Trees

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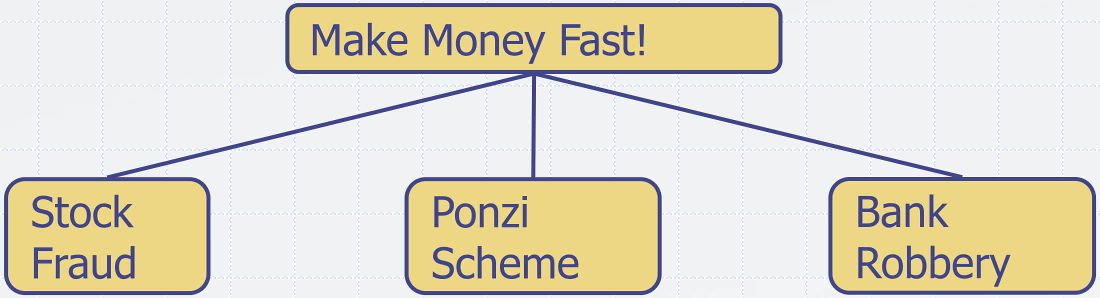
Data Structures and Algorithms in Java, 5th edition. John Wiley& Sons, 2010. ISBN 978-0-470-38326-1.
Data Structures and the Java Collections Framework by William J. Collins, 3rdedition, ISBN 978-0-470-48267-4.
Both books are published by Wiley.

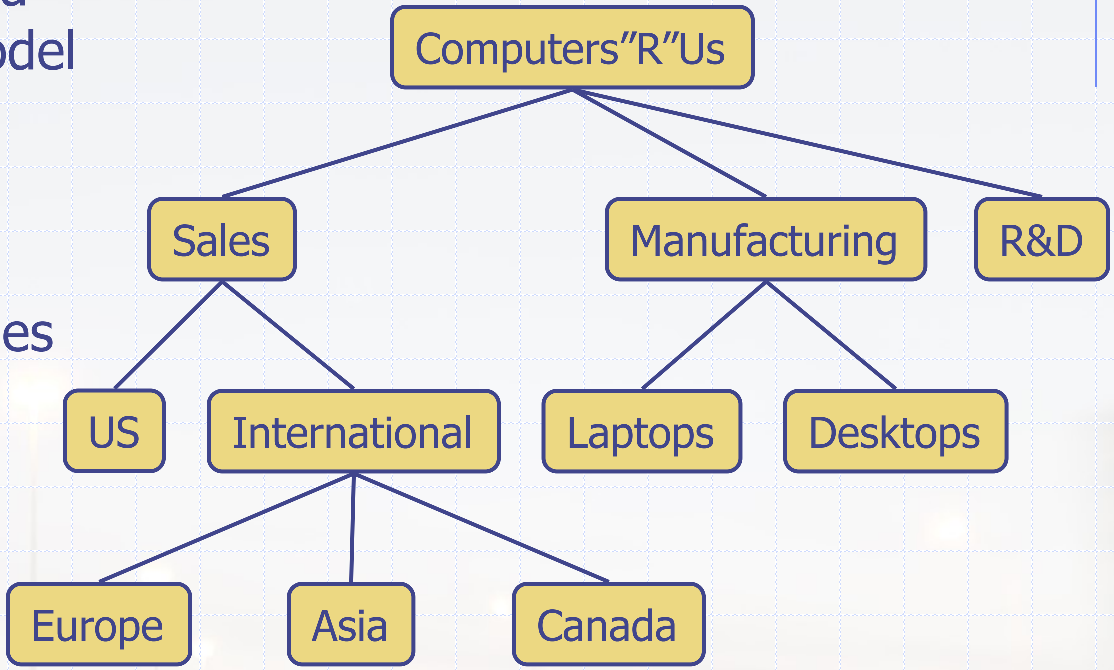
Coverage



- General Trees
- Tree Traversal Algorithms
- Binary Trees

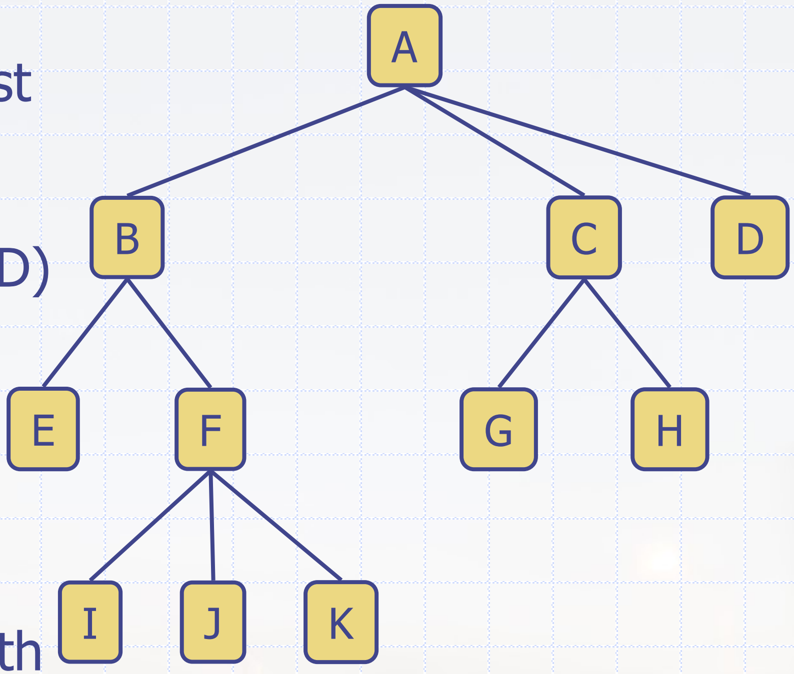
What is a Tree

- In computer science, a tree is an abstract model of a hierarchical structure.
- A tree consists of nodes with a parent-child relation.
- Applications include:
  - Organization charts
  - File systems
  - Programming environments



