COMP 6661 Combinatorial Algorithms Winter 2023
Assignment 2
Submitted by: Rajat Sharma (40196467)

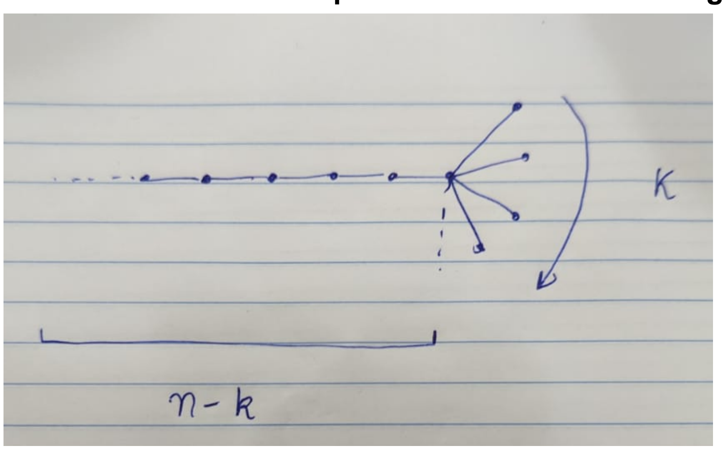
1)Given the following graphs:

- **The fork graph $F_{n,k}$, the graph containing a path with n - k vertices, where one of the leaves of the path is the center of a star graph with k leaves.**
- **The complete bipartite graph $K_{m,n}$ on m + n vertices.**
- **The n-vertex wheel W_n.**
- **The hypercube Q_n.**
- **The Spider graph: k paths of lengths p1 $\geq$ p2 … $\geq$ pk originated from a single vertex.**
- **Two-dimensional grid and torus (m by n).**

a) Give the broadcast time for each of the graphs above. Prove your answers.
b) For each graph indicate the set of the worst originators (vertices for which the broadcast time of the graph is achieved). Describe the broadcast center of each graph.

a)

- **The fork graph $F_{n,k}$, the graph containing a path with n - k vertices, where one of the leaves of the path is the center of a star graph with k leaves.**



Consider a fork graph as such:

The broadcast time of the graph will be when we choose the originator as the end vertex of the path (n-k) which is not the center of the star graph.

The time taken for the information to travel from that vertex to the center node is (n-k-1) as (n-k) was the number of vertices of the path graph.

And we need additional time k to inform all the k leaves of the star graph.

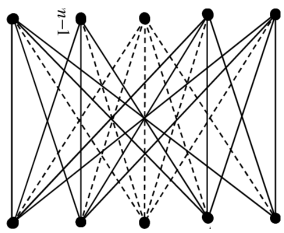
Hence, the total time taken is = n-k-1+k => n-1

$b(F_{n,k})$ = n-1 (Notice that it is independent of value k)

- **The complete bipartite graph $K_{m,n}$ on m + n vertices.**

Let us see for the case when m==n:

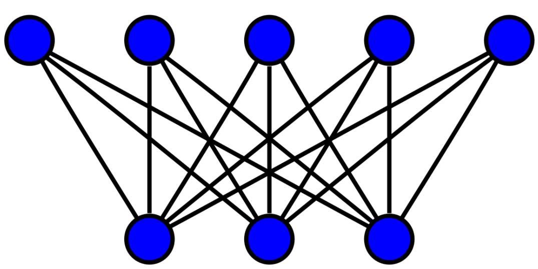
Here we have a graph that looks like this



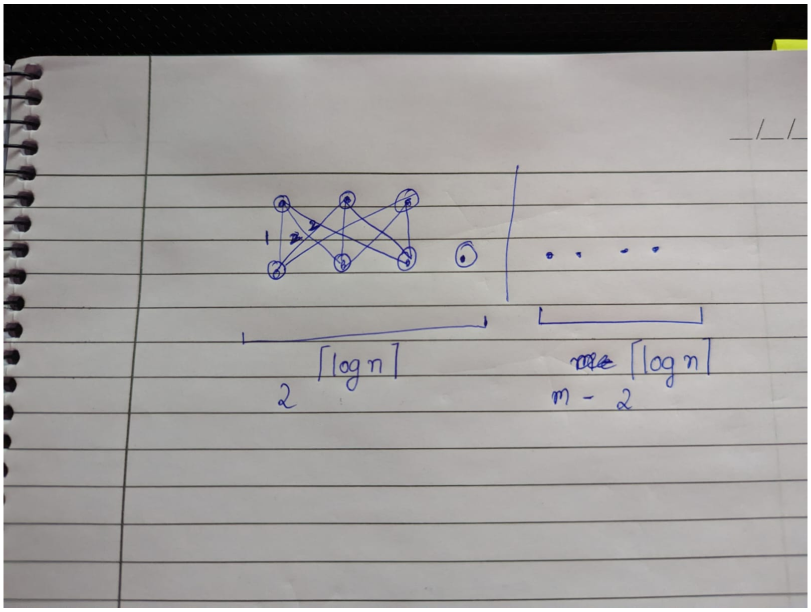
Number of nodes informed based on time
For Time0: 1
For Time1: 2
For Time2: 4 (Notice in this previously 2 were the informed nodes and only they can be the new broadcasters so the informed nodes will double in each step)
.
.
.
For Timek: 2^k

