CSE 202: Design and Analysis of Algorithms

Lecture 7

Instructor: Kamalika Chaudhuri

Announcements

- Pick up graded HW1 after class
- HW2 due on Thu Feb 2

Last class: Three steps of Dynamic Programming

Main Steps:

1. Divide the problem into **subtasks**

2. Define the subtasks **recursively** (express larger subtasks in terms of smaller ones)

3. Find the **right order** for solving the subtasks (but do not solve them recursively!)

DP: Common Subtasks

Case 1: Input: $x_1, x_2,..,x_n$ Subproblem: $x_1, .., x_i$.

$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$

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Case 2: Input: $x_1, x_2,..,x_n$ and $y_1, y_2,..,y_m$ Subproblem: $x_1, .., x_i$ and $y_1, y_2,..,y_j$

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Case 3: Input: $x_1, x_2,..,x_n$. Subproblem: $x_i, .., x_j$

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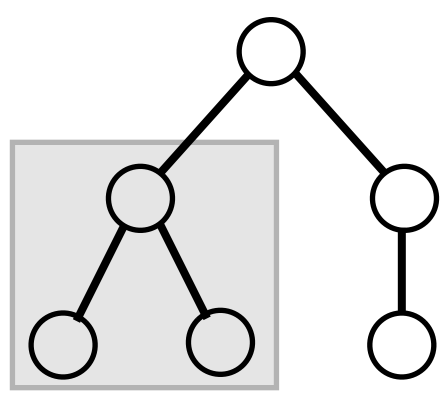
Case 3: Input: $x_1, x_2,..,x_n$. Subproblem: $x_i, .., x_j$

$$x_1 \quad x_2 \quad \boxed{x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

Case 4: Input: a rooted tree. Subproblem: a subtree



Dynamic Programming

- String Reconstruction
- Longest Common Subsequence
- Edit Distance
- Subset Sum

Subset Sum

Problem: Given a list of positive integers a[1..n] and an integer t, is there some subset of a that sums to exactly t?

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For i=1..n, s=1..t,

S(i,s) = True, if some subset of a[1..i] adds to s

= False, ow

Output = S(n, t)

s

i	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
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Row by row, increasing column

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Running Time = O(nt)

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STEP 3: Order of subtasks

Row by row, increasing column

Reconstructing the subset:

Define an array D(i, s).

If S(i, s) = True, and S(i-1,s-a[i]) = True

D(i, s) = (i - 1, s - a[i])
= (i - 1, s) ow.

Reconstruct the subset by following the pointers from D(n,t)

Running Time = O(nt)

Dynamic Programming

- String Reconstruction
- Longest Common Subsequence
- Edit Distance
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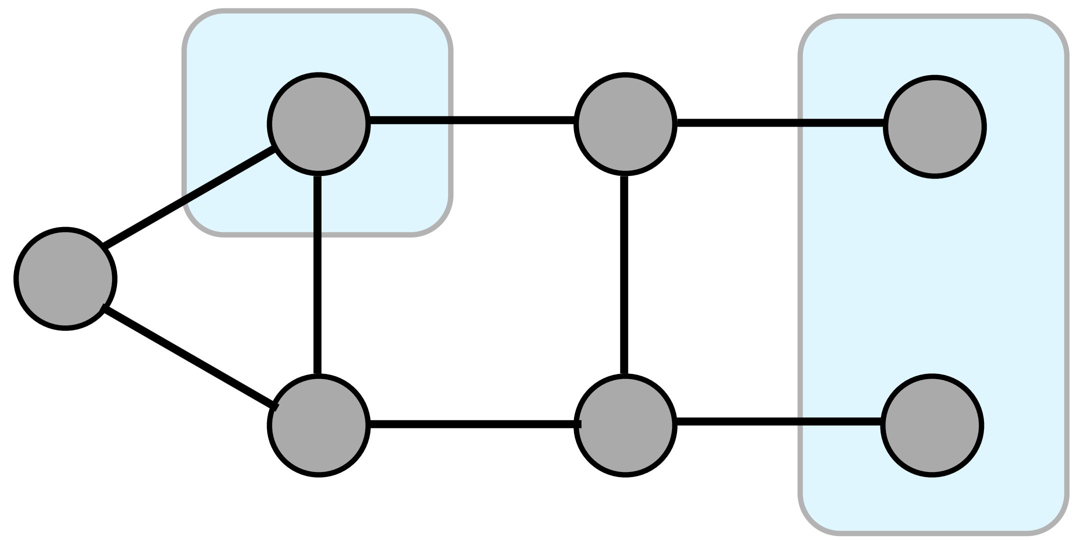
- Independent Set in a Tree

