COMP 6651: Solutions to Assignment 8

Fall 2020

Submission through Moodle is due by November 15th at 23:55

1. (a) Variable x_1 stands for the amount of light beer to be produced, x_2 – regular beer, and x_3 – strong beer. The LP in standard form is

$$\begin{aligned} \max \quad & 30x_1 + 10x_2 + 30x_3 \\ & 2x_1 + x_2 + x_3 \leq 10 \\ & x_1 + 2x_2 + 3x_3 \leq 25 \\ & 2x_1 + 2x_2 + x_3 \leq 30 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(b) The initial slack form:

$$\begin{aligned} z &= 30x_1 + 10x_2 + 30x_3 \\ x_4 &= 10 - 2x_1 - x_2 - x_3 \\ x_5 &= 25 - x_1 - 2x_2 - 3x_3 \\ x_6 &= 30 - 2x_1 - 2x_2 - x_3 \end{aligned}$$

Basic feasible solution: $(0, 0, 0, 10, 25, 30)$ (here and below, format: $(x_1, x_2, x_3, x_4, x_5, x_6)$). Entering variable: x_1. Leaving variable: x_4. New slack form:

$$\begin{aligned} z &= 150 - 5x_2 + 15x_3 - 15x_4 \\ x_1 &= 5 - \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 &= 20 - \frac{3}{2}x_2 - \frac{5}{2}x_3 + \frac{1}{2}x_4 \\ x_6 &= 20 - x_2 + x_4 \end{aligned}$$

Basic feasible solution: $(5, 0, 0, 0, 20, 20)$. Entering variable: x_3. Leaving variable: x_5. New slack form:

$$\begin{aligned} z &= 270 - 14x_2 - 12x_4 - 6x_5 \\ x_1 &= 1 - \frac{1}{5}x_2 - \frac{3}{5}x_4 + \frac{1}{5}x_5 \\ x_3 &= 8 - \frac{3}{5}x_2 + \frac{1}{5}x_4 - \frac{2}{5}x_5 \\ x_6 &= 20 - x_2 + x_4 \end{aligned}$$

Basic feasible solution: $(1, 0, 8, 0, 0, 6)$. No entering variable – simplex terminates. Maximum daily profit is \$270 and is obtained by brewing 1 gallon of light beer and 8 gallons of strong beer (and no regular beer) daily.

(c) The dual LP is

$$\begin{aligned}\min \quad & 10y_1 + 25y_2 + 30y_3 \\ & 2y_1 + y_2 + 2y_3 \geq 30 \\ & y_1 + 2y_2 + 2y_3 \geq 10 \\ & y_1 + 3y_2 + y_3 \geq 30 \\ & y_1, y_2, y_3 \geq 0\end{aligned}$$

2. (a) Let $Ax \leq b$ be the constraints of the LP (if they are not in this form, they can be converted into an equivalent constraint matrix in this form). Let x_1 and x_2 be two distinct feasible solutions. Then for any $\alpha \in (0,1)$ solution $x_\alpha = \alpha x_1 + (1-\alpha)x_2$ is also feasible since $Ax_\alpha = A(\alpha x_1 + (1-\alpha)x_2) = \alpha A x_1 + (1-\alpha)Ax_2 \leq \alpha b + (1-\alpha)b = b$.

(b) Consider the following integer program:

PRIMAL	DUAL
$\max \quad y$	$\min \quad 3\beta$
$2y - 3x \leq 0$	$2\alpha + 2\beta \geq 1$
$2y + 3x \leq 3$	$-3\alpha + 3\beta \geq 0$
$x, y \geq 0$	$\alpha, \beta \geq 0$
$x, y \in \mathbb{Z}$	$\alpha, \beta \in \mathbb{Z}$

It is easy to verify that the only feasible integral solutions for the PRIMAL are $y = 0, x = 0$ and $y = 0, x = 1$. Thus, integral optimal value of PRIMAL is 0.

Observe that in the DUAL if we set $\beta = 0$ we are forced to set $\alpha \geq 1$ by the first constraint, but such choices of α and β would then violate the second constraint. Thus, the smallest feasible integral value of $\beta = 1$. Setting $\alpha = 0$ makes this solution feasible, therefore we conclude that integral optimal value of DUAL is 3.

(c) Let x be a single variable and consider the LP $\max -x$ subject to $x \geq 0$. Observe that the feasible region is $[0, \infty)$, so it is unbounded. Yet the optimal value of LP is 0, which is bounded.

(d) Consider a particular iteration of simplex. Suppose that $z = s_0 + c_1 x_1 + \cdots + c_n x_n$ and x_i is chosen as an entering variable. This means that $c_i > 0$. Furthermore suppose that x_j is chosen as a leaving variable. Let the constraint corresponding to x_j prior to pivot be

$$x_j = s_j - \sum_{k \neq j} \beta_{j,k} x_k,$$

Since x_j is chosen as a leaving variable, we have $\beta_{j,i} > 0$. After the pivot, we have

$$x_i = \frac{s_j}{\beta_{j,i}} - \frac{1}{\beta_{j,i}} x_j + \sum_{k \neq i,j} \frac{\beta j, k}{\beta_{j,i}}.$$

After plugging x_i into z we obtain that the coefficient of x_j in z is $-\frac{c_i}{\beta_{j,i}} < 0$. Therefore, in the next step of simplex x_j has a negative coefficient in front of it, so it won't be chosen as an entering variable.