Broadcasting

General Definitions, Models and Problems in Broadcasting

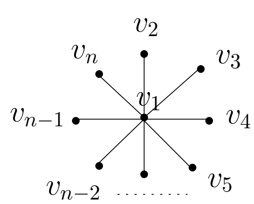
See the doctoral seminar slides by Calin D. Morosan

Broadcasting in Some Known Topologies

Path on n vertices $\boldsymbol{P_n}$:

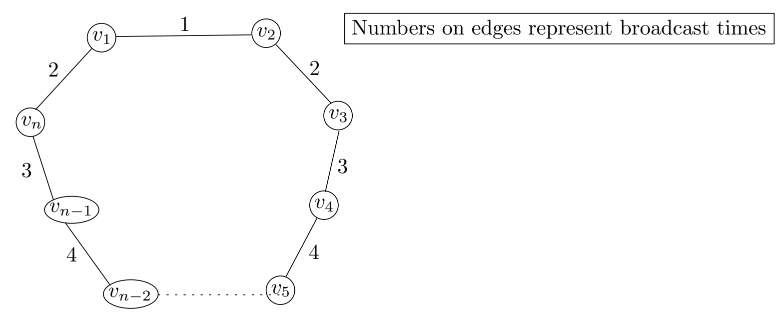
$$\left.\begin{aligned} b(v_1, P_n) &= n-1 \\ b(v_2, P_n) &= n-2 \\ &\dots \\ b(v_{n-1}, P_n) &= n-2 \\ b(v_n, P_n) &= n-1 \end{aligned}\right\} b(P_n) = n-1$$

Star on n vertices $\boldsymbol{S_n}$:



For all i, $b(v_i, S_n) = n-1$ hence $b(S_n) = n-1$

Cycle on n vertices $\boldsymbol{C_n}$:



For all i, $b(v_i, C_n) = \lceil \frac{n}{2} \rceil$ hence $b(C_n) = \lceil \frac{n}{2} \rceil$

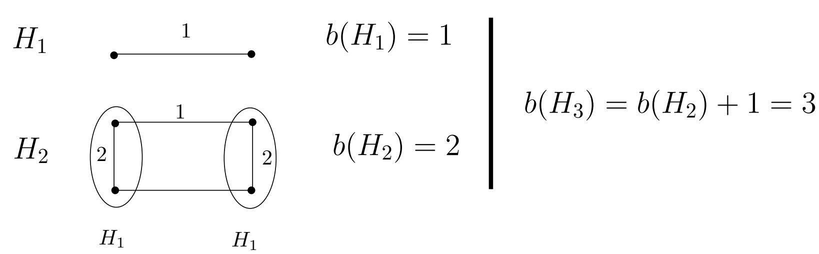
Complete graph on n vertices $\boldsymbol{K_n}$:

Let the vertices be labelled $1, 2, \ldots, n-1, n$. Following is the broadcast scheme (protocol) in K_n where $\rightarrow$ represents the action of a vertex informing another one.

Time 0 : 1 (Number of informed vertices is 1)
Time 1 : $1 \rightarrow 2$ (Number of informed vertices is $1 + 2^0$)
Time 2: $1 \rightarrow 3$, $2 \rightarrow 4$ (Number of informed vertices is $1 + 2^1$)
Time 3: $1 \rightarrow 5$, $2 \rightarrow 6$, $3 \rightarrow 7$, $4 \rightarrow 8$ (Number of informed vertices is $1 + 2^2$)

At time k, there are $1 + 2^{k-1}$ informed vertices, also after time k all vertices between 1 and 2^k are informed.
Hence for all i, $b(v_i, K_n) = b(K_n) = \lceil \log_2 n \rceil$

Hypercube of dimension n $\boldsymbol{H_n}$ (also known as n-cube Q_n):



Note that:

- If 01001 represents a vertex of H_5 then its first dimensional neighbour is 11001, its second dimensional neighbour is 00001, its third dimensional neighbour is 01101, etc
- H_k is formed by 2^p copies of H_{k-p} that are connected through a hypercube H_p

A broadcast scheme for H_k that achieves $b(H_k) = k$:
Given the originator $00\ldots0$, at each time unit i, every informed vertex sends the message to its i^{th} dimensional neighbour

Time 1: $00\dots0 \to 10\dots0$
Time 2: $00\dots0 \to 010\dots0$ and $10\dots0 \to 110\dots0$
...