Independent Set



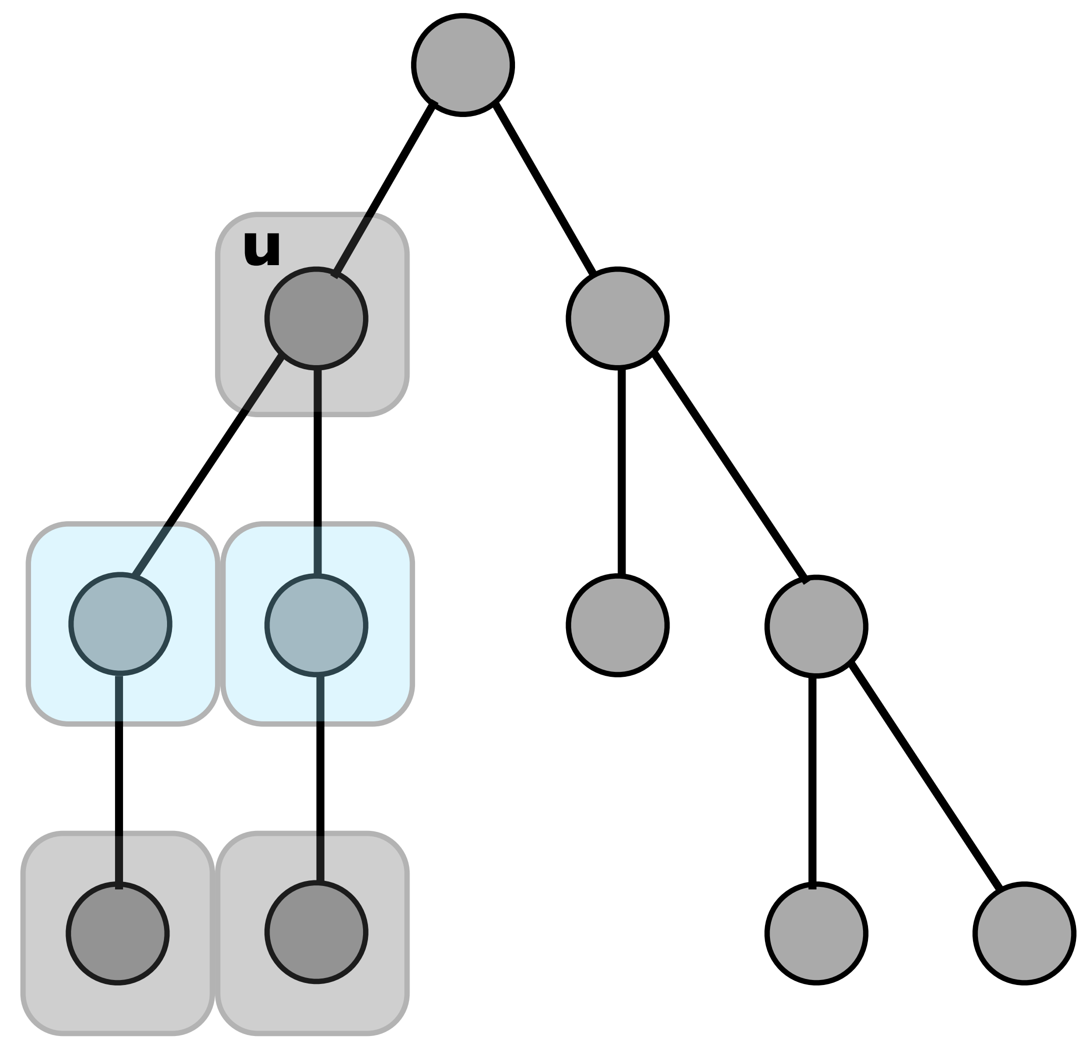
Independent Set: Given a graph G = (V, E), a subset of vertices S is an independent set if there are no edges between them

Max Independent Set Problem: Given a graph G = (V, E), find the largest independent set in G

Max Independent Set is a notoriously hard problem!
We will look at a restricted case, when G is a **tree**

Max. Independent Set in a Tree

A set of nodes is an **independent set** if there are no edges between the nodes



Two Cases at node u:

1. Don't include u
2. Include u, and don't include its children

Max. Independent Set in a Tree

A set of nodes is an **independent set** if there are no edges between the nodes

STEP 1: Define subtask

I(u) = size of largest independent set in subtree rooted at u

We want I(r), where r = root

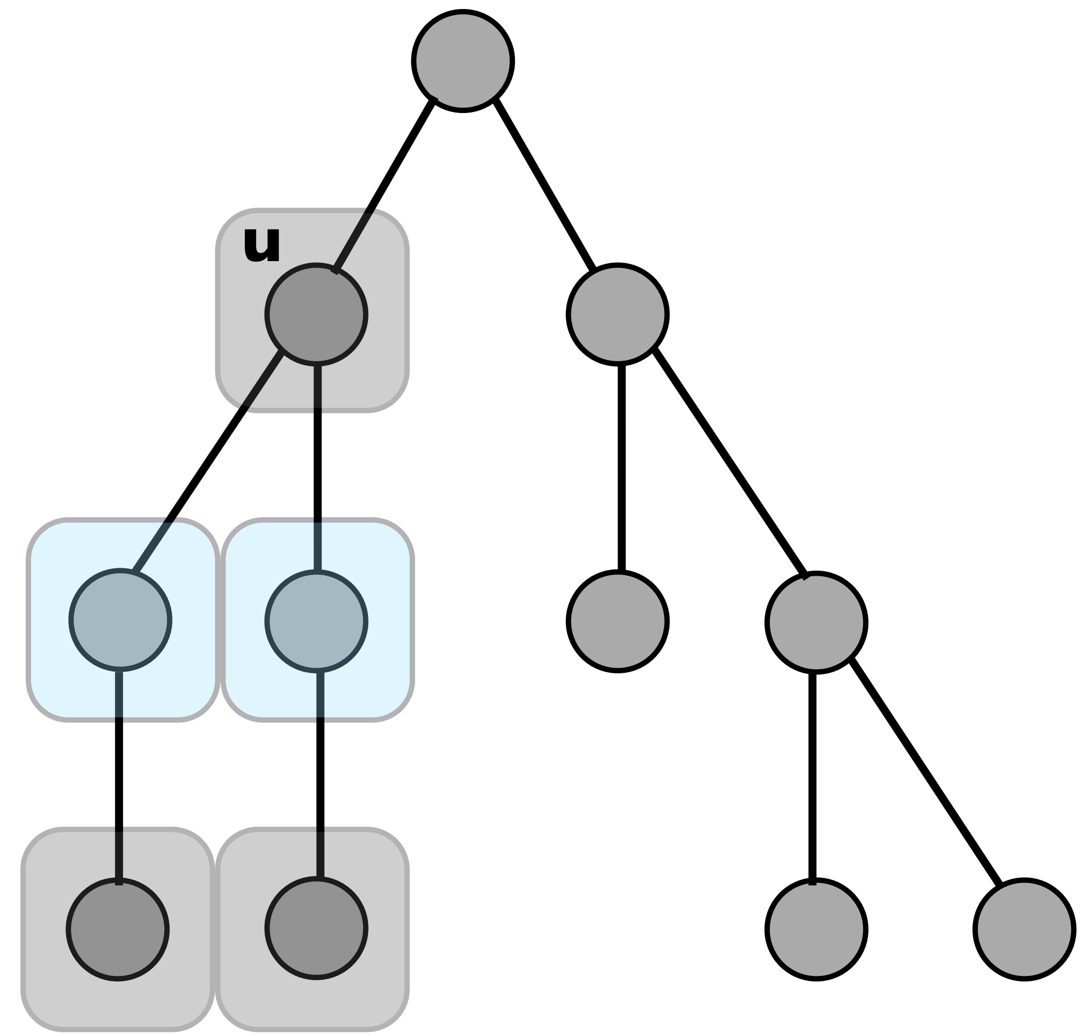
STEP 2: Express recursively

$$I(u) = \max \begin{cases} \sum_{\substack{\text{children} \\ w \text{ of } u}} I(w) \\ 1 + \sum_{\substack{\text{grandchildren} \\ w \text{ of } u}} I(w) \end{cases}$$

Base case: for leaf nodes, I(u) = 1

STEP 3: Order of subtasks

Reverse order of distance from root; use BFS!



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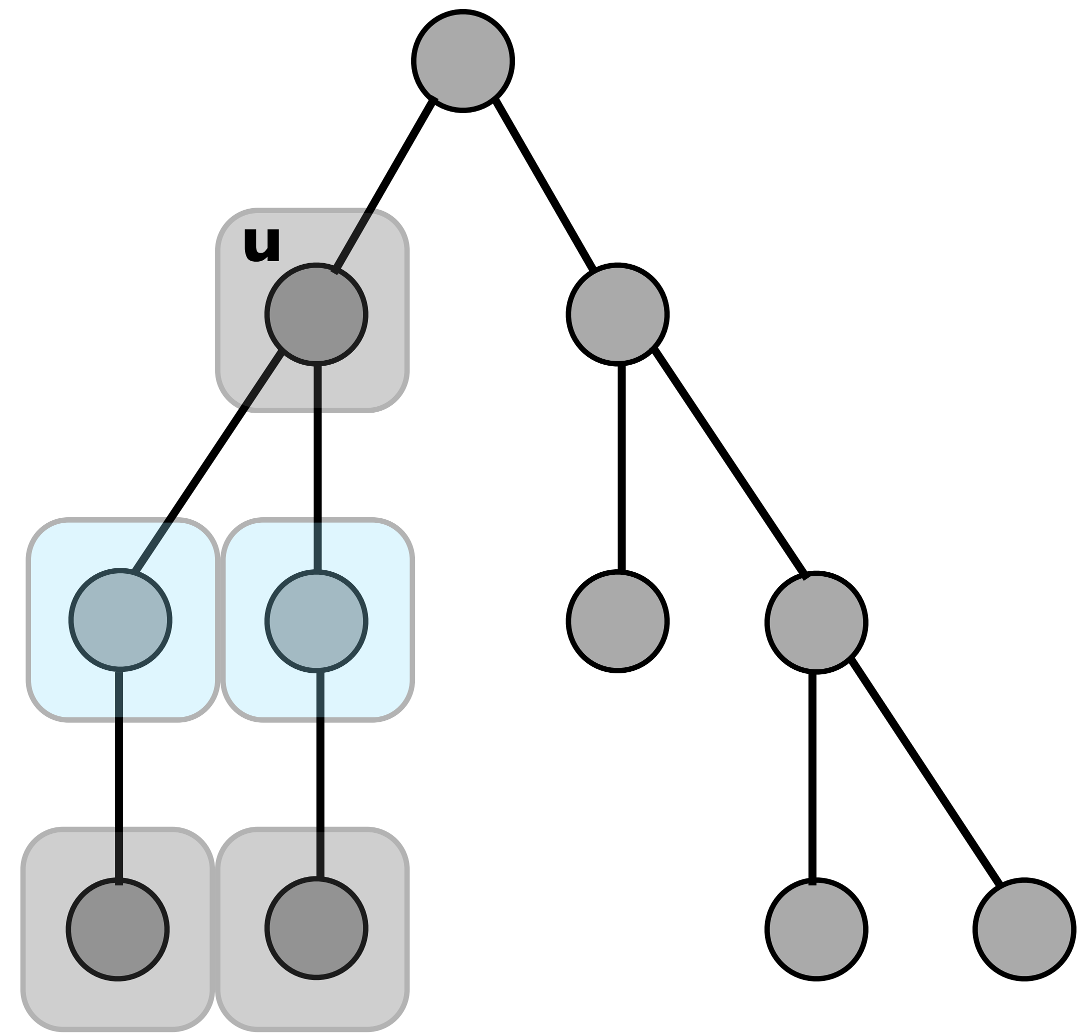

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Base case: for leaf nodes, I(u) = 1

STEP 3: Order of subtasks

Reverse order of distance from root; use BFS!

Running Time: O(n)

Edge (u, v) is examined in Step 2 at most twice:

(1) v is a child of u

(2) v is a grandchild of u's parent

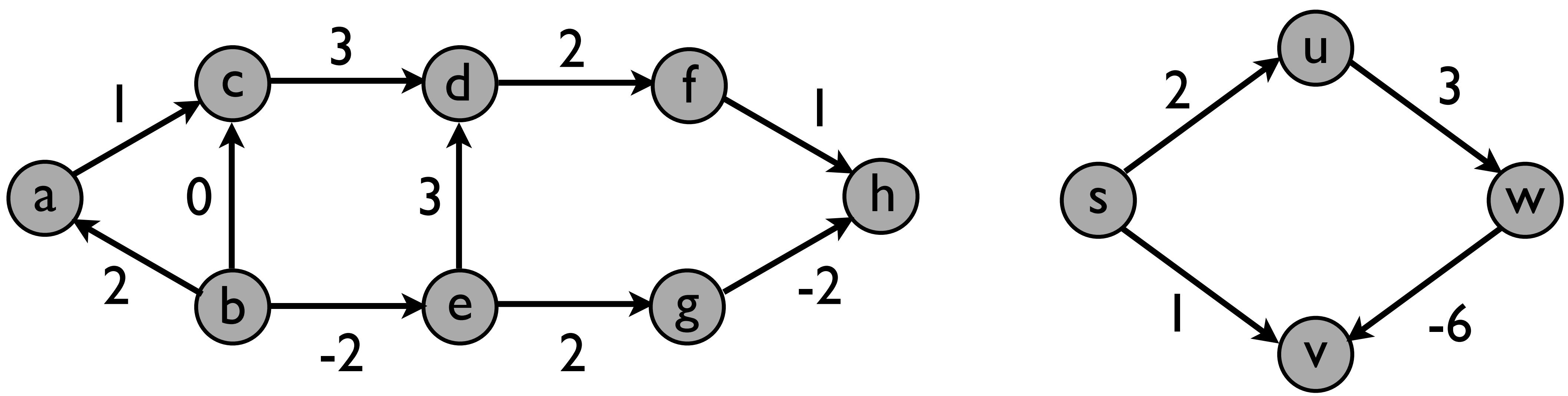
There are n-1 edges in a tree on n nodes

Dynamic Programming

- String Reconstruction
- Longest Common Subsequence
- Edit Distance
- Subset Sum
- Independent Set in a Tree
- All Pairs Shortest Paths

All Pairs Shortest Paths

Problem: Given n nodes and distances d_{ij} (which could be negative, or 0, or positive) on all edges, find shortest path distances between all pairs of nodes.



Does Dijkstra's algorithm work?

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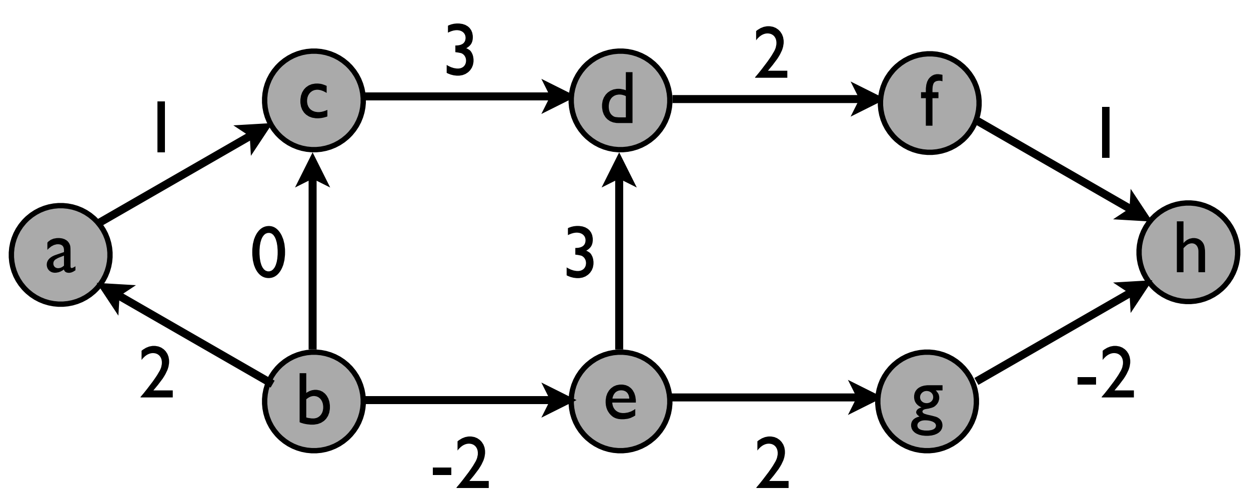
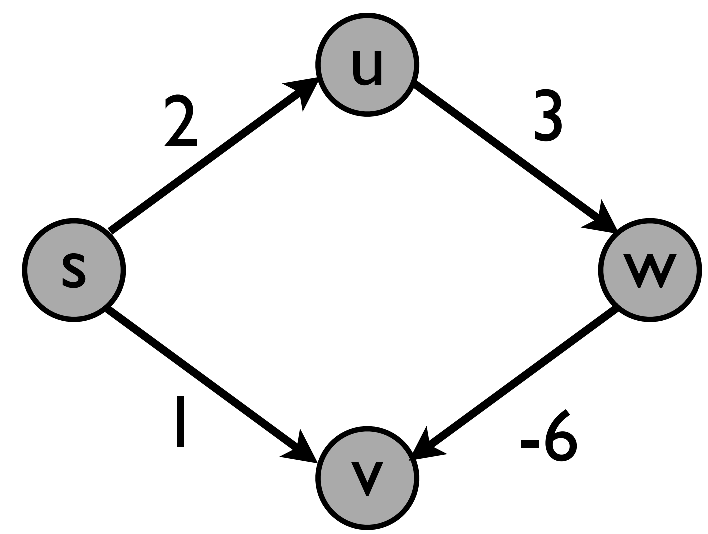




Does Dijkstra's algorithm work?

Ans: No! Example: s-v Shortest Paths

All Pairs Shortest Paths (APSP)

Problem: Given n nodes and distances d_{ij} (which could be negative, or 0, or positive) on all edges, find shortest path distances between all pairs of nodes.

