A Note on Orientations of Mixed Graphs*

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Abstract

We consider orientation problems on mixed graphs in which the goal is to obtain a directed graph satisfying certain connectivity requirements.

Keywords: Mixed graphs, orientations, NP-complete.

1 Introduction

Let $G = (V, E, A)$ be a mixed graph with a set of vertices V, a set of (undirected) edges E and a set of (directed) arcs A. For vertices s and t, an $s - t$ path is a sequence $s = v_0, a_1, v_1, a_2, v_2, ..., a_k, v_k = t$ such that for $i = 1, ..., k$ $v_i \in V$, a_i is either an edge $a_i = \{v_{i-1}, v_i\} \in E$ or the arc $a_i = (v_{i-1}, v_i) \in A$. By *orienting* an edge $e = \{v_i, v_j\} \in E$ we mean replacing e by exactly one of the two arcs (v_i, v_j) or (v_j, v_i). An *orientation* of G is an orientation of all the edges in E. In this paper we refer by 'disjoint paths' to 'edge/arc internally disjoint paths'.

This paper considers several orientation problems on mixed graphs. The objective is to obtain a directed graph satisfying certain connectivity requirements. We begin, in Section 2, with *pair connectivity* problems, in which a list of pairs of vertices is given, and we require the resulting directed graph to have a directed path between each pair of them. This problem is polynomially solvable for undirected graphs [4], however, we prove that it is NP-complete for mixed graphs. In the case of two pairs of vertices we give a polynomial time algorithm based on a set of necessary and sufficient conditions. In Section 3 we consider higher connectivity requirements between pairs of vertices and show that if k-connectivity is required between one pair and n-connectivity between the other pair, then the problem is NP-complete. The problem remains

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NP-complete even if $n = 1$. However, we show that this problem is polynomially solvable if $n = 1$ and the graph is undirected.

Throughout we mention several natural generalizations to our results, and show that they are false. Our concluding section contains a list of open problems.

Previous work on orientations that satisfy connectivity requirements focuses on global connectivity. A mixed graph is said to have a k-*orientation* if its edges can be oriented so that the resulting digraph is k-connected. Nash Williams [6] gave a necessary and sufficient condition for an undirected graph to have a k-orientation. In [3], Frank showed that the problem of deciding whether a mixed graph has a k-orientation is polynomially solvable, by formulating the problem as a submodular flow problem. Jackson [5] gave a sufficient condition for mixed graphs to have a k-orientation. Boesch and Tindell [1] provide a necessary and sufficient condition for a mixed graph to have a 1-orientation.

2 Pair connectivity

Given a mixed graph G and a collection $P = \{(s_j, t_j) \in V \times V \mid j = 1, ..., m\}$, we say that G is P-*connected* if it contains an $s_j - t_j$ path for $j = 1, ..., m$. G has a P-*orientation* if the edges in E can be oriented so that the resulting digraph is P-connected.

Theorem 2.1 *The problem of deciding whether a mixed graph G has a P-orientation is NP-complete.*

Proof: We reduce the Satisfiability problem (SAT) to the P-orientation problem. Given clauses $C_1, C_2, ...C_m$, each consisting of literals among the variables $x_1, x_2, ...x_n$ we construct a graph G as follows: Each variable x_i is represented by an edge $\{u_i, v_i\}$. Each clause C_j consists of a pair of vertices s_j, t_j and two arcs for each literal in the clause: If $x_i \in C_j$ we have arcs (s_j, u_i) and (v_i, t_j). If $\bar{x}_i \in C_j$ we have the arcs (s_j, v_i) and (u_i, t_j). Clearly this construction is polynomial in the size of the SAT problem.

We now show that a formula is satisfiable if and only if the mixed graph G has a P-orientation. Given a truth setting of variables that satisfies the formula, we orient the edges corresponding to true variables from u_i to v_i, and edges corresponding to false variables from v_i to u_i. Since each clause has at least one true literal, this ensures that the resulting directed graph has a path from s_j to t_j for each j, and thus G has a P-orientation. Conversely, given that the graph G has a P-orientation, we set variables to be true (false) if their corresponding edge is oriented from u_i to v_i (from v_i to u_i). The setting of each variable is uniquely determined, given the orientation. Furthermore, since there exists a path from s_j to t_j for each j, this implies that each clause contains at least one true literal. ∎

An obvious necessary and sufficient condition for the existence of a P-orientation for an undirected graph with $|P| \geq 2$ is that there is no cut (X, Y) consisting of a single edge such that for some $i \neq j$ $s_i, t_j \in X$ and $s_j, t_i \in Y$. We call such an edge a *P-bridge.* The problem of deciding whether an undirected graph has a P-orientation can be solved in $O(|P||E|)$ time [4].