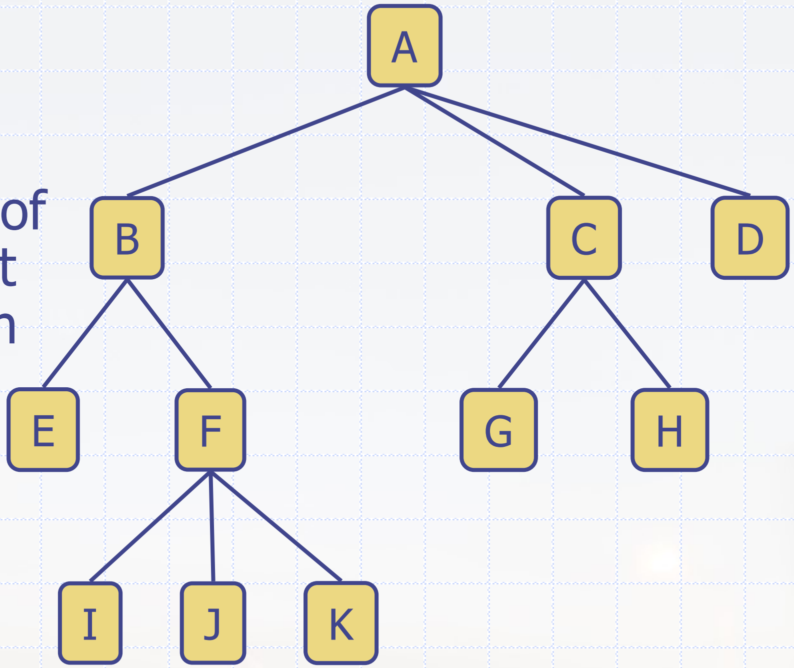
Tree Terminology

- **Root**: node without parent (A)
- **Internal node**: node with at least one child (A, B, C, F)
- **External node** (**leaf**): node without children (E, I, J, K, G, H, D)
- **Ancestors** of a node: parent, grandparent, grand-grandparent, etc.
- **Depth of a node**: number of ancestors
- **Height of a tree**: maximum depth of any node (3, in the shown tree)
- **Descendant** of a node: child, grandchild, grand-grandchild, etc.



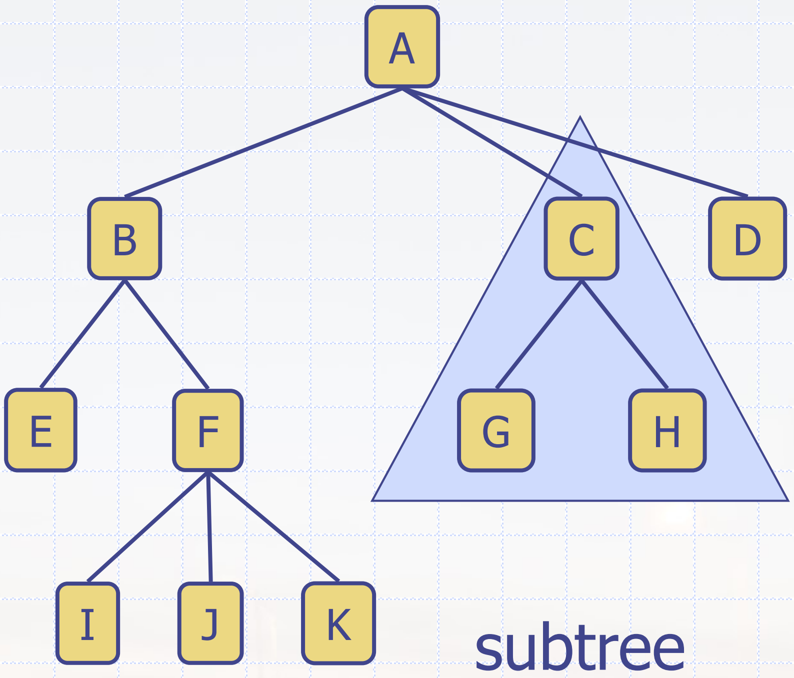
Tree Terminology

- **siblings**: Two nodes that are children of the same parent.
- **edge**: an edge of a tree is a pair of nodes (u, v), where u is the parent and v is the child, or vise versa. In other words, an edge is a connection between a parent and child in the tree.
- **Path**: a sequence of nodes such that any two consecutive nodes in the sequence form an edge (for instance: A, B, F, J).



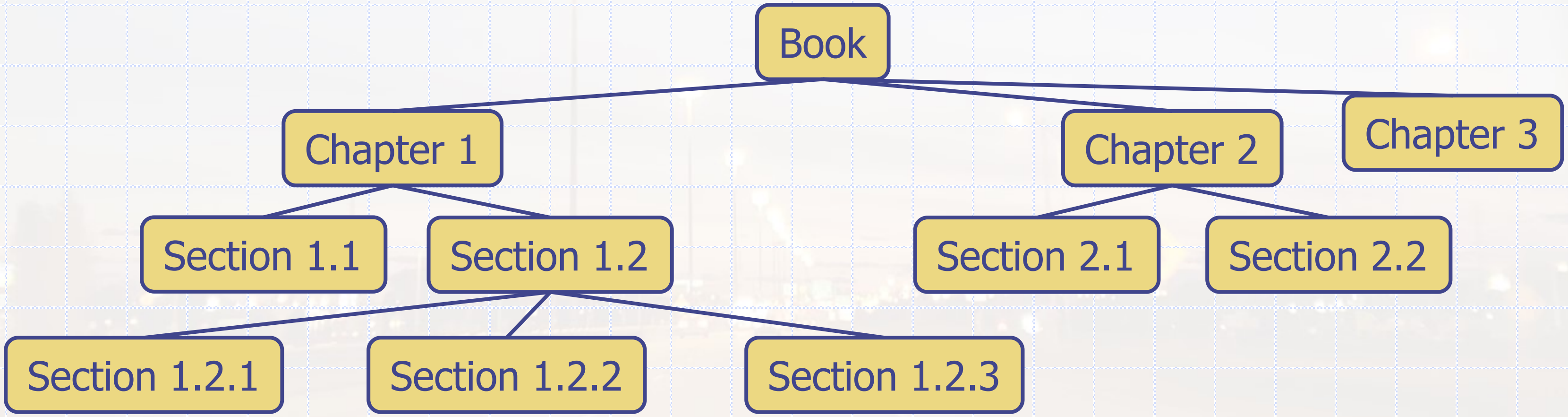
Tree Terminology

- **Subtree**: tree consisting of a node and its descendants.



Ordered Trees

- A tree is ***ordered*** if there is a linear ordering defined for the children of each node.
- That is, with ordered trees we can identify the children of a node as being first, second, third, etc.
- Ordered trees typically indicate the linear order among siblings by listing them in the correct (ordered) order.



Tree ADT

- We can use positions to abstract nodes. *Positions* of a tree are its *nodes*. The terms “position” and “node” is hence used interchangeably. A position object support the following method:
  - element(): Return the object stored in the position.

- Generic methods:
  - integer size(): Return the number of nodes in the tree.
  - boolean isEmpty(): Tests whether or not the tree has nodes.
  - Iterator iterator(): Return an iterator of all the elements stored at nodes of the tree.
  - Iterable positions(): Return an iterable collection of all the nodes of the tree.

Tree ADT

- Accessor methods:
  - position root(): Return the “root” of tree; error if tree is empty.
  - position parent(p): Return the parent of p; error if p is the root.
  - Iterable children(p): Return an iterable collection containing all the children of node p.

    ➔ If the tree is ordered, then the iterable collection returned by children(p) stores the children of p in order.

    ➔ If p is a leaf, then the returned collection is empty.

