$



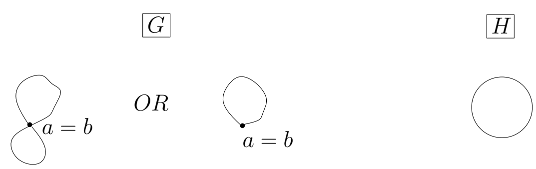

2. Hypothesis: Let $k \in N$ and assume that the result is true for every connected planar graph or multigraph with $e$ edges, where $0 \leq e \leq k$

3. Proof of rank $e = k + 1$: We want to prove the statement for $e = k + 1$ (i.e $v - (k + 1) + r = 2$). Let $G = (V, E)$ be a connected planar graph with $v$ vertices, $r$ regions and let $a, b \in V$ and $(a, b) \in E$. Consider graph $H = G - (a, b)$. There are two cases to consider:

   *Case* 1. $H$ is a connected graph, then $H$ has $v$ vertices, $k$ edges and $r - 1$ regions because the edge $(a, b)$ that is deleted from $G$ was in fact separating two regions in it; this edge is not present in $H$ so previous two regions form only one region in $H$

   Note that even in the case where $a = b$, number of regions in $H$ become one less than the number of regions in $G$

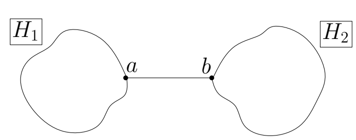
   
   

   As $H$ has less than or equal to $k$ edges, induction hypothesis can be applied to it, resulting in the following equality:

   $$v - k + (r - 1) = 2$$
   $$2 = v - (k + 1) + r$$

   So Euler's theorem is true for $G$

   *Case* 2. $H$ is a disconnected graph, so it has $v$ vertices, $k$ edges and $r$ regions. Also, $H$ has 2 components $H_1$ and $H_2$, where $H_i$ has $v_i$ vertices, $e_i$ edges and $r_i$ regions for $i = 1, 2$

Also we know that:

$v_1 + v_2 = v$
$e_1 + e_2 = k(= e - 1)$
$r_1 + r_2 = r + 1$ (the infinite region is counted twice in $r_1 + r_2$)

By inductive hypothesis the theorem is true for $H_1$ and $H_2$, so

$$\left.\begin{array}{l} v_1 - e_1 + r_1 = 2 \\ v_2 - e_2 + r_2 = 2 \end{array}\right\} 4 = (v_1 + v_2) - (e_1 + e_2) + (r_1 + r_2) = v - k + r + 1$$

So,

$4 = v - (k+1) + r + 2 \Rightarrow v - (k+1) + r = 2$

Hence the theorem is proved

□

*Corollary* *If a connected planar simple graph has $e$ edges and $v$ vertices and no circuits of length $3$, then*

$$e \leq 2v - 4$$

QUESTION

Show that $K_{3,3}$ is nonplanar using the previous corollary

*Definition* For each region $R$ in planar embedding of a (planar) graph, the **degree of $R$**, denoted $deg(R)$ is the number of boundary edges

$$\sum_{i=1}^{r} deg(R_i) = 2|E|$$

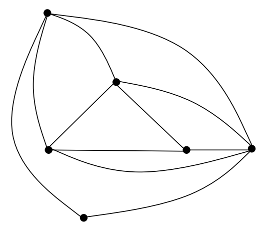
*Corollary* *Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$ and $r$ regions. Then*

$$3r \leq 2e$$

*and*

$$e \leq 3v - 6$$

*Proof*



Since $G$ is loop-free, not a multigraph and $e > 2$, boundary of each region contains at least 3 edges.