For a mixed graph, the condition given above is not sufficient, as shown by Figure 1 with $P = \{(s, t), (t, s)\}$. We modify the concept of a P-bridge as follows: An edge $\{v_i, v_j\} \in E$ is *P-essential* if there is no orientation of it that preserves P-connectedness. In other words, it is essential to keep it undirected. The edge marked e in Figure 1 is P-essential for $P = \{(s, t), (t, s)\}$, but it is not a P-bridge in the underlying undirected graph.

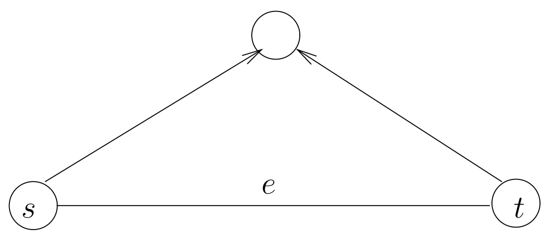

Figure 1: An essential edge e

In the next theorem we consider the case $|P| = 2$.

Theorem 2.2 *A mixed graph $G = (V, E, A)$ has a P-orientation, $P = \{(s_j, t_j) \mid j = 1, 2\}$, if and only if (i) G is P-connected, and (ii) it has no P-essential edges.*

Proof: The conditions are clearly necessary. We will prove that they are also sufficient. The conditions are clearly also sufficient when $E = \emptyset$ so we will assume that $E \neq \emptyset$. By (ii), for each $f \in E$ there is an orientation of f that preserves P-connectivity. We consider two cases. In the first, there is an edge f for which there is an imperative orientation, i.e., orienting the edge otherwise will not preserve (i). In the second case no such edge exists and we let f be an arbitrary edge from E. We will show that in both cases we can orient f so that the two conditions are maintained. The theorem follows by induction on $|E|$.

Case 1. Suppose that the pair s_1, t_1 induces an imperative orientation (w, z) on $f = \{w, z\}$. We fix this imperative orientation. It follows that (i) is preserved. We will show that (ii) is also preserved. We know that every $s_1 - t_1$ path uses f in the chosen orientation. Suppose, by way of contradiction, that there exists an edge $e = \{u, v\}$ that becomes P-essential after the orientation of f is fixed. Suppose that the orientation of e that disconnects all $s_1 - t_1$ paths is (u, v). We conclude that every $s_1 - t_1$ path uses both f in the orientation (w, z) and e in the orientation (v, u). In other words, the pair s_1, t_1 induces an imperative orientation (v, u) on e in G. Furthermore, all $s_1 - t_1$ paths use f and e in the same order since otherwise we can find an $s_1 - t_1$ path that does not use both f and e. Without loss of generality we assume that f is visited first. In fact, there is no path that uses f in the chosen orientation and e in the orientation (u, v) because then we could find an $s_1 - t_1$ path that does not use e.

Now consider $s_2 - t_2$ paths. They must use either f in the orientation (z, w) or e in the orientation (u, v). Since none of these edges was essential, there must be $s_2 - t_2$ paths that uses e but not f in these orientations, and vice versa. By combining two such paths with an $s_1 - t_1$ path we get a path from s_2 to z, to v, to t_2, avoiding both e and f, a contradiction.

Case 2 (no edge has an imperative orientation). We will show that there exists an orientation of f which preserves (ii). Suppose by way of contradiction that one orientation of f, which we denote *positive*, creates an essential edge e, and the other orientation of f denoted *negative* creates an essential edge g.

Every $s_1 - t_1$ path uses either f in its negative orientation or e in some fixed orientation which we denote as negative. Similarly, every $s_1 - t_1$ path uses either f in its positive orientation or g in some fixed orientation which we denote as negative. e has no imperative orientation, and therefore there is at least one path, P_1, that uses f but not e in their negative orientations. Since g has no imperative orientation, there exists an $s_1 - t_1$ path, P_2, that does not use g in its negative orientation. P_2 must use f in its positive orientation. Therefore it cannot use f in its negative orientation and hence it uses e in its negative orientation. P_2 either does not use e in its negative orientation in its part between s_1 and f, or it does not use it between f and t_1. In both cases, by combining parts of P_1 and P_2 one can form an $s_1 - t_1$ path that does not pass through neither e nor f, contradiction the assumption that e became essential. ∎

We observe that the conditions of Theorem 2.2 are not sufficient when $m > 2$. In particular, in Figure 2, in which $m = 3$, the graph is P-connected and has no P-essential edges, but it does not have a P-orientation.

