New Lower Bounds on Broadcast Function

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Abstract. This paper studies the broadcast function $B(n)$. We consider the possible vertex degrees and possible connections between vertices of different degrees in graphs with $b(G) = \lceil \log_2 n \rceil$. Using this, we present improved lower bounds on $B(n)$ when $n = 2^k - 2^p$ and $n = 2^k - 2^p + 1$ $(3 \leq p < k)$. Also, we prove that $B(24) \geq 36$ for graphs with maximum vertex degree at most 4.

Keywords: Broadcasting, minimum broadcast graphs, broadcast function, lower bounds on broadcast function.

1 Introduction

Broadcasting is the process of distributing a message from a node, called the *originator*, to all other nodes of a communication network. Broadcasting is accomplished by placing series of calls over the communication channels of the network and takes place in discrete time units, sometimes called rounds. Each call involves only two nodes (one sender and one receiver), requires one time unit, and each node participates in at most one call at each time unit.

A network can be modeled as a connected graph $G = (V, E)$, where V is the set of all nodes and E is the set of all communication lines. The *broadcast time* $b(v, G)$ or just $b(v)$ of a vertex v in a connected graph G is defined as the minimum time required to inform all the vertices of G from originator v. The broadcast time $b(G)$ of a graph G is defined as $b(G) = max\{b(v) \mid v \in V\}$.

The set of calls used to distribute the message from originator v to all other vertices is called a *broadcast scheme* for vertex v. The broadcast scheme for v is a spanning tree rooted at v where all the communication lines are labeled with the transmission time. Each communication line is used exactly once and the message is always transmitted from a parent to a child.

For any graph G on n vertices, $b(G) \geq \lceil \log n \rceil$ (all logarithms in this paper are base two), since after each time unit the number of informed vertices can at most double. A graph G with $b(G) = \lceil \log n \rceil$, is called a *broadcast graph (bg)*. A broadcast graph with the minimum possible number of edges is called a *minimum broadcast graph (mbg)*. An *mbg* has very important practical implications; it represents the cheapest possible architecture to build a network, in which broadcasting can be accomplished in theoretically minimum possible

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time. The *broadcast function* $B(n)$ is defined as the number of edges in an *mbg* on n vertices.

$B(n)$ is known only for very few values of n, in particular for all $n \leq 32$ except for $n = 23, 24$ and 25. The values of $B(n)$ for $1 \leq n \leq 15$ are presented in [6], also, $B(17) = 22$ [17], $B(18) = 23, B(19) = 25$ [3],[19], $B(20) = 26, B(21) = 28, B(22) = 31$ [16], $B(23) = 33$ or 34 [4],[16], $B(26) = 42$ [18],[20], $B(27) = 44, B(28) = 48, B(29) = 52$ [18], $B(30) = 60$ [3],[13], $B(31) = 65$ [3]. $B(n)$ is also known for $n = 2^k$, $B(2^k) = k2^{k-1}$ [6], for $n = 2^k-2$, $B(2^k-2) = (k-1)(2^{k-1}-1)$ [5],[13], $B(58) = 121$, $B(59) = 124$, $B(60) = 130$, $B(61) = 136$ [18], $B(61) = 136$, $B(63) = 162$ [15], $B(127) = 389$ [8].

Since mbg's seem to be extremely difficult to find, a long sequence of papers presented techniques to construct broadcast graphs and to obtain upper bounds on $B(n)$. Most techniques combine several known mbg's and bg's on smaller sizes to create new ones of a larger size (see e.g. [1],[2],[3],[5],[7],[8],[9],[11],[12],[13]).

However, it is extremely difficult to prove a lower bound on $B(n)$ that matches the obtained upper bound from a broadcast graph construction. When $n = 2^k$ or $n = 2^k - 2$, a simple lower bound based on the minimum vertex degree matches the known upper bounds, thus the known mbg's are for these cases are k-regular and $k-1$-regular graphs respectively. However, when $n < 2^k - 2$, mbg's are not regular graphs, hence the simple lower bounds based only on minimum vertex degree are not very helpful. For small n, an *mbg* can be found by exhaustive case analysis, but when n becomes large, the number of possible graphs grows exponentially and this technique is no longer useful.

