COMP 6661 Combinatorial Algorithms Winter 2023
Assignment 1
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1. Given the following graphs:
- **The complete graph K_n on n vertices.**
- **The complete bipartite graph $K_{m,n}$ on m + n vertices.**
- **The n-vertex wheel W_n.**
- **The hypercube Q_n.**

a) Which of the above graphs is Eulerian? Justify.
b) Which of the above graphs is Hamiltonian? Justify.

a) A graph is **Eulerian** if it is connected and every vertex has an even degree.

- The complete graph **K_n is Eulerian for an odd value of n** and **not Eulerian for an even value of n**. This is because when n is odd each vertex will have a degree of n-1 which will be even.
- The complete bipartite graph **$K_{m,n}$ is Eulerian when both m and n are even** otherwise it's not Eulerian.
- The n-vertex wheel **W_n is not Eulerian**. All nodes except for the one in the center have precisely 3 edges in the wheel graph. Thus, it cannot have an Eulerian Path.
- The hypercube Q_n is Eulerian if and only if it is connected and every vertex has an even degree. The n-dimensional hypercube is connected and every vertex has a degree equal to n. Hence, the **hypercube is Eulerian if and only if n is even.**

b) A graph is a **Hamiltonian** if it has a Hamiltonian cycle, which is a cycle that visits every vertex exactly once.

- The complete graph **K_n is Hamiltonian** for $n \geq 3$ because it is possible to construct a cycle that visits every vertex. The graph C_n is a subgraph of every K_n which has a hamiltonian path as well as a hamiltonian circuit.
- The complete bipartite graph **$K_{m,n}$ is Hamiltonian if and only if $m = n > 1$**. This can be easily seen if we make a complete bipartite graph, the Hamiltonian cycle existing will be of the form $x1,y1,x2,y2,x3,y3\ldots,x_n,y_n,x1$ until it visits all the vertex.
- The n-vertex wheel **W_n is Hamiltonian** for $n \geq 3$ because it is possible to construct a cycle that visits every vertex, by starting from the "hub" vertex and visiting each spoke in turn, and then returning to the "hub" vertex.
- The hypercube **Q_n is Hamiltonian** for $n \geq 2$ because it is possible to construct a Hamiltonian cycle by visiting each vertex exactly once.

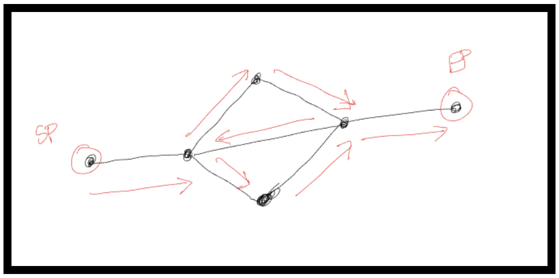
2. The line graph L(G) of graph G has a vertex for each edge of G, and two of these vertices are adjacent if and only if the corresponding edges in G have a vertex in common.
a) Prove that if a simple graph G is Eulerian, then its line graph L(G) is also Eulerian.
b) Prove or disprove. The line graph of any graph is Eulerian.
c) Prove that a graph with more than two vertices of odd degree does not contain an Eulerian path (or trail).

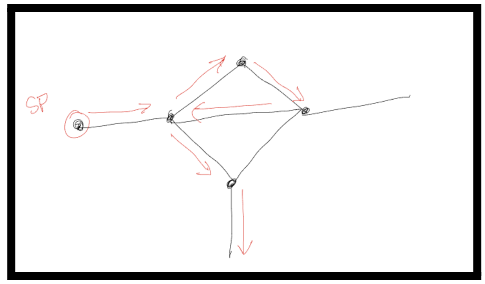
Ans)

A. A graph is Eulerian if and only if it is connected and every vertex has an even degree. The line graph of a graph G is a graph whose vertices correspond to the edges of G and two vertices in L(G) are connected by an edge if and only if the corresponding edges in G share a common endpoint. If G is Eulerian, it is connected and every vertex in G has an even degree. Therefore, every edge in G is connected to two vertices, and since these edges correspond to vertices in L(G), every vertex in L(G) has degree two. Thus, L(G) is a simple graph whose every vertex has an even degree, so L(G) is Eulerian.