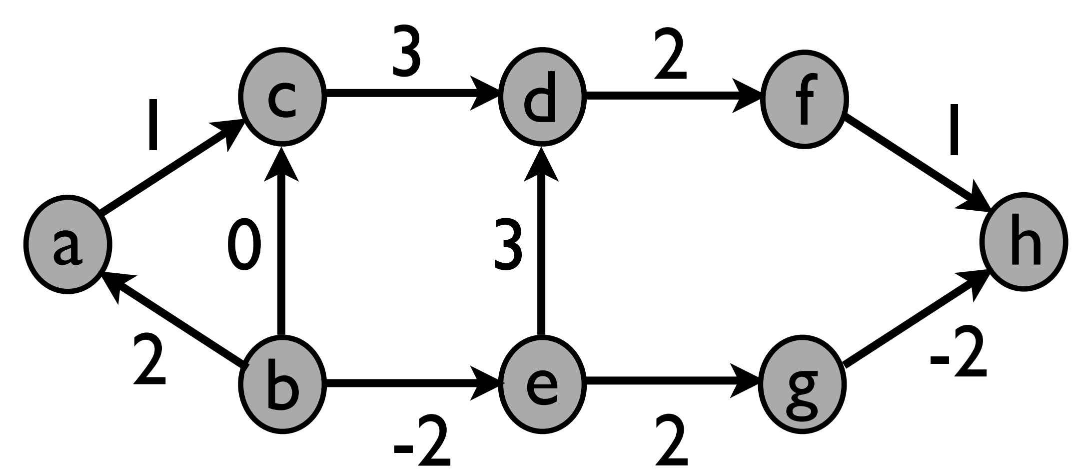
Structure:

For all x, y:

either $SP(x, y) = d_{xy}$

Or there exists some z s.t

$SP(x, y) = SP(x, z) + SP(y, z)$



All Pairs Shortest Paths (APSP)

Problem: Given n nodes and distances d_{ij} (which could be negative, or 0, or positive) on all edges, find shortest path distances between all pairs of nodes.

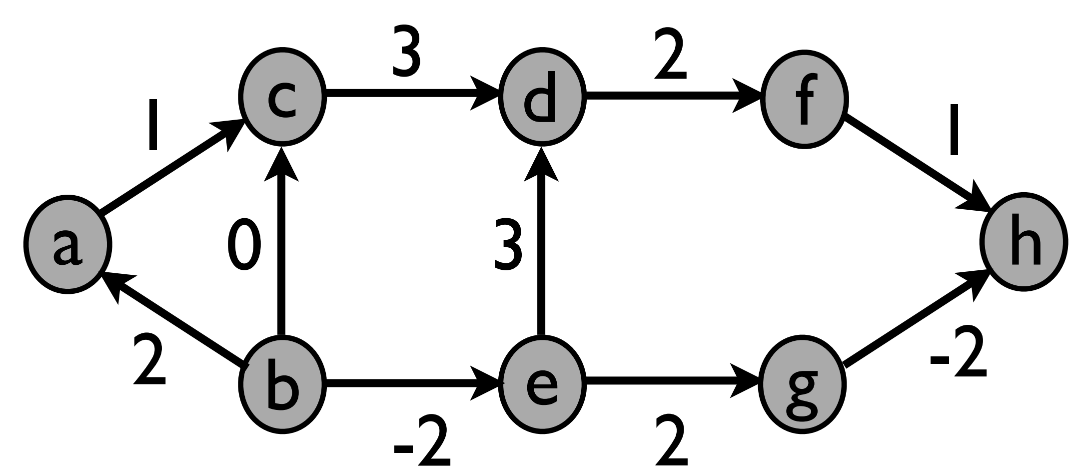
Structure:

For all x, y:

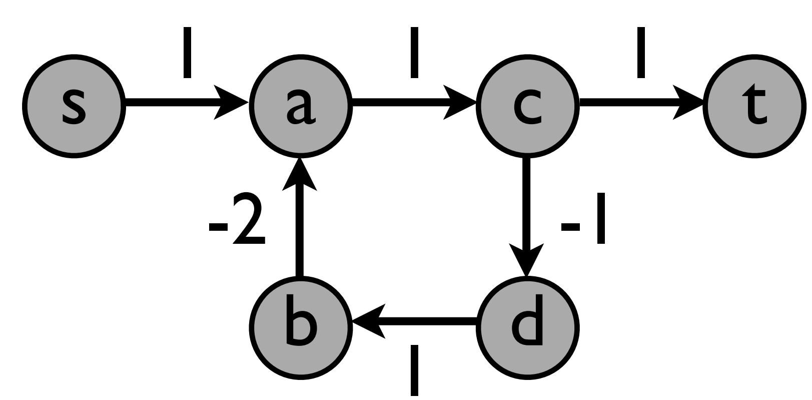
either $SP(x, y) = d_{xy}$

Or there exists some z s.t

$SP(x, y) = SP(x, z) + SP(y, z)$



Property: If there is no negative weight cycle, then for all x, y, SP(x, y) is simple (that is, includes no cycles)



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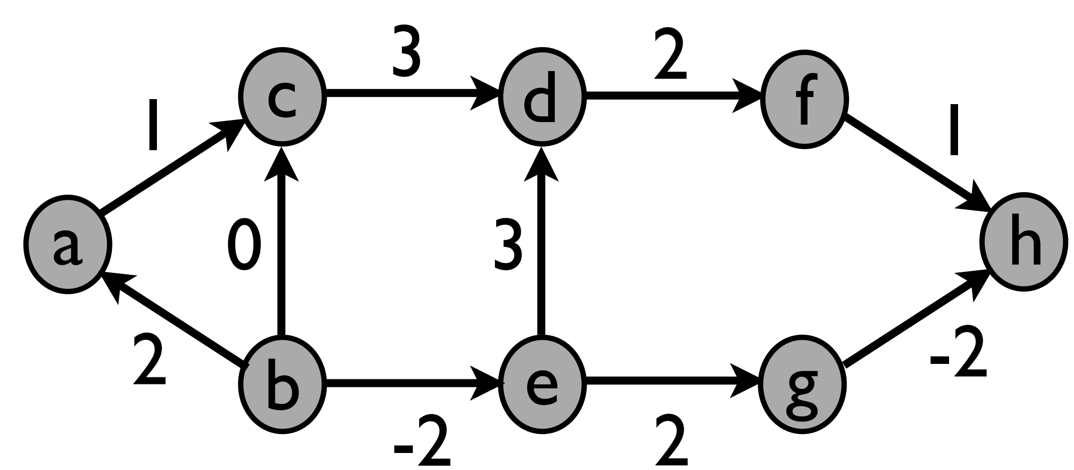
STEP 1: Define Subtasks