$b(K_{m,n})$ = ceil(log (n+m))
Since, n==m
$b(K_{m,n}) = ceil(\log_2 2n) = ceil(\log_2 n) +1$

Let us see for the case when m>n:



In the Example figure we can assume the time taken to inform $n+2^{ceil(\log n)}$ nodes will be **ceil(log n) + 1**



Now we have to calculate time to inform remaining m-$2^{ceil(log\,n)}$.

Since these nodes can only be informed by the upper n nodes the time can be given by.

ceil($\frac{m-2^{ceil(log\,n)}}{n}$)

Total time = b($K_{m,n}$) = ceil(log n) + 1 + ceil($\frac{m-2^{ceil(log\,n)}}{n}$)

Let us see for the case when n>m:

It should be the same as when m>n but with n and m interchanged as both the graphs are isomorphic.

- **The n-vertex wheel W_n.**

The n-vertex wheel, W_n :

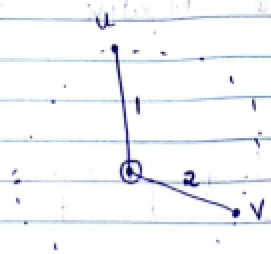
Consider a path graph arbitrary length.

centre node is broadcaster

At time t : $2t$ vertices are informed

0	1
1	2
2	4
3	6
t	$2t$

Now consider the wheel graph



Suppose the center vertex is the originator. And at time $t=1$, it informs u & v at time $t=2$.

So, using the formula above u can inform $2*(t-1)$ nodes. Since 1 unit of time has passed.
v can inform $2*(t-2)$ nodes.

To inform the first edge node → $1 + 2(t-1) + 2(t-2) \; \ldots \; 2(1) + 1$

remaining node informed at time t

$$\Rightarrow \quad 1+1+2\left(\frac{t(t-1)}{2}\right) \Rightarrow t^2 - t + 2 \geqslant n$$

As all the nodes are informed at time t

Also, $t^2 - t + 2 - n \leq b_G$

As this was the total time to inform all the nodes it should be less than the broadcast time of the graph.

Calculating roots of the eq^n

$$\frac{1 \pm \sqrt{1 + 4n - 8}}{2}$$

ignoring the negative root we have

$$\frac{1}{2} + \sqrt{n - \frac{7}{4}} \leq b_G$$

this is the broadcast time for the wheel graph

$$\frac{1}{2} + \sqrt{n - \frac{7}{4}} \leq b(W_n)$$

- **The hypercube Q_n.**

The hypercube Q_n is made by joining the vertices of two hypercubes Q_{n-1}, the broadcast time of $b(Q_n)$ should be $1+b(Q_{n-1})$.

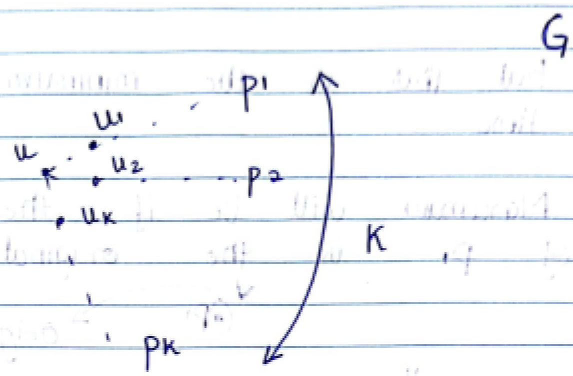
$b(Q_k)=1+b(Q_{k-1})$
$b(Q_k)=1+1+b(Q_{k-2})$
$b(Q_k)=1+1+....+b(Q_1)$
$b(Q_k)=(k-1)+1$ **(Since $b(Q_1)=1$)**

$b(Q_k)=k$ (Note that k is the dimension of hypercube and here $|V|=2^k$)

- **The Spider graph: k paths of lengths p1 $\geq$ p2 … $\geq$ pk originated from a single vertex.**

