CONCORDIA UNIVERSITY DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP 6661 Combinatorial Algorithms Winter 2023

Assignment 1

Due January 30, 2023, 11:55 PM

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1. Given the following graphs:
   - The complete graph K_n on n vertices.
   - The complete bipartite graph $K_{m,n}$ on $m + n$ vertices.
   - The n-vertex wheel W_n.
   - The hypercube Q_n.

   a) Which of the above graphs are Eulerian? Justify.

   b) Which of the above graphs are Hamiltonian? Justify.

2. The line graph $L(G)$ of a graph G has a vertex for each edge of G, and two of these vertices are adjacent iff the corresponding edges in G have a vertex in common.

   a) Prove that if a simple graph G is Eulerian, then its line graph $L(G)$ is also Eulerian.

   b) Prove or disprove. The line graph of any graph is Eulerian.

   c) Prove that a graph with more than two vertices of odd degree does not contain an Eulerian path (or trail).

3. Draw the specified graph or prove that it does not exist:

   a) An 8-vertex simple graph with more than 8 edges that is both Eulerian and Hamiltonian.

   b) An 8-vertex simple graph with more than 8 edges that is Eulerian but not Hamiltonian.

   c) An 8-vertex simple graph with more than 8 edges that is Hamiltonian but not Eulerian.

   d) An 8-vertex simple Hamiltonian graph that does not satisfy the conditions of Ore's theorem.

   e) A 6-vertex simple graph with 10 edges that is not Hamiltonian.

4. Run BFS rooted at the all zero vertex, i.e. $(00 \ldots 0)$ of the k-dimensional hypercube Q_k. What is the number of vertices at distance i from the root, for all $i = 0, 1, 2, \ldots, k, \ldots$? What is the number of edges between level i and $i + 1$? Prove your answers.

5. Prove by induction on e (number of edges) that a planar graph is bipartite iff every face has even length.

6. Prove that if graph G has n vertices then $\chi(G) + \chi(\bar{G}) \leq n + 1$, where $\bar{G}$ is the complimant of graph G.

7. a) Let $G = (V, E)$ be a loop-free undirected graph with $|V| = n \geq 3$, and $\deg(x) + \deg(y) \geq n - 1$ for all nonadjacent vertices x, $y \in V$. Prove that there is a path of length at most 2 between each pair of vertices of G.

   b) Prove that a graph with n vertices and at least $\frac{(n-1)(n-2)}{2} + 1$ edges is connected.