General Bounds on Broadcasting

Let G be a graph on n vertices then:

$$\lceil \log_2 n \rceil \leq b(G) \leq n-1$$

Proof

- $b(G) \leq n-1$ because at every time unit we should have at least one informed vertex ($n-1$ is the broadcast time of a path on n vertices)
- $\lceil \log_2 n \rceil \leq b(G)$: When broadcasting from any originator, the number of informed vertices after time t is at most 2^t. Assume that after time unit s all n vertices of G received the message; it follows that $n \leq 2^s$ so $s \geq \log_2 n$. As s is an integer, $s \geq \lceil \log_2 n \rceil$

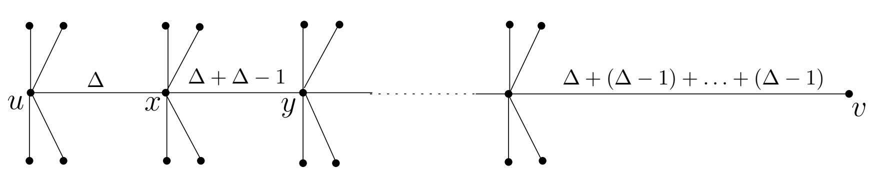
□

Let $b(G)$ be the broadcast time, $D(G)$ the diameter and $\Delta(G)$ the maximum degree of a graph G then:

$$D(G) \leq b(G) \leq D(G) \times (\Delta(G)-1)+1$$

Proof

- $D(G) \leq b(G)$: If you consider diametral vertices u and v ($d(u,v) = D(G)$), then when broadcasting from originator u, one cannot inform vertex v sooner than time unit $D(G)$
- $b(G) \leq D(G) \times (\Delta(G)-1)+1$: to prove this side of the inequality, lets consider the "worst" graph G and the "worst" broadcast scheme possible. Following figure represents a sub-graph of G where u and v are diametral vertices ($d(u,v) = D(G)$)



Suppose that the path that is shown in this figure is the shortest path from u to v and that all vertices of this path (except v itself) is of degree Δ (the maximum degree of G). Moreover suppose that the broadcast scheme applied from originator u is such that all vertices inform their neighbours on this path only after informing all of their other neighbours. Following that broadcast scheme u will inform x at time Δ, x

will inform y at time $\Delta + \Delta - 1$, etc. Hence v will be informed at time $\Delta + (D-1)(\Delta - 1)$ where D represents the diameter of G. So

$$\begin{aligned} b(u,v) &\leq \Delta + (D-1)(\Delta-1) \\ &\leq (\Delta - 1) + 1 + (D-1)(\Delta - 1) \\ &\leq D(\Delta - 1) + 1 \end{aligned}$$

Hence

$$b(G) \leq D(\Delta - 1) + 1$$

$\square$

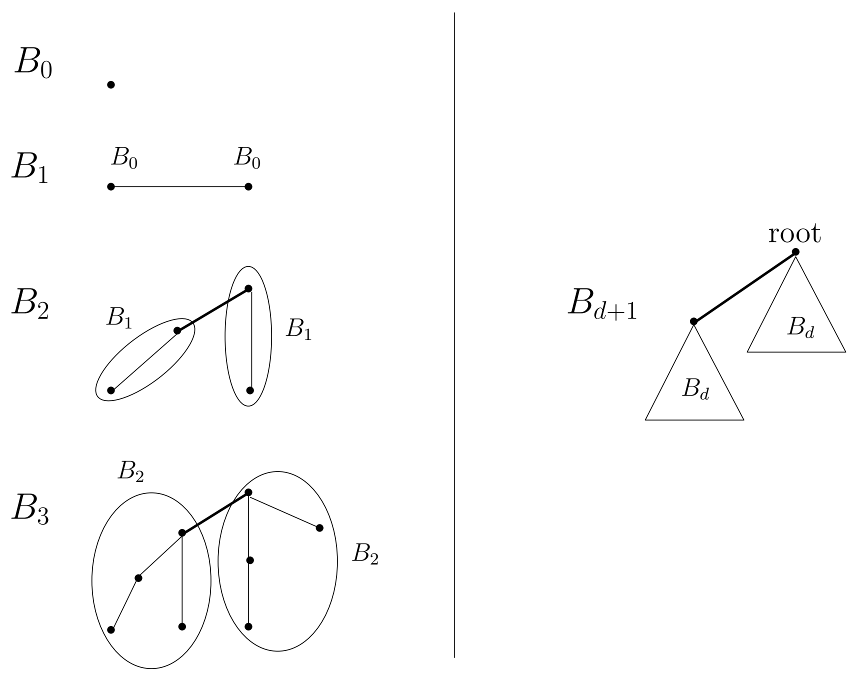
Examples:

- $D(P_n) = b(P_n) = n - 1$
- When n is even, $D(C_n) = b(C_n) = \frac{n}{2}$
  When n is odd, $D(C_n) = \left\lfloor \frac{n}{2} \right\rfloor \leq b(C_n) = \left\lceil \frac{n}{2} \right\rceil$
- $D(H_n) = b(H_n) = n$ (diametral vertices are $00\ldots0$ and $11\ldots1$)