Similar to the approach taken in [16],[18] in this paper we consider the possible vertex degrees in any broadcast graph on n vertices and possible connections between vertices of different degrees.

Our first observation is that in a broadcast graph on $n = 2^k - x$ vertices where $1 \leq x \leq 2^{k-1}$, the minimum vertex degree must be at least $k - \lfloor \log x \rfloor$. A broadcast tree rooted at some vertex v of a smaller degree will contain at most $n = 2^k - x - 1$ vertices. This number is smaller than the total number of vertices. It means that not all vertices will be able to receive the broadcast message by the time k from originator v. From this observation it follows that

$$B(2^k - x) \geq \frac{2^k - x}{2} \cdot (k - \lfloor \log x \rfloor).$$

The fact that a given graph is a broadcast graph determines not only the minimum possible vertex degree in it, but also the possible connections between vertices of different degrees. By making more accurate observations the above mentioned bound can be improved. This approach was used in [16] to obtain lower bounds on $B(n)$ when $n = 2^k - 3, n = 2^k - 4, n = 2^k - 5$ and $n = 2^k - 6$.

The following bounds are presented:

$$B(2^k-3) \geq \left\lceil \frac{2^k-3}{2} \cdot (k-2+\frac{3k-5}{k^2-k-1}) \right\rceil,$$
$$B(2^k-4) \geq \left\lceil \frac{2^k-4}{2} \cdot (k-2+\frac{4}{2k+1}) \right\rceil,$$
$$B(2^k-5) \geq \left\lceil \frac{2^k-5}{2} \cdot (k-2+\frac{2}{2k-1}) \right\rceil,$$
$$B(2^k-6) \geq \left\lceil \frac{2^k-6}{2} \cdot (k-2+\frac{1}{k}) \right\rceil.$$

The same approach is also used in [15] to get a lower bound on $B(n)$ when $n = 2^k - 1$.

$$B(2^k-1) \geq \left\lceil \frac{2^k-1}{2} \cdot (k-1+\frac{1}{k+1}) \right\rceil.$$

We find this method of getting lower bounds on $B(n)$ promising and we will use it to find lower bounds on $B(n)$ when $n = 2^k - 2^p$ and $n = 2^k - 2^p + 1$ ($3 \leq p < k$). The main difficulty in the above approach is that when x increases, the lower bound on the minimum degree presented above decreases and then the number of possibilities of different relations between vertices of different degree increases as well and it becomes more and more difficult to deal with them and derive an improved lower bound on $B(2^k - x)$.

One of the motivations for looking on these two particular forms of n is that the smallest values for which $B(n)$ is not known are $n = 23, n = 24$ and $n = 25$. The latter two have a form $n = 2^k - 7$ and $n = 2^k - 8$ respectively. Where are known broadcast graphs on 24 and 25 vertices having 36 and 40 edges respectively [3] but whether these graphs are *mbg*'s or not is not known. Tight lower bounds on $B(24)$ and $B(25)$ may help to address this problem.

2 Lower Bound on $B(2^k - 7)$

As mentioned above, there are tight lower bounds on $B(n)$ when $n = 2^k-1, 2^k-3, 2^k-4, 2^k-5, 2^k-6$. We continue on this line and in this section we present a new lower bound on $B(n)$ when $n = 2^k-7$. In the following section, we generalize the presented result for $n = 2^k - 2^p + 1$. Thus, the proof in this section will help to follow the proof of Section 3.

In our approach, we extend the technique presented by Sacle in [18].

Theorem 1. $B(2^k - 7) \geq \frac{2^k-7}{2}((k-3) + \frac{5k-11}{(k+1)(k-2)})$.