Spider Graph

k paths of length $p_1 \geq p_2 \dots \geq p_k$



Suppose u is the originator:

$$b(u,G) = \max_{1 \leq i \leq k} \{ b(u_i, G_i) + i \}$$

We know that p_1 is a path graph and for that broadcast time is given by

~~here~~ $b(u_1, G) = p_1 - 2$

$\vdots$

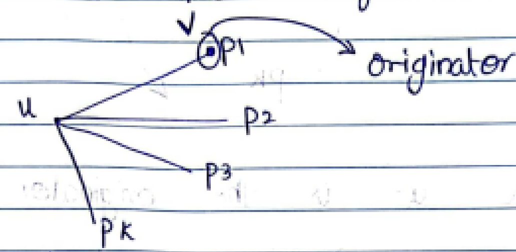
$b(u_k, G) = p_k - 2$

Since $p_1 \geq p_2 \dots \geq p_k$

$$b(u, G) = \max\{1 + P_1 - 2,\ 2 + P_2 - 2,\ \ldots\ldots,\ K + P_K - 2\}$$

But this is the minimum broadcast time

Maximum will be if the end vertex of p_1 is the originator.



In this case time to inform u will be $p_1 - 1$

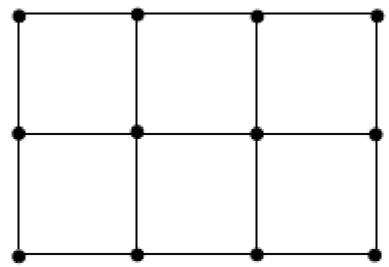
and now our problem simplifies to the one we discussed earlier

$$b(v, G) = p_1 - 1 + \max\{1 + p_2 - 2,\ 2 + p_3 - 2,\ \ldots,\ (K-1) + p_K - 2\}$$

$$b(v, G) = p_1 - 1 + \max\{p_2 - 1,\ p_3,\ p_4 + 1\ \ldots\ (K-3) + p_K\}$$

$S_{k,p} = p_1 - 1 + \max\{ p_2 - 1, p_3, p_4 + 1, \ldots, p_k + (k-3)\}$

- **Two-dimensional grid and torus (m by n)**



2d grid $G_{m,n}$

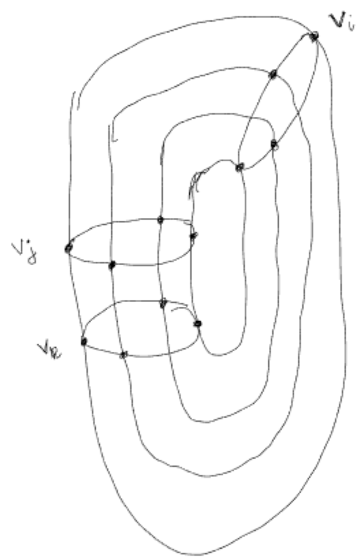
The broadcast time for the grid can be given by (m-1)+(n-1) => m+n-2.
This is the diametrical distance between the top left node and the bottom right node.
So if the originator is a corner vertex **$b(G_{m,n})$ =** m+n-2

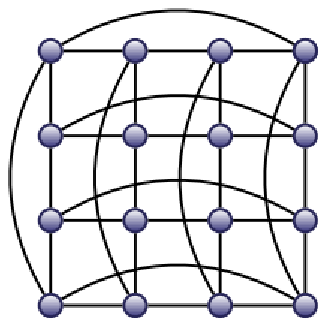
If the broadcaster is some side node in the middle of the graph then the time is taken by the message to reach the top left and bottom right we will calculate the manhattan distance from that point k to xt,yt and xb,yb where xt,yt is the top left corner and xb,yb is the bottom right corner of the grid and the answer will be maximum of both.
So if the originator is a side vertex **$b(G_{m,n})$ =** maximum distance from originator to corner vertex

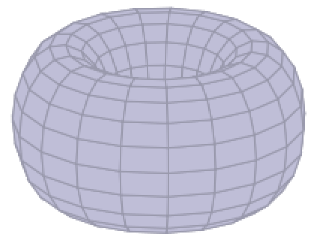
If the broadcaster is some interior vertex not in the corner or on the side then assume v=(i,j) then,

$b(G_{m,n})$ = maximum distance from v to corner vertex

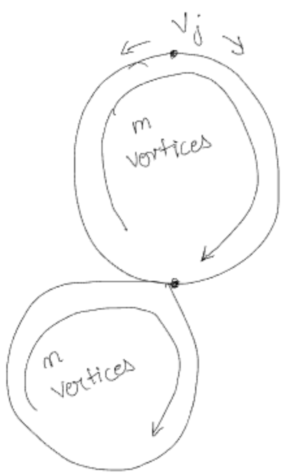
- Plus 1 if j = $\frac{n+1}{2}$
- Plus 2 if j = $\frac{n+1}{2}$ and i = $\frac{m+1}{2}$