$$2e = 2|E| = \sum_{i=1}^{r} deg(R_i) \geq 3r$$

From Euler's Theorem:

$$2 = v - e + r \leq v - e + \frac{2}{3}e = v - \frac{1}{3}e$$

So,

$$6 \leq 3v - e \quad \text{or} \quad e \leq 3v - 6$$

□

EXAMPLE

$K_5$ is a loop-free and connected graph. If $K_5$ was planar, we would have $3v - 6 \geq e$. But $3.5 - 6 = 9 > 10 = e$ is false so $K_5$ is nonplanar

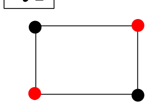
## Graph Colouring and Chromatic Number

*Definition* If $G = (V, E)$ is an undirected graph, a **proper colouring** of $G$ occurs when we colour vertices of $G$ in a way that if $(a, b) \in E$, then $a$ and $b$ are coloured with different colours. The minimum number of colors needed to properly colour $G$ is called the **chromatic number** of $G$ and is written $\boldsymbol{\chi(G)}$

EXAMPLE

Chromatic number of $Q_2$ is $\chi(Q_2) = 2$

$\boxed{Q_2}$



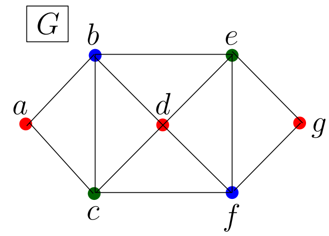
EXAMPLE

For complete graph $K_n$, $\chi(K_n) = n$ since every vertex of $K_n$ is connected to all others and so each vertex should have a different color

For the complete bipartite graph $K_{m,n}$, as for any graph graph with $|E| > 0$, $\chi(K_{m,n}) > 1$. Let $V_1$ and $V_2$ be the two partitions of $K_{m,n}$, then we can color all vertices of $V_1$ in *red* and of $V_2$ in *green* and obtain a proper colouring of $K_{m,n}$. So $\chi(K_{m,n}) = 2$

EXAMPLE

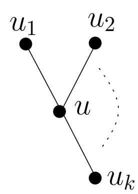
Chromatic number of following graph $G$ is at least 3: $a$,$b$,$c$ must be coloured by different colours, so one need at least 3 colors to colour the graph. Lets try to colour $G$ with three colors: assign *red* to $a$, *blue* to $b$ and *green* to $c$; then $d$ can be coloured *red*, $e$ *green*, $f$ *blue* and $g$ *red*




**Theorem 9** *Let $G = (V, E)$ be a connected simple graph, and let $\Delta = max_{v \in V}\{d(v)\}$ ($\Delta$ is called* ***maxdegree*** *of graph $G$). Then*

$$\chi(G) \leq \Delta + 1$$

*Proof* Pick a random vertex $u \in V$ and color $u$ and all its $k$ neighbours in a different color. For that $\Delta + 1$ colors will be enough as by definition $k \leq \Delta$. Next continue this same operation by picking a non-coloured vertex of




the graph, namely $x$. Let $x_1, \ldots, x_m$ be adjacent vertices of $x$ such that $x_1, \ldots, x_p$ are already coloured and $x$ and $x_{p+1}, \ldots, x_m$ are not coloured. Since $m \leq \Delta$, $m + 1 \leq \Delta + 1$, so there are enough remaining colors to color $x$ and $x_{p+1}, \ldots, x_m$. If we apply this operations till there is no more uncoloured vertex in $G$, then we will color $G$ with less than $\Delta + 1$ colors. $\square$

## Colouring Planar Graphs

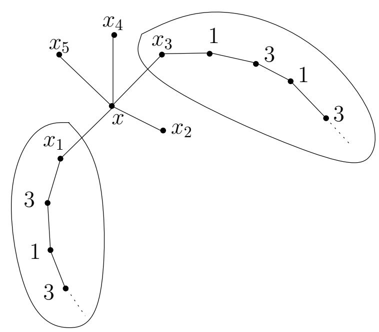
**Theorem 10** *Every planar graph is 5-colorable*