B. This statement is **false**. A line graph of a graph G, L(G), is Eulerian if and only if G is Eulerian. The reason is that if G is Eulerian, it is connected and every vertex in G has an even degree. Therefore, every edge in G is connected to two vertices, and since these edges correspond to vertices in L(G), every vertex in L(G) has degree two. Thus, L(G) is a simple graph whose every vertex has an even degree, so L(G) is Eulerian.
However, if G is not Eulerian, the line graph L(G) may not be Eulerian. For example, consider graph G which is a cycle of 3 vertices. This graph is not Eulerian because it is not connected. The line graph L(G) will have 3 vertices and 3 edges, and each vertex will have degree 2. As all the vertices in L(G) have an even degree, it is tempting to say that L(G) is Eulerian. But it is not connected, it is not Eulerian. In conclusion, the statement "The line graph of any graph is Eulerian" is false. A line graph of a graph G is Eulerian if and only if G is Eulerian.

C. An Eulerian path (or trail) is a path that visits every edge of a graph exactly once. If graph G has more than two vertices of odd degrees, it cannot contain an Eulerian path. Let's suppose G has more than 2 vertices with an odd degree. For a vertex to be part of the Euler path it has to have an even degree, it can have an odd degree if that is the starting or the ending vertex of the path. To see this, consider that the number of edges in the path is equal to the number of edges in the graph. The edges in the path are used twice, once for each endpoint, and each endpoint in the graph should have an even degree. If there are more than two vertices of odd degree, that means that there are more than two vertices that will have an extra edge at the end of the path, and it is not possible to return to the starting vertex.

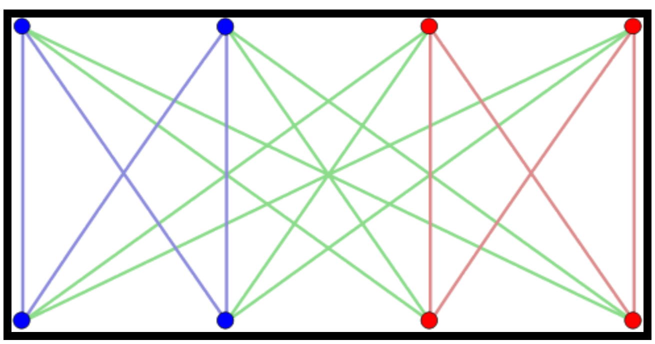




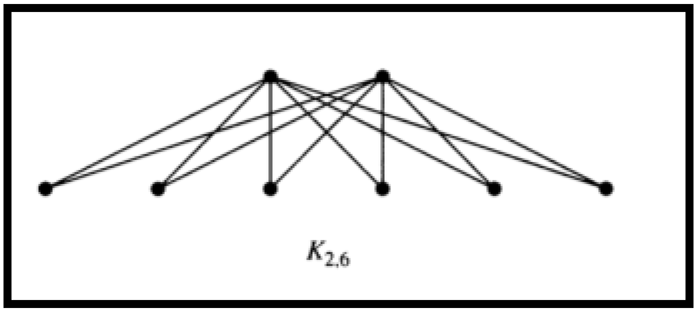
In the above image since the first figure has only 2 vertices with an odd degree there exists an Euler path since these two vertices are the start and end of the Euler path as marked by the red arrow but in the second figure since there is a third vertex with an odd degree it is not possible to traverse one of the odd degree vertex.

