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Minimum k-broadcast graphs

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Abstract

Broadcasting is an information dissemination process in which a message is to be sent from a single originator to all members of a network by placing calls over the communication lines of the network. We study here a variant of broadcasting called k-broadcast, where a processor can call several of its neighbours in one time unit. A k-broadcast graph is an n-vertex communication network that supports a broadcast from any one vertex to all other vertices in optimal time $\lceil \log_{k+1} n \rceil$. A minimum k-broadcast graph is a k-broadcast graph having the minimum number of edges. In this paper, after giving simple results generalizing previously known results on 1-broadcasting, we find minimum k-broadcast graphs for $k + 3 \leqslant n \leqslant 2k + 3$.

Key words: Broadcasting; Graphs; Networks

1. Introduction

Broadcasting refers to the process of message dissemination in a communication network whereby a message, originated by one node, is transmitted to all nodes of the network. Broadcasting is accomplished by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible subject to the constraints that each call involves only one informed node and some of its neighbours, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call its neighbours.

The original definition of broadcasting restricted the calls to involve only two vertices. Here we extend this definition to calls involving several vertices. That is, we shall say that each call involves only one originator and k or fewer of its neighbours.

Given a connected graph G and a message originator, vertex u, we define the *broadcast time of vertex* u, $b_k(u)$, to be the minimum number of time units required to complete broadcasting from vertex u. It is easy to see that for any vertex u in

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a connected graph G with n vertices, $b_k(u) \geqslant \lceil \log_{k+1} n \rceil$, since the number of informed vertices can at most be multiplied by $k + 1$ during each time unit.

We define the *broadcast time of a graph G*, $b_k(G)$, to be the maximum broadcast time of any vertex u in G, i.e. $b_k(G) = \max\{b_k(u) | u \in V(G)\}$. For the complete graph K_n with $n \geqslant 2$ vertices, $b_k(K_n) = \lceil \log_{k+1} n \rceil$, yet K_n is not minimal with respect to this property for any $n > 3$, while $n > k + 1$. That is, we can remove edges from K_n and still have a graph G with n vertices such that $b_k(G) = \lceil \log_{k+1} n \rceil$. We use the term *k-broadcast graph* to refer to any graph G on n vertices with $b_k(G) = \lceil \log_{k+1} n \rceil$.

We define the *broadcast function*, $B_k(n)$, to be the minimum number of edges in any k-broadcast graph on n vertices. A *minimum k-broadcast graph* (mbg) is a k-broadcast graph G on n vertices having $B_k(n)$ edges. From an application perspective, minimum k-broadcast graphs represent the cheapest possible communication networks (having the fewest communication lines) in which broadcasting can be accomplished, from any vertex, as fast as theoretically possible.

2. Previous results

Most of the previous work in this area has been on the processor-bound problem, that is $k = 1$. For a survey of results on broadcasting and related problems, see Hedetniemi et al. [6]. Johnson and Garey [7] showed that the problem of determining $b_1(v)$ for a vertex v in an arbitrary graph G is NP-complete. In [4] Farley et al. studied $B_1(n)$. In particular, they determined the values of $B_1(n)$ for $n \leqslant 15$ and noted that $B_1(2^p) = p2^{p-1}$ (the p-cube is an mbg on $n = 2^p$ vertices). Mitchell and Hedetniemi [10] determined the value for $B_1(17)$, Wang [12] found the value of $B_1(18)$, and Bermond et al. [1] found the values of $B_1(19)$, $B_1(30)$ and $B_1(31)$. Recently, Mahéo and Saclé [11] found the values of $B_1(20)$, $B_1(21)$ and $B_1(22)$. On the directed broadcasting problem, that is broadcasting in digraphs, Liestman and Peters [9] studied $\boldsymbol{B}(n)$, the minimum number of arcs in a broadcast digraph on n vertices.