*Proof* Proof by induction on number of vertices $n$

1. Basis: For $n = 1, 2, 3, 4, 5$, $G$ on $n$ vertices is 5-colorable
2. Hypothesis: We assume that for all $n \leq k$, $G$ on $n$ vertices is 5-colorable

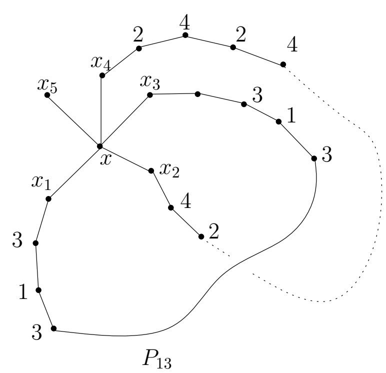
3. <u>Proof of rank $k+1$</u>: Let $G$ be a graph on $v = k+1$ vertices and $e$ edges. By Euler's corollary, $e \leq 3v - 6$. So
$$\sum_{u \in V} d(u) = 2e \leq 6v - 12 < 6v$$
By Pigeonhole Principle, there exist a vertex $x \in V$ such that $d(x) < \frac{6v}{v} = 6$. So $d(x) \leq 5$. Consider graph $H = G - x$ (graph obtained by removing $x$ and all of its incident edges from graph $G$). $H$ has $k$ vertices and by induction hypothesis it is 5-colorable. Assume that $x$ is adjacent in $G$ to vertices $x_1, x_2, \ldots, x_k$ with $k \leq 5$. If in $H$, less than 5 colors are used to color vertices $x_1, x_2, \ldots, x_k$, then in $G$ the $5^{th}$ color can be used to color $x$ and $G$ will be 5-colorable, we will be done. So lets assume that $k = 5$ and that all vertices $x_1, x_2, \ldots, x_5$ have different colors in a 5-colouring of $H$. Assume that $x_1, \ldots, x_5$ are in a cyclic order around $x$ and that the color of $x_i$ is $i$ for $i = 1, \ldots, 5$. Let $H(i,j)$ denote the subgraph of $H$ spanned by vertices of color $i$ and $j$. By definition $x_1$ and $x_3$ belong to $H(1,3)$. There are 2 possibilities:

   - If $x_1$ and $x_3$ belong to distinct components of $H(1,3)$

     Then interchanging the colors 1 and 3 in the component containing $x_1$ won't affect the colouring of the component containing $x_3$. So if we interchange the colors 1 and 3 in the component containing $x_1$, $x_1$ will have color 3 (as $x_3$). So $x$ can now be coloured 1 and we will obtain a 5-colouring of $G$

   - If $x_1$ and $x_3$ belong to the same component of $H(1,3)$

     Then there exists an $x_1$-$x_3$ path $P_{13}$ in $H$ whose vertices are coloured 1 and 3. Any path between $x_2$ and $x_4$ formed only by vertices coloured by 2 and 4 must go through $P_{13}$, which is impossible. So $x_2$ and $x_4$ belong to distinct components of $H(2,4)$ and interchanging the colors 2 and 4 in the component containing $x_2$ won't affect the colouring of the component containing $x_4$. So if we interchange the colors 2 and 4 in the component containing $x_2$, $x_2$ will have color 4 (as $x_4$). So $x$ can now be coloured 2 and we will obtain a 5-colouring of $G$

$\square$

It is clear that for any planar graph $G$, $\chi(G) \leq 4$ because $\chi(K_4) = 4$.