3. Draw the specified graph or prove that it does not exist:
a) An 8-vertex simple graph with more than 8 edges that is both Eulerian and Hamiltonian.
b) An 8-vertex simple graph with more than 8 edges that is Eulerian but not Hamiltonian.
c) An 8-vertex simple graph with more than 8 edges that is Hamiltonian but not Eulerian.
d) An 8-vertex simple Hamiltonian graph that does not satisfy the conditions of Ore's theorem.
e) A 6-vertex simple graph with 10 edges that is not Hamiltonian.

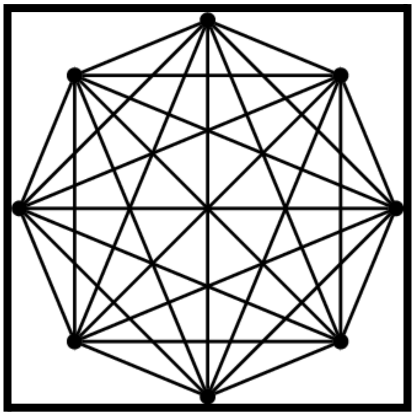
a) Graph $K_{4,4}$ is a complete bipartite graph with 8-vertex and more than 8 edges which is both eulerian and hamiltonian.



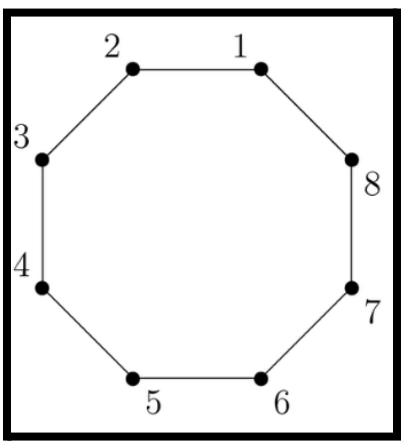
b) The complete bipartite graph $K_{2,6}$ is Eulerian but not Hamiltonian. It has 8 vertices and more than 8 edges.



c) Complete graph K_8 is Hamiltonian but not Eulerian. It has 8 vertices and more than 8 edges. Each vertex has an odd degree.



d) There are many graphs that do not satisfy Ore's theorem in fact all C_n graphs don't satisfy ores theorem, but all C_n are hamiltonian hence C_8 is the graph with 8 vertices which doesn't satisfy Ore's theorem and is hamiltonian.



e) Let's assume that G is not hamiltonian hence by Ore's theorem converse of this should be true:

$\deg(x) + \deg(y) \geq 6 \quad (|V| = 6)$

So for graph G: $\deg(x) + \deg(y) \leq 5$

We also know that sum of the degree of all vertices is 2 times the edge count which gives:

$$\sum_{v \in V} d(v) = 20 = 2*|E|$$

Considering the degree of all 6 vertices to be a,b,c,d,e,f respectively.