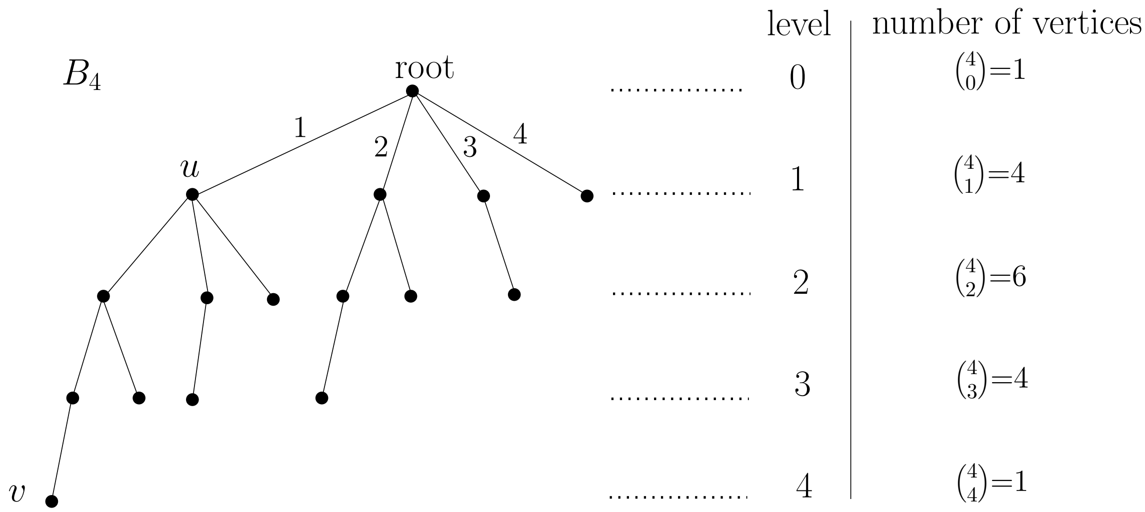
Broadcasting in Specific Trees

Binomial tree of dimension d, $\boldsymbol{B_b}$

As shown in the following figure, binomial tree has a recursive definition



Following figure represents a B_4 and will be used as an example



OBSERVATION

The number of vertices at level i of d-dimensional binomial tree is $\binom{d}{i}$ for all $0 \leq i \leq d$. Hence the total number of vertices of a B_d is $n = 2^d \leftrightarrow d = \lceil \log_2 n \rceil$

It is easy to see that to obtain the minimum broadcast time from originator *root*, one should start by sending the message to vertex u, then to the root of the next biggest subtree etc. It can be proved by induction that

$$b(root, B_d) = d$$

An important think to note is that

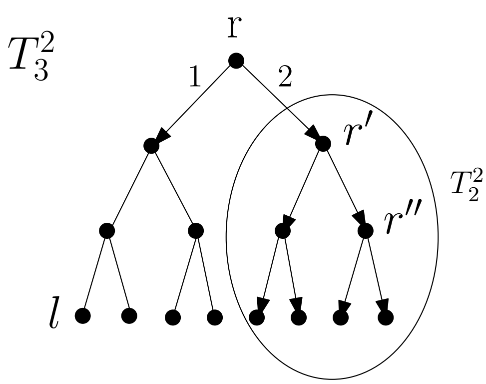
$$b(B_d) = b(v, B_d) \neq b(root, B_d)$$

In fact the worst broadcast time happens when v is the originator. While broadcasting from originator v, all vertices on the path from v to *root* first send the message towards the *root* and only then continue informing other vertices of the subtree containing v. Once the *root* is informed, it will start informing the subtree that does not contain v. This subtree will finish being informed latter than the subtree containing v and will take $b(root, B_{d-1}) = d - 1$ time units. Hence the total broadcast time will be the time for the message to reach the root (d) plus the time that it takes to inform the right subtree from the *root* ($d - 1$). Hence

$$b(B_d) = b(v, B_d) = 2d - 1 = 2\lceil \log_2 n \rceil - 1$$

Complete binary tree of height k, T_k^2

A T_k^2 is a tree of height k where every internal node including the root has exactly 2 children. The leaves have no child. Following figure represents a T_3^2 and will be used as an example



$$\boxed{b(r, T_3^2) = 2 + b(r', T_2^2)}$$

To broadcast from the root r, r first sends the message to the left subtree then to the right subtree. Both subtrees being the same, all vertices will be informed when the right subtree will finish broadcasting. Hence a recursive equation appears

$$\begin{aligned}
b(r, T_k^2) &= 2 + b(r', T_{k-1}^2) \\
&= 2 + 2 + b(r'', T_{k-2}^2) \\
&= 2 + 2 + 2 + \ldots + b(root, T_1^2) \\
&= 2(k-1) + b(root, T_1^2) \\
&= 2k
\end{aligned}$$