Proof. Recall that in a broadcast graph on $n = 2^k - 7$ vertices, the minimum possible vertex degree is $k - 3$. Let us look at the broadcast tree rooted at a vertex u of degree $k - 3$. We observe that u must have at least one neighbour of degree at least k, at least two neighbours of degree at least $k - 1$ and at least three neighbours of degree at least $k-2$. We also observe that a vertex cannot have all neighbours of degree $k-3$. In other words each vertex in the graph must have at least one neighbour of degree at least $k-2$. Let v_i denote the number of vertices of degree i. We can write the following inequalities:

$$\sum_{i \geq k}(i-1)v_i \geq v_{k-3},$$

$$\sum_{i \geq k-1}(i-1)v_i \geq 2v_{k-3},$$

$$\sum_{i \geq k-2}(i-1)v_i \geq 3v_{k-3}.$$

For the number of edges in the graph, denoted by m, we will have

$$2m = \sum_{i \geq k-3} iv_i = n + \sum_{i \geq k-3}(i-1)v_i.$$

This implies that

$$\sum_{i \geq k-3}(i-1)v_i = 2m - n.$$

After substituting this in the above three inequalities we will get

$$2m - n - (k-4)v_{k-3} - (k-3)v_{k-2} - (k-2)v_{k-1} \geq v_{k-3},$$

$$2m - n - (k-4)v_{k-3} - (k-3)v_{k-2} \geq 2v_{k-3},$$

$$2m - n - (k-4)v_{k-3} \geq 3v_{k-3}.$$

After rearranging the terms we will have

$$2m - n \geq (k-3)v_{k-3} + (k-3)v_{k-2} + (k-2)v_{k-1},$$

$$2m - n \geq (k-2)v_{k-3} + (k-3)v_{k-2},$$

$$2m - n \geq (k-1)v_{k-3}.$$

After subtracting v_{k-1} and v_{k-3} from the right hand sides of the first and the second inequalities respectively, we will get

$$2m - n \geq (k-3)(v_{k-3} + v_{k-2} + v_{k-1}),$$

$$2m - n \geq (k-3)(v_{k-3} + v_{k-2}),$$

$$2m - n \geq (k-1)v_{k-3}.$$

It follows that

$$v_{k-3} + v_{k-2} + v_{k-1} \leq \frac{2m-n}{k-3},$$

$$v_{k-3} + v_{k-2} \leq \frac{2m-n}{k-3},$$

$$v_{k-3} \leq \frac{2m-n}{k-1}.$$

Alternatively, for the number of edges we also have the following expression

$$2m \geq nk - (v_{k-1} + 2v_{k-2} + 3v_{k-3}) =$$

$$= nk - (v_{k-1} + v_{k-2} + v_{k-3}) - (v_{k-2} + v_{k-3}) - v_{k-3}.$$

By substituting the above 3 inequalities in this inequality we get

$$2m \geq nk - (2m-n)(\frac{2}{k-3} + \frac{1}{k-1}).$$

From which it follows that

$$m \geq \frac{n}{2} \cdot \frac{k + (\frac{2}{k-3} + \frac{1}{k-1})}{1 + (\frac{2}{k-3} + \frac{1}{k-1})} = \frac{n}{2} \cdot \frac{k + (\frac{2}{k-3} + \frac{1}{k-1})}{1 + (\frac{2}{k-3} + \frac{1}{k-1})}.$$

This gives the following lower bound on $B(2^k - 7)$

$$B(2^k - 7) \geq \frac{n}{2} \cdot \frac{k + (\frac{1}{k-1} + \frac{2}{k-3})}{1 + (\frac{1}{k-1} + \frac{2}{k-3})} = \frac{2^k - 7}{2}((k-3) + \frac{5k-11}{(k+1)(k-2)}).$$

3 Lower Bound on $B(2^k - 2^p + 1)$

In this section we obtain a new lower bound on $B(n)$ where $n = 2^k - 2^p + 1$ based on the degree sequence restrictions of any broadcast graph on $2^k - 2^p + 1$ vertices.

Theorem 2. $B(2^k - 2^p + 1) \geq \frac{2^k - 2^p + 1}{2}((k-p) + \frac{k(2p-1)-(p^2+p-1)}{k(k-1)-(p-1)})$.