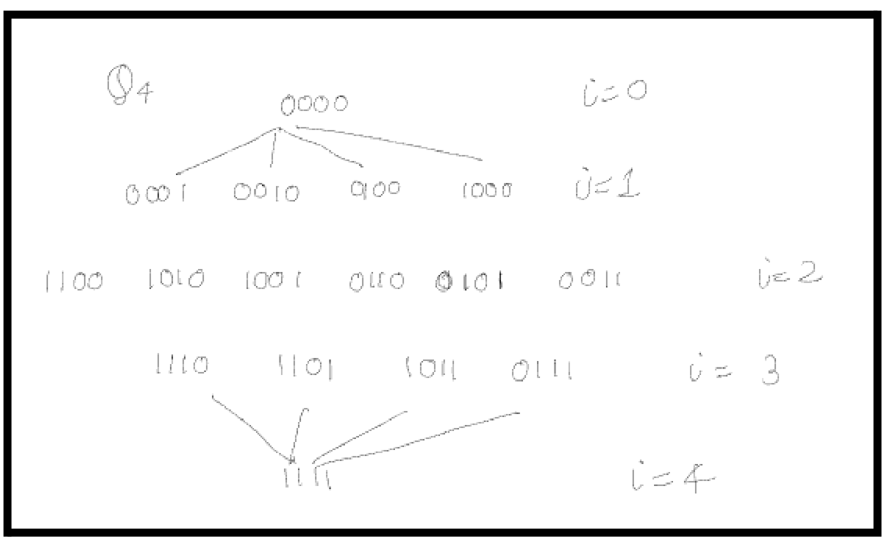
a+b+c+d+e+f = 20

One of the integral solutions to this equation with minimizing the variables is:
=>3,3,3,3,4,4

Considering any two nonadjacent vertices in this we get the sum greater than or equal to 6 which proves that the **graph is hamiltonian**. Hence assumption G is not hamiltonian is false and G is a hamiltonian graph so, **a graph with 6 vertices and 10 edges that is not hamiltonian is not possible.**

4. Run BFS rooted at all zero vertexes, i.e. (00 … 0) of the k-dimensional hypercube Q_k. What is the number of vertices at a distance i from the root, for all i = 0, 1, 2, …, k, …? What is the number of edges between level i and i + 1? Prove your answers.

The number of vertices at a distance i from the root in a k-dimensional hypercube is equal to the number of ways to choose i positions from k positions to flip the bits. This is equal to the number of combinations of k things taken i at a time, C(k, i).



The number of vertices at a distance i from the root is C(k, i) because each bit flip represents a choice of one position from k positions, and the number of ways to choose i positions from k positions is C(k, i).

Proof: The number of edges between level i and i + 1 is equal to the number of vertices at a distance i multiplied by (k-i). This is because each vertex at distance i have exactly k-i neighbors at distance i + 1. Because the elements at distance i+1 have to be bigger than elements at distance i and also differ at the 1-bit position and to do that we have to change the 0th bit at level i since at level i a number of set bits are equal to i remaining bits are (k-i) and hence the edges for each vertex at that level. Thus, the number of edges is C(k, i) * (k-i).

The number of edges between level i and i + 1 is C(k, i) * (k-i) because each vertex at distance i have exactly k-i neighbors at distance i + 1, and there are C(k, i) vertices at distance i.

So, the answer is **C(k, i) is the number of vertices at distance i,** and **C(k, i)*(k-i) is the number of edges between i and i+1**.

5. Prove by induction on e (number of edges) that a planar graph is bipartite if and only if every face has an even length.

Proof by induction on the e (number of edges):
Base case: The graph has no edges. The statement is trivial as the graph has no vertices and is therefore both planar and bipartite.

Inductive step: Suppose the statement holds for all planar graphs with fewer than n edges. Consider a planar graph G with n+1 edges where every face boundary is a cycle of even length. We remove an edge e from G to obtain a new graph G' with n edges. By the induction hypothesis, G' is bipartite. Let (V1, V2) be a bipartition of G'.
If e connects two vertices in V1 (or two vertices in V2), then G is also bipartite and we can obtain a bipartition by simply adding e back to G'.
If e connects a vertex in V1 to a vertex in V2, then G is not bipartite. However, since G is planar and every face boundary is a cycle of even length, it follows that the removal of e splits the original face into two faces, both with even-length boundaries. Hence, G is still bipartite, and we can obtain a bipartition by simply replacing the removed edge e with two edges connecting the two newly created faces.

In either case, we have shown that G is bipartite, which completes the inductive step. By induction, the statement holds for all planar graphs where every face boundary is a cycle of even length.