Again note that

$$b(T_k^2) = b(l, T_k^2) \neq b(r, T_k^2)$$

To broadcast from l, the message should first be sent to the root r and r can then inform r' which in its turn will inform its subtree. The total broadcast time is the time the message takes to reach r (k) plus the time that r takes to inform r' (1) plus the time that r' takes to inform its subtree ($b(r', T_{k-1}^2)$). Please note that all vertices of the subtree containing l are informed by the time the right subtree rooted at r' finishes being informed. Hence

$$b(T_k^2) = b(l, T_k^2) = k + 1 + b(r', T_{k-1}^2) = k + 1 + 2(k-1) = 3k - 1$$

Note that the diameter of T_k^2 is

$$D(T_k^2) = 2k$$

and that

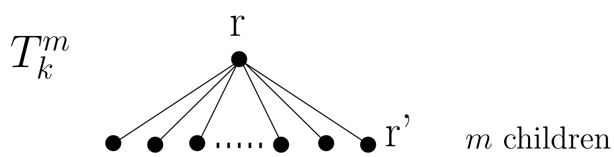
$$b(T_k^2) = b(l, T_k^2) = D(T_k^2) + (k-1)$$

DISCUSSION

k dimensional binomial tree has $n = 2^k$ vertices and $b(B_k) = 2k - 1 = 2\lceil \log n \rceil - 1$
k dimensional binary tree T_k^2 has $n = 2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$ vertices. This number is too big to compare with the vertex number of B_k. So better to compare B_k with T_{k-1}^2 which has in total $n = 2^k - 1$ vertices. And $b(T_{k-1}^2) = 3(k-1) - 1 = 3k - 4 = 3\lceil \log n \rceil - 4$.
B_k and T_{k-1}^2 have nearly the same number of vertices (the difference is just one vertex) and yet B_k has a better broadcast time (the difference is nearly $\lceil \log n \rceil$)
Conclusion: Binomial tree is a better topology than binary tree

Complete m-ary tree of height k, $\boldsymbol{T_k^m}$

A T_k^m is a tree of height k where every internal node including the root has exactly m children. The leaves have no child.



$$b(r, T_k^m) = mk$$

$$\begin{aligned} b(T_k^m) &= k + m - 1 + b(r', T_{k-1}^m) \\ &= k + m - 1 + m(k-1) \\ &= mk + k - 1 \end{aligned}$$

EXERCISE

Try to express $b(T_k^m)$ as a function of total number of vertices n. Note that

$$n = \frac{m^{k+1} - 1}{m - 1}$$

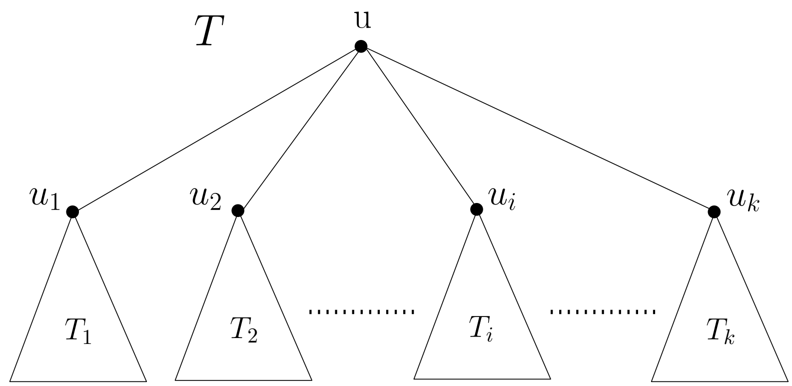
Note that the diameter of m-ary tree is equal to the diameter of binary tree

$$D(T_k^m) = D(T_k^2)$$

Broadcasting in Arbitrary Trees

A) BROADCASTING ALGORITHM IN ARBITRARY TREES

Property *Assume that T is a tree on n vertices rootes at u. Following figure shows T*



Moreover assume that we ordered the subtrees of T in such a way that

$$b(u_1, T_1) \geq b(u_2, T_2) \geq \ldots \geq b(u_i, T_i) \geq \ldots \geq b(u_k, T_k)$$

Then

$$\boldsymbol{b(u, T) = \max_{1 \leq i \leq k} \{b(u_i, T_i) + i\}}$$

Proof

- First we prove that $b(u, T) \leq \max_{1 \leq i \leq k}\{b(u_i, T_i) + i\}$
  Assume that u informs vertex u_i at time i for all $i = 1, \ldots, k$. Then it is clear that every vertex in T_j for any $1 \leq j \leq k$, will be informed at time

  $$b(u_j, T_j) + j \leq \max_{1 \leq i \leq k}\{b(u_i, T_i) + i\}$$

  So

  $$b(u, T) \leq \max_{1 \leq i \leq k}\{b(u_i, T_i) + i\}$$

- In a second time, we prove that $b(u, T) \geq \max_{1 \leq i \leq k}\{b(u_i, T_i) + i\}$
  Assume that