Tree ADT

- Query methods:
  - boolean isInternal(p): Tests whether node p is internal .
  - boolean isExternal(p): Tests whether node p is external.
  - boolean isRoot(p): Tests whether node p is the root.
- Update method:
  - element replace (p, e): Replace the element at node p with e and return the original (old) element.
- Additional update methods may be defined by data structures implementing the Tree ADT

Tree ADT

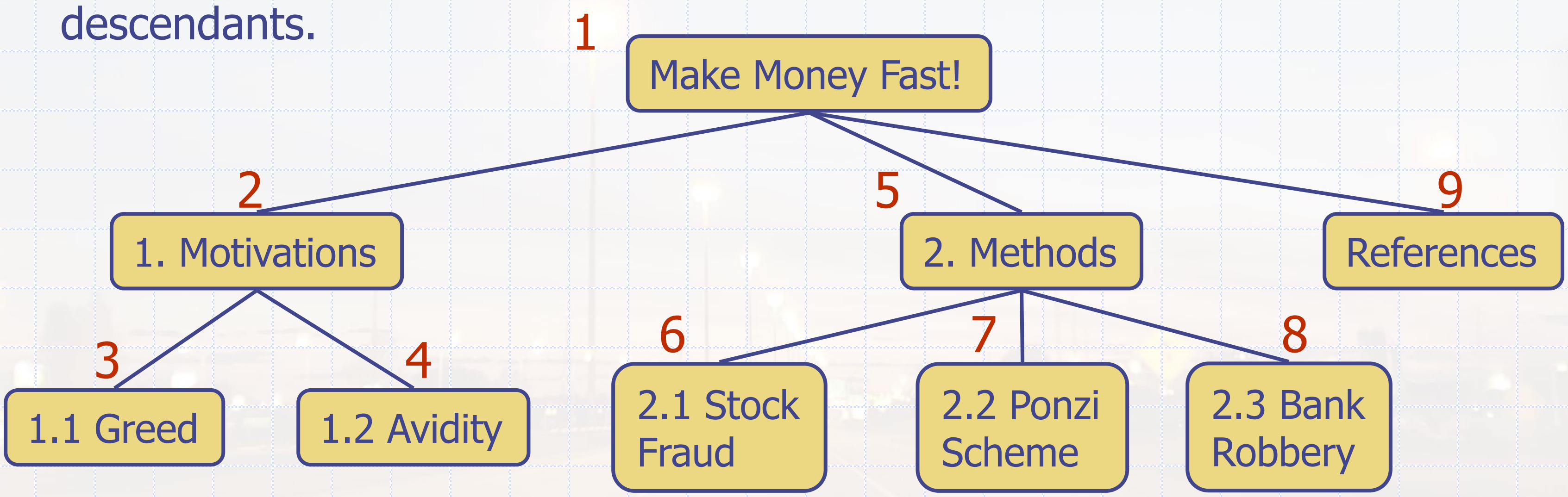
- Performance of the Tree ADT methods:

Operation	Complexity
size, isEmpty	$O(1)$
Iterator, positions	$O(n)$
replace()	$O(1)$
root, parent	$O(1)$
children(v)	$O(c_v)$ c_v *denotes the number of children at node C.*
isInternal, isExternal, isRoot	$O(1)$

Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, the root is visited first and then subtrees rooted as its children are traversed recursively in the same manner.
- That is, a node is visited before its descendants.

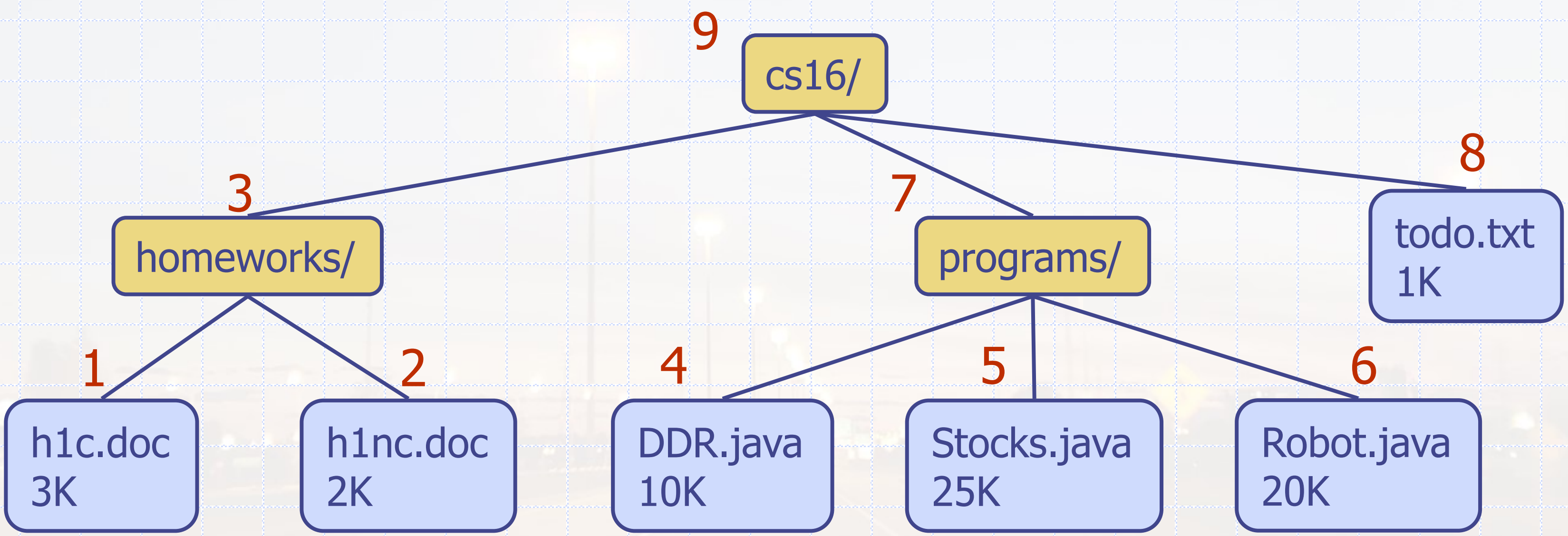
Algorithm *preOrder*(*v*)
    visit(*v*)
    for each child *w* of *v*
        preorder (*w*)



Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

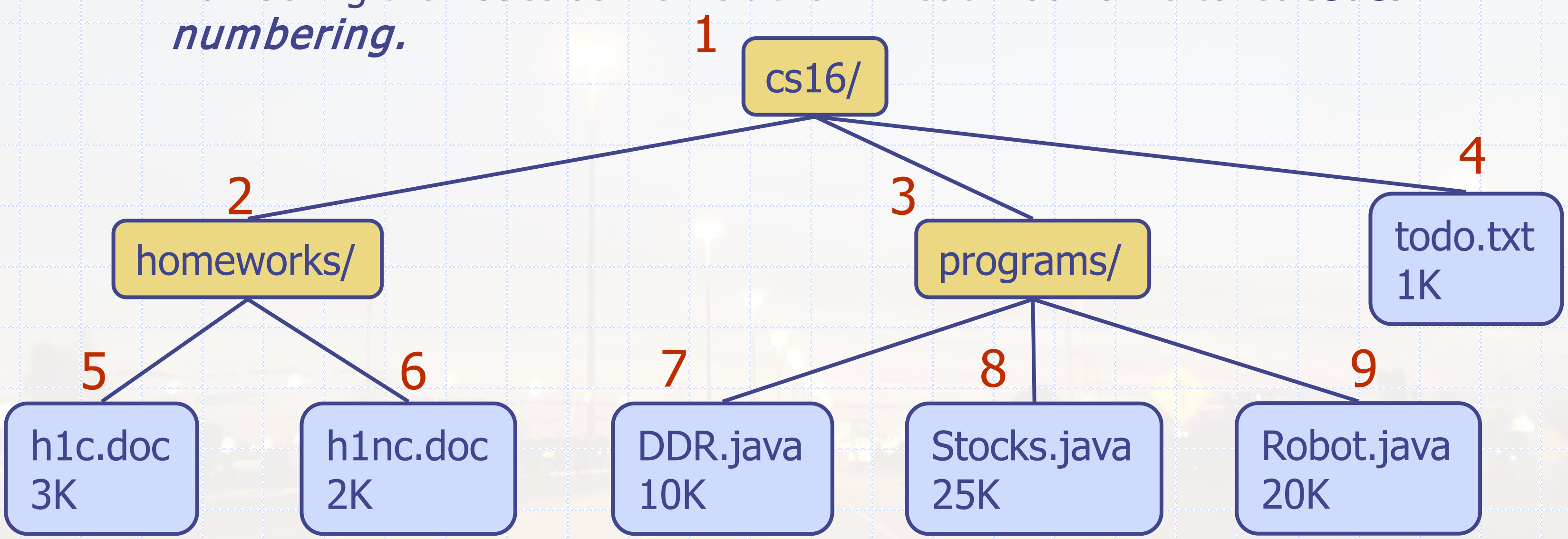
Algorithm *postOrder*(v)
    for each child w of v
        postOrder (w)
    visit(v)



Other Traversals

- Preorder and Postorder traversals are common but other traversals are possible.

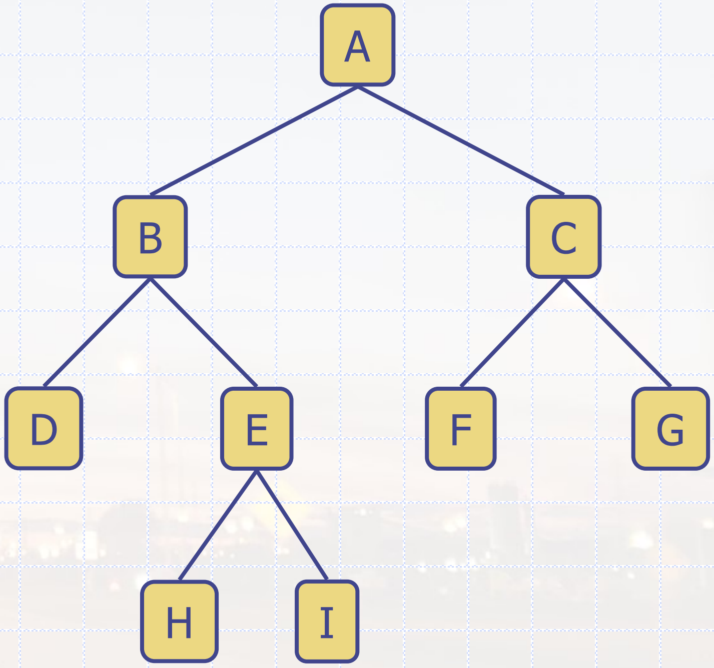
- For instance, we can visit all the nodes at depth d before visiting the ones at depth $d + 1$.
  - Numbering the nodes as we visit them in such fashion is called ***level numbering.***



Binary Trees

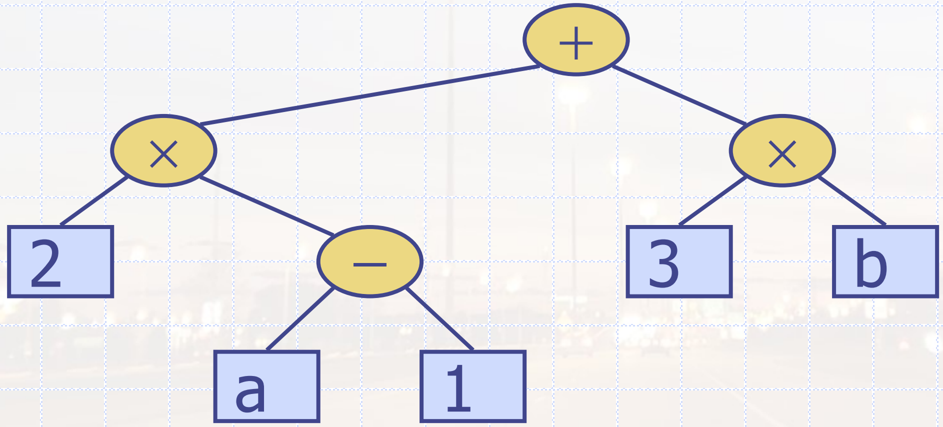
- A binary tree is a tree with the following properties:
  - Each internal node has at most two children (exactly two for proper binary trees; otherwise the tree is improper)
- We call the children of an internal node left child and right child
  - The children of a node are an ordered pair, where the left child precedes the right one
- A binary tree can be defined recursively as either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree

- Applications:
  - arithmetic expressions
  - decision processes
  - searching



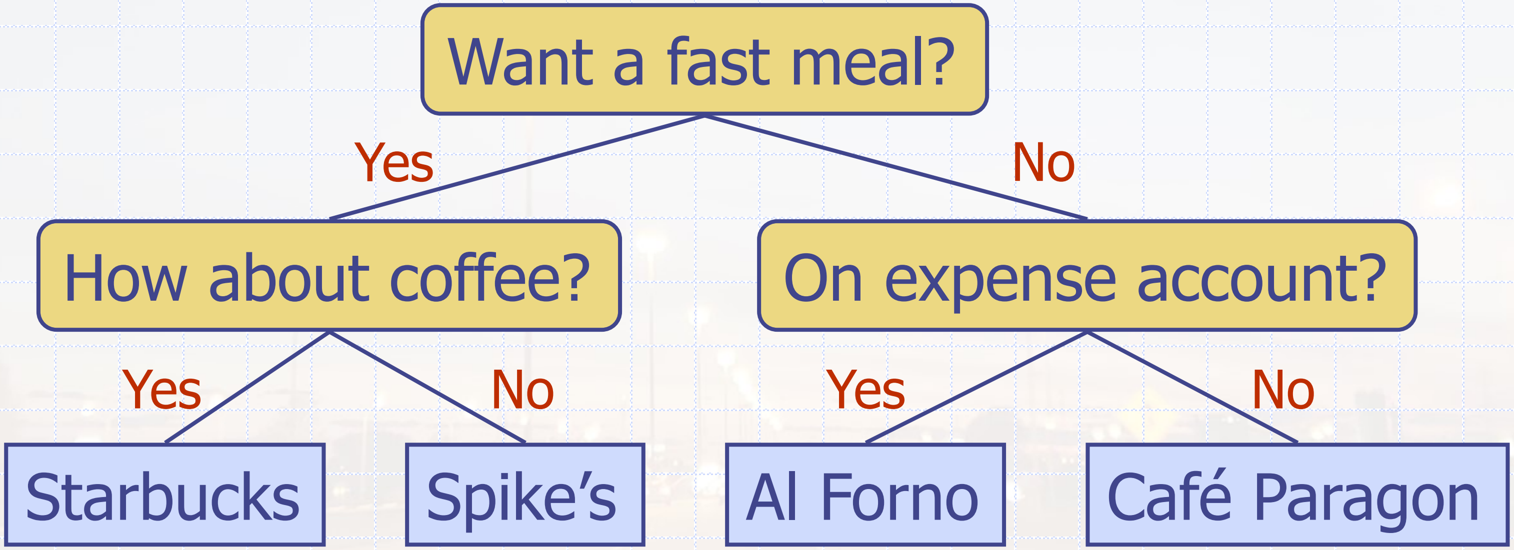
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression is a proper binary tree where:
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



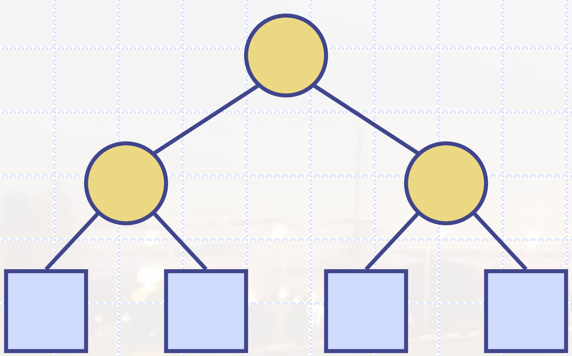
Decision Tree

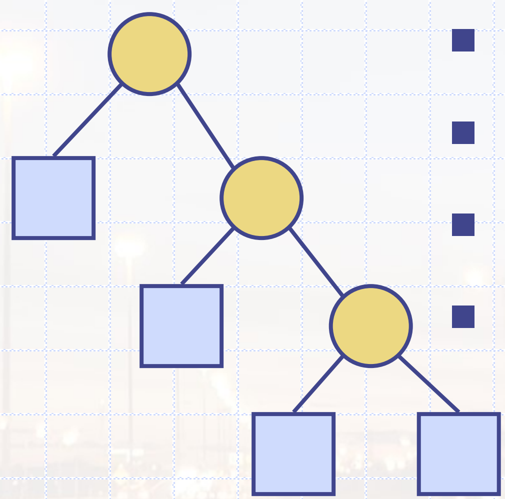
- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



Properties of Proper Binary Trees

- Notation
  - n number of nodes
  - e number of external nodes
  - i number of internal nodes
  - h height





- Properties:
  - $e = i + 1$
  - $n = 2e - 1 = 2i + 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2 (n + 1) - 1$

In a perfectly balanced binary tree:

$n = 2^0 + 2^1 + \ldots + 2^h$

$n = 2^{h+1} - 1$

$2^{h+1} = n + 1$

$h+1 = \log(n + 1)$

$\rightarrow h = \log(n + 1) - 1$

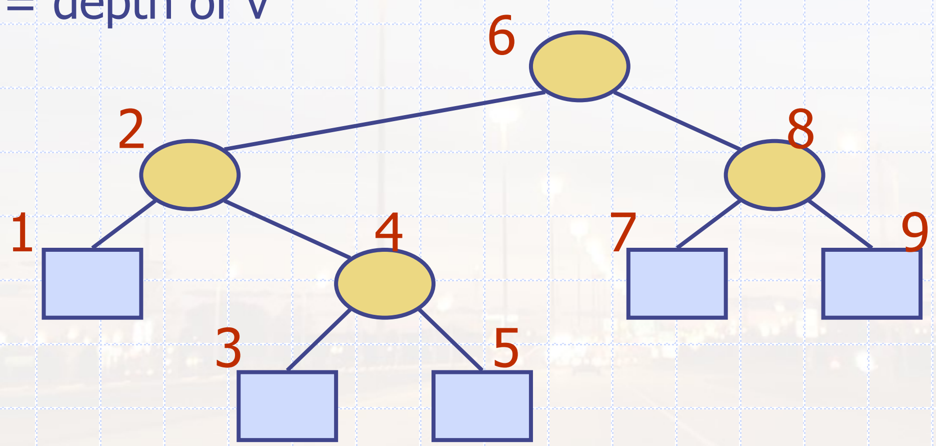
BinaryTree ADT

- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - position left(p): Return the left child of p; error if p has no left child
  - position right(p): Return the right child of p; error if p has no right child
  - boolean hasLeft(p): Test whether p has a left child
  - boolean hasRight(p): Test whether p has a right child
- Update methods may be defined by data structures implementing the BinaryTree ADT

Inorder Traversal of Binary Trees

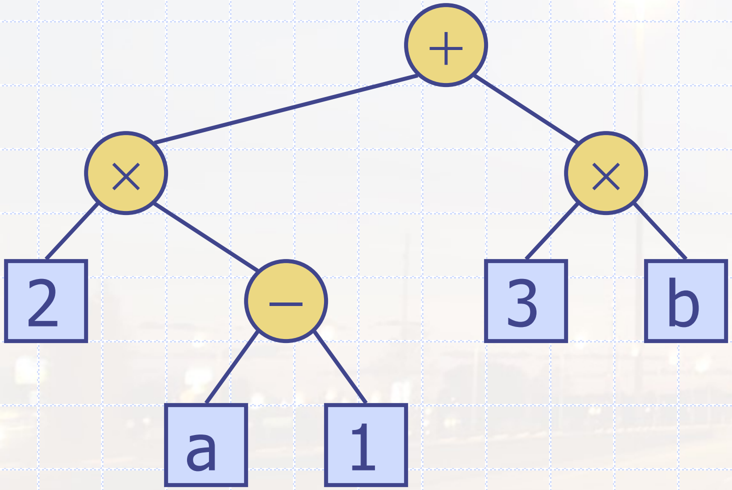
- In an **inorder** traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v

Algorithm *inOrder(v)*
    if *hasLeft* (*v*)
        inOrder (*left* (*v*))
    visit(*v*)
    if *hasRight* (*v*)
        inOrder (*right* (*v*))



Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree

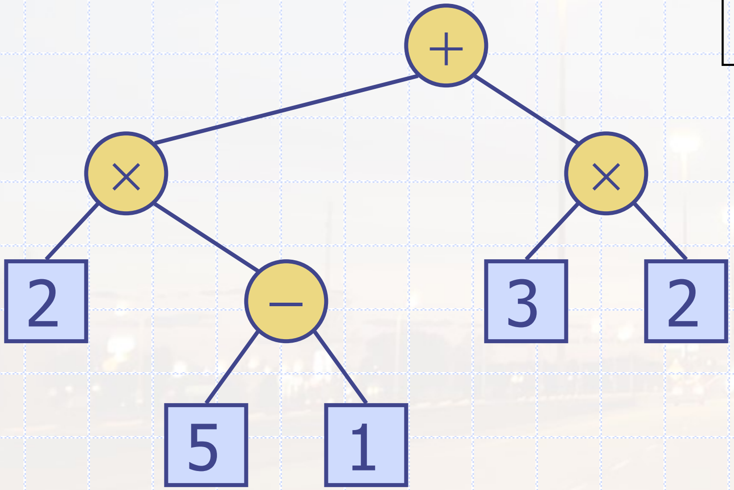


Algorithm *printExpression*(*v*)
    if *hasLeft* (*v*)
        print("(")
        inOrder (*left*(*v*))
    print(*v.element* ())
    if *hasRight* (*v*)
        inOrder (*right*(*v*))
        print (")")

$((2 \times (a - 1)) + (3 \times b))$

Evaluate Arithmetic Expressions

- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees



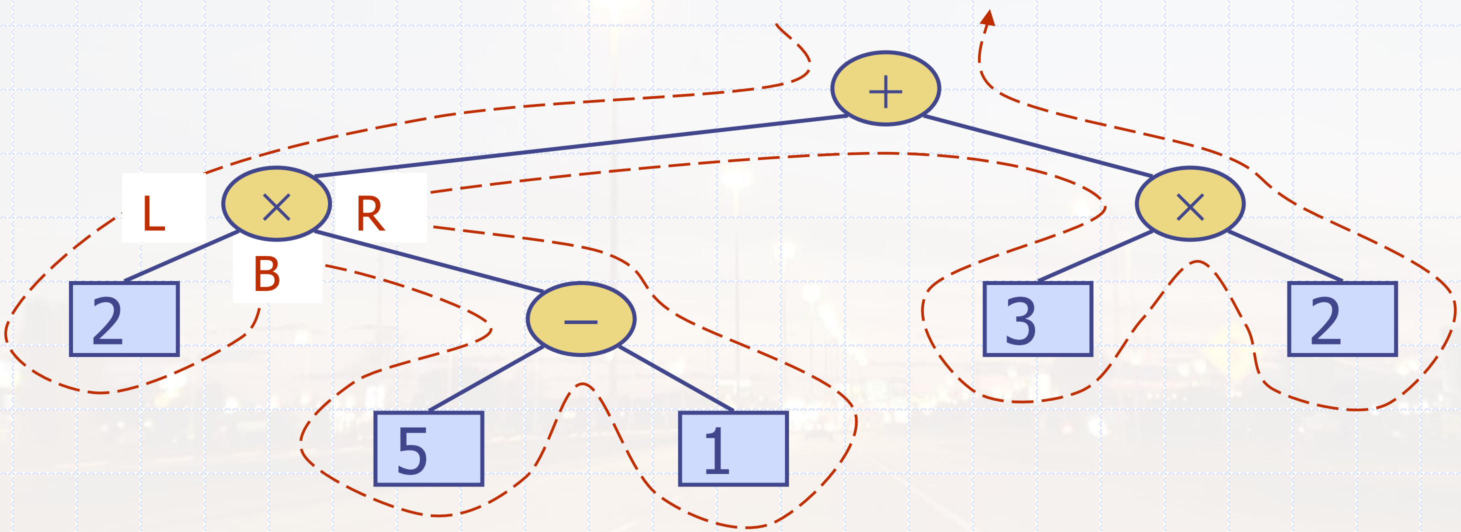
Algorithm *evalExpr*(*v*)
    if *isExternal* (*v*)
        return *v.element* ()
    else
        x ← *evalExpr*(*leftChild* (*v*))
        y ← *evalExpr*(*rightChild* (*v*))
        ◊ ← operator stored at *v*
        return *x* ◊ *y*

Euler Tour Traversal

- All previously discussed traversal algorithms are forms of iterators where each traversal is guaranteed to visit each node in a certain order, and exactly once.
- Euler Tour traversal relaxes that requirement of the single visit to each node.
- The advantage of such algorithms is to allow for more general kinds of algorithms to be expressed easily.
- In other words, it provides a generic traversal of a binary tree which includes the preorder, postorder and inorder traversals.

Euler Tour Traversal

- Euler Tour traversals walks around the tree and visit each node three times:
  - on the left (preorder – root → left → right)
  - from below (inorder – left → root → right)
  - on the right (postorder – left → right → root)
- If the node is external, all three visits actually happen at the same time.