D(i,j,k) = length of shortest path from i to j with intermediate nodes in {1,2,...k}



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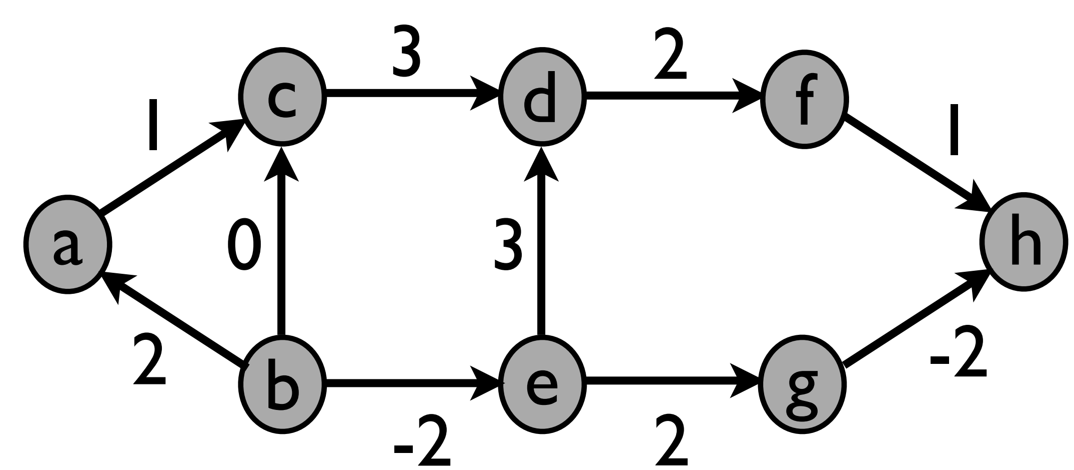

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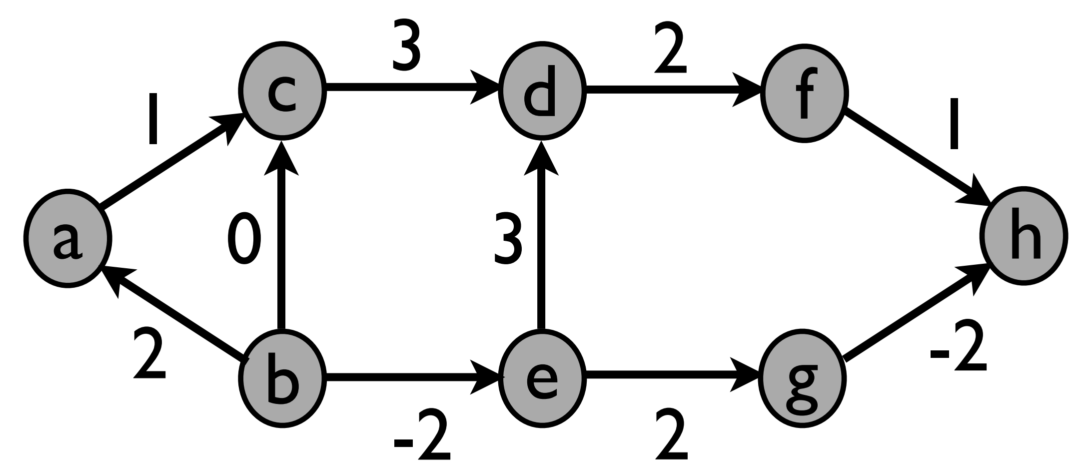

STEP 2: Express Recursively

D(i,j,k) = min{D(i,j,k-1), D(i,k,k-1) + D(k,j,k-1)}

Base case: D(i,j,0) = d_{ij}

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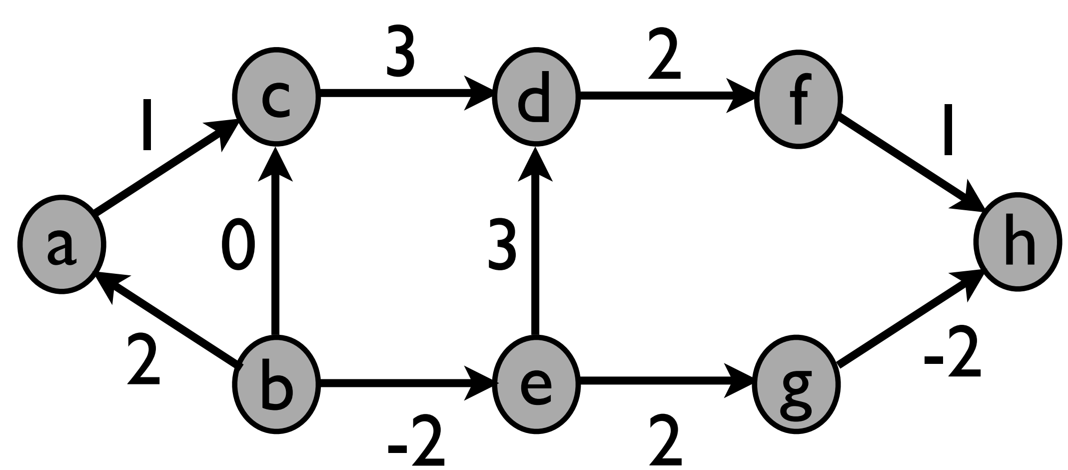