This example leads us to define a P-*essential pair* of edges as a pair of edges such that none of its four possible orientations is P-connected. The following is a natural conjecture: A mixed graph $G = (V, E, A)$ has a P-orientation, $P = \{(s_j, t_j)\ j = 1, 2, 3\}$, if and only if (i) G is P-connected, and (ii) it has no P-essential pair of edges. These conditions are clearly necessary, but as Figure 3 shows, they are not sufficient.

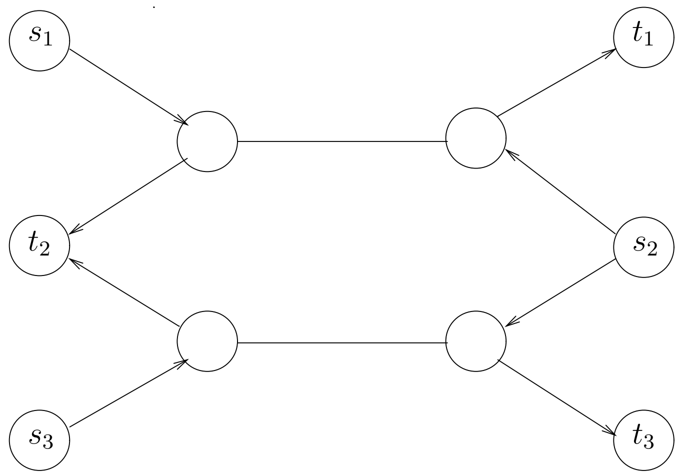

Figure 2: A graph with no P-orientation and no essential edge

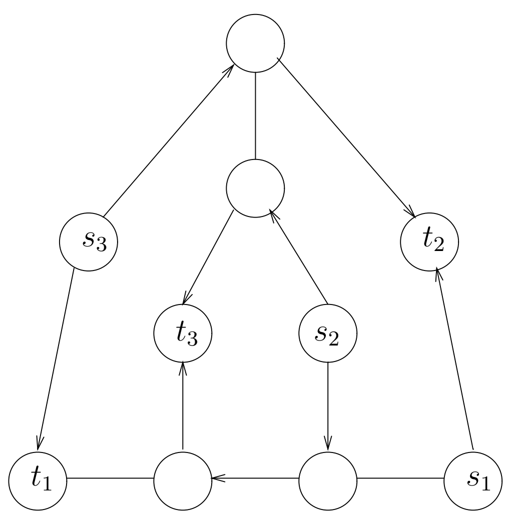

Figure 3: A graph with no P-orientation and no essential pair of edges

3 Higher pair-connectivity

Next we consider higher connectivity requirements between 2 pairs of nodes s_1, t_1 and s_2, t_2. Specifically, we require that the resulting directed graph contain n disjoint $s_1 - t_1$ paths and k disjoint $s_2 - t_2$ paths. Note that the paths from s_1 to t_1 need not be disjoint from the paths from s_2 to t_2. We have shown that the case $k = n = 1$ is polynomially solvable (Theorem 2.2). If we require that all $k + n$ paths be disjoint, the problem is hard even for $k = n = 1$ and $E = \emptyset$ by a result of Fortune Hopcroft and Wyllie [2].

Theorem 3.1 *Given a mixed graph $G = (V, A, E)$, nodes s_1, t_1, s_2, t_2, and integers k and n, the problem of deciding whether there is an orientation of G containing n $s_1 - t_1$ disjoint paths and k $s_2 - t_2$ disjoint paths is NP-complete.*

