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NOTE

MINIMUM BROADCAST GRAPHS

Arthur FARLEY, Stephen HEDETNIEMI, Sandra MITCHELL and Andrzej PROSKUROWSKI
University of Oregon, Eugene, OR 97403, U.S.A.

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1. Introduction

Let a graph $G=(V,E)$ represent a communication network. The set V of vertices corresponds to the members of the network, and the set E of edges corresponds to the communication lines connecting pairs of members. Suppose that a member originates a message which is to be communicated to all other members of the network. This is to be accomplished as quickly as possible by a series of calls placed over lines of the network. We adopt the constraints that (i) each call requires one unit of time, (ii) a member can participate in only one call per unit of time, and (iii) a member can only call an adjacent member. We refer to this one-to-all communication process as *broadcasting*.

Consider the following problem: given a connected graph G and a message originator, vertex u, what is the minimum number of time units required to complete broadcasting from vertex u? We define the *broadcast time of a vertex* u, $b(u)$, to equal this minimum time. It is easy to see that for any vertex u in a connected graph G with n vertices, $b(u) \geq \lceil \log_2 n \rceil$, since during each time unit the number of informed vertices can at most double. We define the *broadcast time of a graph* G, $b(G)$, to equal the maximum broadcast time of any vertex u in G, i.e., $b(G)=\max\{b(u) \mid u \in V(G)\}$. For the complete graph K_n with $n>2$ vertices, $b(K_n)=\lceil \log_2 n \rceil$, yet K_n is not minimal with respect to this property. That is, we can remove several edges from K_n and still have a graph G such that $b(G)=\lceil \log_2 n \rceil$. We define a *minimal broadcast graph* to be a graph G with n vertices such that $b(G)=\lceil \log_2 n \rceil$ but for every proper spanning subgraph $G' \subset G$, $b(G') > \lceil \log_2 n \rceil$.

We define the *broadcast function* $B(n)$ to equal the minimum number of edges in any minimal broadcast graph on n vertices. A *minimum broadcast graph* is a minimal broadcast graph on n vertices having $B(n)$ edges. From the point of view of applications, minimum broadcast graphs represent the cheapest possible communication networks (in terms of number of lines) in which broadcasting can be accomplished from any vertex as fast as theoretically possible.

We define a *minimum broadcast tree* to be a rooted tree with n vertices and root u such that $b(u) = \lceil \log_2 n \rceil$. In any connected graph G, a broadcast from a vertex u determines a rooted spanning tree of G. Thus, every vertex of a minimal broadcast graph G is the root of a minimum broadcast tree which spans the vertices of G. The problem of deciding whether an arbitrary vertex in a graph G is the root of a spanning minimum broadcast tree of G has been shown to be NP-complete (D.S. Johnson, personal communication). Therefore, the recognition problem for minimal broadcast graphs is also NP-complete. We suspect that the problem of determining $B(n)$ for arbitrary n is NP-complete as well.

In this paper we initiate a study of minimum broadcast graphs by determining the value of $B(n)$ for $n \leqslant 15$ and $n = 2^k$. We also construct an example of a minimum broadcast graph for each value of $n \leqslant 15$. Several papers have recently been written on broadcasting. In [7], the broadcast center (the set of vertices having minimum broadcast times) of a tree is determined. In [2], broadcasting in complete networks and techniques for constructing minimal broadcast graphs are discussed. In [6], minimum broadcast trees are studied. A subsequent paper [5] finds all minimum broadcast graphs on n vertices, for all values of $n \leqslant 12$. Efficient techniques for broadcasting in n-dimensional grid graphs have been investigated [1, 3].

2. Values for $B(n)$, for $n = 2^k$

The calculation of $B(n)$ for $n = 2^k$ is a straightforward exercise due to the following observation.

In a minimal broadcast graph G with 2^k vertices ($k > 0$), every vertex must have degree at least k in order to call all 2^k vertices in time k. Thus, G must have at least $\frac{1}{2}(k \cdot 2^k) = k \cdot 2^{k-1}$ edges, i.e. $B(2^k) \geqslant k \cdot 2^{k-1}$.