  $$\max_{1 \leq i \leq k}\{b(u_i, T_i) + i\} = b(u_p, T_p) + p$$

  for some p where $1 \leq p \leq k$. Let consider the vertices $u_1, \ldots, u_p$. Those vertices may be informed in different orders depending on the broadcast scheme. But under any broadcast scheme one of the vertices $u_1, \ldots, u_p$ will be informed by u at time p or latter (we know that each vertex will be informed in different time units; there are p vertices and by pigeon hole principal, one of them should be informed at time unit

p or latter). Assume that vertex u_l, where $1 \leq l \leq p$, receives the message from u at time unit p or latter.Then, the vertices of T_l will be informed no sooner than time unit $p + b(u_l, T_l)$. So

$$b(u,T) \geq p + b(u_l, T_l)$$

We also know that u_l is between u_1 and u_p, hence

$$b(u_l, T_l) \geq b(u_p, T_p)$$

So

$$b(u,T) \geq p + b(u_p, T_p) = \max_{1 \leq i \leq k} \{b(u_i, T_i) + i\}$$

$\square$

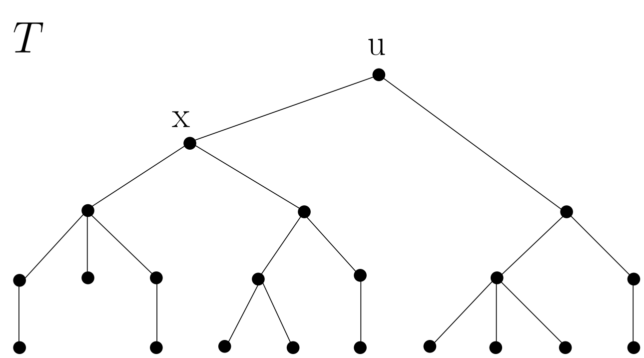
Algorithm Br-Tree

Input: Tree T rooted at u
Output: $b(u,T)$

1. Label every leaf of T with label 0
2. Consider any vertex $v \in T$ such that $v \neq u$
   If all children of v are labelled with $l_1 \geq l_2 \geq \ldots \geq l_p$ then label v with
   $$l_v = \max_{1 \leq i \leq p} \{l_i + i\}$$
3. Continue until all neighbours of u are labelled with $l_1 \geq l_2 \geq \ldots \geq l_k$, then
   $$b(u,T) = l_u = \max_{1 \leq i \leq k} \{l_i + i\}$$

EXERCISE

Run the *Br-Tree algorithm* on the following tree T



There exists an implementation of *Br-Tree* algorithm to find $b(u, T)$ in $\Theta(n)$ time (where n is the total number of vertices). However to find $b(T)$, one should run the previous algorithm for all the originators, which would give a complexity of $\Theta(n^2)$. But a closer analysis shows us that it is possible to improve this complexity. In the previous example of tree T, let assume that we found $b(x, T)$ choosing x as originator and that now we want to find $b(u, T)$ choosing u as the originator. In the left subtree of u, the only label that will be different (from the labels that we obtained choosing x as originator) will be the label of x. Hence one does not need to go through the whole subtree of x to find the label of u and can improve the complexity.

B) BROADCAST PROPERTIES IN ARBITRARY TREES

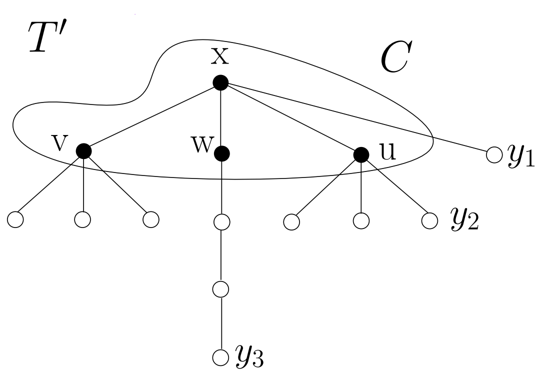
Definition Let T be a graph with the vertex set V. Then $\boldsymbol{b_{min}(T)}$ represents the broadcast time of a vertex in T with minimum possible value

$$b_{min}(T) = \min_{u \in V}\{b(u, T)\}$$

Definition Let T be a graph with the vertex set V. Then $C \subseteq V$ is called the **broadcast center** of T if

$$b(v, T) = b_{min}(T) \quad \text{for all } v \in C$$

EXAMPLE



$b(x, T') = b(v, T') = b(w, T') = b(u, T') = 6$

$b(y, T') > 6$ for any other vertex y

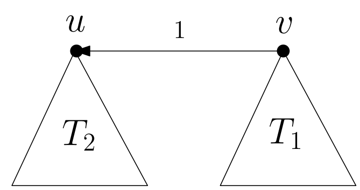
For example $b(y_1) = 1 + 3 + 3 = 7$

For all other originators, it will take at least 2 time units to inform x, so one of the vertices v,w or u will receive the message at time 4 or after.