Proof. We observe that in an *mbg* on $2^k - 2^p + 1$ vertices, each vertex of degree $k - p$ must have at least one neighbour of degree at least k, two neighbours of degree at least $k-1$, three neighbours of degree at least $k-2$, ... , p neighbours of degree at least $k-p+1$. After noticing that a vertex cannot have all its neighbours of degree $k-p$ we are getting the following inequalities

$$\sum_{i \geq k} (i-1)v_i \geq v_{k-p},$$

$$\sum_{i \geq k-1} (i-1)v_i \geq 2v_{k-p},$$

$$\sum_{i \geq k-2} (i-1)v_i \geq 3v_{k-p},$$

$$\dots$$

$$\sum_{i \geq k-p+1} (i-1)v_i \geq pv_{k-p}.$$

For the number of edges in the graph, denoted by m we will have

$$2m = \sum_{i \geq k-p} i v_i = n + \sum_{i \geq k-p} (i-1) v_i.$$

This implies that

$$\sum_{i \geq k-p} (i-1) v_i = 2m - n.$$

After substituting this in the above p inequalities and reversing their order we will get

$$2m - n - (k-p-1)v_{k-p} \geq p v_{k-p},$$

$$2m - n - (k-p-1)v_{k-p} - (k-p)v_{k-p+1} \geq (p-1)v_{k-p},$$

$$2m - n - (k-p-1)v_{k-p} - (k-p)v_{k-p+1} - (k-p+1)v_{k-p+2} \geq (p-2)v_{k-p},$$

$$...$$

$$2m - n - \sum_{j=0}^{i} (k-p-1+j) v_{k-p+j} \geq (p-i) v_{k-p},$$

$$...$$

$$2m - n - \sum_{j=0}^{p-1} (k-p-1+j) v_{k-p+j} \geq v_{k-p}.$$

After rearranging the terms we will have

$$2m - n \geq (k-1) v_{k-p},$$

$$2m - n \geq (k-2) v_{k-p} + (k-p) v_{k-p+1},$$

$$2m - n \geq (k-3) v_{k-p} + (k-p) v_{k-p+1} + (k-p+1) v_{k-p+2},$$

$$...$$

$$2m - n \geq (k-p) v_{k-p} + \sum_{j=1}^{p-1} (k-p-1+j) v_{k-p+j}.$$

By replacing all the $k-2$, $k-3$, ..., $k-p+1$ coefficients on the right side of these inequalities with $k-p$, which is the smallest one, we will get

$$2m - n \geq (k-1) v_{k-p},$$

$$2m - n \geq (k-p)(v_{k-p} + v_{k-p+1}),$$

$$2m - n \geq (k-p)(v_{k-p} + v_{k-p+1} + v_{k-p+2}),$$

$$...$$

$$2m - n \geq (k-p)(v_{k-p} + v_{k-p+1} + v_{k-p+2} + ... + v_{k-1}).$$

Alternatively, we have the following trivial inequality

$$2m \geq nk - (v_{k-1} + 2v_{k-2} + 3v_{k-3} + ... + pv_{k-p}) =$$

$$= nk - (v_{k-1} + ... + v_{k-p}) - (v_{k-2} + ... + v_{k-p}) - (v_{k-3} + ... + v_{k-p}) - ... - v_{k-p}.$$

By substituting the terms in parenthesis with their upper bounds from the previous set of inequalities we will get

$$2m \geq nk - (2m - n)(\frac{1}{k-1} + \frac{p-1}{k-p}).$$

It follows that

$$B(2^k - 2^p + 1) \geq m \geq \frac{n}{2} \cdot \frac{k + (\frac{1}{k-1} + \frac{p-1}{k-p})}{1 + (\frac{1}{k-1} + \frac{p-1}{k-p})} =$$

$$= \frac{2^k - 2^p + 1}{2}((k-p) + \frac{k(2p-1) - (p^2 + p - 1)}{k(k-1) - (p-1)}).$$

Note that by plugging $p = 3$ in Theorem 2 we get the lower bound from Theorem 1.

4 Lower Bound on $B(2^k - 2^p)$

In this section we present a new lower bound on $B(2^k - 2^p)$.

Theorem 3. $B(2^k - 2^p) \geq \frac{2^k - 2^p}{2}((k-p) + \frac{k(2p-2)-(p^2+p-2)}{k(k-2)-(p-2)})$.