Proof: We reduce the Satisfiability problem (SAT) to the above problem. We set k to be the number of clauses and n to be the number of variables in a given instance of SAT. Each variable x_i is represented by $4k+2$ nodes u_i, v_i, l^j_{1i}, l^j_{2i}, r^j_{1i}, and r^j_{2i}, $j = 1, \ldots, k$, edges $\{l^j_{1i}, l^j_{2i}\}$, $\{l^j_{2i}, l^{j+1}_{1i}\}$, and $\{r^j_{1i}, r^j_{2i}\}$, $\{r^j_{2i}, r^{j+1}_{1i}\}$, and arcs (s_1, u_i), (u_i, l^1_{1i}), (u_i, r^1_{1i}), (l^k_{2i}, v_i), (r^k_{2i}, v_i), and (v_i, t_1). Intuitively, nodes l^j_{1i}, l^j_{2i}, form the "left chain", and nodes r^j_{1i}, r^j_{2i} form the "right chain". No other arcs or edges involve s_1 or t_1, therefore n disjoint paths from s_1 to t_1 must be of the following form: For each variable i one of the following 2 paths is used, either s_1, u_i, left chain, v_i, t_1, or s_1, u_i, right chain, v_i, t_1. We intuitively think of the first path as corresponding to a variable x_i being false, and the second path as x_i being true. A clause C_j with t literals is represented by $2(t+1)$ nodes: w_j, z_j and $2t$ nodes which are in the variable gadgets, depending on the literals in the clause. If x_i is a literal in C_j, we consider the nodes l^j_{1i} and l^j_{2i} to also be part of the clause gadget, as well as arcs (w_j, l^j_{2i}), (l^j_{1i}, z_j). The edge $\{l^j_{1i}, l^j_{2i}\}$ which is part of the variable gadget is also considered part of the clause gadget. If $\bar{x}_i$ is a literal in C_j the construction is the same, except that we use nodes r^j_{1i}, r^j_{2i} instead of l^j_{1i}, l^j_{2i}. Finally, we have for each clause C_j the arcs (s_2, w_j) and (z_j, t_2). This completes the construction. Note that in order to obtain k disjoint paths from s_2 to t_2, each of the paths must pass through exactly one clause gadget (recall k is the number of clauses). Given a satisfying truth assignment, we obtain the desired paths by orienting the edges as follows: if a variable x_i is true, orient (r^j_{1i}, r^j_{2i}), (r^j_{2i}, r^{j+1}_{1i}), (right chain points down) (l^j_{2i}, l^j_{1i}), and (l^{j+1}_{1i}, l^j_{2i}) (left chain points up). If a variable x_i is false, we orient the left chain down, and the right chain up. Conversely, given an orientation, we construct a satisfying truth assignment as follows: For each variable gadget i, either we orient the left chain or the right chain are oriented down (or possibly both). In the first case we set x_i to be false, and

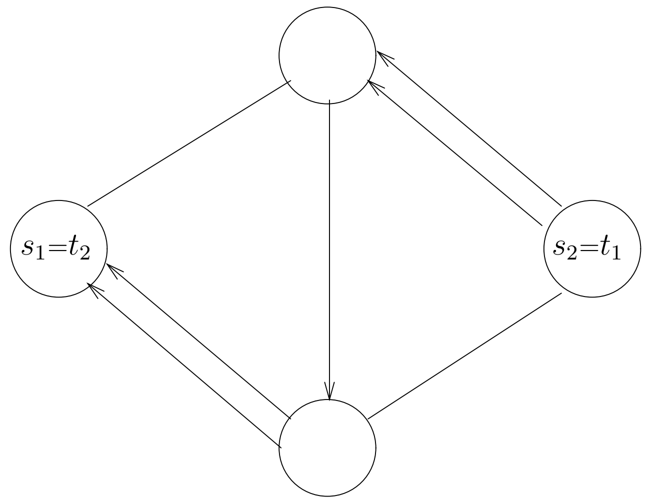

Figure 4: A graph that has no orientation with an $s_1 - t_1$ path and two disjoint $s_2 - t_2$ paths

in the second case to be true (if both, then x_i can be set arbitrarily). Note that since each clause gadget must have a path through it, thus passing through one of its literals, that literal must be true, and hence the formula is satisfied. ■

We can strengthen the previous theorem as follows:

Theorem 3.2 *Given a mixed graph $G = (V, A, E)$, nodes s_1, t_1, s_2, t_2, and an integer k, the problem of deciding whether there is an orientation of G containing one $s_1 - t_1$ path and k $s_2 - t_2$ disjoint paths is NP-complete.*

Proof: We use a similar reduction to the one in Theorem 3.1 except "chaining together" the variable gadgets: Instead of the arcs (s_1, u_i) we have a single arc (s_1, u_1). Instead of the arcs (v_i, t_1) we have a single arc (v_n, t_1). We also have new arcs (v_i, u_{i+1}) for $i = 1, \ldots, n-1$. ■

A natural conjecture for the case $n = 1$ and $k = 2$ is the following: A mixed graph $G = (V, E, A)$ has an orientation such that there is one $s_1 - t_1$ path and 2 disjoint $s_2 - t_2$ paths if and only if (i) G has such paths, (ii) it has no essential edge, and (iii) there is no cut (X, Y) in the underlying undirected graph containing at most two edges, such that $s_1, t_2 \in X$ $s_2, t_1 \in Y$. These conditions are clearly necessary, but as Figure 4 shows, they are not sufficient.