So $C = \{x, v, w, u\}$

Proposition *$|C| \geq 2$ for any tree T on $n \geq 2$ vertices*

Proof Let $v \in C$ be a vertex of the broadcast center of T. We know that $b(v,T) = b_{min}(T)$. Suppose that during the broadcasting (that gives $b_{min}(T)$) v informs a vertex u during the first time unit. This situation is presented by the following figure.



During the rest of broadcasting v informs vertices of T_1 and u informs vertices of T_2. Now let analyze the broadcasting from originator u. If during the first time unit u sends the message to v then we arrive exactly to the same situation as when we were broadcasting from v. Hence broadcasting from originator u can finish in exactly same time unit as when broadcasting from originator v and $b(u,T) = b(v,T) = b_{min}(T)$. So u is also in the broadcasting center of T and $|C| \geq 2$. □

Proposition *Broadcast center of any tree T is a connected subtree of T*

Proposition *Broadcast center of any tree T has diameter at most 2 (diameter in a tree is the longest path). So the broadcast center of any tree T is a star*

Proposition *Let T be a tree with vertex set V and u be a vertex of T. $dist(u,C)$ denotes the smallest distance of u to any vertex in the broadcast center of T. Then*

$$b(u,T) = dist(u,C) + b_{min}(T)$$

It follows that

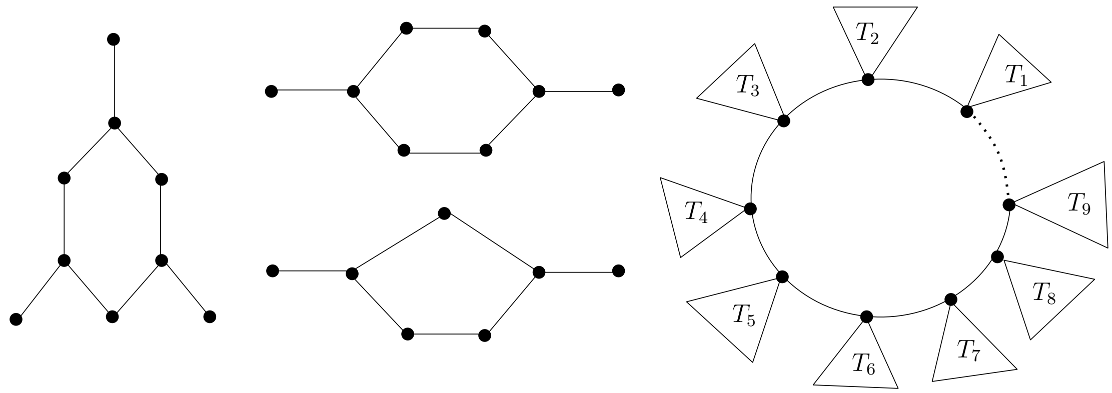
$$b(T) = \max_{u \in V}\{dist(u,C)\} + b_{min}(T)$$

In the previous example of tree T',

$$dist(y_1, C) = 1$$
$$dist(y_2, C) = 1$$
$$dist(y_3, C) = 3$$

EXERCISE

Find the broadcast center of the following graphs



Broadcast Graphs and Minimum Broadcast Graphs

Definition A graph G on n vertices is called a **broadcast graph** (bg) if $\boldsymbol{b(G) = \lceil \log n \rceil}$ (recall that $b(H) \geq \lceil \log n \rceil$ for any graph H on n vertices)

When analyzing the broadcast time of a graph on n vertices, we take $n = 2^{k-1} + x$ where $1 \leq x \leq 2^{k-1}$. This guaranties that $2^{k-1} + 1 \leq n \leq 2^k$ and hence $\lceil \log n \rceil = k$. This trick simplifies the analysis; moreover we can reach all n values by changing the value of k and of x.

DISCUSSION

Complete graph on n vertices K_n is a broadcast graph on n vertices. Following is the broadcast scheme to achieve broadcasting in K_n in $\lceil \log n \rceil$ time units. Let $V(K_n) = 0, 1, \ldots, n-1$ and let vertex 0 be the originator of broadcasting (K_n is vertex transitive hence the minimum broadcast scheme is the same for all originators). We assume that $n = 2^{k-1} + x$ where $1 \leq x \leq 2^{k-1}$. The broadcast scheme is as follows: at time t, every informed vertex i sends the message to vertex $i + 2^{t-1}$

Time 1: $0 \to 2^0 = 1$ (number of informed vertices is $2 = 2^1$)

Time 2: $0 \to 2^1 = 2$ $1 \to 1 + 2^1 = 3$ (number of informed vertices is $4 = 2^2$)

...