In order to show that $B(2^k) \leqslant k \cdot 2^{k-1}$ it suffices to construct a minimal broadcast graph with 2^k vertices and $k \cdot 2^{k-1}$ edges. This is easily done by taking any two minimal broadcast graphs with 2^{k-1} vertices and adding 2^{k-1} edges between the vertices of the two graphs in any one-to-one fashion. This process eventually reduces to the trivial minimum broadcast graph of one vertex.

3. Values for $B(n)$, for $n \leqslant 15$

Table 1 presents the values of $B(n)$ for $n \leqslant 15$. Fig. 1 presents minimum broadcast graphs with n vertices for $7 \leqslant n \leqslant 15$. For $n \leqslant 6$, values of $B(n)$ are easy

Table 1. The known values of $B(n)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$B(n)$	0	1	2	4	5	6	7	12	10	12	13	15	18	21	24

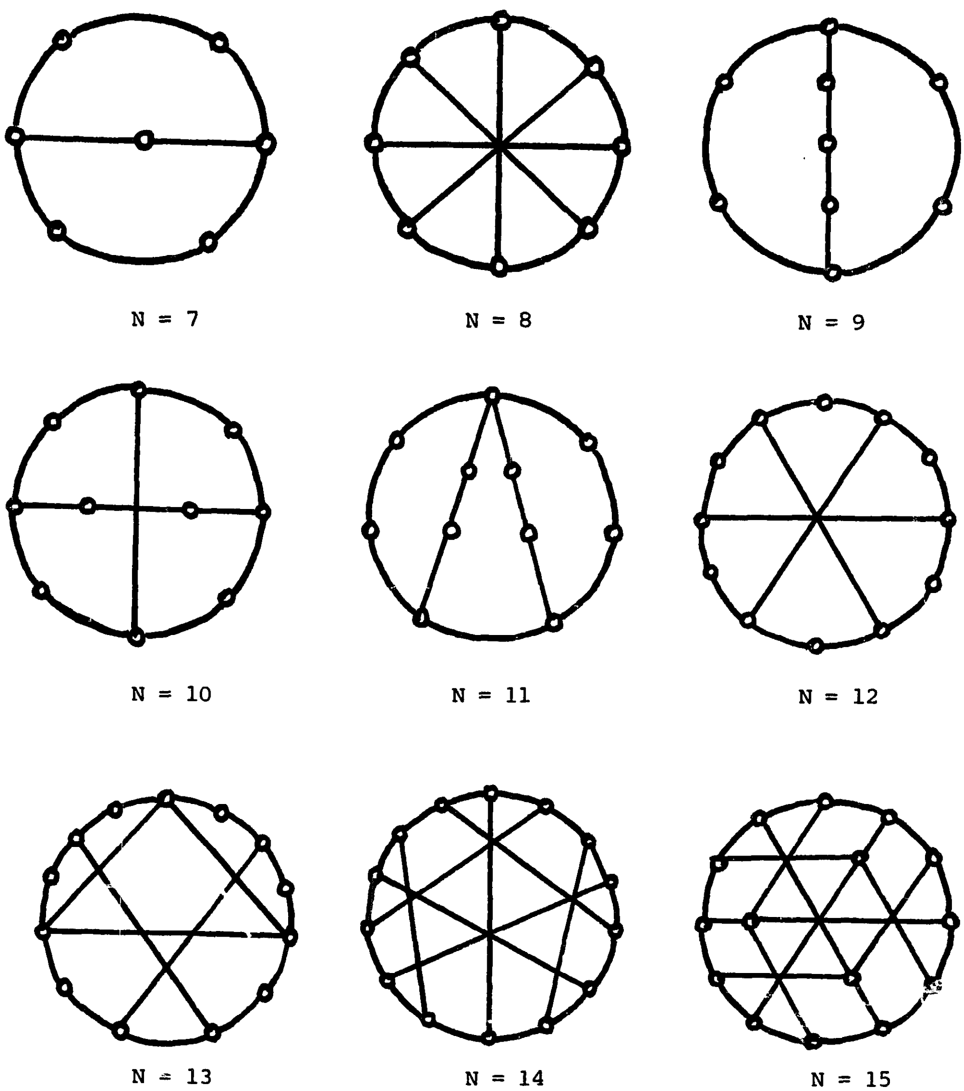

Fig. 1. Minimum broadcast graphs.