**Theorem 11** *Every planar graph is* $4$*-colorable*

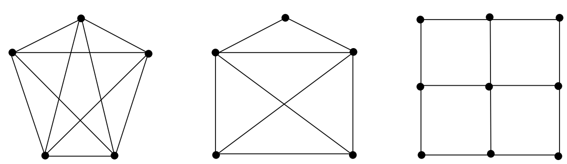
## Euler Paths and Euler Circuits

**Definition** ***(Euler circuit)*** An **Euler circuit** in $G$ is a simple circuit (that does not cross the same edge twice) containing every edge of $G$. *It traverses each edge exactly once and each vertex at least once.* If $G$ contains an Euler circuit, it is called **Eulerian**

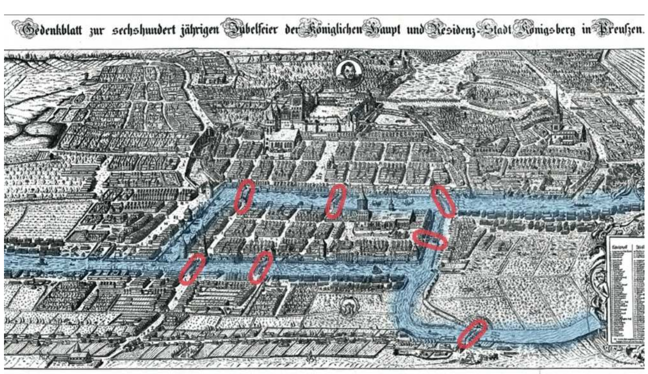
**Definition** ***(Euler path)*** An **Euler path** in $G$ is a simple path containing every edge of $G$. *It traverses every vertex and edge of* $G$ *exactly once*

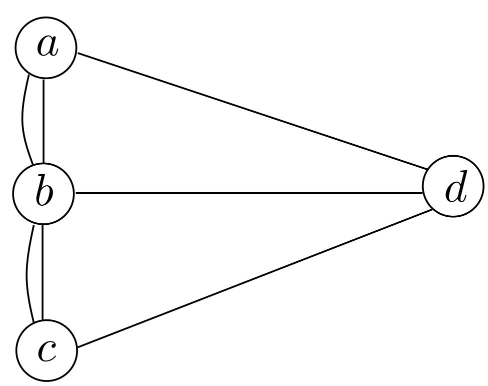
QUESTION

Do those graphs contain any Euler circuit and-or path ?



Königsberg bridges



Necessary and Sufficient conditions for euler circuits and Euler paths

*Theorem 12* *A connected graph $G$ has an Euler circuit iff every vertex has even degree*

*Theorem 13* *A connected graph has an Euler path but not an Euler circuit iff it has exactly two vertices of odd degree*

*Proof* $\Rightarrow$ Let suppose that $G$ contains an Euler path $P_E$. We want to prove that $G$ has 2 vertices of odd degree. Assume that $u$ is the first vertex and $v$ is the last one on $P_E$. By definition $P_E$ must contain all edges of the graph $G$ exactly once. Let $x$ be a vertex between $u$ and $v$. As $P_E$ doesn't finish at $x$, every time $x$ is visited using an edge it should be exited right away using another one. So number of incident edges to $x$ should be even. As for $u$ and $v$: $u$ should be exited once at the beginning and for all other visits to $u$ number of edges that will be used will be even; $v$ should be entered once at the end and for all other visits to $v$ number of edges that will be used will be even. Hence both $u$ and $v$ have odd degree
$\Leftarrow$ Let suppose that $G$ has exactly 2 vertices of odd degree $u$ and $v$. We want to prove that $G$ contains an Euler path. Let $G'$ be the graph built by copying $G$ and adding an extra edge $e$ between $u$ and $v$. Then in $G'$ every vertex has even degree and by theorem 12, $G'$ contains an Euler circuit $v, \dots, u, v$. If we delete the extra edge $e$ between $u$ and $v$ we will obtain the euler path $v, \dots, u$ in $G$ $\square$

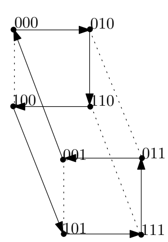
## Hamilton Paths and Circuits

*Definition* ***(Hamilton circuit)*** Let $G = (V, E)$ be a graph or multigraph with $|V| \geq 3$. A **Hamilton circuit** in $G$ is a simple circuit passing through all vertices of $G$ only once. If $G$ contains a Hamilton circuit, it is called **Hamiltonian**