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By increasing order of k

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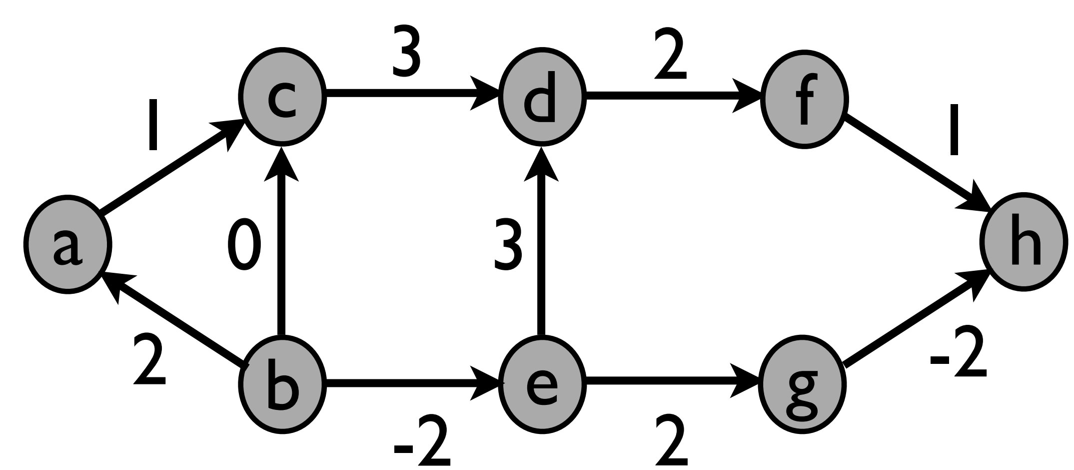

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STEP 3: Order of Subtasks

By increasing order of k

Running Time = $O(n^3)$

Exercise:

Reconstruct the shortest paths

Summary: Dynamic Programming

Main Steps:

1. Divide the problem into **subtasks**

2. Define the subtasks **recursively** (express larger subtasks in terms of smaller ones)

3. Find the **right order** for solving the subtasks (but do not solve them recursively!)

Summary: Dynamic Programming vs Divide and Conquer

Divide-and-conquer

A problem of size n is decomposed into a few subproblems which are significantly smaller (e.g. n/2, 3n/4,...)

Therefore, size of subproblems decreases geometrically.

eg. n, n/2, n/4, n/8, etc

Use a recursive algorithm.

Dynamic programming

A problem of size n is expressed in terms of subproblems that are not much smaller (e.g. n-1, n-2,...)

A recursive algorithm would take exp. time.

Saving grace: in total, there are only polynomially many subproblems.

Avoid recursion and instead solve the subproblems one-by-one, saving the answers in a table, in a clever explicit order.

Summary: Common Subtasks in DP

Case 1: Input: $x_1, x_2,..,x_n$ Subproblem: $x_1, .., x_i$.

$$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \; x_7 \quad x_8 \quad x_9 \quad x_{10}$$

Case 2: Input: $x_1, x_2,..,x_n$ and $y_1, y_2,..,y_m$ Subproblem: $x_1, .., x_i$ and $y_1, y_2,..,y_j$

$$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \; x_7 \quad x_8 \quad x_9 \quad x_{10}$$

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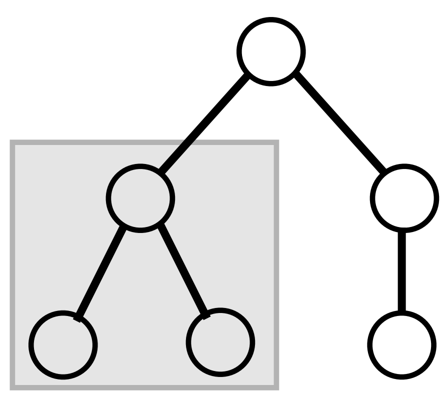
Case 3: Input: $x_1, x_2,..,x_n$. Subproblem: $x_i, .., x_j$

$$x_1 \quad x_2 \; \boxed{x_3 \quad x_4 \quad x_5 \quad x_6} \; x_7 \quad x_8 \quad x_9 \quad x_{10}$$

Case 4: Input: a rooted tree. Subproblem: a subtree



Next: Network Flow

Oil Through Pipelines

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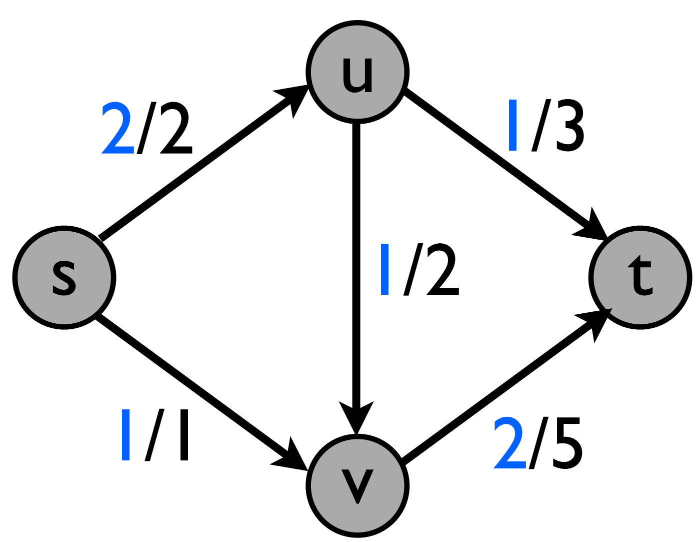
An s-t flow is a function: E $\longrightarrow$R such that:

- 0 <= f(e) <= c(e), for all edges e
- flow into node v = flow out of node v, for all nodes v except s and t,

$$\sum_{e\ into\ v} f(e) = \sum_{e\ out\ of\ v} f(e)$$

Size of flow f = Total flow out of s = total flow into t



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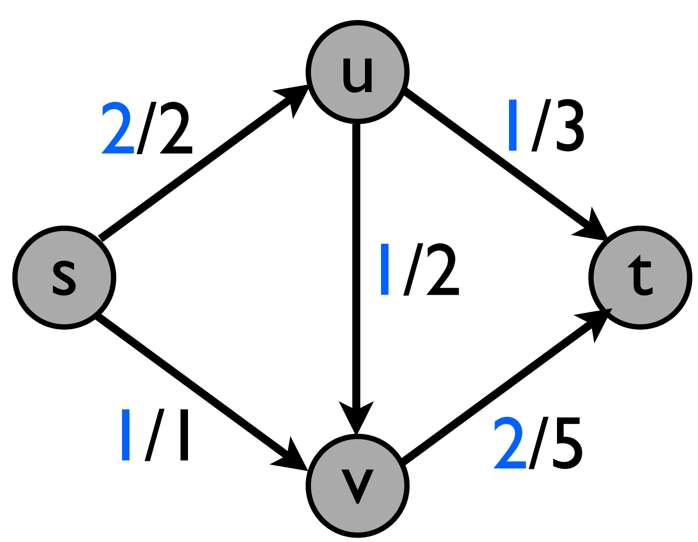