For undirected graphs, $n = 1$ and arbitrary k Theorem 3.3 shows that a modified set of the above conditions is sufficient. Moreover, the proof is constructive, providing in polynomial time an orientation if one exists, in contrast to Theorem 3.2 for mixed graphs.

Theorem 3.3 *Given an undirected graph $G = (V, E)$, nodes s_1, t_1, s_2, t_2, and an integer k, there exists an orientation of G which has one $s_1 - t_1$ path and k disjoint $s_2 - t_2$ paths if and only if (i) G has such paths, and (ii) there is no cut (X, Y) in G containing k edges, such that $s_1, t_2 \in X$ $s_2, t_1 \in Y$.*

Proof: The conditions are clearly necessary and we prove that they are also sufficient. Consider arbitrary k disjoint $s_2 - t_2$ paths in G. Orient the edges of these paths to obtain directed $s_2 - t_2$ paths, and let the resulting graph be G'. We will show that conditions (i) and (ii) imply that every cut (X, Y) such that $s_1 \in X$ and $t_1 \in Y$ in G' contains either at least one arc from X to Y or at least one edge. This, in turn, implies that the edges of G' can be oriented so that the resulting directed graph also has an $s_1 - t_1$ path. There are four cases: (a) $s_2 \in X$ and $t_2 \in Y$. In this case there are at least k arcs in the cut. (b) $s_2, t_2 \in X$. The number of arcs from X to Y is equal to the number of arcs from Y to X. If there are no arcs in the cut then by (i) it must have at least one edge. (c) $s_2, t_2 \in Y$. The proof in this case is as in Case (b). (d) $s_2 \in Y$ and $t_2 \in X$. The number of arcs from Y to X is k plus the number of arcs from X to Y. If there are no arcs from X to Y then there must be an edge in the cut since otherwise (ii) is violated. ∎

Let $G = (V, E)$ be an undirected graph and D an orientation of it. We define $\delta(x, y; G)$ and $\delta(x, y; D)$ as the edge connectivity from x to y in G and D, respectively. Nash Williams [6] proved the following theorem:

Theorem 3.4 *Every undirected graph G has an orientation D such that for every $x, y \in V$ $\delta(x, y; D) \geq \lfloor \delta(x, y; G)/2 \rfloor$.*

We conclude from this theorem that:

Corollary 3.5 *Given an undirected graph $G = (V, E)$, two nodes $a, b \in V$, and an integer k, then there exists an orientation of G containing k disjoint paths from a to b and k disjoint paths from b to a if and only if G contains $2k$ disjoint paths between a and b.*

In view of this result and Theorem 3.3, an interesting open problem is: Given an undirected graph $G = (V, E)$, nodes $s_1, t_1, s_2, t_2 \in V$, and an integer k, is there an orientation of G containing k disjoint paths from s_1 to t_1 and k disjoint paths from s_2 to t_2?

The following is a natural generalization of Theorem 3.3: Given an undirected graph $G = (V, E)$, nodes s_1, t_1, s_2, t_2, and integers n and k, there exists an orientation of G such that there are n $s_1 - t_1$ disjoint paths and k disjoint $s_2 - t_2$ paths if and only if (i) G has such paths, and (ii) there is no cut (X, Y) in G

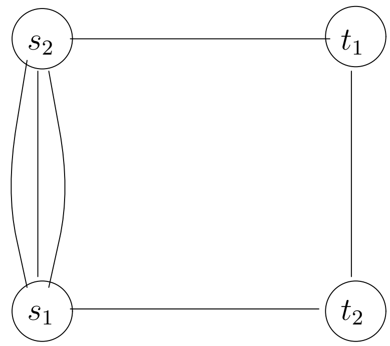

Figure 5: A graph that has no orientation with two $s_1 - t_1$ and two $s_2 - t_2$ disjoint paths

containing at most $n + k - 1$ edges, such that $s_1, t_2 \in X$ $s_2, t_1 \in Y$. These conditions are clearly necessary, but as Figure 5 shows, they are not sufficient even when $n = k = 2$.

4 Open problems

We have proved several results concerning orientations of mixed graphs and showed that some natural generalizations do not hold. We summarize below the 'simplest' remaining open problems.

Given a mixed graph $G = (V, E, A)$, does there exist an orientation of E such that the resulting directed graph is:

- P-connected for $|P| = 3$ (i.e., $s_i - t_i$ connected for $i = 1, 2, 3$).
- $s_1 - t_1$ connected and $s_2 - t_2$ 2-connected.
- $s_1 - t_1$ 2-connected and $s_2 - t_2$ 2-connected, even when $A = \emptyset$.

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