*Definition* ***(Hamilton path)*** Let $G = (V, E)$ be a graph or multigraph with $|V| \geq 3$. A **Hamilton path** in $G$ is a simple path passing through all vertices of $G$ only once

EXAMPLE

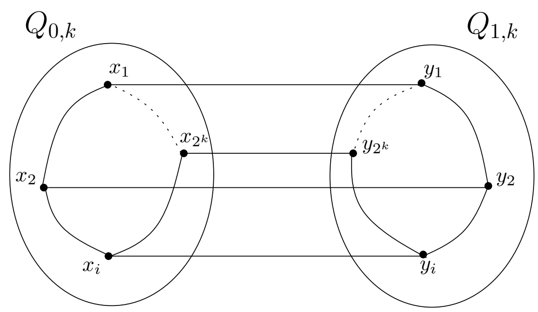
Hamilton cycle on a $Q_3$



*Property* *For all $n \geq 2$, $Q_n$ has a Hamilton cycle*

*Proof* Proof by induction on number of dimension $n$

1. Basis: For $n = 2, 3$, $Q_n$ has a Hamilton cycle
2. Hypothesis: We assume that for all $n \leq k$, $Q_k$ has a Hamilton cycle
3. Proof of rank $k + 1$: Using previous hypothesis we will prove that $Q_{k+1}$ has a Hamilton cycle. $Q_{k+1}$ is represented as 2 copies of $Q_k$ ($Q_{0,k}$ & $Q_{1,k}$) and edges of type $\{x, y\}$, where $x \in Q_{0,k}$, $y \in Q_{1,k}$ and binary labels for $x$,$y$ differ only in the first position.




$(x_i, y_i) \in Q_{k+1}$
$for\ i = 1, 2, \ldots, 2^k$

If the Hamilton cycle in $Q_k$ ($Q_{0,k}$ & $Q_{1,k}$) is

$$x_1 \to x_2 \to x_3 \to \cdots \to x_{2^k} \to x_1$$

then the Hamilton cycle in $Q_{k+1}$ is the following

$$x_1 \to x_2 \to \cdots \to x_{2^k-1} \to x_{2^k} \to y_{2^k} \to y_{2^k-1} \to y_2 \to y_1 \to x_1$$

where $x_i \in Q_{0,k}$ is connected with $y_i \in Q_{1,k}$, for all $i = 1, 2, \ldots, 2^k$

□

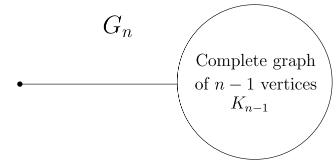
Preliminary discussion about Hamilton circuits

The existence of Hamilton cycle (path) and the existence of an Euler circuit (trail) for a graph are similar problems.
In Hamilton cycle (path) $\rightarrow$ visit each vertex only once
In Euler circuit (trail) $\rightarrow$ travel each edge only once
Unfortunately, there is no helpful connection between the two ideas

| QUESTION |
|---|
| Design a graph with a Hamilton circuit but no Euler circuit and vice versa |

There are no necessary and sufficient conditions for the existence of Hamilton paths and circuits. For sufficient conditions there are many. That is, we know many conditions under which a graph can contain a Hamilton cycle but when a graph contains a Hamilton cycle we cannot say much about what should be its characteristics (If $G$ has a Hamilton cycle then $G$ must be/contain/have ???). Note that in the case of Eulerian circuits necessary condition is well known (If $G$ has an Eulerian circuit then all vertices of $G$ is of even degree (see theorem 12))

Is number of edges a necessary condition for Hamilton circuits ?