Since these studies suggest that mbgs are extremely difficult to find, several authors have devised methods to construct graphs with small numbers of edges which allow minimum time broadcasting from each vertex. In [3] Farley designed several techniques for constructing broadcast graphs with n vertices and approximately $\frac{1}{2}\log_2 n$ edges. Chau and Liestman [2] presented constructions based on Farley's techniques which yield somewhat sparser graphs for most values of n. In [5] Grigni and Peleg showed that $B_1(n) \in \Theta(L(n)n)$ where $L(n)$ denotes the exact number of consecutive leading 1's in the binary representation of $n - 1$. More generally, they showed that $B_k(n) \in \Theta(kL_k(n)n)$ where $L_k(n)$ denotes the exact number of consecutive leading k's in the $(k + 1)$-ary representation of $n - 1$. Asymptotically, Grigni and Peleg's construction (which establishes their upper bound) produces the best results for most values of n. In [8], Lazard gave some results for minimum k-broadcast graphs and especially some values of $B_2(n)$, $B_3(n)$ and $B_4(n)$ for small values of n. He also investigated k-broadcasting in bounded degree graphs.

3. Elementary results

In this section, we give two simple lemmas which generalize previously known results on 1-broadcasting that can be found with the references given for each lemma. Also, setting k equal to one in the formulas will give those 1-broadcasting results.

Lemma 3.1 [1]. *If a k-broadcast graph on n vertices, $(k+1)^{i-1}+1 < n \leqslant (k+1)^i$, with e edges has a vertex of degree Δ, then:*

$$B_k(n-1) \leqslant e + \Delta(\Delta-3)/2.$$

Proof. Let G be a k-broadcast graph on n vertices with e edges where $(k+1)^{i-1}+1 < n \leqslant (k+1)^i$ and a vertex v of degree Δ.

Construct a graph G' on $n-1$ vertices by deleting the vertex v (and the Δ edges incident to it) from G and adding any edges that are necessary to form a clique among the neighbours of v. We claim that G' is a k-broadcast graph on $n-1$ vertices.

If this is true, it follows that

$$B_k(n-1) \leqslant e + \frac{\Delta(\Delta-1)}{2} - \Delta = e + \frac{\Delta(\Delta-3)}{2}.$$

In order to show that G' is a k-broadcast graph, we need only to show that each vertex in G' can broadcast in $\lceil \log_{k+1}(n-1) \rceil$ time units. Note that due to the restriction on n, $\lceil \log_{k+1}(n-1) \rceil = \lceil \log_{k+1} n \rceil$.

Consider any broadcast scheme for G and let u be the vertex that informs vertex v and that this call occurs at time t_0. In this scheme, v may subsequently call some of its neighbours, say at time t_i the set of vertices S_i, where $|S_i| \leqslant k$ and $(1 \leqslant i \leqslant q)$.

A broadcasting scheme for G' can easily be obtained from the scheme for G by deleting the calls involving vertex v and adding

- call from u to $v_1 \in S_1$ at time t_0.
- call from v_1 to $S_1 \setminus \{v_1\} \cup \{v_2\}$ where $v_2 \in S_2$, at time t_1.
- call from v_i to $S_i \setminus \{v_i\} \cup \{v_{i+1}\}$ where $v_{i+1} \in S_{i+1}$, at time t_i.
- call from v_q to $S_q \setminus \{v_q\}$ at time t_q.

Thus every vertex is ready to continue with other calls as in the scheme for G. □

Lemma 3.2 [5].

$$B_k(n) \geqslant \frac{nk}{2}(\lceil \log_{k+1} n \rceil - \lfloor \log_{k+1}((k+1)^{\lceil \log_{k+1} n \rceil} - (n-1)) \rfloor - 1).$$

Proof. Let c_t^{Δ} denote the maximum number of vertices that can be informed in time t in any graph in which the originator u has a degree of Δ. It is clear that a neighbour who receives the message at time i can forward it to no more than $(k+1)^{t-i}-1$ other vertices by time t.