Time $k-1$: all vertices $0, 1, \ldots, 2^{k-1}$ are informed (number of informed vertices is 2^{k-1})

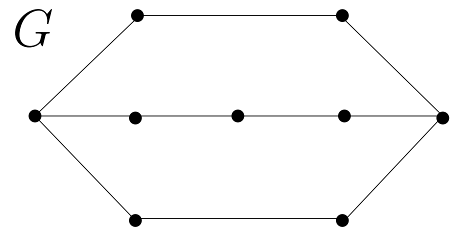
Time k: all the vertices of K_n are informed

Note that, if all informed vertices inform a new vertex at time k, then the number of informed vertices at time k becomes 2^k which is bigger than n. Hence at time k some of the already informed vertices do not inform a new vertex. They are idle vertices at time k.

K_n is not a good topology and has too many edges so it is costly ($|E| = \frac{n(n-1)}{2}$). There are other graphs on the same number of vertices with less edges in which broadcasting from any originator takes $\lceil \log n \rceil$ time units.

EXAMPLE

Graph G is a broadcast graph on 9 vertices and 10 edges. The complete graph on 9 vertices K_9 has $\frac{9\times 8}{2} = 36$ edges



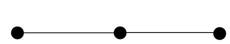
Definition A **minimum broadcast graph** (mbg) is a broadcast graph that has the minimum possible number of edges. This number is denoted by $\boldsymbol{B(n)}$.

B(N) FOR SOME SPECIFIC VALUE OF N

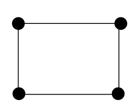
Let n be the total number of vertices

If $n = 2$, we want to determine the minimum possible number of edges such that the broadcasting from any originator can finish in $\lceil \log 2 \rceil = 1$ time unit. The following graph achieves that and with 0 edge it is not possible to do broadcasting. Hence $B(2) = 1$

If $n = 3$, we want to determine the minimum possible number of edges such that the broadcasting from any originator can finish in $\lceil \log 3 \rceil = 2$ time units. The following graph achieves that and with 1 edge it is not possible to even connect all vertices. Hence $B(3) = 2$



If $n = 4$, we want to determine the minimum possible number of edges such that the broadcasting from any originator can finish in $\lceil \log 4 \rceil = 2$ time units. The following graph achieves that and $B(4) = 4$



Some other $B(n)$ values are:

$B(5) = 5$

$B(6) = 6$

$B(7) = 8$

$B(8) = 12$

QUESTIONS

1. Figure out why $B(4) \neq 3$
2. Enumerate possible graphs on 6 vertices and show that $B(6) = 6$. Did you find a technique to enumerate all possible graphs on 6 vertices ? Which arguments can you use to rule out some type of graphs ?
3. Argue why $B(7) > 7$

Property

$$B(2^k) = k2^{k-1}$$
$$B(2^k - 2) = (k-1)(2^{k-1} - 1)$$

Annexe

Graph	$\|V\|$	$\|E\|$	Δ	D	$b(G)$
Path P_n n:vertex #	n	$n-1$	2	$n-1$	$n-1$
Star S_n	n	$n-1$	$n-1$	2	$n-1$
Cycle C_n n:vertex #	n	n	2	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$
Complete K_n	n	$\frac{n(n-1)}{2} = \binom{n}{2}$	$n-1$	1	$\lceil logn \rceil$
Complete bipartite $K_{m,n}$	$m+n$	$\frac{mn}{2}$	$max\{m,n\}$	1	$m=n : \lceil logn \rceil + 1$ $m>n : \lceil logn \rceil + 1 + \lceil \frac{m-2^{\lceil logn \rceil}}{n} \rceil$ $m<n : \lceil logm \rceil + 1 + \lceil \frac{n-2^{\lceil logm \rceil}}{m} \rceil$
Wheel W_n with n vertex # on circle	$n+1$	$2n$	n	2	$bg \geq \left\lceil \sqrt{n-\frac{3}{4}} + \frac{1}{2} \right\rceil$
Spider $S_{k,p}$ k paths of length p	$kp+1$	kp	k	$2p$	$2p+k-2$
Hypercube H_n with n dim	2^n	$n2^{n-1}$	n	n	n
CCC_m with m dim	$m2^m$	$3m2^{m-1}$	3	$2m + \lfloor \frac{m}{2} \rfloor - 2$	$\lceil \frac{5m}{2} \rceil - 1$
Grid $G_{m,n}$	mn	$2mn-n-m$	4	$m+n-2$	$m+n-2$
Binomial B_d d dim=height	2^d with $d = \lceil logn \rceil$	$2^d - 1$	d	$2d-1$	$br(r) = d$ $br(B_d) = 2d-1$
Binary T_k^2 with k=height	$2^{k+1}-1$	$2^{k+1}-2$	3	$2k$	$br(r) = 2k$ $br(T_k^2) = 3k-1$
m-ary T_k^m with k=height m=# of child	$\frac{m^{k+1}-1}{m-1}$	$\frac{m^{k+1}-1}{m-1} - 1$	$m+1$	$2k$	$br(r) = mk$ $br(T_k^m) = mk+k-1$