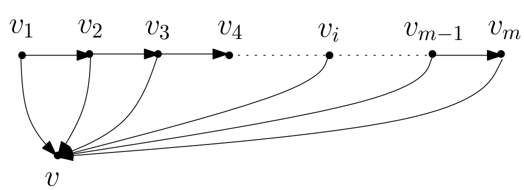



$G_n$ is a critical graph (i.e. Any edge that will be added to $G_n$, will make it to have a Hamilton cycle)

Sufficient conditions for Hamilton path and Hamilton circuit

*Theorem 14* ***(Hamilton path)*** *Let $K_n^*$ be a* ***complete directed graph*** *(i.e.* ***tournament****) - that is, $K_n^*$ has $n$ vertices and for each distinct pair $x$,$y$ of vertices, eaxctly one of the edges $(x, y)$ or $(y, x)$ is in $K_n^*$. Such a graph always contains a (directed) Hamilton path*

*Proof* Let $P_m = (v_1, v_2), (v_2, v_3), \dots, (v_{m-1}, v_m)$ be a longest path in $K_n^*$ and $m \geq 2$. If $m = n$ we are done. If not, let $v$ be a vertex that doesn't appear in $P_m$. Either $(v, v_1) \in K_n^*$ or $(v_1, v) \in K_n^*$. First one is impossible, since if $(v, v_1) \in K_n^*$ then the path $P' = (v, v_1)P_m$ would have been longer than $P_m$ - contradiction. Therefore $(v_1, v) \in K_n^*$.




$v_1, \dots, v_{i-1}, v, v_i, \dots, v_m$
is longer than $P_m$

Now, again $(v, v_2) \notin K_n^*$, otherwise $(v_1, v), (v, v_2), (v_2, v_3), \dots, (v_{m-1}, v_m)$ would have been longer than $P_m$. Therefore $(v_2, v) \in K_n^*$. If we continue this process then we get that $(v_3, v) \in K_n^*$, ..., $(v_m, v) \in K_n^*$. But this situation is impossible since the path $P' = P_m v$ would be longer than $P_m$. Contradiction. So, there is no such vertex $v$ which is not on path $P_m$. Therefore $P_m$ is a Hamilton path □

*Theorem 15* ***(Hamilton path)*** *Let $G = (V, E)$ be a loop-free graph with $|V| = n \geq 2$. If*

$$deg(x) + deg(y) \geq n - 1 \text{ for all } x, y \in V \text{ with } x \neq y$$

*then $G$ has a Hamilton path*

*Proof* The proof is similar to that of theorem 14 □

**Corollary** ***(Hamilton path)*** *Let $G = (V, E)$ be a loop-free graph with $n \geq 2$ vertices. If*

$$deg(v) \geq \frac{n-1}{2} \text{ for all } v \in V$$

*then $G$ has a Hamilton path*

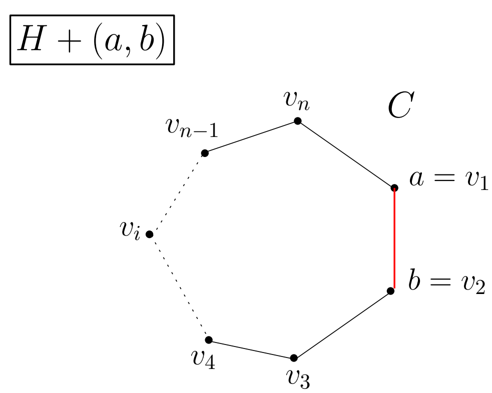
*Proof* If $deg(v) \geq \frac{n-1}{2}$ for $\forall v \in V$, then $deg(x) + deg(v) \geq 2 \times \frac{n-1}{2} = n - 1$ for all $x \in V$ such that $x \neq v$. Hence by theorem 15 $G$ contains a Hamilton path □