Since at most k neighbours of u can be informed at each time from 1 to $\lceil \Delta/k \rceil$, we know that

$$c_t^\Delta \leqslant 1 + k \sum_{i=1}^{\lceil \Delta/k \rceil} (k+1)^{t-i}.$$

Thus if each vertex in the graph must be able to originate a broadcast informing n vertices by time $\lceil \log_{k+1} n \rceil = t$, we have

$$n \leqslant 1 + k \sum_{i=1}^{\lceil \Delta/k \rceil} (k+1)^{t-i}, \quad \frac{n-1}{k} \leqslant \sum_{i=1}^{\lceil \Delta/k \rceil} (k+1)^{t-i},$$

$$\frac{n-1}{k} \leqslant \sum_{j=t-\lceil \Delta/k \rceil}^{t-1} (k+1)^j, \quad \frac{n-1}{k} \leqslant \frac{(k+1)^{t-\lceil \Delta/k \rceil} - (k+1)^t}{1-(k+1)},$$

$$(k+1)^{t-\lceil \Delta/k \rceil} \leqslant (k+1)^t - (n-1), \quad t - \lceil \Delta/k \rceil \leqslant \log_{k+1}((k+1)^t - (n-1)),$$

as $t - \lceil \Delta/k \rceil$ is a natural number, we have

$$t - \lceil \Delta/k \rceil \leqslant \lfloor \log_{k+1}((k+1)^t - (n-1)) \rfloor,$$

and thus each vertex must have a degree at least

$$k(\lceil \log_{k+1} n \rceil - \lfloor \log_{k+1}((k+1)^{\lceil \log_{k+1} n \rceil} - (n-1)) \rfloor - 1)$$

and the result follows. □

We now extend some of Farley's constructions to produce sparse k-broadcast graphs (sbg), that is, graphs capable of broadcasting in minimum time but which have more edges than the minimum possible. The construction is by induction on n.

- First of all, we construct SBG(n) for $n \leqslant k+1$. In this case, an initiator must be able to broadcast to all other vertices in one time unit. Therefore, a vertex must be connected to all others. So SBG(n) = K_n the complete graph on n vertices.
- To construct SBG(n) for all n, select integers $n_1, \ldots, n_{k+1}$ such that

$$\sum_{i=1}^{k+1} n_i = n, \qquad \lceil \log_{k+1} n_i \rceil = \lceil \log_{k+1} n \rceil - 1,$$

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant n_{k+1}, \qquad n_i - n_{k+1} \leqslant 1.$$

To do this, if $n = \lfloor n/(k+1) \rfloor (k+1) + x$ let $n_1 = \cdots = n_x = \lfloor n/(k+1) \rfloor + 1$ and $n_{x+1} = \cdots = n_{k+1} = \lfloor n/(k+1) \rfloor$. Construct each SBG($n_i$). Now, for each member of SBG(n_{k+1}), connect it to a different member in each component so that they form a clique. Form a clique between each member nonconnected of $n_1, \ldots, n_x$. It is easy to see that this graph can broadcast in minimum time. Also, we used

$$\left\lfloor \frac{n}{k+1} \right\rfloor \frac{k(k+1)}{2} + \frac{x(x-1)}{2}$$

edges to connect the different components.

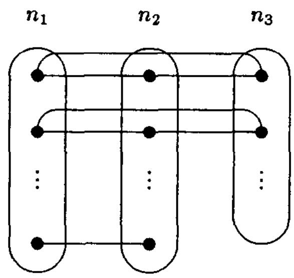

Fig. 1. Example of sbg for $k = 2$ and $n \equiv 2 \pmod 3$.

In Fig. 1, we give an example of an sbg for $k = 2$ and $n \equiv 2 \pmod 3$, which gives $x = 2$.

Lemma 3.3. *The total number of edges used to construct* SBG(n) *is less than or equal to* $\frac{1}{2}nk\lceil \log_{k+1} n \rceil$.

Proof. The proof is also by induction.

