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A 1-Relaxed Minimum Broadcast Graph on 15 Vertices

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Abstract—In 1998, Shastri studied the sparsest possible broadcast graphs in which broadcasting can be accomplished in slightly more than the optimal time of $\lceil \log_2 n \rceil$. In particular, they constructed the sparsest possible time-relaxed broadcast graphs for small n (≤ 14) and very sparse time-relaxed broadcast graphs for larger n (≤ 65). Let $B_t(n)$ be the number of edges in the sparsest possible graph on n vertices in which broadcasting can be accomplished in t additional steps than the optimal (i.e., in $\lceil \log_2 n \rceil + t$ steps), they conjectured that $B_1(15) = 18$. In this paper, we give a 1-relaxed minimum broadcast graph on 15 vertices which shows that $B_1(15) = 17$, thus reject the conjecture.

Keywords—Broadcasting, Networks, Communication, Time-relaxed broadcast.

1. INTRODUCTION

We represent a communication network by a connected graph G, where the vertices of G represent processors and edges represent bidirectional communication channels. The problem of *broadcasting* is to disseminate a piece of information, which originates at one vertex (member) to all the members. This is to be accomplished as quickly as possible by a series of calls under the following constraints:

- each call requires one unit time;
- any member may participate in at most one call per unit time;
- a member can only call an adjacent member.

That is, if u sends a message to v, then neither u nor v can send or receive another message at the same time. A *broadcast protocol* for G allows any originator to send messages to all other vertices in the network.

Let $G(V, E)$ be a graph of order n, representing a communication network. For $v \in V$, we denote by $b(v, G)$ the minimum time needed to broadcast a message from v in G, and $b(G) = \max_v b(v, G)$ the *broadcast time* of G. Since the number of vertices knowing the message can at most double at each step, it is clear that $b(G) \geq \lceil \log_2 n \rceil$. Graphs for which the broadcast time is equal to

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$\lceil \log_2 n \rceil$ are called *broadcast graphs*. Broadcast graphs with the fewest number of edges are called *minimum broadcast graphs* (MBG) or *minimum broadcast networks* (MBN). Let $B(n)$ denote the number of edges of a MBN on n vertices.

Determination of $B(n)$ has been the central focal point in most of the papers written thus far on the subject of broadcasting. The progress in this direction has been excruciatingly slow, and enormous effort of the authors of [1–8] have resulted in exact determination of $B(n)$ for $n \leq 22$ or some special cases of n, and only some bounds for larger n. In [9], the author investigated the sparsest possible broadcast graphs in which broadcasting can be accomplished in slightly more than the optimal time of $\lceil \log_2 n \rceil$. Let $B_t(n)$ be the minimum number of edges in the sparsest possible graph in n vertices in which broadcasting can be accomplished in t additional steps than the optimal (i.e., in $\lceil \log_2 n \rceil + t$ steps). A graph with broadcast time $\lceil \log_2 n \rceil + t$ is called *t-relaxed broadcast graph* (t-RMBG). Graphs with $\lceil \log_2 n \rceil + t$ broadcast time were also considered by Liestman [10] in the context fault-tolerant broadcasting.

In [9], Shastri considers the behavior of $B_t(n)$ and gives out the following conjecture.

CONJECTURE. $B_1(15) = 18$.

In this paper, we give a 1-relaxed minimum broadcast graph on 15 vertices with 17 edges and prove that $B_1(n) = 17$, thus reject the conjecture.

2. MAIN RESULT

THEOREM 1. $B_1(n) = 17$.

PROOF. We shall first show that $B_1(15) \geq 17$ by contradiction. Suppose there is a 1-relaxed broadcast graph G on 15 vertices with 16 edges, and with broadcast time $b = \lceil \log_2 15 \rceil + 1 = 5$.

If $P = v_1 v_2 \cdots v_k$ is path in G, where $d_G(v_i) = 2$ for $i = 1, 2, \ldots, k$, then we say that P is an essential path in G.

CLAIM 1. *If $P = v_1 v_2 \cdots v_k$ is a longest essential path in G, then $k \leq 3$.*

PROOF OF CLAIM 1. Suppose to the contrary that $k = 4$. Since P is an essential path in G, v_1 is adjacent to u and v_k is adjacent to w. Assume first that $u \neq w$, consider the message originating at v_2. If it is passed to v_1 and then to v_3 in the first two steps, then u is informed at Step 2, and v_4 is informed at Step 3. Note that the broadcast time is 5 and there are $5 - 2 = 3$ steps left at u and two steps left at v_4. It follows that

$$|V(G)| \leq 2^{5-2} + 2^{5-4} + 4 = 14 < 15,$$

a contradiction to the fact that $|V(G)| = 15$. If the message originating at v_2 is first passed to v_3 and then to v_1 in the first two steps, a similar argument as above will deduce that

$$|V(G)| \leq 2^{5-3} + 2^{5-3} + 4 = 12 < 15,$$

a contradiction, too. Hence, assume that $u = w$, in this case, it is clear that

$$|V(G)| \leq 2^{5-2} + 4 = 12 < 15,$$

again a contradiction. Therefore, we have $k \leq 3$. ∎

CLAIM 2. *G contains at least one vertex of degree one.*

PROOF OF CLAIM 2. Suppose to the contrary that G does not contain any vertex of degree one. Since G is a connected graph on 15 vertices with 16 edges, then G is a graph obtained by a tree on 15 vertices, adding two additional edges. Let $\Delta = \max_v d(v)$. It is obvious that $3 \leq \Delta \leq 4$.