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Flows and Cuts

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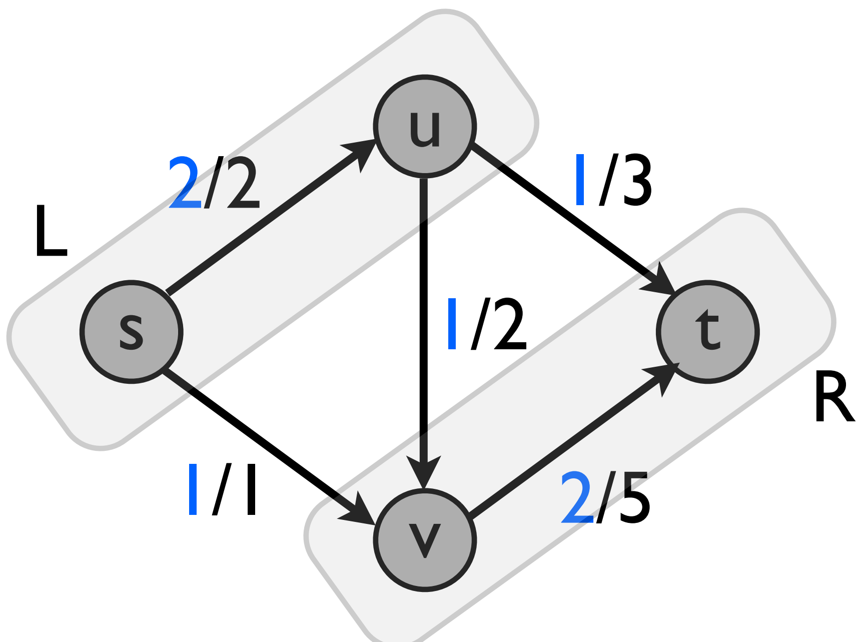

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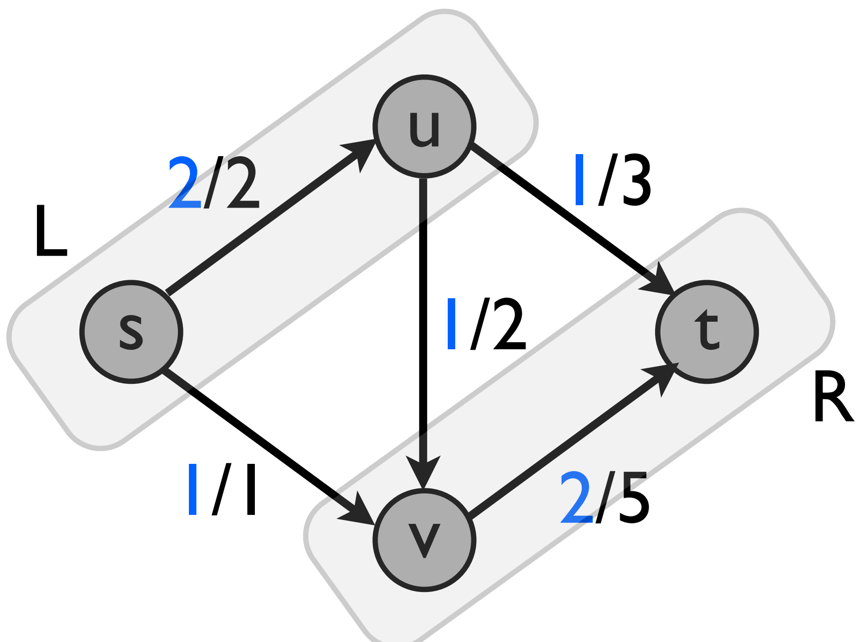

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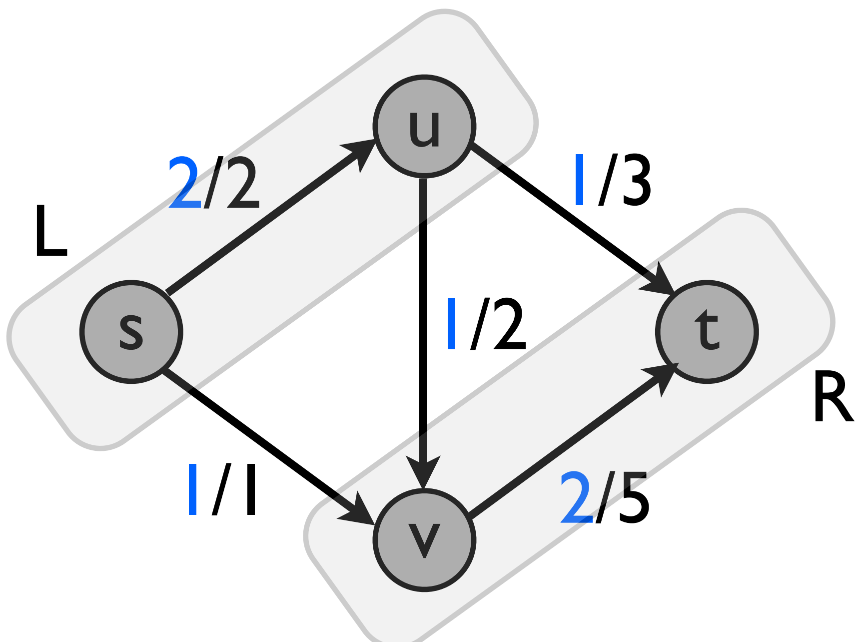

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Property: For any flow f, any s-t cut (L, R), size(f) <= capacity(L, R)