Consider this torus grid graph, in this vi, vj, vk have the same broadcast time. Assume WLOG and the originator vertex to be vj, it can be clearly seen that if vertex vj is the originator the time taken to inform the vertex just opposite to vj is ceil(m/2) if vj was part of the torus having m rings. Also, the vertex opposite to vj is part of the torus having n rings the time taken by that vertex to inform all the other vertex in that ring is ceil(n/2)



So the total broadcast time will be equal to

$$b(T_{n,m}) = \begin{cases} \text{floor}(m/2)+\text{floor}(n/2) & \text{if } n, m \text{ both are even} \\ \text{floor}(m/2)+\text{floor}(n/2)+1 & \text{otherwise} \end{cases}$$

b)

For $F_{n,k}$
The **worst originator** is a leaf node of the path graph with length (n-k) which is not connected to the leaves of the star graph.

The best originator(**broadcast center**) is the center node which is the leaf node of the path graph and is the central node for the star graph and also all the k leaf nodes of the star graph so we have a total of k+1 broadcast centers.

For $K_{m,n}$
All the vertices have the same broadcast time.

For W_n
All the vertices have the same broadcast time.

For Q_n
All the vertices have the same broadcast time.

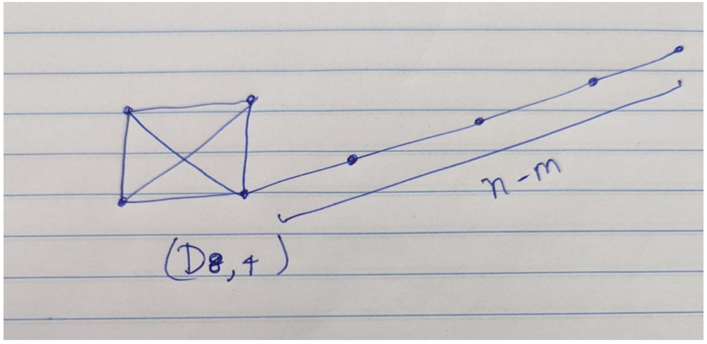
For $S_{k,p}$
The **worst originator** is the leaf node(not the center node) of the path graph with length p1 which is the longest path length.

The best originator(**broadcast center**) is the center node to which all the k-path graphs are connected. Also if any of the p values is equal to 1 that can also act as the broadcast center.

For $G_{m,n}$
All the vertices have the same broadcast time.

For $T_{m,n}$
All the vertices have the same broadcast time.

2. Find the broadcast time of the dipper graph $D_{n,m}$, the graph that has n vertices and $\frac{m(m-1)}{2}$ + n - m edges containing a complete graph K_m and a path of length n - m originated from a vertex of K_m. Indicate the worst originator(s). Also indicate the values of m (as a function of n) that maximizes and minimizes $b(D_{n,m})$ (as a function of n).



Consider this graph,

We know that the time taken by the path graph is n-1 (where n is the number of vertices). In this case also if we take the originator to be **the leaf node of the path graph which is not connected to the complete graph K_m that should be the worst originator** as the message will take the maximum time to reach the complete graph. In any other case, the message will be closer to the vertex of the complete graph which is ready to broadcast the message to the whole complete graph.

Now we can calculate the $b(D_{n,m})$

This can be given by the time taken to travel the length of the path graph plus the time taken for the message to be broadcasted to the whole of the complete graph.

The time taken to travel the path graph will be n-m i.e. length of the path graph, and then it will take additional $\text{ceil}(\log_2 m)$ time units as that is the broadcast time for the complete graph K_m

So the total time taken is **$n-m+\text{ceil}(\log_2 m)$.**

$b(D_{n,m}) = n-m+\text{ceil}(\log_2 m)$

The leaf node of the path graph which is not connected to the complete graph K_m is the worst originator.

Now, to maximize $b(D_{n,m}) = n-m+\text{ceil}(\log_2 m)$ using m.

We can observe that n-m is linear and **if m increases $b(D_{n,m})$ should decrease** the function $\text{ceil}(\log_2 m)$ should not have that much effect for a very large value of m. The vice versa is also true.

This means that the maximum value of $b(D_{n,m})$ is achieved when m is as small as possible (also only integral values of m can be taken), i.e., for the value of m=[1,2,3], $ceil(\log_2 m)-m$ evaluates to be -1 which is the maximum value this function can take.

Similarly, we can observe that the minimum value of $b(D_{n,m})$ is achieved when m is as large as possible, i.e., when m = n.

Therefore, as a function of n, the values of m that maximize and minimize $b(D_{n,m})$ are m = 1, 2, 3, and m = n, respectively.

$$\mathbf{b(D_{n,m}) =} \begin{cases} n-1 & m=1,2,3 \quad \text{(maximum)} \\ ceil(\log_2 n) & m=n \quad \text{(minimum)} \end{cases}$$