If $\Delta = 3$, then G contains exactly two vertices (say u, v) of degree three. Hence, G is either the union of three internal disjoint (u, v)-paths, or a graph consists of two disjoint cycles, connected by a (u, v)-path. In either case, there always exists an essential path of length at least 4, a contradiction to Claim 1.

If $\Delta = 4$, then G is the union of two cycles, having exactly one vertex in common. Thus, G must contain an essential path of length at least 6, which again contradicts Claim 1. ∎

CLAIM 3. *Let v be a vertex in G that is adjacent to a vertex u of degree one, then $d(v) \geq 4$.*

PROOF OF CLAIM 3. The message originating at u is first transferred to v, in the left four steps, the message at v can be disseminated to at most $2^4 - 1 = 15$ vertices (see Figure 1). Hence, G is a graph obtained from the tree T illustrated in Figure 1, deleting two vertices of T except u (such two vertices should be chosen such that by deleting them, the resulting graph is connected), and adding two more additional edges. It is easy to see that $d_G(v) \geq 4$. ∎

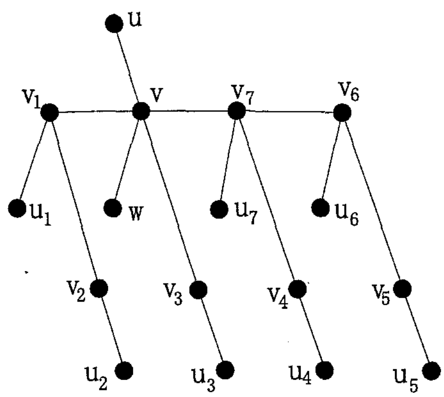

Figure 1. A tree T used to construct G.

Furthermore, we can prove the following.

CLAIM 4. *Let v be a vertex in G that is adjacent to a vertex u of degree one, then $d(v) \geq 5$.*

PROOF OF CLAIM 4. Assume that there is a vertex u of degree one, which is adjacent to a vertex v of degree four. Since G can be constructed from T (Figure 1), as in the proof of Claim 3, note that v_1 and v_7 cannot be deleted, otherwise the resulting graph will not be connected. Assume first that w is deleted. Consider the vertices u_i $(1 \leq i \leq 6)$ in $T - w$. If none of the vertices u_i is deleted, then by Claim 3, in order to construct G, we must introduce either a new edge that is adjacent to u_i, or at least one new edge that is adjacent to v_i. Hence, it is obvious to see that there is an integer j for some $1 \leq j \leq 6$ in G such that $d_G(u_j) = 1$, and $d_G(v_j) \leq 3$, which contradicts Claim 3. Assume now that w is not deleted, then u_3 and v_3 should be deleted, a similar argument as above will introduce a contradiction. ∎

Now note $d_T(u_7) = 1$ and $d_T(v_7) = 4$, hence, in the graph G that is constructed from T, there is a vertex u_k for some $1 \leq k \leq 7$, such that $d_G(u_k) = 1$ and $d_G(v_k) \leq 4$, which is a contradiction to Claim 4. Therefore, we have proved that $B_1(15) \geq 17$.

In order to complete the proof of Theorem 1, we should construct a graph with 15 vertices and 17 edges, whose broadcast time is five. This is illustrated in Figure 2. By symmetry, we need only to check four broadcasting schemes (see Figures 2a–2d) for the broadcast graph. In Figure 2(I), the letter beside a vertex indicates which scheme is used. In a scheme, '+' indicates originator and a label beside a vertex indicates the time unit of the vertex receiving the message (note the vertices labelled with '+' and '1' and use the same scheme by informing each other at the first time unit). ∎

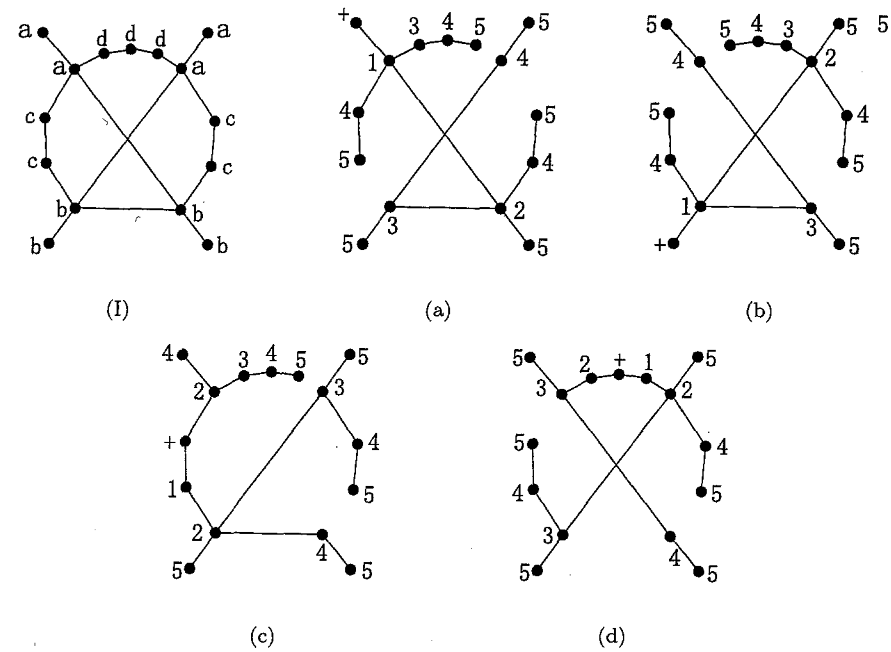

Figure 2. A 1-relaxed minimum broadcast graph on 15 vertices.

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