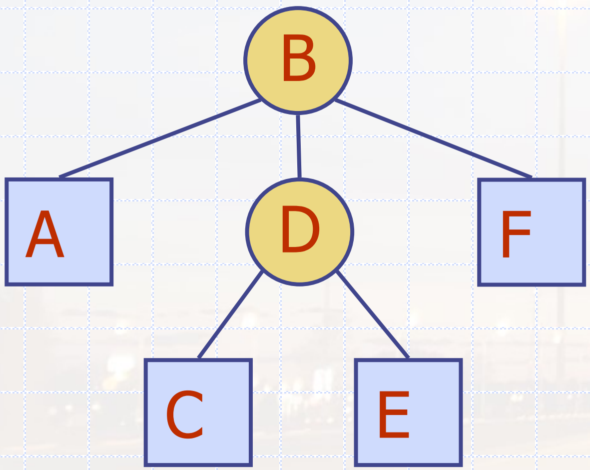


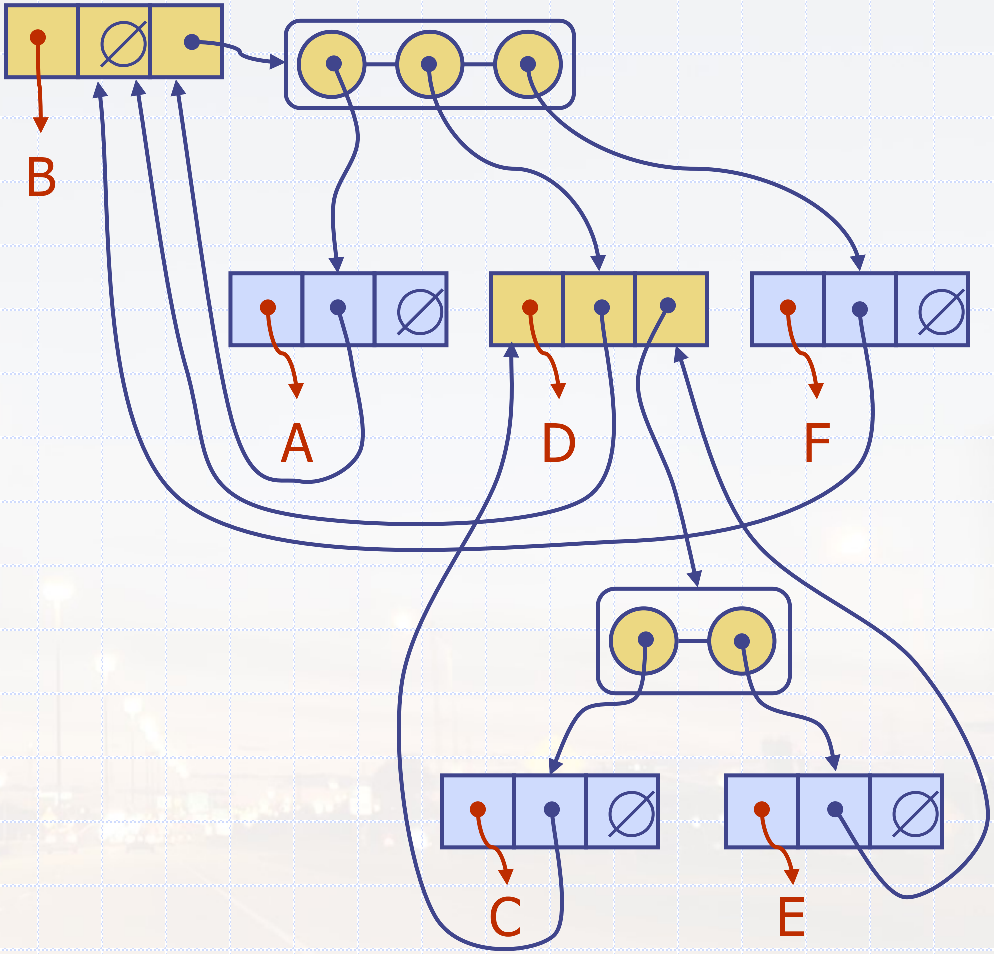
Euler Tour Traversal

- Complexity of Euler Tour is easy to analyze:
  - Assume each visit takes a constant time $O(1)$
  - Since we spend a constant time at each node, the overall running time is $O(n)$
- The Euler Tour can perform other kinds of traversals as well
- For instance, to compute the number of descendants of a node v, we can perform the following:
  - Start the tour by initializing a counter to 0
  - Record the counter when v is visited on the left, say x
  - Record the counter when v is visited on the right, say y
  - Number of descendants = $y - x + 1$

Linked Structure for General Trees

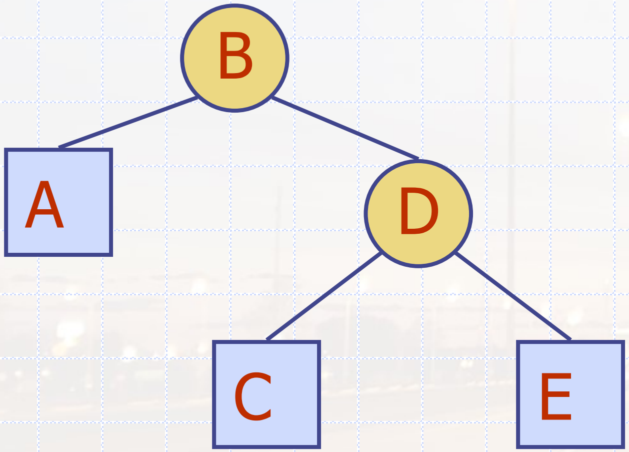
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT

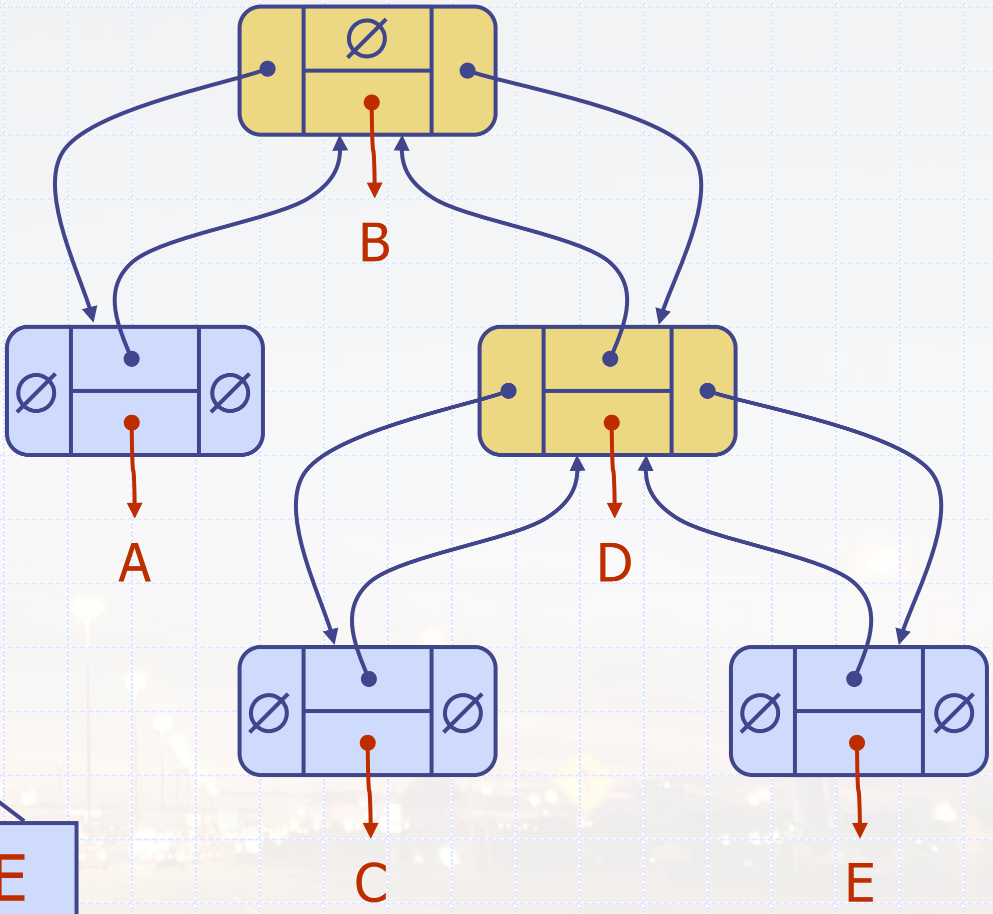




Linked Structure for Binary Trees

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT



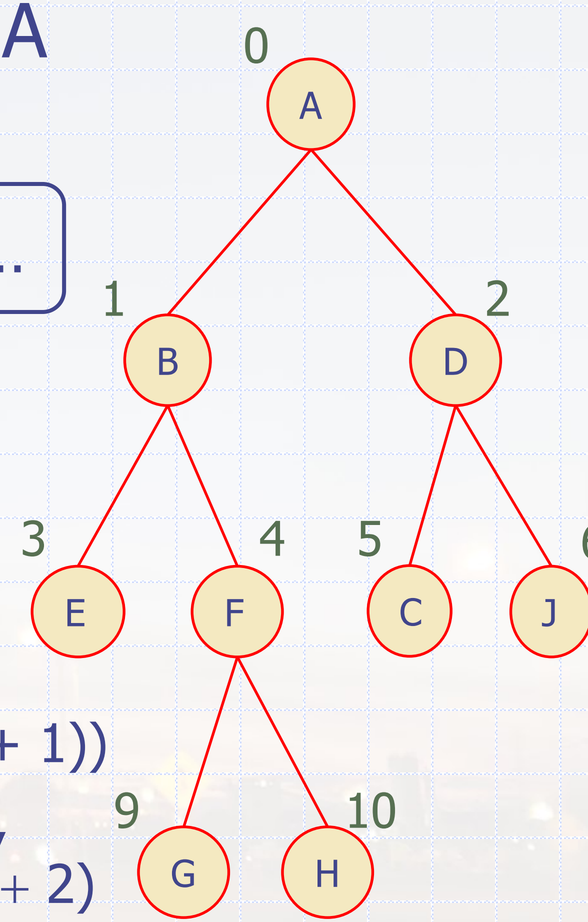


Performance of List Implementation of Binary Trees

Operation	**Complexity**
size, isEmpty	*O(1)*
Iterator, positions	*O(n)*
replace	*O(1)*
root, parent, left, right, sibling, children	*O(1)*
children(v)	*O(1)* *Since there are at most two children of any node in a binary tree.*
hasLeft, hasRight, isInternal, isExternal, isRoot, remove, insertLeft, insertRight, attach	*O(1)*

Array-Based Representation of Binary Trees

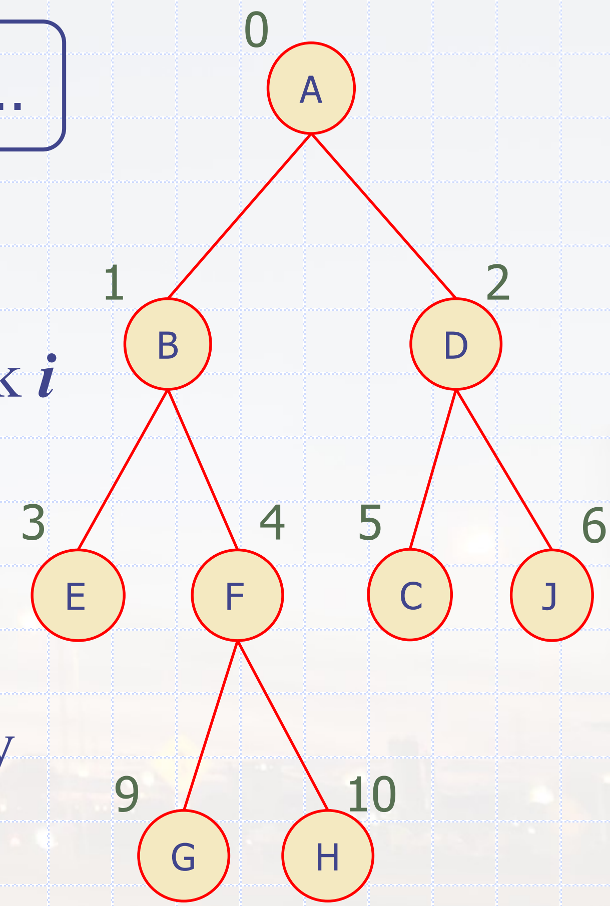
- Nodes are stored in an array A



- Use level numbering
- Node v is stored at A[rank(v)]
  - rank(root) = 0
  - if node is the left child of parent(node), rank(node) = 2 · (rank(parent(node) + 1))
  - if node is the right child of parent(node), rank(node) = 2· (rank(parent(node)) + 2)

Array-Based Representation of Binary Trees

A	B	D	…	G	H	…
0	1	2		9	10	



- So, in general; for any parent/root at rank ***i***
  - the left child is at rank ***2i* + *1***
  - the right child is at rank ***2i* + *2***
  - links between nodes are not explicitly stored

Performance of Array-based Implementation of Binary Trees

Operation	**Complexity**
size, isEmpty	*O(1)*
Iterator, positions	*O(n)*
replace()	*O(1)*
root, parent, left, right, children	*O(1)*
children(v)	*O(1)* *Since there are at most two children of any node in a binary tree.*
hasLeft, hasRight, isInternal, isExternal, isRoot	*O(1)*

Template Method Pattern

- Generic algorithm
- Implemented by abstract Java class
- Visit methods redefined by subclasses
- Template method eulerTour
  - Recursively called on left and right children
  - A TourResult object with fields left, right and out keeps track of the output of the recursive calls to eulerTour

```
public abstract class EulerTour <E, R> {
    protected BinaryTree<E> tree;
    public abstact R execute(BinaryTree<E> T);
    protected void init(BinaryTree<E> T) { tree = T; }
    protected R eulerTour(Position<E> v) {
        TourResult<R> r = new TourResult<R>();
        visitLeft(v, r);
        if (tree.hasLeft(p))
            { r.left=eulerTour(tree.left(v)); }
        visitBelow(v, r);
        if (tree.hasRight(p))
            { r.right=eulerTour(tree.right(v)); }
        return r.out;
    }
    protected void visitLeft(Position<E> v, TourResult<R> r) {}
    protected void visitBelow(Position<E> v, TourResult<R> r) {}
    protected void visitRight(Position<E> v, TourResult<R> r) {}
}
```

Specializations of EulerTour

- Specialization of class EulerTour to evaluate arithmetic expressions
- Assumptions
  - Nodes store ExpressionTerm objects with method getValue
  - ExpressionVariable objects at external nodes
  - ExpressionOperator objects at internal nodes with method setOperands(Integer, Integer)

```
public class EvaluateExpressionTour
        extends EulerTour<ExpressionTerm, Integer> {
    public Integer execute
            (BinaryTree<ExpressionTerm> T) {
        init(T);
        return eulerTour(tree.root());
    }
    protected void visitRight
            (Position<ExpressionTerm> v,
             TourResult<Integer> r) {
        ExpressionTerm term = v.element();
        if (tree.isInternal(v)) {
            ExpressionOperator op = (ExpressionOperator) term;
            op.setOperands(r.left, r.right); }
        r.out = term.getValue();
    }
}
```