**Theorem 16** ***(Hamilton cycle - Ore 1960)*** *Let $G = (V, E)$ be a loop-free undirected graph with $|V| = n \geq 2$. If*

$$deg(x) + deg(y) \geq n \text{ for all nonadjacent } x, y \in V$$

*then $G$ has a Hamilton cycle*

*Proof* Following is a proof by contradiction. Assume that $G$ does not contain a Hamilton cycle. Add edges to $G$ until we arrive at a graph $H$ (which is a subgraph of $K_n$) such that $H$ does not have a Hamilton cycle but for any edge $e$ (of $K_n$) not in $H$, graph $H + e$ does have a Hamilton cycle. Since $H \neq K_n$, there are vertices $a, b \in V$ where $(a, b)$ is not in $H$ but $H + (a, b)$ has a Hamilton cycle $C$. Lets label vertices of $H$ such that $v_1, v_2, \ldots, v_i, \ldots, v_{n-1}, v_n$ represents the Hamilton cycle $C$ in $H + (a, b)$ where $a$ will be $v_1$ and $b$ will be $v_2$. Such labelling is presented in the following figure. As there is no Hamilton cycle in $H$, for each $3 \leq i \leq n$, if




the edge $(b, v_i)$ is in $H$, then $(a, v_{i-1})$ is not in $H$ otherwise $b, v_i, v_{i+1}, \ldots, v_n, a, v_{i-1}, v_{i-2}, \ldots, v_4, v_3$ would have been a Hamilton cycle for graph $H$. Therefore for each $3 \leq i \leq n$ at most one of $(b, v_i)$, $(a, v_{i-1})$ is in $H$. Consequently,

$$deg_H(a) + deg_H(b) < n$$

where $deg_H(a)$ represents the degree of $a$ in $H$ and $deg_H(b)$ the degree of $b$ in $H$. It is clear that for $\forall v \in V$,

$$deg_H(v) \geq deg_G(v) = deg(v)$$

so,

$$deg(a) + deg(b) < n$$

This contradicts the hypothesis that

$$deg(x) + deg(y) \geq n \text{ for all nonadjacent } x, y \in V$$

Thus $G$ contains a Hamilton path. $\square$

An immediate corollary of this theorem is the following

> *Corollary* ***(Hamilton cycle)*** *If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$, and if*
>
> $$deg(v) \geq \frac{n}{2} \text{ for all } v \in V$$
>
> *then $G$ has a Hamilton cycle*

> *Corollary* ***(Hamilton cycle)*** *If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$, and if*
>
> $$|E| \geq \binom{n-1}{2} + 2$$
>
> *then $G$ has a Hamilton cycle*

*Proof* Let $a, b \in V$ such that $(a, b) \notin E$. We want to show that

$$deg(a) + deg(b) \geq n$$

Remove the followings from $G$

(i) all edges of the form $(a, x)$ where $x \in V$

(ii) all edges of the form $(y, b)$ where $y \in V$

(iii) the vertices $a$ & $b$

Let $H = (V', E')$ denote the resulting subgraph. Then

$$|E| = |E'| + deg(a) + deg(b)$$

because $(a, b) \notin E$. Since $|V'| = |V| - 2 = n - 2$, $H$ is a subgraph of $K_{n-2}$, so $|E'| \leq \binom{n-2}{2}$. Hence

$$\binom{n-1}{2} + 2 \leq |E| = |E'| + deg(a) + deg(b) \leq \binom{n-2}{2} + deg(a) + deg(b)$$

Therefore

$$
\begin{aligned}
& deg(a) + deg(b) \geq \binom{n-1}{2} + 2 - \binom{n-2}{2} \\
& = \frac{1}{2}(n-1)(n-2) + 2 - \frac{1}{2}(n-2)(n-3) \\
& = \frac{1}{2}(n-2)((n-1)-(n-3)) + 2 \\
& = \frac{1}{2}(n-2).2 + 2 \\
& = n
\end{aligned}
$$

Therefore it follows from theorem 16 that $G$ has a Hamilton cycle $\square$