- If $n \leqslant k + 1$, SBG$(n) = K_n$ and has $n(n-1)/2$ edges, which is less than or equal to $nk/2$.
- The total number of edges used in SBG(n) is less than or equal to:

$$\sum_{1}^{k+1} \frac{n_i k}{2}(\lceil \log_{k+1} n \rceil - 1) + \left\lfloor \frac{n}{k+1} \right\rfloor \frac{k(k+1)}{2} + \frac{x(x-1)}{2}$$

$$\leqslant \frac{nk}{2}\lceil \log_{k+1} n \rceil - \frac{nk}{2} + \left\lfloor \frac{n}{k+1} \right\rfloor \frac{k(k+1)}{2} + \frac{x(x-1)}{2}$$

$$\leqslant \frac{nk}{2}\lceil \log_{k+1} n \rceil - \frac{nk}{2} + \frac{k(n-x)}{2} + \frac{x(x-1)}{2}$$

$$\leqslant \frac{nk}{2}\lceil \log_{k+1} n \rceil - \frac{xk}{2} + \frac{x(x-1)}{2}$$

$$\leqslant \frac{nk}{2}\lceil \log_{k+1} n \rceil. \qquad \square$$

4. Construction of minimum k-broadcast graphs

Let $n = k + r + 1$, $2 \leqslant r \leqslant k + 2$ and δ be the minimum degree.

We wish to find minimum k-broadcast graphs. That is graphs which broadcast in time 2 (because $k + 1 \leqslant n \leqslant (k+1)^2$) and which have the minimum number of edges.

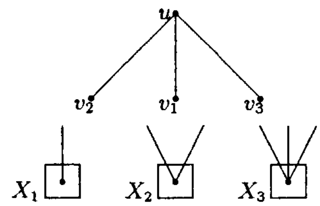

Fig. 2.

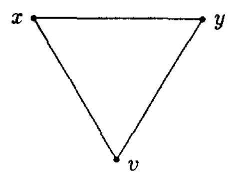

Fig. 3.

We can see that δ cannot be equal to 1 because a vertex of degree one can broadcast, in two time units, to only $k + 2$ vertices.

Let G be a mbg on n vertices with E edges (the diameter of G is at most 2). We wish to give a tight lower bound on $E = B_k(n)$.

4.1. *The case $\delta \geqslant 3$*

If $\delta \geqslant 4$ then $E \geqslant 4n/2 = 2n$.

Now $\delta = 3$ so let u be a vertex of degree 3 and v_1, v_2, v_3 its neighbours (see Fig. 2). A vertex is in X_i if it is connected to only i of the v_j $(1 \leqslant j \leqslant 3)$.

$$E \geqslant 3 + |X_1| + 2|X_2| + 3|X_3| + \tfrac{1}{2}(2|X_1| + |X_2|),$$

$$E \geqslant 2n - 5 + |X_3| + \tfrac{1}{2}|X_2|.$$

We obtain $E \geqslant 2n - 4$ except if X_2 and X_3 are empty and if all other vertices are of degree 3. In this case, we can easily show that $n = 10$ and we obtain the Petersen graph which can broadcast in time 2, with $2n - 5$ edges.

4.2. *The case $\delta = 2$*

If there is only one vertex of degree 2, then by the same demonstration as above, $E \geqslant 2n - 4$.

If there are at least two vertices x and y of degree 2, then we present several cases.

Case 1: If there is an edge between x and y, we have two possibilities.

Case 1.1: Either x and y have the same second neighbour (see Fig. 3), then $n = k + 3$ because v must be able to call all the other vertices in one time unit. Then

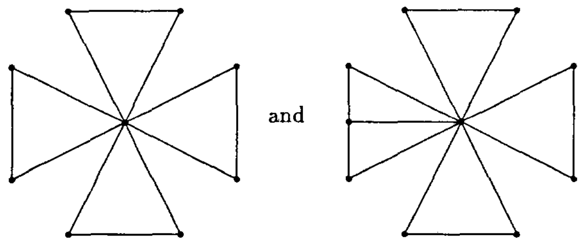

Fig. 4.