Proof. Most of the proof is omitted due to its similarity to the proof for $B(2^k - 2^p - 1)$. We observe that in an *mbg* on $2^k - 2^p$ vertices, each vertex of degree $k-p$ must have at least two neighbours of degree at least $k-1$, three neighbours of degree at least $k-2$, ... , p neighbours of degree at least $k-p+1$. See Fig. 1. This will give the following inequalities:

$$\sum_{i \geq k-1} (i-1)v_i \geq 2v_{k-p},$$

$$\sum_{i \geq k-2} (i-1)v_i \geq 3v_{k-p},$$

$$...$$

$$\sum_{i \geq k-p+1} (i-1)v_i \geq pv_{k-p}.$$

Note that the first inequality from Theorem 2 is missing here. The reason is that in a broadcast graph on $2^k - 2^p + 1$ vertices, unlike for the $2^k - 2^p$ case, a vertex of degree $k-p$ can have a neghbour of degree $k-1$. Using these inequalities in a similar way as in Theorem 2, we were able to prove the presented lower bound on $B(2^k - 2^p)$.

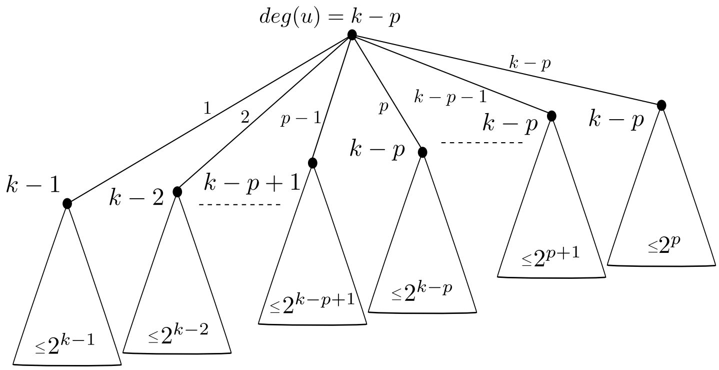

Fig. 1. Broadcast tree rooted at vertex u of degree $k-p$. The neighbours of u are sorted in the decreasing order of their degree and labeled $1, 2, ..., k-p$. The number in each triangle is the maximum possible number of vertices in that subtree. The number on each vertex is the minimum required degreed of that vertex.

5 About the Value of $B(24)$

A broadcast graph on 24 vertices and 36 edges was constructed by Bermond et al. [3]. This gives $B(24) \leq 36$.

We will prove the $B(24) \geq 36$ inequality for graphs G with $\Delta(G) \leq 4$, i.e. for graphs with maximum vertex degree at most 4. It will follow that, if it exists, a broadcast graph on 24 vertices and less than 36 edges, must have at least one vertex of degree at least 5.

Let v_i denote the number of vertices of degree i, and α_{ij} denote the number of all edges between vertices of degree i and j. By our definition $\alpha_{ij} = \alpha_{ji}$.

Lemma 1. *For a broadcast graph G on 24 vertices, $\delta(G) \geq 2$, i.e. $v_1 = 0$.*

Proof. Let G be a broadcast graph on 24 vertices, i.e $b(G) = \lceil \log 24 \rceil = 5$. Let u be the broadcast originator and $deg(u) = 1$. In the first round, it will inform its only neighbor v. In the remaining 4 rounds, v will be able to inform at most $16 = 2^4$ vertices. It follows that in 5 rounds u can inform only at most 17 vertices, as shown in Fig. 2.

Theorem 4. *A broadcast graph G on 24 vertices and $\Delta(G) \leq 4$, must have at least 36 edges.*

Proof. From Lemma 1, it follows that $v_1 = 0$. By counting the number of edges adjacent to vertices of degree 4, we will have

$$4v_4 = \alpha_{42} + \alpha_{43} + 2\alpha_{44}.$$

We observe that the broadcast tree rooted at a vertex u of degree 2 must have a form shown in Figure 3, otherwise in 5 rounds it will not be possible

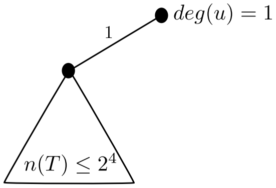

Fig. 2. Broadcast tree rooted at a vertex of degree 1

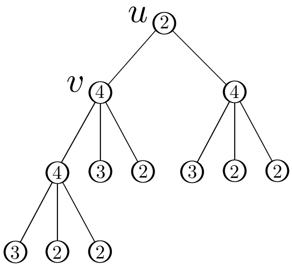

Fig. 3. Subtree of a broadcast tree rooted at a vertex of degree 2

to inform 24 vertices. The number next on each vertex indicates the minimal possible degree for that vertex. For example, a vertex with label 3 may actually have degree 4.