Graph	M_1	M_2	M_3	Euler	Hamil	$\boldsymbol{BC}$
Path P_n n:vertex #	$n-1$	$n-1$	$2(n-1)$	No	No	n even: $\frac{n}{2}$, $\frac{n}{2}+1$ n odd: $\lceil\frac{n}{2}\rceil+1$, $\lceil\frac{n}{2}\rceil-1$, $\lceil\frac{n}{2}\rceil$
Star S_n	n	n	$n+1$	No	No	All
Cycle C_n n:vertex #	$\lceil\frac{n}{2}\rceil$	$\lceil\frac{n}{2}\rceil$	$\lceil n\rceil$	Yes	Yes	All
Complete K_n	$n-1$	$n-1$	$n-1$	n odd : Yes n even : No	Yes	All
Complete bipartite $K_{m,n}$	$max\{m,n\}$	$max\{m,n\}$	$max\{m,n\}$+1	m and n even:Yes m or n odd:No	$m=n$: Yes $m\neq n$: No	All
Wheel W_n with n vertex # on circle	$\lceil\frac{n}{2}\rceil+1$	$t_2\geq\lceil\frac{2}{3}(n+1)\rceil$	$n+1$	No	Yes	All
Spider $S_{k,p}$ k paths of length p	$2p+k-2$	$2p+k-2$	$2p+k-3$	No	No	Middle star
Hypercube H_n with n dim	$\frac{3}{2}\leq t_1 \leq\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}$	n even : Yes n odd : No	Yes (proof by induction on n)	All
CCC_m with m dim	$2\left\lfloor\frac{5m}{2}\right\rfloor-1$	$2\left\lfloor\frac{5m}{2}\right\rfloor-1$	$3\left\lfloor\frac{5m}{2}\right\rfloor-3$	No	Yes	All
Grid $G_{m,n}$	$m+n-2$	$m+n-2$	$3m+3n-3$	No	m&n odd:No	?
Binomial B_d d dim=height	$n+(n+1)^2$	$n+(n+1)^2$	n^2+n-1	No	No	Root + Root of B_{d-1}
Binary T_k^2 with k=height	$4k-2$	$4k-2$	$6k-3$	No	No	Root and two children
m-ary T_k^m with k=height m=# of child	$2km-k$	$2km-k$	$(k+1)(2m-1)$	No	No	Root and all of its children

height of root in trees is taken as 0

$\boldsymbol{M_1}$: At any time unit, every vertex knows the state of its neighbours

$\boldsymbol{M_2}$: At any time unit, every informed vertex knows which vertex transmitted the information to it and the vertices to whom it transmitted the message

$\boldsymbol{M_3}$: At any time unit, every informed vertex knows only the vertices to which it has transmitted the message

⇒ For any tree: $t_1(T)=t_2(T)$

⇒ For binomial tree: $t_3(B_n)=t_1(B_n)+D(B_n)-1$

⇒ General case: $t_3(T)=t_1(T)+dist(u,v)-1$ where $t_1(T)=t_1(u,v)$ and u and v are diametral vertices

graph	M_3 upper bound	M_1 and M_2 upper bound	broadcast lower bound	broadcast upper bound
K_n	$n-1$	$n-1$	$\lceil \log n \rceil$	$\lceil \log n \rceil$
P_n	$2n-3$	$n-1$	$n-1$	$n-1$
C_n	$n-1$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$
Q_d	$\frac{d(d+1)}{2}$	$\frac{d(d-1)}{2}+1$	d	d
$T_{d,h}$	$(d+1)(2h-3)$	$d(2h-3)$	$(d+1)(h-1)-1$	$(d+1)(h-1)-1$
CCC_d	$3\lfloor \frac{5d}{2} \rfloor-3$	$2\lfloor \frac{5d}{2} \rfloor-1$	$\lceil \frac{5d}{2} \rceil-1$	$\lceil \frac{5d}{2} \rceil-1$
SE_d	$6d-3$	$4d-1$	$2d-1$	$2d-1$
BF_d	$4\lfloor \frac{3d}{2} \rfloor$	$3\lfloor \frac{3d}{2} \rfloor+1$	$1.7417d$	$2d-1$
DB_d	$4d$	$3d+1$	$1.3131d$	$\frac{3}{2}(d+1)$

In addition, we have given the upper bounds obtained from Corollary 2.1 for the cube-connected cycles network of dimension d (CCC_d), the shuffle-exchange network of dimension d (SE_d), the butterfly network of dimension d (BF_d), and the DeBruijn network of dimension d (DB_d). The bounds for broadcasting time in