to prove by exhaustive case analyses; corresponding minimum broadcast graphs are K_1, $K_{1,1}$, $K_{1,2}$, C_4, C_5, and C_6.

The remaining values of $B(n)$ are more difficult to prove. An exhaustive argument is no longer attractive, due to the increasing number of possible graphs. We use a different method of proof which takes into consideration the topology of minimum broadcast trees with n vertices. This technique allows us to immediately eliminate a large number of candidate graphs.

Each proof proceeds by first presenting a minimal broadcast graph G with n vertices and l edges. This establishes that $B(n) \leq l$. Verification that each graph in Fig. 1 is a minimal broadcast graph is left as an exercise for the reader (cf. [4]). We then show that no minimal broadcast graph with n vertices exists which has $l-1$ edges. Therefore, $B(n) \geq l$, completing the proof that $B(n) = l$.

To prove that $B(n) \geq l$, we show that in a connected graph G with n vertices and $l-1$ edges not every vertex can be the root of a spanning minimum broadcast tree. In some cases, only the degree of the root must be considered. The minimum degree of a vertex in a minimum broadcast graph can not be less than the minimum degree of the roots of the corresponding minimum broadcast trees. For example, the graph with seven vertices in Fig. 1 establishes that $B(7) \leq 8$. However, one can see that $B(7) > 7$, since the root of a minimum broadcast tree with 7 vertices must have degree 2 and C_7, the only connected graph with 7 vertices, 7 edges and minimum degree of 2, is not a minimal broadcast graph. Similarly, the graph with 14 vertices in Fig. 1 shows that $B(14) \leq 21$. However, the minimum degree of the root of a minimum broadcast tree with 14 vertices is 3. Hence, $B(14) \geq 21$.

Fig. 1 shows that $B(9) \leq 10$. This result may come as something of a surprise since $B(9) < B(8)$! However, it is less surprising when one realizes that an additional time unit is required to complete broadcasting. The fact that $B(9) > 9$ follows from two observations. In a minimal broadcast graph with 9 vertices, any vertex of degree 1 must be adjacent to a vertex of degree 4. This property cannot be satisfied by the addition of one edge to the minimum broadcast tree with 9 vertices and root of degree 1. On the other hand, the only connected graph with a minimum vertex degree of 2 containing 9 vertices and 9 edges is C_9, which is not a minimal broadcast graph.

The proofs establishing $B(n)$ for the other values of $n \leq 15$ all employ methods similar to those above. As n increases, we must deal with an increasing number of graphs which satisfy the minimum vertex degree requirements of the minimum broadcast trees. We can represent these candidate graphs by their vertex degree sequences. Each degree sequence implies that the graphs consist of an associated set of paths, cycles and stars. These can be parameterized and eliminated in an orderly fashion through case analyses. For some values of n, there are very few minimum broadcast trees. In these cases, we examine the possible distributions of the remaining $l-n$ edges, showing that at least one more edge is required. Detailed proofs for all $n \leq 15$ are presented in [4].

4. Conclusion

Neither the value of $B(n)$ nor a technique for constructing a minimum broadcast graph is known for any $n \geq 17$ (except for $n = 2^k$). However, the graphs presented here can be used to improve the cost of minimal broadcast graphs obtained by the recursive construction algorithm defined in [2]. This efficient algorithm will thus produce approximations of minimum broadcast graphs. Attempts to construct reasonable approximations in an efficient manner are justified due to the likely NP-completeness of the problem of determining $B(n)$.

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