From the figure we observe that a vertex of degree 2 must have both its neighbours of degree 4. Therefore,

$$\alpha_{42} = 2v_2.$$

From the fact that a vertex of degree 4 cannot have all its neighbours having degree 2, it follows that it has at least one adjacent is edge going to vertex of degree 3 or 4. Also we note that a vertex of degree 2 must have a neighbour v (left child in Figure 3) of degree 4 having at least 2 edges going to a vertex of degree 3 or 4. That is v cannot have three neghbours of degree 2. Vertex v can be shared between at most 2 vertices of degree 2. It follows that there are at least $\lceil \frac{v_2}{2} \rceil$ such vertices "v", i.e. vertices of degree 4 having at least 2 edges going to a vertex of degree 3 or 4. Thus, we can claim that that

$$\alpha_{43} + \alpha_{44} \geq v_4 + \left\lceil \frac{v_2}{2} \right\rceil.$$

From the observation that an edge between vertices of degree 4 in Figure 3 can be shared among at most 4 vertices of degree 2 we have that

$$\alpha_{44} \geq \left\lceil \frac{v_2}{4} \right\rceil.$$

Finally, by using the expressions above we will have

$$4v_4 = \alpha_{42} + \alpha_{43} + 2\alpha_{44} = \alpha_{42} + (\alpha_{43} + \alpha_{44}) + \alpha_{44} \geq$$

$$\geq 2v_2 + v_4 + \left\lceil \frac{v_2}{2} \right\rceil + \left\lceil \frac{v_2}{4} \right\rceil \geq (2 + \frac{1}{2} + \frac{1}{4})v_2 + v_4 = \frac{11}{4}v_2 + v_4.$$

It follows that

$$v_4 \geq \frac{11}{12}v_2.$$

To prove that $b(24) \geq 36 = \frac{24\cdot 3}{2}$, we must show that in any broadcast graph of on 24 vertices, the average vertex degree is at least 3. In our case, this means that in any broadcast graph G with $\Delta(G) = 4, |G| = 24$, we must show that $v_4 \geq v_2$. From $v_4 \geq \frac{11}{12}v_2$ it almost always follows that $v_4 \geq v_2$. The only pair of values for which it is not the case is $v_2 = 12, v_4 = 11$, but this would mean that $v_3 = 24 - v_2 - v_4 = 1$. We observe that this is not possible, since in any graph the number of vertices of odd degree must be even.

6 Summary

In [7], it was shown that $B(n) \geq \frac{n}{2}(\lfloor \log n \rfloor - \log(1 + 2^{\lfloor \log n \rfloor} - n))$. Let k be the index of the leftmost 0 bit in the binary representation $(\alpha_{p-1}\alpha_{p-2}...\alpha_1\alpha_0)$ of $n-1$. In [14], the following bound was obtained $B(n) \geq \frac{n}{2}(p - k - 1)$. This bound was later improved in [10] to $B(n) \geq \frac{n}{2}(p - k - 1 + \beta)$ where $\beta = 0$ if $k = 0$ or if $\alpha_0 = \alpha_1 = ... = \alpha_{k-1} = 0$, otherwise $\beta = 1$. For cases $n = 2^p - 2^k + 1$ and $n = 2^p - 2^k$, these bounds give $B(n) \geq \frac{n}{2}(p - k)$. It follows that the bounds from Theorems 2 and 3 are obviously better.

Note that the best known upper bounds on $B(n)$ for both $n = 2^p - 2^k + 1$ and $n = 2^p - 2^k$ is $B(n) \leq \frac{2^p - 2^k}{2}(p - \frac{k+1}{2})$ [9]. So, still there is a gap between the best known lower and upper bounds.

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