6. Prove that if graph G has n vertices then χ(G) + χ(G’) <= n + 1, where G’ is the complement of graph G.

Lemma: For every graph G such that V(G)=n

$$\chi(G) \leq 1+\max\{\triangle(H)\} \text{ (Brooks' Theorem)}$$

where $\max\{\triangle(H)\}$ is the maximum degree of all vertices in the subgraph H of graph G and the maximum is taken over all the subgraphs H of G

Let $q=\max\{\triangle(H)\}$

Then, by the above lemma, we have $\chi(G)\leq 1+q$

Next we determine $\max\{\triangle(G')\}$ which is $n-q-1$. Assuming the contrary, that is, there exists a subgraph H' of G' such that $\triangle(H')\geq n-q$. This implies that every vertex of H' has a degree less than or equal to q-1 in the complement graph G.

Let K be a subgraph of G such that $\triangle(K)=q$ (note such a subgraph exists since $q=\max\{\triangle(H)\}$). Clearly, no vertex in K is in H, meaning that the degree of every vertex in K in the complement graph G' is less than or equal to n-q-1.

Since $|V(K)|\geq q+1$ (since $\triangle(K)=q$), it follows that $|V(H)|\leq n-(q-1)=n-q-1$, which contradicts the fact that $\triangle(H')\geq n-q$.

Therefore, $\max\{\triangle(G')\}\leq n-q-1$, and by the lemma $\chi(G')\leq 1+(n-q-1)=n-q$.

Putting this all together gives

$$\chi(G)+\chi(G')\leq(1+q)+(n-q)=n+1.$$

7. a) Let G = (V, E) be a loop-free undirected graph with |V| = n >= 3, and deg(x) + deg(y) >= n - 1 for all nonadjacent vertices x, y ∈ V . Prove that there is a path of length at most 2 between each pair of vertices of G.
b) Prove that a graph with n vertices and at least $\frac{(n-1)*(n-2)}{2}$ + 1 edges is connected.

a) Proof by contradiction. Suppose for some vertices u,v in V, there is no path of length at most 2 between them. Then, neither u nor v is a neighbor of the other.

Since deg(u) + deg(v) >= n - 1, at least one of them, say u, has a degree of at least (n-1)/2. Let N(u) denote the set of neighbors of u. Then, |N(u)| >= (n-1)/2.

Since v is not a neighbor of u, it must be adjacent to some vertex in N(u). This means that there exists a vertex w in N(u) such that (u,w) and (v,w) are edged in E, forming a path of length 2 between u and v, which contradicts our assumption.

Therefore, for any two vertices u,v in V, there exists a path of length at most 2 between them, which proves the theorem.

b) If $|E| \geq {}^{n-1}C_2 + 2$ with $|V| >= 3$ we can prove that deg(a)+deg(b)>=n

Remove the following from G
(i) all edges of the form (a, x) where $x \in V$
(ii) all edges of the form (y, b) where $y \in V$
(iii) the vertices a & b

Let $H = (V', E')$ denote the resulting subgraph. Then
$$|E| = |E'| + \deg(a) + \deg(b)$$
because $(a, b) \notin E$. Since $|V'| = |V| - 2 = n - 2$, H is a subgraph of K_{n-2}, so $|E'| \leq {}^{n-1}C_2$. Hence
$${}^{n-1}C_2 + 2 \leq |E| = |E'| + \deg(a) + \deg(b) \leq {}^{n-2}C_2 + \deg(a) + \deg(b)$$
Therefore
$\deg(a) + \deg(b) \geq {}^{n-1}C_2 + 2 - {}^{n-2}C_2$
$=\frac{1}{2}(n - 1)(n - 2) + 2 - \frac{1}{2}(n - 2)(n - 3)$
$=\frac{1}{2}(n - 2)((n - 1) - (n - 3)) + 2$
$=\frac{1}{2}(n - 2).2 + 2$
$= n$
Therefore it follows from Ore's theorem that G has a Hamilton cycle.

This was the condition taken for the edges $|E| \geq {}^{n-1}C_2 + 2$
If we expand the choose and remove 1 edge from it we will get $\frac{(n-1)*(n-2)}{2} + 1$
So if we remove 1 edge from G the remaining graph contains a path with all the remaining vertices. Since a path is connected this proves the claim.