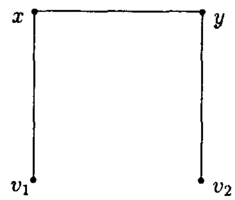

Fig. 5.

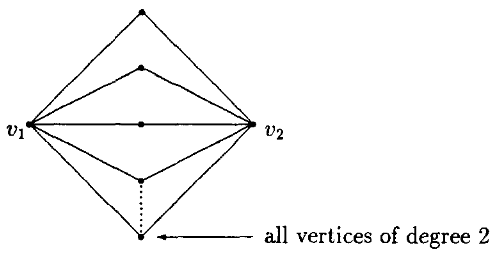

Fig. 6.

the graphs are all mbg and if n is odd, $E = 3(n-1)/2$, if n is even, $E = 3n/2 - 1$ (see Fig. 4).

Case 1.2: Or they have different second neighbours, see Fig. 5. Then $k + 3 \leqslant n \leqslant k + 4$. Any other vertex must be connected to v_1 and to v_2, to be able to broadcast to x and y in two time units. Therefore $E \geqslant 2n - 5$.

Case 2: We must now consider the case when no two vertices of degree two are connected. Any two vertices of degree two have at least one neighbour in common. There are two possibilities.

Case 2.1: Any pair of vertices of degree two have both of their neighbours, v_1 and v_2 in common. Any other vertex is connected to v_1 or v_2. $E \geqslant 2n - 4$ (see Fig. 6).

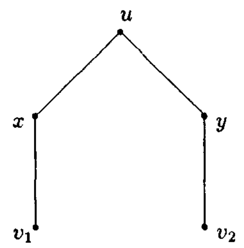

Fig. 7.

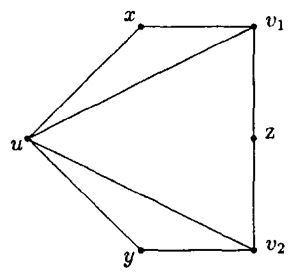

Fig. 8.

Case 2.2: There are two vertices, x and y, of degree two connected as with $v_1 \neq v_2$. Again we have two possibilities (see Fig. 7).

Case 2.2.1: At least one of the two edges $\{v_1, u\}$ and $\{v_2, u\}$ does not exist (for example $\{v_1, u\}$). Then the edge $\{v_1, v_2\}$ must be present so that the broadcast can be done in 2 time units (otherwise there is no path of length 2 between v_1 and y). All the other vertices must be linked to u or (v_1 and v_2). Now we partition these vertices into three sets.

- $X_1 = \{$vertices linked only to $u\}$,
- $X_2 = \{$vertices linked only to v_1 and $v_2\}$,
- $X_3 = \{$vertices linked only to u and atleast one of v_1 and $v_2\}$.

Each vertex in X_1 must have a neighbour outside of X_1 and different than u because it must be at a distance of 2 from v_1 and v_2 and because one of the edges $\{u, v_1\}$, $\{u, v_2\}$ does not exist. Therefore,

$$E \geqslant 5 + 2|X_1| + 2|X_2| + 2|X_3| \geqslant 5 + 2(n-5) \geqslant 2n - 5.$$

Using this construction, it has been possible to build k-minimum broadcast graphs using exactly $2n - 5$ edges. See Fig. 10.

Case 2.2.2: Now, we suppose that both edges $\{u, v_1\}$, $\{u, v_2\}$ exist.

(a) $\{v_1, v_2\}$ does not exist.

(i) There is a vertex of degree two between v_1 and v_2 (see Fig. 8). This case has already been considered in Case 2.2.1, with vertex z corresponding to vertex y in Case 2.2.1, v_1 to u and u to v_1.

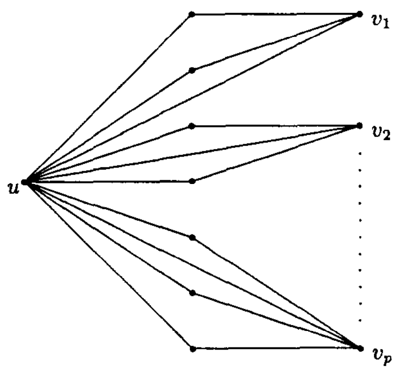

Fig. 9.

(ii) There is no vertex of degree two between v_1 and v_2. Then every vertex of degree two must be linked to u (otherwise we would have two vertices of degree two without one neighbour in common).

We have p different vertices, apart from u, which are neighbours of vertices of degree two (see Fig. 9).

$$E \geqslant 2|D| + 2|R| + p \geqslant 2(n-1) - p. \tag{1}$$

If we start broadcasting from a vertex of degree two, it calls u and v_i in one time unit. Then, u can call at most k neighbours, therefore v_i must call the remaining ones, and there are $n-k-3 = r-2$ of them. So the degree of v_i is r, because it is also linked to u (otherwise we are in the case (i)) and to the initiator. So

$$E \geqslant rp + |D| + |R| \geqslant rp + n - 1 - p \geqslant (r-1)p + (n-1). \tag{2}$$

As E must satisfy equations (1) and (2), the minimum is obtained when both terms are equal i.e. $p = (n-1)/r$.

As p is a natural number, we have $p = \lceil (n-1)/r \rceil$ or $p = \lfloor (n-1)/r \rfloor$ but, we have

$$2(n-1) - \left\lfloor \frac{n-1}{r} \right\rfloor \leqslant (r-1)\left(\left\lfloor \frac{n-1}{r} \right\rfloor + 1\right) + (n-1),$$

because $n - 1 \leqslant r\lfloor (n-1)/r \rfloor + r - 1$. Therefore $E \geqslant 2(n-1) - \lfloor (n-1)/r \rfloor$.

(b) $\{v_1, v_2\}$ exists. Let x and y be two vertices of degree two: either they have the same neighbours, or they have a common neighbour and the other neighbours are adjacent (otherwise we are in the previous cases). So the vertices of the graph can be partitioned into three sets:

- $X_1 = \{$the vertices of degree 2$\}$,
- $X_2 = \{$the vertices adjacent to at least one vertex of $X_1\}$,
- $X_3 = \{$the other vertices $\}$.

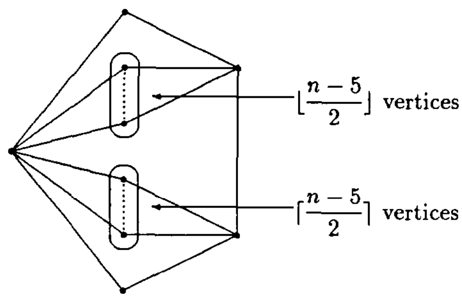

Fig. 10.

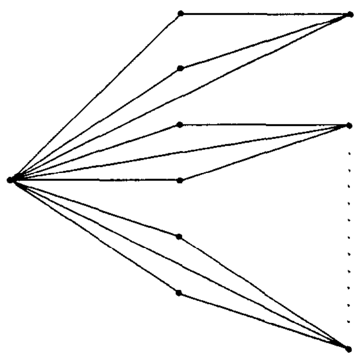

Fig. 11.

We can remark the vertices of X_2 are all linked together and each vertex of X_3 has degree at least three and has an adjacent vertex in X_2. So we have

$$E \geqslant 2|X_1| + \frac{|X_2|*(|X_2|-1)}{2} + 2|X_3| = 2n + \frac{|X_2|*(|X_2|-5)}{2}.$$

As $|X_2| \geqslant 3$, $E \geqslant 2n-3$.

5. Conclusion

For $n \geqslant k+5$ (see Figs. 10 and 11), the first series of mbg has $2n-5$ edges and the second series of mbg has $2n-2-\lfloor (n-1)/r \rfloor$ edges.

Therefore if $r \leqslant \lfloor k/3 \rfloor$, $\lfloor (n-1)/r \rfloor \geqslant 4$ then the second series of mbg is the best one. If $r > \lfloor k/3 \rfloor$, then the first series of mbg is the best one.

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